AIMS Mathematics, 8(2): 4806–4813.
DOI:10.3934/math.2023238
Received: 19 October 2022
Revised: 29 November 2022
Accepted: 02 December 2022
Published: 08 December 2022

Mathematics

http://www.aimspress.com/journal/Math

Research article

New proofs for three identities of seventh order mock theta functions

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Abstract: Using the three-term Weierstrass relation for theta functions and the properties of Hecke-type double sums and Appell-Lerch sums, we give new and simple proofs for the seventh order mock theta conjectures.

Keywords: mock theta functions; Hecke-type double sums; Appell-Lerch sums; the three-term Weierstrass relation
Mathematics Subject Classification: 11F27, 33D15

1. Introduction

The mock theta functions were named and studied by the Indian mathematician Ramanujan. Four months before he died, he summarized his results in a letter to G.H. Hardy [7]. He provided four separate classes of mock theta functions: one class of third order, two of fifth order, and one of seventh order together with identities satisfied by them. Watson laid the foundations of this subject in the twentieth century. He showed many elegant and significant theorems. In particular, he made important contributions to the third order mock theta functions, see [9, 10]. Since then, many great mathematicians made remarkable achievement of mock theta functions, such as Andrews, Hickerson, Ono, Duke, and so on. We first introduce the basic definitions and notation. Then we provide the background of our research.

Throughout this paper, let q denote a complex number with $|q| < 1$. Here and in what follows, we adopt the standard q-series notation [3]. For any positive integer n,

$$(a;q)_0 := 1, \qquad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \qquad (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k).$$

For convenience, we use $(a)_n$ to denote $(a;q)_n$.

The Jacobi triple product identity is stated as follows.

$$j(x;q) := (x)_\infty (q/x)_\infty (q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} x^n. \tag{1.1}$$

From the definition of $j(x;q)$, we have

$$j(x;q) = j(q/x;q) \tag{1.2}$$

and

$$j(q^n x;q) = (-1)^n q^{-\binom{n}{2}} x^{-n} j(x;q), \quad n \in \mathbb{Z}. \tag{1.3}$$

Let m and a be integers with m positive. Define

$$\begin{aligned}
&J_{a,m} := j(q^a;q^m), \quad \overline{J}_{a,m} := j(-q^a;q^m),\\
&J_m := \prod_{i\geq 1}(1-q^{mi}), \quad \overline{J}_m := \prod_{i\geq 1}(1+q^{mi}), \text{ and}\\
&j(b_1,b_2,\ldots,b_m;q) := j(b_1;q)j(b_2;q)\ldots j(b_m;q).
\end{aligned}$$

Ramanujan's seventh order mock theta functions [7, p.355] are defined by

$$\begin{aligned}
\mathcal{F}_0(q) &:= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n+1};q)_n},\\
\mathcal{F}_1(q) &:= \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q^{n};q)_n},\\
\mathcal{F}_2(q) &:= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^{n+1};q)_{n+1}}.
\end{aligned}$$

In 1988, by using the constant term method, Hickerson [5] proved the seventh order mock theta conjectures which can be written as follows.

$$\mathcal{F}_0(q) = 2 + 2qg(q,q^7) - \frac{J_{3,7}^2}{J_1}, \tag{1.4}$$

$$\mathcal{F}_1(q) = 2q^2 g(q^2,q^7) + \frac{qJ_{1,7}^2}{J_1}, \tag{1.5}$$

$$\mathcal{F}_2(q) = 2q^2 g(q^3,q^7) + \frac{J_{2,7}^2}{J_1}, \tag{1.6}$$

where $g(z,q)$ is a universal mock theta function introduced by Hickerson [4] as

$$g(z;q) := z^{-1}\left(-1 + \sum_{n=0}^{\infty} \frac{q^{n^2}}{(z)_{n+1}(qz^{-1})_n}\right) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(z)_{n+1}(qz^{-1})_{n+1}}.$$

Furthermore, from [6, Proposition 4.2], we have

$$g(z;q) = -z^{-1}m(q^2z^{-3},q^3,z^2) - z^{-2}m(qz^{-3},q^3,z^2), \tag{1.7}$$

where the Appell-Lerch sum $m(x,q,z)$ is defined by Hickerson and Mortenson [6] in the following.

Definition 1.1. *Let generic $x, z \in \mathbb{C}^* := \mathbb{C}\backslash\{0\}$ with neither z nor xz an integer power of q. Then*

$$m(x,q,z) := \frac{-z}{j(z;q)} \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{\binom{r+1}{2}} z^r}{1-q^r xz}.$$

Following [6], the term "generic" means that the parameters do not cause poles in the Appell-Lerch sums or in the quotients of theta functions. In this paper, employing the three-term Weierstrass relation for theta functions and the relationships between Appell-Lerch sums and Hecke -type double sums, we provide new and simple proofs for the above three identities (1.4)–(1.6).

2. Main results

First, we recall some notation and definitions. The Hecke-type double sums are defined as follows.

Definition 2.1. *Let $x, y \in \mathbb{C}^*$ and define $sg(r) := 1$ for $r \geq 0$ and $sg(r) := -1$ for $r < 0$. Then*

$$f_{a,b,c}(x,y,q) := \sum_{sg(r)=sg(s)} sg(r)(-1)^{r+s} x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}.$$

The three-term Weierstrass relation for theta functions is stated in the following.

Proposition 2.1. *[1, Theorem 1.20] For generic $a, b, c, d \in \mathbb{C}^*$,*

$$j(ac, a/c, bd, b/d; q) = j(ad, a/d, bc, b/c; q) + b/c \cdot j(ab, a/b, cd, c/d; q).$$

Next, we review some propositions for Appell-Lerch sums and Hecke-type double sums.

Proposition 2.2. *[6, Proposition 3.1] For generic $x, z, z_0, z_1 \in \mathbb{C}^*$,*

$$m(x,q,z) = m(x,q,qz), \tag{2.1}$$
$$m(x,q,z) = x^{-1}m(x^{-1},q,z^{-1}), \tag{2.2}$$
$$m(qx,q,z) = 1 - xm(x,q,z), \tag{2.3}$$
$$m(x,q,z) = m(x,q,x^{-1}z^{-1}), \text{ and} \tag{2.4}$$
$$m(x,q,z_1) = m(x,q,z_0) + \frac{z_0 J_1^3 j(z_1/z_0;q) j(xz_0z_1;q)}{j(z_0;q)j(z_1;q)j(xz_0;q)j(xz_1;q)}. \tag{2.5}$$

Lemma 2.1. *[6, Theorem 1.3] Let n and p be positive integers with $(n,p) = 1$. For generic $x, y \in \mathbb{C}^*$,*

$$f_{n,n+p,n}(x,y,q) = g_{n,n+p,n}(x,y,q,-1,-1) + \frac{1}{\overline{J}_{0,np(2n+p)}}\theta_{n,p}(x,y,q),$$

where

$$g_{a,b,c}(x,y,q,z_1,z_0) :=$$
$$\sum_{t=0}^{a-1}(-y)^t q^{c\binom{t}{2}} j(q^{bt}x;q^a) m(-q^{a\binom{b+1}{2}-c\binom{a+1}{2}-t(b^2-ac)}(-x)^{-b}(-y)^a, q^{a(b^2-ac)}, z_0)$$

$$+\sum_{t=0}^{c-1}(-x)^t q^{a\binom{t}{2}} j(q^{bt}x;q^c) m(-q^{c\binom{b+1}{2}-a\binom{c+1}{2}-t(b^2-ac)}(-x)^c(-y)^b, q^{c(b^2-ac)}, z_1),$$

and

$$\theta_{n,p}(x,y,q) := \sum_{r^*=0}^{p-1}\sum_{s^*=0}^{p-1} q^{n\binom{r-(n-1)/2}{2}+(n+p)(r-(n-1)/2)(s-(n+1)/2)n\binom{s+(n+1)/2}{2}}(-x)^{r-(n-1)/2}$$
$$\times \frac{(-y)^{s+(n+1)/2} J^3_{p^2(2n+p)} j(-q^{np(s-r)}x^n/y^n; q^{np^2}) j(q^{p(2n+p)(s+r)+p(n+p)}x^p y^p; q^{p^2(2n+p)})}{j(q^{p(2n+p)r+p(n+p)/2}(-y)^{n+p}/(-x)^n, q^{p(2n+p)s+p(n+p)/2}(-x)^{n+p}/(-y)^n; q^{p^2(2n+p)})}.$$

Here $r := r^* + \{(n-1)/2\}$ *and* $s := s^* + \{(n-1)/2\}$, *with* $0 \le \{\alpha\} < 1$ *denoting the fractional part of* α.

Taking the $n = 3$, $p = 1$ specialization of the above lemma, we derive the following result.

Proposition 2.3. *For generic* $x, y \in \mathbb{C}^*$,

$$\begin{aligned} f_{3,4,3}(x,y,q) =& j(x;q^3)m(q^{12}x^{-4}y^3,q^{21},-1) - y j(q^4x;q^3)m(q^5x^{-4}y^3,q^{21},-1)\\ &+ q^3y^2 j(q^8x;q^3)m(q^{-2}x^{-4}y^3,q^{21},-1) + j(y;q^3)m(q^{12}x^3y^{-4},q^{21},-1)\\ &- x j(q^4y;q^3)m(q^5x^3y^{-4},q^{21},-1) + q^3x^2 j(q^8y;q^3)m(q^{-2}x^3y^{-4},q^{21},-1)\\ &- \frac{y^2J_7^3 j(-x^3y^{-3};q^3) j(q^4xy;q^7)}{q^2 x j(-q^2x^{-3}y^4, -q^2x^4y^{-3};q^7)\overline{J}_{0,21}}. \end{aligned} \tag{2.6}$$

Then we have the following lemma.

Lemma 2.2. *We have*

$$f_{3,4,3}(q^2,q^2,q) = 2J_{1,3}m(q^{10},q^{21},-1) - 2q^{-1}J_{1,3}m(q^4,q^{21},-1) + \frac{q^{-1}J_7^3\overline{J}_{0,3}J_{1,7}}{\overline{J}^2_{4,7}\overline{J}_{0,21}}, \tag{2.7}$$

$$f_{3,4,3}(q^3,q^3,q) = -2q^{-2}J_{1,3}m(q^2,q^{21},-1) - 2q^{-1}J_{1,3}m(q^5,q^{21},-1) + \frac{q^{-2}J_7^3\overline{J}_{0,3}J_{3,7}}{\overline{J}^2_{2,7}\overline{J}_{0,21}}, \tag{2.8}$$

$$f_{3,4,3}(q^4,q^4,q) = -2q^{-1}J_{1,3}m(q^8,q^{21},-1) - 2q^{-3}J_{1,3}m(q,q^{21},-1) + \frac{q^{-3}J_7^3\overline{J}_{0,3}J_{2,7}}{\overline{J}^2_{1,7}\overline{J}_{0,21}}. \tag{2.9}$$

Proof. Based on the definition of $j(x;q)$, for an integer m, we arrive at

$$j(q^m;q) = j(q^{-m+1};q) = 0. \tag{2.10}$$

From (2.6) and (2.10), we have

$$f_{3,4,3}(q^2,q^2,q) = 2J_{2,3}m(q^{10},q^{21},-1) + 2q^7J_{10,3}m(q^{-4},q^{21},-1) - \frac{J_7^3\overline{J}_{0,3}J_{8,7}}{\overline{J}^2_{4,7}\overline{J}_{0,21}}.$$

Then applying (1.2), (1.3) and (2.2), we derive (2.7). The proofs of (2.8) and (2.9) are similar. □

In the following, we begin to prove (1.4)–(1.6).

From [5, Theorem 2.0], we obtain the Hecke-type double sums for $\mathcal{F}_0(q)$, $\mathcal{F}_1(q)$, and $\mathcal{F}_2(q)$.

$$J_1\mathcal{F}_0(q) = f_{3,4,3}(q^2,q^2,q), \tag{2.11}$$
$$J_1\mathcal{F}_1(q) = qf_{3,4,3}(q^4,q^4,q), \tag{2.12}$$
$$J_1\mathcal{F}_2(q) = f_{3,4,3}(q^3,q^3,q). \tag{2.13}$$

Therefore, combining Lemma 2.2, (2.11)–(2.13) and the fact that J_1 and $J_{1,3}$ are the same from the definitions of them, we have

$$\begin{aligned}
\mathcal{F}_0(q) =&2m(q^{10},q^{21},-1) - 2q^{-1}m(q^4,q^{21},-1) + \frac{q^{-1}J_7^3\overline{J}_{0,3}J_{1,7}}{J_1\overline{J}_{4,7}^2\overline{J}_{0,21}}\\
=&2m(q^{10},q^{21},q^{-2}) - 2q^{-1}m(q^4,q^{21},q^2)\\
&- \frac{2J_{21}^3 j(-q^2,-q^8;q^{21})}{j(-1,q^2,q^8,-q^{10};q^{21})} - \frac{2q^{-1}J_{21}^3 j(-q^2,-q^6;q^{21})}{j(-1,q^2,-q^4,q^6;q^{21})} + \frac{q^{-1}J_7^3\overline{J}_{0,3}J_{1,7}}{J_1\overline{J}_{4,7}^2\overline{J}_{0,21}}\\
=&2m(q^{10},q^{21},q^{-2}) - 2q^{-1}m(q^4,q^{21},q^2)\\
&- \frac{2J_{21}^3 j(-q^2;q^{21})}{j(-1,q^2;q^{21})}\left(\frac{j(-q^8;q^{21})}{j(q^8,-q^{10};q^{21})} + \frac{q^{-1}j(-q^6;q^{21})}{j(-q^4,q^6;q^{21})}\right) + \frac{q^{-1}J_7^3\overline{J}_{0,3}J_{1,7}}{J_1\overline{J}_{4,7}^2\overline{J}_{0,21}}\\
=&2m(q^{10},q^{21},q^{-2}) - 2q^{-1}m(q^4,q^{21},q^2)\\
&- \frac{2J_{21}^3 j(-q^2;q^{21})}{j(-1,q^2;q^{21})}\left(\frac{j(q^{-2},-q^5,q^7,-q^9;q^{21})}{j(q^{-1},-q^4,q^6,q^8,-q^{10};q^{21})}\right) + \frac{q^{-1}J_7^3\overline{J}_{0,3}J_{1,7}}{J_1\overline{J}_{4,7}^2\overline{J}_{0,21}}\\
=&2m(q^{10},q^{21},q^{-2}) - 2q^{-1}m(q^4,q^{21},q^2)\\
&- \frac{2q^{-1}J_{21}^3 j(-q^2,-q^5,q^7,-q^9;q^{21})}{j(-1,q,-q^4,q^6,q^8,-q^{10};q^{21})} + \frac{q^{-1}J_7^3\overline{J}_{0,3}J_{1,7}}{J_1\overline{J}_{4,7}^2\overline{J}_{0,21}}\\
=&2m(q^{10},q^{21},q^{-2}) - 2q^{-1}m(q^4,q^{21},q^2)\\
&- \frac{2q^{-1}J_7^2 j(-q^2;q^7)j(-q^3;q^{21})}{j(q,-q^3;q^7)j(-1;q^{21})} + \frac{q^{-1}J_7^3\overline{J}_{0,3}J_{1,7}}{J_1\overline{J}_{4,7}^2\overline{J}_{0,21}}\\
=&2q^{-10}m(q^{-10},q^{21},q^2) - 2q^{-1}m(q^4,q^{21},q^2)\\
&- \frac{2q^{-1}J_7^2 j(-q^2;q^7)j(-q^3;q^{21})}{j(q,-q^3;q^7)j(-1;q^{21})} + \frac{q^{-1}J_7^3\overline{J}_{0,3}J_{1,7}}{J_1\overline{J}_{4,7}^2\overline{J}_{0,21}}\\
=&2(1 - m(q^{11},q^{21},q^{-2})) - 2q^{-1}m(q^4,q^{21},q^2)\\
&- \frac{2q^{-1}J_7^2 j(-q^2;q^7)j(-q^3;q^{21})}{j(q,-q^3;q^7)j(-1;q^{21})} + \frac{q^{-1}J_7^3\overline{J}_{0,3}J_{1,7}}{J_1\overline{J}_{4,7}^2\overline{J}_{0,21}}\\
=&2 + 2qg(q,q^7) - \frac{2q^{-1}J_7^2 j(-q^2;q^7)j(-q^3;q^{21})}{j(q,-q^3;q^7)j(-1;q^{21})} + \frac{q^{-1}J_7^3\overline{J}_{0,3}J_{1,7}}{J_1\overline{J}_{4,7}^2\overline{J}_{0,21}},
\end{aligned} \tag{2.14}$$

where in the fourth equality, we replace q, a, b, c, and d by q^{21}, $-q^6$, $-q^7$, $-q^8$, and q^2, respectively, in Proposition 2.1.

For $\mathcal{F}_1(q)$, we have

$$\begin{aligned}
\mathcal{F}_1(q) =& -2m(q^8,q^{21},-1)-2q^{-2}m(q,q^{21},-1)+\frac{q^{-2}J_7^3\overline{J}_{0,3}J_{2,7}}{J_1\overline{J}_{1,7}^2\overline{J}_{0,21}}\\
=& -2m(q^8,q^{21},q^4)-2q^{-2}m(q,q^{21},q^4)\\
&-\frac{2J_{21}^3 j(-q^4,-q^{12};q^{21})}{j(-1,q^4,-q^8,q^{12};q^{21})}-\frac{2q^{-2}J_{21}^3 j(-q^4,-q^5;q^{21})}{j(-1,-q,q^4,q^5;q^{21})}+\frac{q^{-2}J_7^3\overline{J}_{0,3}J_{2,7}}{J_1\overline{J}_{1,7}^2\overline{J}_{0,21}}\\
=& -2m(q^8,q^{21},q^4)-2q^{-2}m(q,q^{21},q^4)\\
&-\frac{2J_{21}^3 j(-q^4;q^{21})}{j(-1,q^4;q^{21})}\left(\frac{j(-q^9;q^{21})}{j(-q^{12},q^{13};q^{21})}+\frac{q^{-2}j(-q^5;q^{21})}{j(-q,q^{16};q^{21})}\right)+\frac{q^{-2}J_7^3\overline{J}_{0,3}J_{2,7}}{J_1\overline{J}_{1,7}^2\overline{J}_{0,21}}\\
=& -2m(q^8,q^{21},q^4)-2q^{-2}m(q,q^{21},q^4)\\
&-\frac{2J_{21}^3 j(-q^4;q^{21})}{j(-1,q^4;q^{21})}\left(\frac{j(-q^3,q^{-4},-q^{11},q^{14};q^{21})}{j(q^{-2},-q,q^{16},q^{12},-q^{13};q^{21})}\right)+\frac{q^{-2}J_7^3\overline{J}_{0,3}J_{2,7}}{J_1\overline{J}_{1,7}^2\overline{J}_{0,21}}\\
=& -2m(q^8,q^{21},q^4)-2q^{-2}m(q,q^{21},q^4)\\
&-\frac{2q^{-2}J_{21}^3 j(-q^3,-q^4,q^7,-q^{11};q^{21})}{j(-1,-q,q^2,q^5,-q^8,q^9;q^{21})}+\frac{q^{-2}J_7^3\overline{J}_{0,3}J_{2,7}}{J_1\overline{J}_{1,7}^2\overline{J}_{0,21}}\\
=& -2m(q^8,q^{21},q^4)-2q^{-2}m(q,q^{21},q^4)\\
&-\frac{2q^{-2}J_7^2 j(-q^3;q^7)j(-q^6;q^{21})}{j(-q,q^2;q^7)j(-1;q^{21})}+\frac{q^{-2}J_7^3\overline{J}_{0,3}J_{2,7}}{J_1\overline{J}_{1,7}^2\overline{J}_{0,21}}\\
=&2q^2g(q^2,q^7)-\frac{2q^{-2}J_7^2 j(-q^3;q^7)j(-q^6;q^{21})}{j(-q,q^2;q^7)j(-1;q^{21})}+\frac{q^{-2}J_7^3\overline{J}_{0,3}J_{2,7}}{J_1\overline{J}_{1,7}^2\overline{J}_{0,21}},
\end{aligned}\tag{2.15}$$

where in the fourth equality, we replace q, a, b, c, and d by q^{21}, $-q^5$, $-q^7$, $-q^9$, and q^4, respectively, in Proposition 2.1.

Moreover, we have

$$\begin{aligned}
\mathcal{F}_2(q) =& -2q^{-1}m(q^5,q^{21},-1)-2q^{-2}m(q^2,q^{21},-1)+\frac{q^{-2}J_7^3\overline{J}_{0,3}J_{3,7}}{J_1\overline{J}_{2,7}^2\overline{J}_{0,21}}\\
=& -2q^{-1}m(q^5,q^{21},q^6)-2q^{-4}m(q^{-2},q^{21},q^6)\\
&-\frac{2q^{-1}J_{21}^3 j(-q^6,-q^{11};q^{21})}{j(-1,-q^5,q^6,q^{11};q^{21})}-\frac{2q^{-4}J_{21}^3 j(-q^4,-q^6;q^{21})}{j(-1,-q^{-2},q^4,q^6;q^{21})}+\frac{q^{-2}J_7^3\overline{J}_{0,3}J_{3,7}}{J_1\overline{J}_{2,7}^2\overline{J}_{0,21}}\\
=& -2q^{-1}m(q^5,q^{21},q^6)-2q^{-4}m(q^{-2},q^{21},q^6)\\
&-\frac{2q^{-1}J_{21}^3 j(-q^6;q^{21})}{j(-1,q^6;q^{21})}\left(\frac{j(-q^{11};q^{21})}{j(-q^5,q^{11};q^{21})}+\frac{q^{-1}j(-q^4;q^{21})}{j(-q^2,q^4;q^{21})}\right)+\frac{q^{-2}J_7^3\overline{J}_{0,3}J_{3,7}}{J_1\overline{J}_{2,7}^2\overline{J}_{0,21}}
\end{aligned}$$

$$
\begin{aligned}
&= -2q^{-1}m(q^5, q^{21}, q^6) - 2q^{-4}m(q^{-2}, q^{21}, q^6)\\
&\quad - \frac{2q^{-1}J_{21}^3 j(-q^6;q^{21})}{j(-1,q^6;q^{21})}\left(\frac{q^{-1}j(-q,q^6,q^7,-q^8;q^{21})}{j(-q^2,q^3,q^4,-q^5,q^{11};q^{21})}\right) + \frac{q^{-2}J_7^3\overline{J}_{0,3}J_{3,7}}{J_1\overline{J}_{2,7}^2\overline{J}_{0,21}}\\
&= -2q^{-1}m(q^5, q^{21}, q^6) - 2q^{-4}m(q^{-2}, q^{21}, q^6)\\
&\quad - \frac{2q^{-2}J_{21}^3 j(-q,-q^6,-q^8,q^7;q^{21})}{j(-1,-q^2,q^3,q^4,-q^5,q^{11};q^{21})} + \frac{q^{-2}J_7^3\overline{J}_{0,3}J_{3,7}}{J_1\overline{J}_{2,7}^2\overline{J}_{0,21}}\\
&= -2q^{-1}m(q^5, q^{21}, q^6) - 2q^{-4}m(q^{-2}, q^{21}, q^6)\\
&\quad - \frac{2q^{-2}J_7^2 j(-q;q^7)j(-q^9;q^{21})}{j(-q^2,q^3;q^7)j(-1;q^{21})} + \frac{q^{-2}J_7^3\overline{J}_{0,3}J_{3,7}}{J_1\overline{J}_{2,7}^2\overline{J}_{0,21}}\\
&=2q^2g(q^3,q^7) - \frac{2q^{-2}J_7^2 j(-q;q^7)j(-q^9;q^{21})}{j(-q^2,q^3;q^7)j(-1;q^{21})} + \frac{q^{-2}J_7^3\overline{J}_{0,3}J_{3,7}}{J_1\overline{J}_{2,7}^2\overline{J}_{0,21}},
\end{aligned} \tag{2.16}
$$

where in the fourth equality, we replace q, a, b, c, and d by q^{21}, $-q^{15}$, q^{10}, q^7, and q^4, respectively, in Proposition 2.1.

Furthermore, applying the standard computational techniques from the theory of modular forms, we can prove the following identities. Those unfamiliar with this method might consult the work of Garvan and Liang [2] and Robins [8].

$$
\frac{\eta_{21,0}(\tau)\eta_{14,4}(\tau)\eta_{42,6}(\tau)}{\eta_{14,6}(\tau)\eta_{21,3}(\tau)\eta_{42,0}(\tau)} - \frac{\eta_{6,0}(\tau)\eta_{7,1}(\tau)\eta_{21,0}^{\frac{1}{2}}(\tau)}{\eta_{3,0}^{\frac{1}{2}}(\tau)\eta_{14,6}^2(\tau)\eta_{42,0}(\tau)} = 1,
$$

$$
\frac{\eta_{6,0}(\tau)\eta_{7,2}(\tau)\eta_{21,0}^{\frac{1}{2}}(\tau)}{\eta_{3,0}^{\frac{1}{2}}(\tau)\eta_{14,2}^2(\tau)\eta_{42,0}(\tau)} - \frac{\eta_{14,6}(\tau)\eta_{21,0}(\tau)\eta_{42,12}(\tau)}{\eta_{14,2}(\tau)\eta_{21,6}(\tau)\eta_{42,0}(\tau)} = 1,
$$

$$
\frac{\eta_{6,0}(\tau)\eta_{7,3}(\tau)\eta_{21,0}^{\frac{1}{2}}(\tau)}{\eta_{3,0}^{\frac{1}{2}}(\tau)\eta_{14,4}^2(\tau)\eta_{42,0}(\tau)} - \frac{\eta_{14,2}(\tau)\eta_{21,0}(\tau)\eta_{42,18}(\tau)}{\eta_{14,4}(\tau)\eta_{21,9}(\tau)\eta_{42,0}(\tau)} = 1.
$$

Here, $\eta_{\delta,g}(\tau)$ is the generalized Dedekind η-function defined by

$$
\eta_{\delta,g}(\tau) = q^{P_2(g/\delta)\delta/2}\prod_{\substack{n>0\\ n\equiv g \pmod{\delta}}}(1-q^n)\prod_{\substack{n>0\\ n\equiv -g \pmod{\delta}}}(1-q^n),
$$

where $\tau \in \mathcal{H} := \{\tau \in C : Im\tau > 0\}$, $q = e^{2\pi i\tau}$, $P_2(t) = \{t\}^2 - \{t\} + \frac{1}{6}$ is the second Bernoulli function, and $\{t\}$ is the fractional part of t.

The above three identities can be rewritten as

$$
\frac{2q^{-1}J_7^2\overline{J}_{2,7}\overline{J}_{3,21}}{J_{1,7}\overline{J}_{3,7}\overline{J}_{0,21}} - \frac{q^{-1}J_7^3\overline{J}_{0,3}J_{1,7}}{J_1\overline{J}_{4,7}^2\overline{J}_{0,21}} = \frac{J_{3,7}^2}{J_1}, \tag{2.17}
$$

$$\frac{q^{-2}J_7^3\overline{J}_{0,3}J_{2,7}}{J_1\overline{J}_{1,7}^2\overline{J}_{0,21}}-\frac{2q^{-2}J_7^2\overline{J}_{3,7}\overline{J}_{6,21}}{\overline{J}_{1,7}J_{2,7}\overline{J}_{0,21}}=\frac{qJ_{1,7}^2}{J_1}, \tag{2.18}$$

$$\frac{q^{-2}J_7^3\overline{J}_{0,3}J_{3,7}}{J_1\overline{J}_{2,7}^2\overline{J}_{0,21}}-\frac{2q^{-2}J_7^2\overline{J}_{1,7}\overline{J}_{9,21}}{\overline{J}_{2,7}J_{3,7}\overline{J}_{0,21}}=\frac{J_{2,7}^2}{J_1}. \tag{2.19}$$

Then combining (2.14) (resp. (2.15) and (2.16)), (2.17) (resp. (2.18) and (2.19)) and (1.7), we arrive at (1.4) (resp. (1.5) and (1.6)).

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