AIMS Mathematics, 8(2): 2961–2994.
DOI: 10.3934/math.2023154
Received: 15 June 2022
Revised: 22 October 2022
Accepted: 09 November 2022
Published: 14 November 2022

Mathematics

http://www.aimspress.com/journal/Math

Research article

Modified subgradient extragradient algorithms for systems of generalized equilibria with constraints

Lu-Chuan Ceng[1], **Li-Jun Zhu**[2,*] **and Tzu-Chien Yin**[3,*]

[1] Department of Mathematics, Shanghai Normal University, Shanghai 200234, China
[2] The Key Laboratory of Intelligent Information and Big Data Processing of NingXia, North Minzu University, Yinchuan 750021, China
[3] Research Center for Interneural Computing, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan

* **Correspondence:** Email: zljmath@outlook.com, yintzuchien@mail.cmuh.org.tw.

Abstract: In this paper, we introduce the modified Mann-like subgradient-like extragradient implicit rules with linear-search process for finding a common solution of a system of generalized equilibrium problems, a pseudomonotone variational inequality problem and a fixed-point problem of an asymptotically nonexpansive mapping in a real Hilbert space. The proposed algorithms are based on the subgradient extragradient rule with linear-search process, Mann implicit iteration approach, and hybrid deepest-descent technique. Under mild restrictions, we demonstrate the strong convergence of the proposed algorithms to a common solution of the investigated problems, which is a unique solution of a certain hierarchical variational inequality defined on their common solution set.

Keywords: system of generalized equilibrium problems; variational inequality; subgradient extragradient method; fixed point; convergence
Mathematics Subject Classification: 47H09, 47H10, 47J20, 47J25

1. Introduction

Let H be a real Hilbert space with its inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let $\emptyset \neq C \subset H$ be a closed convex set, and P_C be the metric projection (or nearest point) from H onto C. Let $S : C \to H$ be a nonlinear mapping. Let the $\mathrm{Fix}(S)$ and $\mathbf{R}$ indicate the fixed-point set of S and the real-number set, respectively. We denote by the $\to$ and $\rightharpoonup$ the strong convergence and weak convergence in H, respectively. A mapping $S : C \to C$ is referred to as being asymptotically nonexpansive if $\exists\{\theta_r\}_{r=1}^{\infty} \subset [0, +\infty)$ s.t. $\lim_{r\to\infty} \theta_r = 0$ and $\theta_r\|u - v\| + \|u - v\| \geq \|S^r u - S^r v\|,\ \forall r \geq 1,\ u, v \in C$. In particular, if $\theta_r = 0,\ \forall r \geq 1$, then S is known as being nonexpansive.

Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction. The equilibrium problem (EP) for Θ is to determine its equilibrium points, that is, the set $\mathbf{EP}(\Theta) = \{u \in C : \Theta(u, v) \geq 0, \ \forall v \in C\}$. Under the theory framework of equilibrium problems, there is a unified way to investigate a wide number of problems arising in the physics, optimization, structural analysis, transportation, finance and economics. In order to find an element in $\mathbf{EP}(\Theta)$, one assumes that the following hold:

(H1) $\Theta(u, u) = 0, \quad \forall u \in C$;

(H2) Θ is monotone, that is, $\Theta(u, v) + \Theta(v, u) \leq 0, \quad \forall u, v \in C$;

(H3) $\lim_{\lambda \to 0^+} \Theta((1 - \lambda)u + \lambda w, v) \leq \Theta(u, v), \quad \forall u, v, w \in C$;

(H4) $v \mapsto \Theta(u, v)$ is convex and lower semicontinuous (l.s.c.) for each $u \in C$.

In 1994, Blum and Oettli [34] gave the following lemma, which plays a key role in solving the equilibrium problems.

Lemma 1.1 ([34]). *Let $\Theta : C \times C \to \mathbf{R}$ satisfy the hypotheses (H1)–(H4). For any $u \in C$ and $\ell > 0$, let $S_\ell : H \to C$ be the mapping formulated below:*

$$T_\ell^\Theta(u) := \{w \in C : \Theta(w, v) + \frac{1}{\ell}\langle v - w, w - u\rangle \geq 0, \ \forall v \in C\}.$$

Then T_ℓ^Θ is well defined and the following hold: (i) T_ℓ^Θ is single-valued, and firmly nonexpansive, that is, $\|T_\ell^\Theta u - T_\ell^\Theta v\|^2 \leq \langle T_\ell^\Theta u - T_\ell^\Theta v, u - v\rangle, \ \forall u, v \in H$; and (ii) $\mathrm{Fix}(T_\ell^\Theta) = \mathrm{EP}(\Theta)$, and $\mathrm{EP}(\Theta)$ is convex and closed.

It is worth pointing out that the variational inequality problem (VIP) is a special case of the EP. In particular, if $\Theta(u, v) = \langle Au, v - u\rangle, \ \forall u, v \in C$, then the EP reduces to the classical VIP of finding $u \in C$ s.t. $\langle Au, v - u\rangle \geq 0, \ \ \forall v \in C$, where A is a self-mapping on H. The solution set of the VIP is denoted by $\mathrm{VI}(C, A)$. It is well known that, one of the most popular techniques for solving the VIP is the extragradient one put forth by Korpelevich [26] in 1976, that is, for any starting point $u_0 \in C$, let $\{u_p\}$ be the sequence constructed below

$$\begin{cases} v_p = P_C(u_p - \mu A u_p), \\ u_{p+1} = P_C(u_p - \mu A v_p), \quad \forall p \geq 0, \end{cases}$$

where $\mu \in (0, \frac{1}{L})$ and L is Lipschitz constant of A. Whenever $\mathrm{VI}(C, A) \neq \emptyset$, the sequence $\{u_p\}$ converges weakly to a point in $\mathrm{VI}(C, A)$. Till now, the vast literature on the Korpelevich extragradient technique shows that many authors have paid great attention to it and enhanced it in different manners; see e.g., [1–7, 9, 10, 12–18, 20–25, 27–31, 36–41] and references therein.

Let $\Theta_1, \Theta_2 : C \times C \to \mathbf{R}$ be two bifunctions and let $B_1, B_2 : C \to H$ be two nonlinear mappings. In 2010, Ceng and Yao [35] considered the following problem of finding $(u^*, v^*) \in C \times C$ such that

$$\begin{cases} \Theta_1(u^*, u) + \langle B_1 v^*, u - u^*\rangle + \frac{1}{\mu_1}\langle u^* - v^*, u - u^*\rangle \geq 0, \quad \forall u \in C, \\ \Theta_2(v^*, v) + \langle B_2 u^*, v - v^*\rangle + \frac{1}{\mu_2}\langle v^* - u^*, v - v^*\rangle \geq 0, \quad \forall v \in C, \end{cases} \tag{1.1}$$

with $\mu_1, \mu_2 > 0$, which is called a system of generalized equilibrium problems (SGEP). In particular, if $\Theta_1 = \Theta_2 = 0$, then the SGEP reduces to the following general system of variational inequalities (GSVI) considered in [6]: Find $(u^*, v^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 v^* + u^* - v^*, u - u^*\rangle \geq 0, \quad \forall u \in C, \\ \langle \mu_2 B_2 u^* + v^* - u^*, v - v^*\rangle \geq 0, \quad \forall v \in C. \end{cases} \tag{1.2}$$

Note that SGEP (1.1) can be transformed into the fixed-point problem.

Lemma 1.2 ([35]). *Let $\Theta_1, \Theta_2 : C \times C \to \mathbf{R}$ be two bifunctions satisfying the hypotheses (H1)–(H4) and let the mappings $B_1, B_2 : C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\beta)$, respectively. Then, for given $u^*, v^* \in C$, (u^*, v^*) is a solution of SGEP* (1.1) *if and only if $x^* \in \mathrm{Fix}(\mathcal{G})$, where* $\mathrm{Fix}(\mathcal{G})$ *is the fixed point set of the mapping $\mathcal{G} := T^{\Theta_1}_{\mu_1}(I - \mu_1 B_1)T^{\Theta_2}_{\mu_2}(I - \mu_2 B_2)$, and $v^* = T^{\Theta_2}_{\mu_2}(I - \mu_2 B_2)u^*$.*

On the other hand, suppose that the mappings $B_1, B_2 : C \to H$ are α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let $f : C \to C$ be contractive with constant $\delta \in [0, 1)$ and $F : C \to H$ be κ-Lipschitzian and η-strongly monotone with constants $\kappa, \eta > 0$ such that $\delta < \zeta := 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$ for $\mu \in (0, \frac{2\eta}{\kappa^2})$. Let $S : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{\theta_r\}$ such that $\Omega := \mathrm{Fix}(S) \cap \mathrm{Fix}(\mathcal{G}) \neq \emptyset$, where $\mathrm{Fix}(\mathcal{G})$ is the fixed-point set of the mapping $\mathcal{G} := P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$ for $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\beta)$. Recently, Cai, Shehu and Iyiola [13] proposed the modified viscosity implicit rule for finding an element of Ω, that is, for any starting point $x_1 \in C$, let $\{x_p\}$ be the sequence constructed below

$$\begin{cases} u_p = \beta_p x_p + (1 - \beta_p) y_p, \\ v_p = P_C(u_p - \mu_2 B_2 u_p), \\ y_p = P_C(v_p - \mu_1 B_1 v_p), \\ x_{p+1} = P_C[\alpha_p f(x_p) + (I - \alpha_p \mu F) T^p y_p], \quad \forall p \geq 1, \end{cases} \tag{1.3}$$

where $\{\alpha_p\}, \{\beta_p\} \subset (0, 1]$ s.t. (i) $\sum_{p=1}^{\infty} |\alpha_{p+1} - \alpha_p| < \infty$, $\sum_{p=1}^{\infty} \alpha_p = \infty$; (ii) $\lim_{p\to\infty} \alpha_p = 0$, $\lim_{p\to\infty} \frac{\theta_p}{\alpha_p} = 0$; (iii) $0 < \varepsilon \leq \beta_p \leq 1$, $\sum_{p=1}^{\infty} |\beta_{p+1} - \beta_p| < \infty$; and (iv) $\sum_{p=1}^{\infty} \|T^{p+1} y_p - T^p y_p\| < \infty$. It was proved in [13] that the sequence $\{x_p\}$ converges strongly to an element $u^* \in \Omega$, which is a unique solution of the hierarchical variational inequality (HVI): $\langle (\mu F - f)u^*, u - u^* \rangle \geq 0$, $\forall u \in \Omega$. Very recently, Reich et al. [29] suggested the modified projection-type method for solving the pseudomonotone VIP with uniform continuity mapping A. Let $\{\alpha_p\} \subset (0, 1)$ and $f : C \to C$ be contractive with constant $\delta \in [0, 1)$. Given any initial $x_1 \in C$.

Algorithm 1.3 ([29]). **Initial step:** *Let $\nu > 0$, $\ell \in (0, 1)$, $\lambda \in (0, \frac{1}{\nu})$.*

Iterations: *Given the current iterate x_p, calculate x_{p+1} as follows:*

Step 1. Compute $y_p = P_C(x_p - \lambda A x_p)$ and $R_\lambda(x_p) := x_p - y_p$. If $R_\lambda(x_p) = 0$, then stop; x_p is a solution of $\mathrm{VI}(C, A)$. *Otherwise,*

Step 2. Compute $w_p = x_p - \tau_p R_\lambda(x_p)$, where $\tau_p := \ell^{j_p}$ and j_p is the smallest nonnegative integer j s.t. $\frac{\nu}{2}\|R_\lambda(x_p)\|^2 \geq \langle A x_p - A(x_p - \ell^j R_\lambda(x_p)), R_\lambda(x_p) \rangle$.

Step 3. Compute $x_{p+1} = \alpha_p f(x_p) + (1 - \alpha_p) P_{C_p}(x_p)$, where $C_p := \{x \in C : \hbar_p(x_p) \leq 0\}$ and $\hbar_p(x) = \langle A w_p, x - x_p \rangle + \frac{\tau_p}{2\lambda}\|R_\lambda(x_p)\|^2$. Again set $p := p + 1$ and go to Step 1.

It was proven in [29] that under mild conditions, $\{x_p\}$ converges strongly to an element of $\mathrm{VI}(C, A)$. In a real Hilbert space H, we always assume that the SGEP, VIP, HVI and FPP represent a system of generalized equilibrium problems, a pseudomonotone variational inequality problem, a hierarchical variational inequality and a fixed-point problem of an asymptotically nonexpansive mapping, respectively. We introduce the modified Mann-like subgradient-like extragradient implicit rules with linear-search process for finding a common solution of the SGEP, VIP and FPP. The proposed algorithms are based on the subgradient extragradient rule with linear-search process, Mann implicit iteration approach, and hybrid deepest-descent technique. Under mild restrictions, we demonstrate the strong convergence of the proposed algorithms to a common solution of the SGEP, VIP and FPP, which is a unique solution of a certain HVI defined on their common solution set. In addition, an illustrated example is provided to illustrate the feasibility and implementability of our suggested rules.

The architecture of this article is constituted below: In Section 2, we present some concepts and basic tools for further use. Section 3 treats the convergence analysis of the suggested algorithms. Last, Section 4 applies our main results to solve the SGEP, VIP and FPP in an illustrated example. Our results improve and extend the ones associated with very recent literature, e.g., [13, 17, 29].

2. Preliminaries

Let H be a real Hilbert space and $\emptyset \neq C \subset H$ be a convex and closed set. Given a sequence $\{u_k\} \subset H$. We denote by the $u_k \to u^*$ (resp., $u_k \rightharpoonup u^*$) the strong (resp., weak) convergence of $\{u_k\}$ to u^*. For all $u, v \in C$, an operator $\Psi : C \to H$ is referred to as being

(a) L-Lipschitzian (or L-Lipschitz continuous) if $\exists L > 0$ s.t. $\|\Psi u - \Psi v\| \leq L\|u - v\|$;

(b) pseudomonotone if $\langle \Psi u, v - u\rangle \geq 0 \Rightarrow \langle \Psi v, v - u\rangle \geq 0$;

(c) monotone if $\langle \Psi u - \Psi v, u - v\rangle \geq 0$;

(d) α-strongly monotone if $\exists \alpha > 0$ s.t. $\langle \Psi u - \Psi v, u - v\rangle \geq \alpha\|u - v\|^2$;

(e) β-inverse-strongly monotone if $\exists \beta > 0$ s.t. $\langle \Psi u - \Psi v, u - v\rangle \geq \beta\|\Psi u - \Psi v\|^2$;

(f) sequentially weakly continuous if $\forall \{v_k\} \subset C$, the relation holds: $v_k \rightharpoonup v \Rightarrow \Psi v_k \rightharpoonup \Psi v$. It is clear that each monotone mapping is pseudomonotone but the converse is not true. Also, $\forall v \in H$, $\exists|$ (nearest point) $P_C(v) \in C$ s.t. $\|v - P_C(v)\| \leq \|v - w\|\ \forall w \in C$. P_C is called a nearest point (or metric) projection of H onto C. The following conclusions hold (see [19]):

(a) $\langle v - w, P_C(v) - P_C(w)\rangle \geq \|P_C(v) - P_C(w)\|^2, \ \ \forall v, w \in H$;

(b) $w = P_C(v) \Leftrightarrow \langle v - w, u - w\rangle \leq 0, \ \ \forall v \in H, u \in C$;

(c) $\|v - w\|^2 \geq \|v - P_C(v)\|^2 + \|w - P_C(v)\|^2, \ \ \forall v \in H, w \in C$;

(d) $\|v - w\|^2 = \|v\|^2 - \|w\|^2 - 2\langle v - w, w\rangle, \ \ \forall v, w \in H$;

(e) $\|sv + (1 - s)w\|^2 = s\|v\|^2 + (1 - s)\|w\|^2 - s(1 - s)\|v - w\|^2, \ \ \forall v, w \in H, s \in [0, 1]$.

The following inequality is an immediate consequence of the subdifferential inequality of the function $\frac{1}{2}\|\cdot\|^2$:

$$\|u + v\|^2 \leq \|u\|^2 + 2\langle v, u + v\rangle, \quad \forall u, v \in H.$$

The following lemmas will be used for demonstrating our main results in the sequel.

Lemma 2.1. *Let the mapping $B : C \to H$ be γ-inverse-strongly monotone. Then, for a given $\lambda \geq 0$,*

$$\|(I - \lambda B)u - (I - \lambda B)v\|^2 \leq \|u - v\|^2 - \lambda(2\gamma - \lambda)\|Bu - Bv\|^2.$$

In particular, if $0 \leq \lambda \leq 2\gamma$, then $I - \lambda B$ is nonexpansive.

Using Lemma 1.1 and Lemma 2.1, we immediately derive the following lemma.

Lemma 2.2 ([35]). *Let $\Theta_1, \Theta_2 : C \times C \to \mathbf{R}$ be two bifunctions satisfying the hypotheses (H1)–(H4), and the mappings $B_1, B_2 : C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let the mapping $\mathcal{G} : C \to C$ be defined as $\mathcal{G} := T^{\Theta_1}_{\mu_1}(I - \mu_1 B_1)T^{\Theta_2}_{\mu_2}(I - \mu_2 B_2)$. Then $\mathcal{G} : C \to C$ is a nonexpansive mapping provided $0 < \mu_1 \leq 2\alpha$ and $0 < \mu_2 \leq 2\beta$.*

In particular, if $\Theta_1 = \Theta_2 = 0$, using Lemma 1.1 we deduce that $T^{\Theta_1}_{\mu_1} = T^{\Theta_2}_{\mu_2} = P_C$. Thus, from Lemma 2.2 we obtain the corollary below.

Corollary 2.3 ([6]). *Let the mappings $B_1, B_2 : C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let the mapping $\mathcal{G} : C \to C$ be defined as $\mathcal{G} := P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$. If $0 < \mu_1 \le 2\alpha$ and $0 < \mu_2 \le 2\beta$, then $\mathcal{G} : C \to C$ is nonexpansive.*

Lemma 2.4 ([6]). *Let $A : C \to H$ be pseudomonotone and continuous. Then $u \in C$ is a solution to the VIP $\langle Au, \upsilon - u\rangle \ge 0,\ \forall \upsilon \in C$, if and only if $\langle A\upsilon, \upsilon - u\rangle \ge 0,\ \forall \upsilon \in C$.*

Lemma 2.5 ([8]). *Let $\{a_l\}$ be a sequence of nonnegative numbers satisfying the conditions: $a_{l+1} \le (1 - \lambda_l)a_l + \lambda_l \gamma_l\ \forall l \ge 1$, where $\{\lambda_l\}$ and $\{\gamma_l\}$ are sequences of real numbers such that (i) $\{\lambda_l\} \subset [0, 1]$ and $\sum_{l=1}^{\infty} \lambda_l = \infty$, and (ii) $\limsup_{l\to\infty} \gamma_l \le 0$ or $\sum_{l=1}^{\infty} |\lambda_l \gamma_l| < \infty$. Then $\lim_{l\to\infty} a_l = 0$.*

Later on, we will make use of the following lemmas to demonstrate our main results.

Lemma 2.6 ([32]). *Let H_1 and H_2 be two real Hilbert spaces. Suppose that $A : H_1 \to H_2$ is uniformly continuous on bounded subsets of H_1 and M is a bounded subset of H_1. Then, $A(M)$ is bounded.*

Lemma 2.7 ([33]). *Let h be a real-valued function on H and define $K := \{x \in C : h(x) \le 0\}$. If K is nonempty and h is Lipschitz continuous on C with modulus $\theta > 0$, then $\mathrm{dist}(x, K) \ge \theta^{-1} \max\{h(x), 0\}\ \forall x \in C$, where $\mathrm{dist}(x, K)$ denotes the distance of x to K.*

Lemma 2.8 ([11]). *Let X be a Banach space which admits a weakly continuous duality mapping, C be a nonempty closed convex subset of X, and $T : C \to C$ be an asymptotically nonexpansive mapping with $\mathrm{Fix}(T) \ne \emptyset$. Then $I - T$ is demiclosed at zero, i.e., if $\{u_k\}$ is a sequence in C such that $u_k \rightharpoonup u \in C$ and $(I - T)u_k \to 0$, then $(I - T)u = 0$, where I is the identity mapping of X.*

The following lemmas are very crucial to the convergence analysis of the proposed algorithms.

Lemma 2.9 ([30]). *Let $\{\Lambda_m\}$ be a sequence of real numbers that does not decrease at infinity in the sense that, $\exists\{\Lambda_{m_k}\} \subset \{\Lambda_m\}$ s.t. $\Lambda_{m_k} < \Lambda_{m_k+1}\ \forall k \ge 1$. Let the sequence $\{\phi(m)\}_{m\ge m_0}$ of integers be formulated below:*

$$\phi(m) = \max\{k \le m : \Lambda_k < \Lambda_{k+1}\},$$

with integer $m_0 \ge 1$ satisfying $\{k \le m_0 : \Lambda_k < \Lambda_{k+1}\} \ne \emptyset$. Then there hold the statements below:

(i) $\phi(m_0) \le \phi(m_0 + 1) \le \cdots$ and $\phi(m) \to \infty$;

(ii) $\Lambda_{\phi(m)} \le \Lambda_{\phi(m)+1}$ and $\Lambda_m \le \Lambda_{\phi(m)+1},\ \forall m \ge m_0$.

Lemma 2.10 ([8]). *Let $\lambda \in (0, 1]$ and Let $S : C \to C$ be a nonexpansive mapping. Let $S^\lambda : C \to H$ be the mapping formulated by $S^\lambda u := (I - \lambda\mu F)Su\ \forall u \in C$ with $F : C \to H$ being κ-Lipschitzian and η-strongly monotone. Then S^λ is a contraction provided $0 < \mu < \frac{2\eta}{\kappa^2}$, i.e., $\|S^\lambda u - S^\lambda v\| \le (1 - \lambda\tau)\|u - v\|,\ \forall u, v \in C$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$.*

3. Main results

In this section, let the feasible set C be a nonempty closed convex subset of a real Hilbert space H, and assume always that the following conditions hold:

(1) $S : C \to C$ is an asymptotically nonexpansive mapping with a sequence $\{\theta_n\}$, and $A : H \to H$ is pseudomonotone and uniformly continuous on C, s.t. $\|Az\| \leq \liminf_{n\to\infty} \|Au_n\|$ for each $\{u_n\} \subset C$ with $u_n \rightharpoonup z$.

(2) $\Theta_1, \Theta_2 : C \times C \to \mathbf{R}$ are two bifunctions satisfying the hypotheses (H1)–(H4), and $B_1, B_2 : C \to H$ are α-inverse-strongly monotone and β-inverse-strongly monotone, respectively.

(3) $\Omega = \mathrm{Fix}(S) \cap \mathrm{Fix}(\mathcal{G}) \cap \mathrm{VI}(C, A) \neq \emptyset$ where $\mathcal{G} := T^{\Theta_1}_{\mu_1}(I - \mu_1 B_1)T^{\Theta_2}_{\mu_2}(I - \mu_2 B_2)$ for $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\beta)$.

(4) $f : C \to H$ is a contraction with constant $\delta \in [0, 1)$, and $F : C \to H$ is η-strongly monotone and κ-Lipschitzian such that $\delta < \tau := 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$ for $\mu \in (0, \frac{2\eta}{\kappa^2})$.

(5) $\{\sigma_n\}, \{\alpha_n\} \subset (0, 1]$ and $\{\beta_n\} \subset [0, 1]$ are three real number sequences satisfying

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\limsup_{n\to\infty} \sigma_n < 1$.

Algorithm 3.1. Initial step: *Given $\nu > 0$, $\ell \in (0, 1)$, $\lambda \in (0, \frac{1}{\nu})$. Let $x_1 \in C$ be arbitrary.*

Iterations: *Given the current iterate x_n, calculate x_{n+1} below:*

Step 1. Calculate $w_n = \sigma_n x_n + (1 - \sigma_n)u_n$ with

$$v_n = T^{\Theta_2}_{\mu_2}(w_n - \mu_2 B_2 w_n),$$

$$u_n = T^{\Theta_1}_{\mu_1}(v_n - \mu_1 B_1 v_n).$$

Step 2. Calculate $y_n = P_C(w_n - \lambda A w_n)$ and $R_\lambda(w_n) := w_n - y_n$.

Step 3. Calculate $t_n = w_n - \tau_n R_\lambda(w_n)$, where $\tau_n := \ell^{j_n}$ and j_n is the smallest nonnegative integer j satisfying

$$\langle Aw_n - A(w_n - \ell^j R_\lambda(w_n)), w_n - y_n\rangle \leq \frac{\nu}{2}\|R_\lambda(w_n)\|^2. \tag{3.1}$$

Step 4. Compute $z_n = P_{C_n}(w_n)$ and $x_{n+1} = \beta_n x_n + (1 - \beta_n)P_C[\alpha_n f(x_n) + (I - \alpha_n\mu F)S^n z_n]$, where $C_n := \{u \in C : \hbar_n(u) \leq 0\}$ and

$$\hbar_n(u) = \langle At_n, u - w_n\rangle + \frac{\tau_n}{2\lambda}\|R_\lambda(w_n)\|^2. \tag{3.2}$$

Again put $n := n + 1$ and return to Step 1.

Lemma 3.2. *The Armijo-type search approach* (3.1) *is well formulated, and the relation holds: $\lambda^{-1}\|R_\lambda(w_n)\|^2 \leq \langle R_\lambda(w_n), Aw_n\rangle$.*

Proof. Since $\ell \in (0, 1)$ and A is of uniform continuity on C, it is clear that $\lim_{j\to\infty}\langle Aw_n - A(w_n - \ell^j R_\lambda(w_n)), R_\lambda(w_n)\rangle = 0$. If $R_\lambda(w_n) = 0$, one gets $j_n = 0$. Otherwise, from $R_\lambda(w_n) \neq 0$, it follows that $\exists$ (integer) $j_n \geq 0$ fulfilling (3.1). It is readily known that the firm nonexpansivity of P_C implies $\langle u - P_C v, u - v\rangle \geq \|u - P_C v\|^2$, $\forall u \in C, v \in H$. Setting $v = w_n - \lambda A w_n$ and $u = w_n$, one has $\lambda\langle w_n - P_C(w_n - \lambda A w_n), Aw_n\rangle \geq \|w_n - P_C(w_n - \lambda A w_n)\|^2$. Hence the relation holds. □

Lemma 3.3. *Suppose that $\hbar_n$ is the function formulated in* (3.2). *Then, $\hbar_n(\upsilon) \leq 0$ $\forall \upsilon \in \Omega$. In addition, when $R_\lambda(w_n) \neq 0$, one has $\hbar_n(w_n) > 0$.*

Proof. It suffices to show the former claim of Lemma 3.3 because the latter claim is clear. In fact, pick an arbitrary $\upsilon \in \Omega$. By Lemma 2.4 one gets $\langle At_n, t_n - \upsilon\rangle \geq 0$. Thus, one has

$$\begin{aligned}\hbar_n(\upsilon) &= \langle At_n, \upsilon - w_n\rangle + \frac{\tau_n}{2\lambda}\|R_\lambda(w_n)\|^2 \\ &= \langle At_n, t_n - w_n\rangle + \langle At_n, \upsilon - t_n\rangle + \frac{\tau_n}{2\lambda}\|R_\lambda(w_n)\|^2 \\ &\leq -\tau_n\langle At_n, R_\lambda(w_n)\rangle + \frac{\tau_n}{2\lambda}\|R_\lambda(w_n)\|^2.\end{aligned} \tag{3.3}$$

Meanwhile, from (3.1) it follows that $\frac{\nu}{2}\|R_\lambda(w_n)\|^2 \geq \langle Aw_n - At_n, R_\lambda(w_n)\rangle$. So, from Lemma 3.2 one gets

$$\langle At_n, R_\lambda(w_n)\rangle \geq -\frac{\nu}{2}\|R_\lambda(w_n)\|^2 + \langle R_\lambda(w_n), Aw_n\rangle \geq (-\frac{\nu}{2} + \frac{1}{\lambda})\|R_\lambda(w_n)\|^2. \tag{3.4}$$

This along with (3.3), arrives at

$$\hbar_n(\upsilon) \leq -\frac{\tau_n}{2}(\frac{1}{\lambda} - \nu)\|R_\lambda(w_n)\|^2. \tag{3.5}$$

Therefore, we derive the desired result. □

Lemma 3.4. *Let $\{w_n\}, \{x_n\}, \{y_n\}, \{z_n\}$ be the bounded sequences constructed in Algorithm 3.1. Assume that $x_n - x_{n+1} \to 0$, $x_n - \mathcal{G}w_n \to 0$, $w_n - y_n \to 0$ and $x_n - z_n \to 0$. If $S^n x_n - S^{n+1}x_n \to 0$ and $\exists\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup z \in C$, then $z \in \Omega$.*

Proof. From Algorithm 3.1, we get $w_n - x_n = (1 - \sigma_n)(u_n - x_n)$ $\forall n \geq 1$, and hence $\|w_n - x_n\| = (1 - \sigma_n)\|u_n - x_n\| \leq \|u_n - x_n\|$. Utilizing the assumption $u_n - x_n \to 0$, we have

$$\lim_{n\to\infty} \|w_n - x_n\| = 0. \tag{3.6}$$

Putting $q_n := \alpha_n f(x_n) + (I - \alpha_n\mu F)S^n z_n$, by Algorithm 3.1 we know that $x_{n+1} = \beta_n x_n + (1 - \beta_n)P_C(q_n)$ and $q_n - S^n z_n = \alpha_n f(x_n) - \alpha_n\mu F S^n z_n$. Hence one gets

$$\begin{aligned}\|x_n - S^n z_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - S^n z_n\| \\ &\leq \|x_n - x_{n+1}\| + \beta_n\|x_n - S^n z_n\| + (1 - \beta_n)\|q_n - S^n z_n\| \\ &\leq \|x_n - x_{n+1}\| + \beta_n\|x_n - S^n z_n\| + \alpha_n\|f(x_n)\| + \alpha_n\|\mu F S^n z_n\|.\end{aligned}$$

This immediately ensures that

$$(1 - \beta_n)\|x_n - S^n z_n\| \leq \|x_n - x_{n+1}\| + \alpha_n\|f(x_n)\| + \alpha_n\|\mu F S^n z_n\|.$$

Since $x_n - x_{n+1} \to 0$, $\alpha_n \to 0$ and $\liminf_{n\to\infty}(1 - \beta_n) > 0$, by the boundedness of $\{x_n\}, \{z_n\}$ we obtain

$$\lim_{n\to\infty} \|x_n - S^n z_n\| = 0.$$

We claim that $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$. In fact, using the asymptotical nonexpansivity of S, one deduces that

$$\begin{aligned}\|x_n - Sx_n\| &\leq \|x_n - S^n z_n\| + \|S^n z_n - S^n x_n\| + \|S^n x_n - S^{n+1}x_n\| \\ &\quad + \|S^{n+1}x_n - S^{n+1}z_n\| + \|S^{n+1}z_n - Sx_n\| \\ &\leq \|x_n - S^n z_n\| + (1 + \theta_n)\|z_n - x_n\| + \|S^n x_n - S^{n+1}x_n\| \\ &\quad + (1 + \theta_{n+1})\|x_n - z_n\| + (1 + \theta_1)\|S^n z_n - x_n\| \\ &= (2 + \theta_1)\|x_n - S^n z_n\| + (2 + \theta_n + \theta_{n+1})\|z_n - x_n\| + \|S^n x_n - S^{n+1}x_n\|.\end{aligned}$$

Since $x_n - z_n \to 0$, $x_n - S^n z_n \to 0$ and $S^n x_n - S^{n+1} x_n \to 0$, we obtain

$$\lim_{n\to\infty} \|x_n - S x_n\| = 0. \tag{3.7}$$

Also, let us show that $\lim_{n\to\infty} \|x_n - \mathcal{G} x_n\| = 0$. In fact, by Lemma 2.2 we know that $\mathcal{G} : C \to C$ is nonexpansive for $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\beta)$. Again from Algorithm 3.1, we have $u_n = \mathcal{G} w_n$. Since

$$\|\mathcal{G} x_n - x_n\| \le \|\mathcal{G} x_n - \mathcal{G} w_n\| + \|\mathcal{G} w_n - x_n\| \le \|x_n - w_n\| + \|u_n - x_n\|,$$

Noticing $u_n - x_n \to 0$ and $x_n - w_n \to 0$ (due to (3.6)), we obtain

$$\lim_{n\to\infty} \|\mathcal{G} x_n - x_n\| = 0. \tag{3.8}$$

Next, let us show $z \in \mathrm{VI}(C, A)$. Indeed, noticing $x_n - w_n \to 0$ and $x_{n_k} \rightharpoonup z$, we know that $w_{n_k} \rightharpoonup z$. Since C is convex and closed, from $\{w_n\} \subset C$ and $w_{n_k} \rightharpoonup z$ we get $z \in C$. In what follows, we consider two cases. In the case of $Az = 0$, it is clear that $z \in \mathrm{VI}(C, A)$ because $\langle Az, y - z\rangle \ge 0,\ \forall y \in C$. In the case of $Az \ne 0$, it follows from $w_n - x_n \to 0$ and $x_{n_k} \rightharpoonup z$ that $w_{n_k} \rightharpoonup z$ as $k \to \infty$. Utilizing the assumption on A, instead of the sequentially weak continuity of A, we get $0 < \|Az\| \le \liminf_{k\to\infty} \|Aw_{n_k}\|$. So, we might assume that $\|Aw_{n_k}\| \ne 0\ \forall k \ge 1$. On the other hand, from $y_n = P_C(w_n - \lambda A w_n)$, one has $\langle w_n - \lambda A w_n - y_n, x - y_n\rangle \le 0,\ \forall x \in C$, and hence

$$\frac{1}{\lambda}\langle w_n - y_n, x - y_n\rangle + \langle Aw_n, y_n - w_n\rangle \le \langle Aw_n, x - w_n\rangle, \quad \forall x \in C. \tag{3.9}$$

In the light of the uniform continuity of A on C, one knows that $\{Aw_n\}$ is bounded (due to Lemma 2.6). Note that $\{y_n\}$ is bounded as well. Thus, from (3.9) we get $\liminf_{k\to\infty}\langle Aw_{n_k}, x - w_{n_k}\rangle \ge 0\ \forall x \in C$.

To show that $z \in \mathrm{VI}(C, A)$, we now choose a sequence $\{\gamma_k\} \subset (0, 1)$ satisfying $\gamma_k \downarrow 0$ as $k \to \infty$. For each $k \ge 1$, we denote by l_k the smallest positive integer such that

$$\langle Aw_{n_j}, x - w_{n_j}\rangle + \gamma_k \ge 0, \quad \forall j \ge l_k. \tag{3.10}$$

Because $\{\gamma_k\}$ is decreasing, it is readily known that $\{l_k\}$ is increasing. Note that $Aw_{l_k} \ne 0\ \forall k \ge 1$ (due to $\{Aw_{l_k}\} \subset \{Aw_{n_k}\}$). Then one puts $\upsilon_{l_k} = \frac{Aw_{l_k}}{\|Aw_{l_k}\|^2}$, one gets $\langle Aw_{l_k}, \upsilon_{l_k}\rangle = 1,\ \forall k \ge 1$. So, using (3.10) one has $\langle Aw_{l_k}, x + \gamma_k \upsilon_{l_k} - w_{l_k}\rangle \ge 0,\ \forall k \ge 1$. Again from the pseudo-monotonicity of A one has $\langle A(x + \gamma_k \upsilon_{l_k}), x + \gamma_k \upsilon_{l_k} - w_{l_k}\rangle \ge 0,\ \forall k \ge 1$. This immediately arrives at

$$\langle Ax, x - w_{l_k}\rangle \ge \langle Ax - A(x + \gamma_k \upsilon_{l_k}), x + \gamma_k \upsilon_{l_k} - w_{l_k}\rangle - \gamma_k \langle Ax, \upsilon_{l_k}\rangle, \quad \forall k \ge 1. \tag{3.11}$$

We claim that $\lim_{k\to\infty} \gamma_k \upsilon_{l_k} = 0$. In fact, from $x_{n_k} \rightharpoonup z \in C$ and $w_n - x_n \to 0$, we obtain $w_{n_k} \rightharpoonup z$. Note that $\{w_{l_k}\} \subset \{w_{n_k}\}$ and $\gamma_k \downarrow 0$ as $k \to \infty$. So it follows that

$$0 \le \limsup_{k\to\infty} \|\gamma_k \upsilon_{l_k}\| = \limsup_{k\to\infty} \frac{\gamma_k}{\|Aw_{l_k}\|} \le \frac{\limsup_{k\to\infty} \gamma_k}{\liminf_{k\to\infty} \|Aw_{n_k}\|} = 0.$$

Hence one gets $\gamma_k \upsilon_{l_k} \to 0$ as $k \to \infty$. Thus, letting $k \to \infty$, we deduce that the right-hand side of (3.11) tends to zero by the uniform continuity of A, the boundedness of $\{w_{l_k}\}, \{\upsilon_{l_k}\}$ and the limit $\lim_{k\to\infty} \gamma_k \upsilon_{l_k} = 0$. Therefore, $\langle Ax, x - z\rangle = \liminf_{k\to\infty}\langle Ax, x - w_{l_k}\rangle \ge 0\ \forall x \in C$. Using Lemma 2.4 one has $z \in \mathrm{VI}(C, A)$.

Last, we claim that $z \in \Omega$. In fact, because (3.7) yields $x_{n_k} - Sx_{n_k} \to 0$. By Lemma 2.8 one knows that $I - S$ is demiclosed at zero. So, from $x_{n_k} \rightharpoonup z$ it follows that $(I - S)z = 0$, i.e., $z \in \mathrm{Fix}(S)$. Besides, let us claim that $z \in \mathrm{Fix}(\mathcal{G})$. Actually, by Lemma 2.8 we deduce that $I - \mathcal{G}$ is demiclosed at zero. Thus, from (3.8) and $x_{n_k} \rightharpoonup z$ one has $(I - \mathcal{G})z = 0$, i.e., $z \in \mathrm{Fix}(\mathcal{G})$. Accordingly, $z \in \mathrm{Fix}(S) \cap \mathrm{Fix}(\mathcal{G}) \cap \mathrm{VI}(C, A) = \Omega$. This completes the proof. □

Lemma 3.5. *Let $\{w_n\}$ be the sequence constructed in Algorithm 3.1. Then,*

$$\lim_{n\to\infty} \tau_n \|R_\lambda(w_n)\|^2 = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \|R_\lambda(w_n)\| = 0. \tag{3.12}$$

Proof. We claim that $\limsup_{n\to\infty} \|R_\lambda(w_n)\| = 0$. Conversely, suppose that $\limsup_{n\to\infty} \|R_\lambda(w_n)\| = d > 0$. Then, $\exists\{n_p\} \subset \{n\}$ s.t. $\lim_{p\to\infty} \|R_\lambda(w_{n_p})\| = d > 0$. Note that $\lim_{p\to\infty} \tau_{n_p}\|R_\lambda(w_{n_p})\|^2 = 0$. First, if $\liminf_{p\to\infty} \tau_{n_p} > 0$, we might assume that $\exists\xi > 0$ s.t. $\tau_{n_p} \geq \xi > 0,\ \forall p \geq 1$. So it follows that

$$\|R_\lambda(w_{n_p})\|^2 = \frac{1}{\tau_{n_p}}\tau_{n_p}\|R_\lambda(w_{n_p})\|^2 \leq \frac{1}{\xi}\cdot\tau_{n_p}\|R_\lambda(w_{n_p})\|^2 = \frac{1}{\xi}\cdot\tau_{n_p}\|R_\lambda(w_{n_p})\|^2, \tag{3.13}$$

which immediately leads to

$$0 < d^2 = \lim_{p\to\infty} \|R_\lambda(w_{n_p})\|^2 \leq \lim_{p\to\infty}\{\frac{1}{\xi}\cdot\tau_{n_p}\|R_\lambda(w_{n_p})\|^2\} = 0.$$

So, this reaches at a contradiction.

If $\liminf_{p\to\infty} \tau_{n_p} = 0$, there exists a subsequence of $\{\tau_{n_p}\}$, still denoted by $\{\tau_{n_p}\}$, s.t. $\lim_{p\to\infty} \tau_{n_p} = 0$. We now set

$$\upsilon_{n_p} := \frac{1}{\ell}\tau_{n_p}y_{n_p} + (1 - \frac{1}{\ell}\tau_{n_p})w_{n_p} = w_{n_p} - \frac{1}{\ell}\tau_{n_p}(w_{n_p} - y_{n_p}).$$

Then, from $\lim_{p\to\infty} \tau_{n_p}\|R_\lambda(w_{n_p})\|^2 = 0$ we infer that

$$\lim_{p\to\infty} \|\upsilon_{n_p} - w_{n_p}\|^2 = \lim_{p\to\infty} \frac{1}{\ell^2}\tau_{n_p}\cdot\tau_{n_p}\|R_\lambda(w_{n_p})\|^2 = 0. \tag{3.14}$$

Using the stepsize rule (3.1), one gets $\langle Aw_{n_p} - A\upsilon_{n_p}, R_\lambda(w_{n_p})\rangle > \frac{\nu}{2}\|R_\lambda(w_{n_p})\|^2$. Since A is uniformly continuous on bounded subsets of C, (3.14) guarantees that

$$\lim_{p\to\infty} \|Aw_{n_p} - A\upsilon_{n_p}\| = 0, \tag{3.15}$$

which hence attains $\lim_{p\to\infty} \|R_\lambda(w_{n_p})\| = 0$. So, this reaches a contradiction. Therefore, $R_\lambda(w_n) \to 0$ as $n \to \infty$. This completes the proof. □

Theorem 3.6. *Suppose that $\{x_n\}$ is the sequence constructed in Algorithm 3.1. Then $x_n \to x^* \in \Omega$ provided $S^n x_n - S^{n+1}x_n \to 0$, with $x^* \in \Omega$ being only a solution to the HVI*

$$\langle(\mu F - f)x^*, y - x^*\rangle \geq 0,\ \forall y \in \Omega.$$

Proof. First of all, noticing $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$ and $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n} = 0$, we may assume, without loss of generality, that $\{\sigma_n\} \subset [a,b] \subset (0,1)$ and $\theta_n \le \frac{\alpha_n(\tau-\delta)}{2}$, $\forall n \ge 1$. We claim that $P_\Omega(I - \mu F + f) : C \to C$ is contractive map. In fact, using Lemma 2.10, one has

$$\|P_\Omega(I-\mu F+f)u - P_\Omega(I-\mu F+f)v\| \le [1-(\tau-\delta)]\|u-v\|, \quad \forall u,v \in C.$$

This ensures that $P_\Omega(I-\mu F+f)$ is contractive. Banach's Contraction Mapping Principle guarantees that there exists a unique fixed point of $P_\Omega(I-\mu F+f)$ in C. Say $x^* \in C$ s.t. $x^* = P_\Omega(I-\mu F+f)x^*$. That is, $\exists\,|$ (solution) $x^* \in \Omega = \mathrm{Fix}(S) \cap \mathrm{Fix}(\mathcal{G}) \cap \mathrm{VI}(C,A)$ of the HVI

$$\langle(\mu F - f)x^*, y - x^*\rangle \ge 0, \quad \forall y \in \Omega. \tag{3.16}$$

Next we demonstrate the conclusion of the theorem. To the goal, we divide the remainder of the proof into several aspects.

Aspect 1. We assert that $\{x_n\}$ is of boundedness. Indeed, for $x^* \in \Omega = \mathrm{Fix}(S) \cap \mathrm{Fix}(\mathcal{G}) \cap \mathrm{VI}(C,A)$ we have $Sx^* = x^*$, $\mathcal{G}x^* = x^*$ and $P_C(x^* - \lambda Ax^*) = x^*$. We observe that

$$\begin{aligned}\|z_n - x^*\|^2 = \|P_{C_n}(w_n) - x^*\|^2 &\le \|w_n - x^*\|^2 - \|w_n - P_{C_n}(w_n)\|^2 \\ &= \|w_n - x^*\|^2 - \mathrm{dist}^2(w_n, C_n),\end{aligned} \tag{3.17}$$

which hence leads to

$$\|z_n - x^*\| \le \|w_n - x^*\|, \quad \forall n \ge 1. \tag{3.18}$$

Using Lemma 2.2, one knows that $\mathcal{G} = T^{\theta_1}_{\mu_1}(I - \mu_1 B_1)T^{\theta_2}_{\mu_2}(I - \mu_2 B_2)$ is nonexpansive for $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\beta)$. Thus, by the definition of w_n, one gets

$$\begin{aligned}\|w_n - x^*\| &\le \sigma_n\|x_n - x^*\| + (1-\sigma_n)\|\mathcal{G}w_n - x^*\| \\ &\le \sigma_n\|x_n - x^*\| + (1-\sigma_n)\|w_n - x^*\|,\end{aligned}$$

which immediately yields

$$\|w_n - x^*\| \le \|x_n - x^*\|, \quad \forall n \ge 1.$$

This together with (3.18), yields

$$\|z_n - x^*\| \le \|w_n - x^*\| \le \|x_n - x^*\|, \quad \forall n \ge 1. \tag{3.19}$$

Thus, using (3.19), from Lemma 2.10 we obtain

$$\begin{aligned}\|x_{n+1} - x^*\| &\le \beta_n\|x_n - x^*\| + (1-\beta_n)\|\alpha_n f(x_n) + (I - \alpha_n\mu F)S^n z_n - x^*\| \\ &= \beta_n\|x_n - x^*\| + (1-\beta_n)\|\alpha_n(f(x_n) - f(x^*)) + (I - \alpha_n\mu F)S^n z_n - (I - \alpha_n\mu F)x^* \\ &\quad + \alpha_n(f - \mu F)x^*\| \\ &\le \beta_n\|x_n - x^*\| + (1-\beta_n)\{\alpha_n\delta\|x_n - x^*\| + (1-\alpha_n\tau)(1+\theta_n)\|z_n - x^*\| \\ &\quad + \alpha_n\|(f - \mu F)x^*\|\} \\ &\le \beta_n\|x_n - x^*\| + (1-\beta_n)\{[\alpha_n\delta + (1-\alpha_n\tau) + \theta_n]\|x_n - x^*\| + \alpha_n\|(f - \mu F)x^*\|\}\end{aligned}$$

$$
\begin{aligned}
&\le \beta_n\|x_n - x^*\| + (1-\beta_n)\{[1-\alpha_n(\tau-\delta) + \frac{\alpha_n(\tau-\delta)}{2}]\|x_n - x^*\| + \alpha_n\|(f-\mu F)x^*\|\} \\
&= \beta_n\|x_n - x^*\| + (1-\beta_n)\{[1-\frac{\alpha_n(\tau-\delta)}{2}]\|x_n - x^*\| + \alpha_n\|(f-\mu F)x^*\|\} \\
&= [1-\frac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n - x^*\| + \frac{\alpha_n(1-\beta_n)(\tau-\delta)}{2} \cdot \frac{2\|(f-\mu F)x^*\|}{\tau-\delta} \\
&\le \max\{\|x_n - x^*\|, \frac{2\|(f-\mu F)x^*\|}{\tau-\delta}\}.
\end{aligned}
$$

By induction, we get

$$
\|x_n - x^*\| \le \max\{\|x_1 - x^*\|, \frac{2\|(f-\mu F)x^*\|}{\tau-\delta}\}, \ \forall n \ge 1.
$$

Thus, $\{x_n\}$ is bounded, and so are the sequences $\{w_n\}, \{y_n\}, \{z_n\}, \{f(x_n)\}, \{At_n\}, \{\mathcal{G}w_n\}, \{S^n z_n\}$.

Aspect 2. We assert that

$$
\begin{aligned}
(1-\beta_n)\{[1-\tfrac{\alpha_n(\tau+\delta)}{2}]\|w_n - z_n\|^2 + \|q_n - P_C(q_n)\|^2\} + \beta_n(1-\beta_n)\|x_n - P_C(q_n)\|^2 \\
\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n M_1,
\end{aligned}
$$

for some $M_1 > 0$. In fact, noticing $z_n = P_{C_n}(w_n)$ and $w_n = \sigma_n x_n + (1-\sigma_n)u_n$, we obtain

$$
\begin{aligned}
\|z_n - x^*\|^2 &\le \|w_n - x^*\|^2 - \|w_n - z_n\|^2 \\
&\le \sigma_n\|x_n - x^*\|^2 + (1-\sigma_n)\|u_n - x^*\|^2 - \|w_n - z_n\|^2.
\end{aligned}
$$

Since $x_{n+1} = \beta_n x_n + (1-\beta_n)P_C(q_n)$ where $q_n = \alpha_n f(x_n) + (I - \alpha_n\mu F)S^n z_n$, by Lemma 2.10 and the convexity of the function $h(s) = s^2, \ \forall s \in \mathbf{R}$ we deduce that

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 = \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|P_C(q_n) - x^*\|^2 - \beta_n(1-\beta_n)\|x_n - P_C(q_n)\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\|q_n - x^*\|^2 - \|q_n - P_C(q_n)\|^2\} - \beta_n(1-\beta_n)\|x_n - P_C(q_n)\|^2 \\
&= \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\|\alpha_n(f(x_n) - f(x^*)) + (I-\alpha_n\mu F)S^n z_n - (I-\alpha_n\mu F)x^* \\
&\quad + \alpha_n(f-\mu F)x^*\|^2 - \|q_n - P_C(q_n)\|^2\} - \beta_n(1-\beta_n)\|x_n - P_C(q_n)\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\|\alpha_n(f(x_n) - f(x^*)) + (I-\alpha_n\mu F)S^n z_n - (I-\alpha_n\mu F)x^*\|^2 \\
&\quad + 2\alpha_n\langle(f-\mu F)x^*, q_n - x^*\rangle - \|q_n - P_C(q_n)\|^2\} - \beta_n(1-\beta_n)\|x_n - P_C(q_n)\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{[\alpha_n\|f(x_n) - f(x^*)\| + \|(I-\alpha_n\mu F)S^n z_n - (I-\alpha_n\mu F)x^*\|]^2 \\
&\quad + 2\alpha_n\langle(f-\mu F)x^*, q_n - x^*\rangle - \|q_n - P_C(q_n)\|^2\} - \beta_n(1-\beta_n)\|x_n - P_C(q_n)\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{[\alpha_n\delta\|x_n - x^*\| + (1-\alpha_n\tau)(1+\theta_n)\|z_n - x^*\|]^2 \\
&\quad + 2\alpha_n\langle(f-\mu F)x^*, q_n - x^*\rangle - \|q_n - P_C(q_n)\|^2\} - \beta_n(1-\beta_n)\|x_n - P_C(q_n)\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{[\alpha_n\delta\|x_n - x^*\| + [(1-\alpha_n\tau) + \theta_n]\|z_n - x^*\|]^2 \\
&\quad + 2\alpha_n\langle(f-\mu F)x^*, q_n - x^*\rangle - \|q_n - P_C(q_n)\|^2\} - \beta_n(1-\beta_n)\|x_n - P_C(q_n)\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n - x^*\|^2 + [(1-\alpha_n\tau) + \theta_n]\|z_n - x^*\|^2 \\
&\quad + 2\alpha_n\langle(f-\mu F)x^*, q_n - x^*\rangle - \|q_n - P_C(q_n)\|^2\} - \beta_n(1-\beta_n)\|x_n - P_C(q_n)\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n - x^*\|^2 + [(1-\alpha_n\tau) + \theta_n][\sigma_n\|x_n - x^*\|^2 \\
&\quad + (1-\sigma_n)\|u_n - x^*\|^2 - \|w_n - z_n\|^2] + 2\alpha_n\langle(f-\mu F)x^*, q_n - x^*\rangle \\
&\quad - \|q_n - P_C(q_n)\|^2\} - \beta_n(1-\beta_n)\|x_n - P_C(q_n)\|^2,
\end{aligned} \tag{3.20}
$$

(due to $\alpha_n\delta + (1-\alpha_n\tau) + \theta_n \le 1 - \alpha_n(\tau-\delta) + \frac{\alpha_n(\tau-\delta)}{2} = 1 - \frac{\alpha_n(\tau-\delta)}{2} \le 1$), which together with $u_n = \mathcal{G}w_n$ and (3.19), ensures that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \beta_n\|x_n-x^*\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n-x^*\|^2 + 2\alpha_n\langle(f-\mu F)x^*, q_n-x^*\rangle \\
&\quad + [(1-\alpha_n\tau) + \frac{\alpha_n(\tau-\delta)}{2}][\sigma_n\|x_n-x^*\|^2 + (1-\sigma_n)\|x_n-x^*\|^2 \\
&\quad - \|w_n-z_n\|^2] - \|q_n-P_C(q_n)\|^2\} - \beta_n(1-\beta_n)\|x_n-P_C(q_n)\|^2 \\
&\le [1-\frac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n-x^*\|^2 - (1-\beta_n)\{\|q_n-P_C(q_n)\|^2 \\
&\quad + [1-\frac{\alpha_n(\tau+\delta)}{2}]\|w_n-z_n\|^2\} - \beta_n(1-\beta_n)\|x_n-P_C(q_n)\|^2 \\
&\quad + 2\alpha_n(1-\beta_n)\langle(f-\mu F)x^*, q_n-x^*\rangle \\
&\le \|x_n-x^*\|^2 - (1-\beta_n)\{[1-\frac{\alpha_n(\tau+\delta)}{2}]\|w_n-z_n\|^2 + \|q_n-P_C(q_n)\|^2\} \\
&\quad - \beta_n(1-\beta_n)\|x_n-P_C(q_n)\|^2 + \alpha_n M_1,
\end{aligned} \tag{3.21}$$

where $\sup_{n\ge1} 2\|(f-\mu F)x^*\|\|q_n-x^*\| \le M_1$ for some $M_1>0$. This attains the desired assertion.

Aspect 3. We assert that

$$(1-\beta_n)[1-\frac{\alpha_n(\tau+\delta)}{2}][\frac{\tau_n}{2\lambda\bar{L}}\|R_\lambda(w_n)\|^2]^2 \le \|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2 + \alpha_n M_1.$$

In fact, we claim that for some $\bar{L}>0$,

$$\|z_n-x^*\|^2 \le \|w_n-x^*\|^2 - [\frac{\tau_n}{2\lambda\bar{L}}\|R_\lambda(w_n)\|^2]^2. \tag{3.22}$$

Noticing the boundedness of $\{At_n\}$, one knows that $\exists\bar{L}>0$ s.t. $\|At_n\| \le \bar{L}, \ \forall n\ge1$. This implies that

$$|\hbar_n(u) - \hbar_n(v)| = |\langle At_n, u-v\rangle| \le \|At_n\|\|u-v\| \le \bar{L}\|u-v\|, \quad \forall u,v\in C_n,$$

which hence guarantees that $\hbar_n(\cdot)$ is $\bar{L}$-Lipschitz continuous on C_n. By Lemmas 2.7 and 3.3, we have

$$\mathrm{dist}(w_n, C_n) \ge \frac{1}{\bar{L}}\hbar_n(w_n) = \frac{\tau_n}{2\lambda\bar{L}}\|R_\lambda(w_n)\|^2. \tag{3.23}$$

Combining (3.17) and (3.23) yields

$$\|z_n-x^*\|^2 \le \|w_n-x^*\|^2 - [\frac{\tau_n}{2\lambda\bar{L}}\|R_\lambda(w_n)\|^2]^2.$$

From (3.20), (3.19) and (3.22) it follows that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \beta_n\|x_n-x^*\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n-x^*\|^2 + [(1-\alpha_n\tau)+\theta_n]\|z_n-x^*\|^2 \\
&\quad + 2\alpha_n\langle(f-\mu F)x^*, q_n-x^*\rangle\} \\
&\le \beta_n\|x_n-x^*\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n-x^*\|^2 + [(1-\alpha_n\tau)+\theta_n][\|w_n-x^*\|^2 \\
&\quad - [\frac{\tau_n}{2\lambda\bar{L}}\|R_\lambda(w_n)\|^2]^2] + 2\alpha_n\langle(f-\mu F)x^*, q_n-x^*\rangle\}
\end{aligned}$$

$$
\begin{aligned}
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n - x^*\|^2 + [(1-\alpha_n\tau) + \frac{\alpha_n(\tau-\delta)}{2}][\|x_n - x^*\|^2 \\
&\quad - [\frac{\tau_n}{2\lambda\bar{L}}\|R_\lambda(w_n)\|^2]^2] + 2\alpha_n\langle(f-\mu F)x^*, q_n - x^*\rangle\} \\
&= [1 - \frac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n - x^*\|^2 - (1-\beta_n)[1 - \frac{\alpha_n(\tau+\delta)}{2}][\frac{\tau_n}{2\lambda\bar{L}}\|R_\lambda(w_n)\|^2]^2 \\
&\quad + 2\alpha_n(1-\beta_n)\langle(f-\mu F)x^*, q_n - x^*\rangle \\
&\le \|x_n - x^*\|^2 - (1-\beta_n)[1 - \frac{\alpha_n(\tau+\delta)}{2}][\frac{\tau_n}{2\lambda\bar{L}}\|R_\lambda(w_n)\|^2]^2 + \alpha_n M_1.
\end{aligned}
$$

This hence leads to

$$
(1-\beta_n)[1 - \frac{\alpha_n(\tau+\delta)}{2}][\frac{\tau_n}{2\lambda\bar{L}}\|R_\lambda(w_n)\|^2]^2 \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n M_1.
$$

Aspect 4. We assert that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 \le [1 - \alpha_n(1-\beta_n)(\tau-\delta)]\|x_n - x^*\|^2 + \alpha_n(1-\beta_n)(\tau-\delta) \\
\times [\frac{2\langle(f-\mu F)x^*, q_n - x^*\rangle}{\tau-\delta} + \frac{\theta_n}{\alpha_n}\cdot\frac{M}{\tau-\delta}]
\end{aligned}
\tag{3.24}
$$

for some $M > 0$. In fact, from Lemma 2.10 and (3.19), one obtains

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|q_n - x^*\|^2 \\
&= \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|\alpha_n(f(x_n) - f(x^*)) + \alpha_n(f-\mu F)x^* \\
&\quad + (I - \alpha_n\mu F)S^n z_n - (I - \alpha_n\mu F)x^*\|^2 \\
&\le (1-\beta_n)\{\|\alpha_n(f(x_n) - f(x^*)) + (I - \alpha_n\mu F)S^n z_n - (I - \alpha_n\mu F)x^*\|^2 \\
&\quad + \beta_n\|x_n - x^*\|^2 + 2\alpha_n\langle(f-\mu F)x^*, q_n - x^*\rangle\} \\
&\le (1-\beta_n)\{[\alpha_n\|f(x_n) - f(x^*)\| + \|(I - \alpha_n\mu F)S^n z_n - (I - \alpha_n\mu F)x^*\|]^2 \\
&\quad + \beta_n\|x_n - x^*\|^2 + 2\alpha_n\langle(f-\mu F)x^*, q_n - x^*\rangle\} \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{[\alpha_n\delta\|x_n - x^*\| + (1-\alpha_n\tau)(1+\theta_n)\|z_n - x^*\|]^2 \\
&\quad + 2\alpha_n\langle(f-\mu F)x^*, q_n - x^*\rangle\} \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{[\alpha_n\delta\|x_n - x^*\| + (1-\alpha_n\tau+\theta_n)\|z_n - x^*\|]^2 \\
&\quad + 2\alpha_n\langle(f-\mu F)x^*, q_n - x^*\rangle\} \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n - x^*\|^2 + (1-\alpha_n\tau+\theta_n)\|z_n - x^*\|^2 \\
&\quad + 2\alpha_n\langle(f-\mu F)x^*, q_n - x^*\rangle\} \\
&\le [1 - \alpha_n(1-\beta_n)(\tau-\delta)]\|x_n - x^*\|^2 + (1-\beta_n)\{\theta_n\|x_n - x^*\|^2 \\
&\quad + 2\alpha_n\langle(f-\mu F)x^*, q_n - x^*\rangle\} \\
&\le [1 - \alpha_n(1-\beta_n)(\tau-\delta)]\|x_n - x^*\|^2 + \alpha_n(1-\beta_n)(\tau-\delta) \\
&\quad \times [\frac{2\langle(f-\mu F)x^*, q_n - x^*\rangle}{\tau-\delta} + \frac{\theta_n}{\alpha_n}\cdot\frac{M}{\tau-\delta}],
\end{aligned}
$$

where $\sup_{n\ge1}\|x_n - x^*\|^2 \le M$ for some $M > 0$.

Aspect 5. We assert that $x_n \to x^* \in \Omega$, which is only a solution of the HVI (3.16).

In fact, from (3.24), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leq [1-\alpha_n(1-\beta_n)(\tau-\delta)]\|x_n-x^*\|^2+\alpha_n(1-\beta_n)(\tau-\delta)\\ &\quad\times[\frac{2\langle(f-\mu F)x^*,q_n-x^*\rangle}{\tau-\delta}+\frac{\theta_n}{\alpha_n}\cdot\frac{M}{\tau-\delta}].\end{aligned}\tag{3.25}$$

Setting $\Lambda_n = \|x_n - x^*\|^2$, we demonstrate the convergence of $\{\Lambda_n\}$ to zero by the following two situations.

Situation 1. $\exists$ (integer) $n_0 \geq 1$ s.t. $\{\Lambda_n\}$ is nonincreasing. It is clear that the limit $\lim_{n\to\infty}\Lambda_n = k < +\infty$ and $\lim_{n\to\infty}(\Lambda_n - \Lambda_{n+1}) = 0$. From Aspect 2 and $\{\beta_n\} \subset [a, b] \subset (0, 1)$ we obtain

$$\begin{aligned}&(1-b)\{[1-\frac{\alpha_n(\tau+\delta)}{2}]\|w_n-z_n\|^2+\|q_n-P_C(q_n)\|^2\}+a(1-b)\|x_n-P_C(q_n)\|^2\\ &\leq(1-\beta_n)\{[1-\frac{\alpha_n(\tau+\delta)}{2}]\|w_n-z_n\|^2+\|q_n-P_C(q_n)\|^2\}+\beta_n(1-\beta_n)\|x_n-P_C(q_n)\|^2\\ &\leq\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_nM_1=\Lambda_n-\Lambda_{n+1}+\alpha_nM_1.\end{aligned}$$

Thanks to the facts that $\alpha_n \to 0$ and $\Lambda_n - \Lambda_{n+1} \to 0$, from $\frac{\tau+\delta}{2} \in (0, 1)$ one deduces that

$$\lim_{n\to\infty}\|w_n-z_n\|=\lim_{n\to\infty}\|q_n-P_C(q_n)\|=\lim_{n\to\infty}\|x_n-P_C(q_n)\|=0.\tag{3.26}$$

Hence it is readily known that

$$\|x_n-q_n\|\leq\|x_n-P_C(q_n)\|+\|P_C(q_n)-q_n\|\to0\quad(n\to\infty),$$

$$\|x_{n+1}-x_n\|=(1-\beta_n)\|P_C(q_n)-x_n\|\leq\|q_n-x_n\|\to0\quad(n\to\infty),$$

and

$$\begin{aligned}\|S^nz_n-x_n\|&=\|q_n-x_n-\alpha_n(f(x_n)-\mu FS^nz_n)\|\\ &\leq\|q_n-x_n\|+\alpha_n(\|f(x_n)\|+\mu\|FS^nz_n\|)\to0\quad(n\to\infty).\end{aligned}$$

Next, we show that $\|x_n - u_n\| \to 0$ as $n \to \infty$. Indeed, note that $y^* = T_{\mu_2}^{\Theta_2}(x^* - \mu_2 B_2 x^*)$, $v_n = T_{\mu_2}^{\Theta_2}(w_n - \mu_2 B_2 w_n)$ and $u_n = T_{\mu_1}^{\Theta_1}(v_n - \mu_1 B_1 v_n)$. Then $u_n = \mathcal{G}w_n$. By Lemma 2.1 we have

$$\|v_n-y^*\|^2\leq\|w_n-x^*\|^2-\mu_2(2\beta-\mu_2)\|B_2w_n-B_2x^*\|^2$$

and

$$\|u_n-x^*\|^2\leq\|v_n-y^*\|^2-\mu_1(2\alpha-\mu_1)\|B_1v_n-B_1y^*\|^2.$$

Combining the last two inequalities, from (3.19) we obtain

$$\|u_n-x^*\|^2\leq\|x_n-x^*\|^2-\mu_2(2\beta-\mu_2)\|B_2w_n-B_2x^*\|^2-\mu_1(2\alpha-\mu_1)\|B_1v_n-B_1y^*\|^2.$$

This together with (3.20), implies that

$$\begin{aligned}\|x_{n+1}-x^*\|^2&\leq\beta_n\|x_n-x^*\|^2+(1-\beta_n)\{\alpha_n\delta\|x_n-x^*\|^2+[(1-\alpha_n\tau)+\theta_n][\sigma_n\|x_n-x^*\|^2\\ &\quad+(1-\sigma_n)\|u_n-x^*\|^2]+2\alpha_n\langle(f-\mu F)x^*,q_n-x^*\rangle\}\end{aligned}$$

$$
\begin{aligned}
&\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n - x^*\|^2 + [(1-\alpha_n\tau) + \frac{\alpha_n(\tau-\delta)}{2}] \\
&\quad \times [\sigma_n\|x_n - x^*\|^2 + (1-\sigma_n)(\|x_n - x^*\|^2 - \mu_2(2\beta-\mu_2)\|B_2w_n - B_2x^*\|^2 \\
&\quad - \mu_1(2\alpha-\mu_1)\|B_1v_n - B_1y^*\|^2)] + \alpha_n M_1\} \\
&\leq [1 - \frac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n - x^*\|^2 - (1-\beta_n)(1-\sigma_n)[1 - \frac{\alpha_n(\tau+\delta)}{2}] \\
&\quad \times \{\mu_2(2\beta-\mu_2)\|B_2w_n - B_2x^*\|^2 \\
&\quad + \mu_1(2\alpha-\mu_1)\|B_1v_n - B_1y^*\|^2\} + \alpha_n M_1 \\
&\leq \|x_n - x^*\|^2 - (1-\beta_n)(1-\sigma_n)[1 - \frac{\alpha_n(\tau+\delta)}{2}]\{\mu_2(2\beta-\mu_2)\|B_2w_n - B_2x^*\|^2 \\
&\quad + \mu_1(2\alpha-\mu_1)\|B_1v_n - B_1y^*\|^2\} + \alpha_n M_1,
\end{aligned}
$$

which immediately arrives at

$$
\begin{aligned}
&(1-\beta_n)(1-\sigma_n)[1 - \frac{\alpha_n(\tau+\delta)}{2}]\{\mu_2(2\beta-\mu_2)\|B_2w_n - B_2x^*\|^2 + \mu_1(2\alpha-\mu_1)\|B_1v_n - B_1y^*\|^2\} \\
&\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n M_1 = \Lambda_n - \Lambda_{n+1} + \alpha_n M_1.
\end{aligned}
$$

Since $\beta_n \leq b < 1$, $\mu_1 \in (0, 2\alpha)$, $\mu_2 \in (0, 2\beta)$, $\lim_{n\to\infty}\alpha_n = 0$ and $\limsup_{n\to\infty}\sigma_n < 1$, we obtain from $\Lambda_n - \Lambda_{n+1} \to 0$ that

$$
\lim_{n\to\infty}\|B_2w_n - B_2x^*\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|B_1v_n - B_1y^*\| = 0. \tag{3.27}
$$

On the other hand, by Lemma 1.1 one has

$$
\begin{aligned}
\|u_n - x^*\|^2 &\leq \langle v_n - y^*, u_n - x^*\rangle + \mu_1\langle B_1y^* - B_1v_n, u_n - x^*\rangle \\
&\leq \frac{1}{2}[\|v_n - y^*\|^2 + \|u_n - x^*\|^2 - \|v_n - u_n + x^* - y^*\|^2] + \mu_1\|B_1y^* - B_1v_n\|\|u_n - x^*\|,
\end{aligned}
$$

which hence leads to

$$
\|u_n - x^*\|^2 \leq \|v_n - y^*\|^2 - \|v_n - u_n + x^* - y^*\|^2 + 2\mu_1\|B_1y^* - B_1v_n\|\|u_n - x^*\|.
$$

Similarly, one gets

$$
\|v_n - y^*\|^2 \leq \|w_n - x^*\|^2 - \|w_n - v_n + y^* - x^*\|^2 + 2\mu_2\|B_2x^* - B_2w_n\|\|v_n - y^*\|.
$$

Combining the last two inequalities, from (3.19) we get

$$
\begin{aligned}
\|u_n - x^*\|^2 \leq{}& \|x_n - x^*\|^2 - \|w_n - v_n + y^* - x^*\|^2 - \|v_n - u_n + x^* - y^*\|^2 \\
&+ 2\mu_1\|B_1y^* - B_1v_n\|\|u_n - x^*\| + 2\mu_2\|B_2x^* - B_2w_n\|\|v_n - y^*\|.
\end{aligned}
$$

This together with (3.20), ensures that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 \leq{}& \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n - x^*\|^2 + [(1-\alpha_n\tau) + \theta_n] \\
&\times [\sigma_n\|x_n - x^*\|^2 + (1-\sigma_n)\|u_n - x^*\|^2] + 2\alpha_n\langle(f - \mu F)x^*, q_n - x^*\rangle\}
\end{aligned}
$$

$$\begin{aligned}
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n - x^*\|^2 + [(1-\alpha_n\tau) + \frac{\alpha_n(\tau-\delta)}{2}][\sigma_n\|x_n - x^*\|^2 \\
&\quad + (1-\sigma_n)(\|x_n - x^*\|^2 - \|w_n - v_n + y^* - x^*\|^2 - \|v_n - u_n + x^* - y^*\|^2 \\
&\quad + 2\mu_1\|B_1y^* - B_1v_n\|\|u_n - x^*\| + 2\mu_2\|B_2x^* - B_2w_n\|\|v_n - y^*\|)] + \alpha_n M_1\} \\
&\le [1 - \frac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n - x^*\|^2 - (1-\beta_n)(1-\sigma_n)[1 - \frac{\alpha_n(\tau+\delta)}{2}]\{\|w_n - v_n \\
&\quad + y^* - x^*\|^2 + \|v_n - u_n + x^* - y^*\|^2\} + 2\mu_1\|B_1y^* - B_1v_n\|\|u_n - x^*\| \\
&\quad + 2\mu_2\|B_2x^* - B_2w_n\|\|v_n - y^*\| + \alpha_n M_1 \\
&\le \|x_n - x^*\|^2 - (1-\beta_n)(1-\sigma_n)[1 - \frac{\alpha_n(\tau+\delta)}{2}]\{\|w_n - v_n + y^* - x^*\|^2 \\
&\quad + \|v_n - u_n + x^* - y^*\|^2\} + 2\mu_1\|B_1y^* - B_1v_n\|\|u_n - x^*\| \\
&\quad + 2\mu_2\|B_2x^* - B_2w_n\|\|v_n - y^*\| + \alpha_n M_1,
\end{aligned}$$

which immediately leads to

$$\begin{aligned}
&(1-\beta_n)(1-\sigma_n)[1 - \frac{\alpha_n(\tau+\delta)}{2}]\{\|w_n - v_n + y^* - x^*\|^2 + \|v_n - u_n + x^* - y^*\|^2\} \\
&\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\mu_1\|B_1y^* - B_1v_n\|\|u_n - x^*\| \\
&\quad + 2\mu_2\|B_2x^* - B_2w_n\|\|v_n - y^*\| + \alpha_n M_1 \\
&= \Lambda_n - \Lambda_{n+1} + 2\mu_1\|B_1y^* - B_1v_n\|\|u_n - x^*\| + 2\mu_2\|B_2x^* - B_2w_n\|\|v_n - y^*\| + \alpha_n M_1.
\end{aligned}$$

Since $\beta_n \le b < 1$, $\lim_{n\to\infty}\alpha_n = 0$, $\limsup_{n\to\infty}\sigma_n < 1$ and $\limsup_{n\to\infty}(\Lambda_n - \Lambda_{n+1}) = 0$, we deduce from (3.27) and the boundedness of $\{u_n\}, \{v_n\}$ that

$$\lim_{n\to\infty}\|w_n - v_n + y^* - x^*\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|v_n - u_n + x^* - y^*\| = 0.$$

Therefore,

$$\begin{aligned}
\|w_n - \mathcal{G}w_n\| &= \|w_n - u_n\| \\
&\le \|w_n - v_n + y^* - x^*\| + \|v_n - u_n + x^* - y^*\| \to 0 \quad (n \to \infty).
\end{aligned} \tag{3.28}$$

Noticing $w_n = \sigma_n x_n + (1-\sigma_n)u_n$, we get

$$\begin{aligned}
\|w_n - x^*\|^2 &= \sigma_n\langle x_n - x^*, w_n - x^*\rangle + (1-\sigma_n)\langle u_n - x^*, w_n - x^*\rangle \\
&\le \sigma_n\langle x_n - x^*, w_n - x^*\rangle + (1-\sigma_n)\|w_n - x^*\|^2,
\end{aligned}$$

which immediately yields

$$\|w_n - x^*\|^2 \le \langle x_n - x^*, w_n - x^*\rangle \le \frac{1}{2}[\|x_n - x^*\|^2 + \|w_n - x^*\|^2 - \|x_n - w_n\|^2].$$

So it follows that

$$\|w_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|x_n - w_n\|^2,$$

which together with (3.20), leads to

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\{\alpha_n\delta\|x_n-x^*\|^2+[(1-\alpha_n\tau)+\theta_n]\\&\quad\times[\sigma_n\|x_n-x^*\|^2+(1-\sigma_n)\|w_n-x^*\|^2]+2\alpha_n\langle(f-\mu F)x^*,q_n-x^*\rangle\}\\&\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\{\alpha_n\delta\|x_n-x^*\|^2+[(1-\alpha_n\tau)+\frac{\alpha_n(\tau-\delta)}{2}]\\&\quad\times[\sigma_n\|x_n-x^*\|^2+(1-\sigma_n)(\|x_n-x^*\|^2-\|x_n-w_n\|^2)]+\alpha_nM_1\}\\&\le[1-\frac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n-x^*\|^2-(1-\beta_n)(1-\sigma_n)[1-\frac{\alpha_n(\tau+\delta)}{2}]\\&\quad\times\|x_n-w_n\|^2+\alpha_nM_1\\&\le\|x_n-x^*\|^2-(1-\beta_n)(1-\sigma_n)[1-\frac{\alpha_n(\tau+\delta)}{2}]\|x_n-w_n\|^2+\alpha_nM_1.\end{aligned}$$

This hence arrives at

$$\begin{aligned}(1-\beta_n)(1-\sigma_n)[1-\frac{\alpha_n(\tau+\delta)}{2}]\|x_n-w_n\|^2&\le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_nM_1\\&=\Lambda_n-\Lambda_{n+1}+\alpha_nM_1.\end{aligned}$$

Since $\beta_n\le b<1$, $\lim_{n\to\infty}\alpha_n=0$, $\limsup_{n\to\infty}\sigma_n<1$ and $\limsup_{n\to\infty}(\Lambda_n-\Lambda_{n+1})=0$, we deduce from $\frac{\tau+\delta}{2}\in(0,1)$ that

$$\lim_{n\to\infty}\|x_n-w_n\|=0.$$

So it follows from (3.26) and (3.28) that

$$\|x_n-z_n\|\le\|x_n-w_n\|+\|w_n-z_n\|\to0\quad(n\to\infty),\tag{3.29}$$

and

$$\|x_n-\mathcal{G}w_n\|\le\|x_n-w_n\|+\|w_n-\mathcal{G}w_n\|\to0\quad(n\to\infty).\tag{3.30}$$

Meanwhile, from Aspect 3 we obtain

$$\begin{aligned}(1-\beta_n)[1-\frac{\alpha_n(\tau+\delta)}{2}][\frac{\tau_n}{2\lambda\bar{L}}\|R_\lambda(w_n)\|^2]^2&\le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_nM_1\\&=\Lambda_n-\Lambda_{n+1}+\alpha_nM_1.\end{aligned}$$

Noticing $\beta_n\le b<1$, $\alpha_n\to0$ and $\Lambda_n-\Lambda_{n+1}\to0$, one gets

$$\lim_{n\to\infty}[\frac{\tau_n}{2\lambda\bar{L}}\|R_\lambda(w_n)\|^2]^2=0,$$

which together with Lemma 3.5, yields

$$\lim_{n\to\infty}\|w_n-y_n\|=0.\tag{3.31}$$

By the boundedness of $\{x_n\}$, we know that $\exists$ subsequence $\{x_{n_p}\}\subset\{x_n\}$ s.t.

$$\limsup_{n\to\infty}\langle(f-\mu F)x^*,x_n-x^*\rangle=\lim_{p\to\infty}\langle(f-\mu F)x^*,x_{n_p}-x^*\rangle.\tag{3.32}$$

Since H is reflexive and $\{x_n\}$ is bounded, we might assume that $x_{n_p} \rightharpoonup \bar{x}$. Thus, from (3.32) one has

$$\begin{aligned}\limsup_{n\to\infty}\langle(f-\mu F)x^*, x_n - x^*\rangle &= \lim_{p\to\infty}\langle(f-\mu F)x^*, x_{n_p} - x^*\rangle \\ &= \langle(f-\mu F)x^*, \bar{x} - x^*\rangle.\end{aligned} \tag{3.33}$$

Since $x_n - x_{n+1} \to 0,\ x_n - \mathcal{G}w_n \to 0,\ w_n - y_n \to 0,\ x_n - z_n \to 0$ and $x_{n_p} \rightharpoonup \bar{x}$, by Lemma 3.4 we infer that $\bar{x} \in \Omega$. Thus, using (3.16) and (3.33) one has

$$\limsup_{n\to\infty}\langle(f-\mu F)x^*, x_n - x^*\rangle = \langle(f-\mu F)x^*, \bar{x} - x^*\rangle \le 0, \tag{3.34}$$

which together with (3.26), arrives at

$$\begin{aligned}&\limsup_{n\to\infty}\langle(f-\mu F)x^*, q_n - x^*\rangle \\ &= \limsup_{n\to\infty}[\langle(f-\mu F)x^*, q_n - P_C(q_n) + P_C(q_n) - x_n\rangle + \langle(f-\mu F)x^*, x_n - x^*\rangle] \\ &\le \limsup_{n\to\infty}[\|(f-\mu F)x^*\|(\|q_n - P_C(q_n)\| + \|P_C(q_n) - x_n\|) + \langle(f-\mu F)x^*, x_n - x^*\rangle] \le 0.\end{aligned} \tag{3.35}$$

Note that $\{\alpha_n(1-\beta_n)(\tau-\delta)\} \subset [0,1],\ \sum_{n=1}^\infty \alpha_n(1-\beta_n)(\tau-\delta) = \infty$, and

$$\limsup_{n\to\infty}\left[\frac{2\langle(f-\mu F)x^*, q_n - x^*\rangle}{\tau-\delta} + \frac{\theta_n}{\alpha_n}\cdot\frac{M}{\tau-\delta}\right] \le 0.$$

Consequently, applying Lemma 2.5 to (3.25), one has $\lim_{n\to\infty}\|x_n - x^*\|^2 = 0$.

Situation 2. $\exists\{\Lambda_{n_p}\} \subset \{\Lambda_n\}$ s.t. $\Lambda_{n_p} < \Lambda_{n_p+1}\ \forall p \in \mathcal{N}$, with $\mathcal{N}$ being the set of all natural numbers. Let $\phi : \mathcal{N} \to \mathcal{N}$ be formulated as

$$\phi(n) := \max\{p \le n : \Lambda_p < \Lambda_{p+1}\}.$$

Using Lemma 2.9, we have

$$\Lambda_{\phi(n)} \le \Lambda_{\phi(n)+1} \quad \text{and} \quad \Lambda_n \le \Lambda_{\phi(n)+1}.$$

By Aspect 2 one gets

$$\begin{aligned}&(1-b)\{[1-\frac{\alpha_{\phi(n)}(\tau+\delta)}{2}]\|w_{\phi(n)} - z_{\phi(n)}\|^2 + \|q_{\phi(n)} - P_C(q_{\phi(n)})\|^2\} \\ &\quad + a(1-b)\|x_{\phi(n)} - P_C(q_{\phi(n)})\|^2 \\ &\le (1-\beta_{\phi(n)})\{[1-\frac{\alpha_{\phi(n)}(\tau+\delta)}{2}]\|w_{\phi(n)} - z_{\phi(n)}\|^2 + \|q_{\phi(n)} - P_C(q_{\phi(n)})\|^2\} \\ &\quad + \beta_{\phi(n)}(1-\beta_{\phi(n)})\|x_{\phi(n)} - P_C(q_{\phi(n)})\|^2 \\ &\le \|x_{\phi(n)} - x^*\|^2 - \|x_{\phi(n)+1} - x^*\|^2 + \alpha_{\phi(n)}M_1 = \Lambda_{\phi(n)} - \Lambda_{\phi(n)+1} + \alpha_{\phi(n)}M_1,\end{aligned} \tag{3.36}$$

which immediately ensures that

$$\lim_{n\to\infty}\|w_{\phi(n)} - z_{\phi(n)}\| = \lim_{n\to\infty}\|q_{\phi(n)} - P_C(q_{\phi(n)})\| = \lim_{n\to\infty}\|x_{\phi(n)} - P_C(q_{\phi(n)})\| = 0.$$

By Aspect 3 we have

$$
\begin{aligned}
(1-\beta_{\phi(n)})[1-\frac{\alpha_{\phi(n)}(\tau+\delta)}{2}][\frac{\tau_{\phi(n)}}{2\lambda\bar{L}}\|R_\lambda(w_{\phi(n)})\|^2]^2 &\le \|x_{\phi(n)}-x^*\|^2-\|x_{\phi(n)+1}-x^*\|^2+\alpha_{\phi(n)}M_1 \\
&= \Lambda_{\phi(n)}-\Lambda_{\phi(n)+1}+\alpha_{\phi(n)}M_1,
\end{aligned}
$$

which hence leads to

$$\lim_{n\to\infty}[\frac{\tau_{\phi(n)}}{2\lambda\bar{L}}\|R_\lambda(w_{\phi(n)})\|^2]^2 = 0.$$

Using the similar arguments to those of Situation 1, we infer that $\lim_{n\to\infty}\|x_{\phi(n)+1}-x_{\phi(n)}\| = 0$,

$$\lim_{n\to\infty}\|x_{\phi(n)}-\mathcal{G}w_{\phi(n)}\| = \lim_{n\to\infty}\|w_{\phi(n)}-y_{\phi(n)}\| = \lim_{n\to\infty}\|x_{\phi(n)}-z_{\phi(n)}\| = 0,$$

and

$$\limsup_{n\to\infty}\langle(f-\mu F)x^*, q_{\phi(n)}-x^*\rangle \le 0. \tag{3.37}$$

On the other hand, from (3.25) we obtain

$$
\begin{aligned}
&\alpha_{\phi(n)}(1-\beta_{\phi(n)})(\tau-\delta)\Lambda_{\phi(n)} \\
&\le \Lambda_{\phi(n)}-\Lambda_{\phi(n)+1}+\alpha_{\phi(n)}(1-\beta_{\phi(n)})(\tau-\delta)[\frac{2\langle(f-\mu F)x^*, q_{\phi(n)}-x^*\rangle}{\tau-\delta}+\frac{\theta_{\phi(n)}}{\alpha_{\phi(n)}}\cdot\frac{M}{\tau-\delta}] \\
&\le \alpha_{\phi(n)}(1-\beta_{\phi(n)})(\tau-\delta)[\frac{2\langle(f-\mu F)x^*, q_{\phi(n)}-x^*\rangle}{\tau-\delta}+\frac{\theta_{\phi(n)}}{\alpha_{\phi(n)}}\cdot\frac{M}{\tau-\delta}],
\end{aligned}
$$

which immediately attains

$$\limsup_{n\to\infty}\Lambda_{\phi(n)} \le \limsup_{n\to\infty}[\frac{2\langle(f-\mu F)x^*, q_{\phi(n)}-x^*\rangle}{\tau-\delta}+\frac{\theta_{\phi(n)}}{\alpha_{\phi(n)}}\cdot\frac{M}{\tau-\delta}] \le 0.$$

Thus, $\lim_{n\to\infty}\|x_{\phi(n)}-x^*\|^2 = 0$. In addition, observe that

$$
\begin{aligned}
\|x_{\phi(n)+1}-x^*\|^2-\|x_{\phi(n)}-x^*\|^2 &= 2\langle x_{\phi(n)+1}-x_{\phi(n)}, x_{\phi(n)}-x^*\rangle+\|x_{\phi(n)+1}-x_{\phi(n)}\|^2 \\
&\le 2\|x_{\phi(n)+1}-x_{\phi(n)}\|\|x_{\phi(n)}-x^*\|+\|x_{\phi(n)+1}-x_{\phi(n)}\|^2.
\end{aligned} \tag{3.38}
$$

Thanks to $\Lambda_n \le \Lambda_{\phi(n)+1}$, one gets

$$\|x_n-x^*\|^2 \le \|x_{\phi(n)}-x^*\|^2+2\|x_{\phi(n)+1}-x_{\phi(n)}\|\|x_{\phi(n)}-x^*\|+\|x_{\phi(n)+1}-x_{\phi(n)}\|^2 \to 0(n\to\infty).$$

That is, $x_n \to x^*$ as $n\to\infty$. This completes the proof. □

Theorem 3.7. *If $S : C \to C$ is nonexpansive and $\{x_n\}$ is the sequence constructed in the modified version of Algorithm 3.1, that is, for any initial $x_1 \in C$,*

$$
\begin{cases}
w_n = \sigma_n x_n+(1-\sigma_n)u_n, \\
v_n = T_{\mu_2}^{\Theta_2}(w_n-\mu_2 B_2 w_n), \\
u_n = T_{\mu_1}^{\Theta_1}(v_n-\mu_1 B_1 v_n), \\
y_n = P_C(w_n-\lambda A w_n), \\
t_n = (1-\tau_n)w_n+\tau_n y_n, \\
z_n = P_{C_n}(w_n), \\
x_{n+1} = \beta_n x_n+(1-\beta_n)P_C[\alpha_n f(x_n)+(I-\alpha_n\mu F)Sz_n] \quad \forall n\ge 1,
\end{cases} \tag{3.39}
$$

where for each $n \geq 1$, C_n and τ_n are picked as in Algorithm 3.1, then $x_n \to x^* \in \Omega$ with $x^* \in \Omega$ being only a solution to the HVI: $\langle(\mu F - f)x^*, y - x^*\rangle \geq 0, \ \ \forall y \in \Omega$.

Proof. We divide the proof of the theorem into several aspects.

Aspect 1. We assert the boundedness of $\{x_n\}$. Indeed, using the same reasonings as in Aspect 1 of the proof of Theorem 3.6, one derives the desired assertion.

Aspect 2. We assert that

$$(1-\beta_n)\{(1-\alpha_n\tau)\|w_n - z_n\|^2 + \|q_n - P_C(q_n)\|^2\} + \beta_n(1-\beta_n)\|x_n - P_C(q_n)\|^2 \\ \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n M_1,$$

for some $M_1 > 0$. In fact, putting $\theta_n = 0$, from (3.20) we get

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \leq\ & \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n - x^*\|^2 + (1-\alpha_n\tau)[\sigma_n\|x_n - x^*\|^2 \\ & + (1-\sigma_n)\|u_n - x^*\|^2 - \|w_n - z_n\|^2] + 2\alpha_n\langle(f-\mu F)x^*, q_n - x^*\rangle \\ & - \|q_n - P_C(q_n)\|^2\} - \beta_n(1-\beta_n)\|x_n - P_C(q_n)\|^2 \\ \leq\ & [1-\alpha_n(1-\beta_n)(\tau-\delta)]\|x_n - x^*\|^2 - (1-\beta_n)\{(1-\alpha_n\tau)\|w_n - z_n\|^2 \\ & + \|q_n - P_C(q_n)\|^2\} + \alpha_n M_1 - \beta_n(1-\beta_n)\|x_n - P_C(q_n)\|^2 \\ \leq\ & \|x_n - x^*\|^2 - (1-\beta_n)\{(1-\alpha_n\tau)\|w_n - z_n\|^2 + \|q_n - P_C(q_n)\|^2\} \\ & + \alpha_n M_1 - \beta_n(1-\beta_n)\|x_n - P_C(q_n)\|^2,\end{aligned}$$

where $\sup_{n\geq 1} 2\|(f-\mu F)x^*\|\|q_n - x^*\| \leq M_1$ for some $M_1 > 0$. This attains the desired assertion.

Aspect 3. We assert that

$$(1-\beta_n)(1-\alpha_n\tau)[\frac{\tau_n}{2\lambda\bar{L}}\|R_\lambda(w_n)\|^2]^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n M_1.$$

Indeed, using the same reasonings as in Aspect 3 of the proof of Theorem 3.6, one deduces the desired assertion.

Aspect 4. We assert that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \leq\ & [1-\alpha_n(1-\beta_n)(\tau-\delta)]\|x_n - x^*\|^2 \\ & + \alpha_n(1-\beta_n)(\tau-\delta)\cdot\frac{2\langle(f-\mu F)x^*, q_n - x^*\rangle}{\tau-\delta}.\end{aligned} \tag{3.40}$$

Indeed, using the same reasonings as in Aspect 4 of the proof of Theorem 3.6, one obtains the desired assertion.

Aspect 5. We assert that $\{x_n\}$ converges strongly to the unique solution $x^* \in \Omega$ of the HVI (3.16). Indeed, using the same reasonings as in Aspect 5 of the proof of Theorem 3.6, one gets the desired assertion. □

On the other hand, we put forth another modification of Mann-like subgradient-like extragradient implicit rule with linear-search process.

Algorithm 3.8. Initial step: *Given $\nu > 0$, $\ell \in (0,1)$, $\lambda \in (0, \frac{1}{\nu})$. Let $x_1 \in C$ be arbitrary.*

Iterations: *Given the current iterate x_n, calculate x_{n+1} below:*

Step 1. Calculate $w_n = \sigma_n x_n + (1-\sigma_n)u_n$ *with*

$$v_n = T^{\Theta_2}_{\mu_2}(w_n - \mu_2 B_2 w_n),$$

$$u_n = T^{\Theta_1}_{\mu_1}(v_n - \mu_1 B_1 v_n).$$

Step 2. Calculate $y_n = P_C(w_n - \lambda A w_n)$ *and* $R_\lambda(w_n) := w_n - y_n$.

Step 3. Calculate $t_n = w_n - \tau_n R_\lambda(w_n)$, *where* $\tau_n := \ell^{j_n}$ *and* j_n *is the smallest nonnegative integer* j *satisfying*

$$\langle Aw_n - A(w_n - \ell^j R_\lambda(w_n)), w_n - y_n\rangle \le \frac{\nu}{2}\|R_\lambda(w_n)\|^2.$$

Step 4. Compute $z_n = P_{C_n}(w_n)$ *and* $x_{n+1} = \beta_n w_n + (1-\beta_n)P_C[\alpha_n f(z_n) + (I - \alpha_n\mu F)S^n z_n]$, *where* $C_n := \{u \in C : \hbar_n(u) \le 0\}$ *and*

$$\hbar_n(u) = \langle At_n, u - w_n\rangle + \frac{\tau_n}{2\lambda}\|R_\lambda(w_n)\|^2.$$

Again put $n := n+1$ *and return to Step 1.*

It is worth mentioning that (3.16)–(3.19) and Lemmas 3.2–3.5 remain true for Algorithm 3.8.

Theorem 3.9. *Suppose that* $\{x_n\}$ *is the sequence constructed in Algorithm 3.8. Then* $x_n \to x^* \in \Omega$ *provided* $S^n x_n - S^{n+1}x_n \to 0$, *with* $x^* \in \Omega$ *being only a solution to the HVI:* $\langle(\mu F - f)x^*, y - x^*\rangle \ge 0,\ \forall y \in \Omega$.

Proof. In what follows, under the assumption $S^n x_n - S^{n+1}x_n \to 0$, one divides the proof into several aspects.

Aspect 1. We assert that $\{x_n\}$ is of boundedness. Indeed, for $x^* \in \Omega = \mathrm{Fix}(S) \cap \mathrm{Fix}(\mathcal{G}) \cap \mathrm{VI}(C, A)$ we have $Sx^* = x^*$, $\mathcal{G}x^* = x^*$ and $P_C(x^* - \lambda Ax^*) = x^*$. Using (3.19), from Lemma 2.10 we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\| &\le \beta_n\|w_n - x^*\| + (1-\beta_n)\{\alpha_n\delta\|z_n - x^*\| + (1-\alpha_n\tau)(1+\theta_n)\|z_n - x^*\| + \alpha_n\|(f-\mu F)x^*\|\} \\
&\le \beta_n\|x_n - x^*\| + (1-\beta_n)\{[\alpha_n\delta + (1-\alpha_n\tau) + \theta_n]\|x_n - x^*\| + \alpha_n\|(f-\mu F)x^*\|\} \\
&\le \beta_n\|x_n - x^*\| + (1-\beta_n)\{[1 - \alpha_n(\tau-\delta) + \frac{\alpha_n(\tau-\delta)}{2}]\|x_n - x^*\| + \alpha_n\|(f-\mu F)x^*\|\} \\
&= [1 - \frac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n - x^*\| + \frac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}\cdot\frac{2\|(f-\mu F)x^*\|}{\tau-\delta} \\
&\le \max\{\|x_n - x^*\|, \frac{2\|(f-\mu F)x^*\|}{\tau-\delta}\}.
\end{aligned}$$

By induction, we get $\|x_n - x^*\| \le \max\{\|x_1 - x^*\|, \frac{2\|(f-\mu F)x^*\|}{\tau-\delta}\}\ \forall n \ge 1$. Thus, $\{x_n\}$ is bounded, and so are the sequences $\{w_n\}, \{y_n\}, \{z_n\}, \{f(z_n)\}, \{At_n\}, \{\mathcal{G}w_n\}, \{S^n z_n\}$.

Aspect 2. We assert that

$$\begin{aligned}
&(1-\beta_n)\{[1 - \frac{\alpha_n(\tau+\delta)}{2}]\|w_n - z_n\|^2 + \|\bar{q}_n - P_C(\bar{q}_n)\|^2\} + \beta_n(1-\beta_n)\|w_n - P_C(\bar{q}_n)\|^2 \\
&\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n M_1,
\end{aligned}$$

for some $M_1 > 0$. In fact, it is clear that

$$\|z_n - x^*\|^2 \le \|w_n - x^*\|^2 - \|w_n - z_n\|^2 \le \sigma_n\|x_n - x^*\|^2 + (1-\sigma_n)\|u_n - x^*\|^2 - \|w_n - z_n\|^2.$$

Since $x_{n+1} = \beta_n w_n + (1-\beta_n)P_C(\bar{q}_n)$ where $\bar{q}_n = \alpha_n f(z_n) + (I - \alpha_n\mu F)S^n z_n$, Using Lemma 2.10 and the convexity of the function $h(s) = s^2\ \forall s \in \mathbf{R}$, from (3.19) we obtain that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \beta_n\|w_n - x^*\|^2 + (1-\beta_n)\|P_C(\bar{q}_n) - x^*\|^2 - \beta_n(1-\beta_n)\|w_n - P_C(\bar{q}_n)\|^2 \\
&\le \beta_n\|w_n - x^*\|^2 + (1-\beta_n)\{\|\bar{q}_n - x^*\|^2 - \|\bar{q}_n - P_C(\bar{q}_n)\|^2\} - \beta_n(1-\beta_n)\|w_n - P_C(\bar{q}_n)\|^2 \\
&= \beta_n\|w_n - x^*\|^2 + (1-\beta_n)\{\|\alpha_n(f(z_n) - f(x^*)) + (I-\alpha_n\mu F)S^n z_n - (I-\alpha_n\mu F)x^* \\
&\quad + \alpha_n(f-\mu F)x^*\|^2 - \|\bar{q}_n - P_C(\bar{q}_n)\|^2\} - \beta_n(1-\beta_n)\|w_n - P_C(\bar{q}_n)\|^2 \\
&\le \beta_n\|w_n - x^*\|^2 + (1-\beta_n)\{\|\alpha_n(f(z_n) - f(x^*)) + (I-\alpha_n\mu F)S^n z_n - (I-\alpha_n\mu F)x^*\|^2 \\
&\quad + 2\alpha_n\langle (f-\mu F)x^*, \bar{q}_n - x^*\rangle - \|\bar{q}_n - P_C(\bar{q}_n)\|^2\} - \beta_n(1-\beta_n)\|w_n - P_C(\bar{q}_n)\|^2 \\
&\le \beta_n\|w_n - x^*\|^2 + (1-\beta_n)\{[\alpha_n\|f(z_n) - f(x^*)\| + \|(I-\alpha_n\mu F)S^n z_n - (I-\alpha_n\mu F)x^*\|]^2 \\
&\quad + 2\alpha_n\langle (f-\mu F)x^*, \bar{q}_n - x^*\rangle - \|\bar{q}_n - P_C(\bar{q}_n)\|^2\} - \beta_n(1-\beta_n)\|w_n - P_C(\bar{q}_n)\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n - x^*\|^2 + [(1-\alpha_n\tau) + \theta_n]\|z_n - x^*\|^2 \\
&\quad + 2\alpha_n\langle (f-\mu F)x^*, \bar{q}_n - x^*\rangle - \|\bar{q}_n - P_C(\bar{q}_n)\|^2\} - \beta_n(1-\beta_n)\|w_n - P_C(\bar{q}_n)\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n - x^*\|^2 + [(1-\alpha_n\tau) + \theta_n][\sigma_n\|x_n - x^*\|^2 \\
&\quad + (1-\sigma_n)\|u_n - x^*\|^2 - \|w_n - z_n\|^2] + 2\alpha_n\langle (f-\mu F)x^*, \bar{q}_n - x^*\rangle \\
&\quad - \|\bar{q}_n - P_C(\bar{q}_n)\|^2\} - \beta_n(1-\beta_n)\|w_n - P_C(\bar{q}_n)\|^2
\end{aligned} \tag{3.41}$$

(due to $\alpha_n\delta + (1-\alpha_n\tau) + \theta_n \le 1 - \alpha_n(\tau-\delta) + \frac{\alpha_n(\tau-\delta)}{2} = 1 - \frac{\alpha_n(\tau-\delta)}{2} \le 1$), which together with $u_n = \mathcal{G}w_n$, guarantees that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n - x^*\|^2 + [(1-\alpha_n\tau) + \frac{\alpha_n(\tau-\delta)}{2}][\sigma_n\|x_n - x^*\|^2 \\
&\quad + (1-\sigma_n)\|x_n - x^*\|^2 - \|w_n - z_n\|^2] + 2\alpha_n\langle (f-\mu F)x^*, \bar{q}_n - x^*\rangle \\
&\quad - \|\bar{q}_n - P_C(\bar{q}_n)\|^2\} - \beta_n(1-\beta_n)\|w_n - P_C(\bar{q}_n)\|^2 \\
&\le \|x_n - x^*\|^2 - (1-\beta_n)\{[1 - \frac{\alpha_n(\tau+\delta)}{2}]\|w_n - z_n\|^2 + \|\bar{q}_n - P_C(\bar{q}_n)\|^2\} \\
&\quad - \beta_n(1-\beta_n)\|w_n - P_C(\bar{q}_n)\|^2 + \alpha_n M_1,
\end{aligned} \tag{3.42}$$

where $\sup_{n\ge1} 2\|(f-\mu F)x^*\|\|\bar{q}_n - x^*\| \le M_1$ for some $M_1 > 0$. This attains the desired assertion.

Aspect 3. We assert that

$$(1-\beta_n)[1 - \frac{\alpha_n(\tau+\delta)}{2}][\frac{\tau_n}{2\lambda\bar{L}}\|R_\lambda(w_n)\|^2]^2 \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n M_1.$$

In fact, using the similar arguments to those of (3.22) in the proof of Theorem 3.6, we can deduce that for some $\bar{L} > 0$,

$$\|z_n - x^*\|^2 \le \|w_n - x^*\|^2 - [\frac{\tau_n}{2\lambda\bar{L}}\|R_\lambda(w_n)\|^2]^2. \tag{3.43}$$

From (3.41), (3.19) and (3.43) it follows that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\{\alpha_n\delta\|x_n-x^*\|^2+[(1-\alpha_n\tau)\\&\quad+\theta_n]\|z_n-x^*\|^2+2\alpha_n\langle(f-\mu F)x^*,\bar{q}_n-x^*\rangle\}\\&\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\{\alpha_n\delta\|x_n-x^*\|^2+[(1-\alpha_n\tau)+\theta_n][\|w_n-x^*\|^2\\&\quad-[\frac{\tau_n}{2\lambda\bar{L}}\|R_\lambda(w_n)\|^2]^2]+2\alpha_n\langle(f-\mu F)x^*,\bar{q}_n-x^*\rangle\}\\&\le[1-\frac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n-x^*\|^2-(1-\beta_n)[1-\frac{\alpha_n(\tau+\delta)}{2}][\frac{\tau_n}{2\lambda\bar{L}}\|R_\lambda(w_n)\|^2]^2\\&\quad+2\alpha_n(1-\beta_n)\langle(f-\mu F)x^*,\bar{q}_n-x^*\rangle\\&\le\|x_n-x^*\|^2-(1-\beta_n)[1-\frac{\alpha_n(\tau+\delta)}{2}][\frac{\tau_n}{2\lambda\bar{L}}\|R_\lambda(w_n)\|^2]^2+\alpha_nM_1,\end{aligned}$$

which hence yields the desired assertion.

Aspect 4. We assert that

$$\begin{aligned}\|x_{n+1}-x^*\|^2\le[1-\alpha_n(1-\beta_n)(\tau-\delta)]\|x_n-x^*\|^2+\alpha_n(1-\beta_n)(\tau-\delta)\\\times[\frac{2\langle(f-\mu F)x^*,\bar{q}_n-x^*\rangle}{\tau-\delta}+\frac{\theta_n}{\alpha_n}\cdot\frac{M}{\tau-\delta}]\end{aligned}\tag{3.44}$$

for some $M>0$. In fact, from Lemma 2.10 and (3.19), one obtains

$$\begin{aligned}\|x_{n+1}-x^*\|^2&\le\beta_n\|w_n-x^*\|^2+(1-\beta_n)\|\bar{q}_n-x^*\|^2\\&=\beta_n\|w_n-x^*\|^2+(1-\beta_n)\|\alpha_n(f(z_n)-f(x^*))+(I-\alpha_n\mu F)S^nz_n\\&\quad-(I-\alpha_n\mu F)x^*+\alpha_n(f-\mu F)x^*\|^2\\&\le\beta_n\|w_n-x^*\|^2+(1-\beta_n)\{\|\alpha_n(f(z_n)-f(x^*))+(I-\alpha_n\mu F)S^nz_n\\&\quad-(I-\alpha_n\mu F)x^*\|^2+2\alpha_n\langle(f-\mu F)x^*,\bar{q}_n-x^*\rangle\}\\&\le\beta_n\|w_n-x^*\|^2+(1-\beta_n)\{[\alpha_n\delta\|z_n-x^*\|+(1-\alpha_n\tau)(1+\theta_n)\|z_n-x^*\|]^2\\&\quad+2\alpha_n\langle(f-\mu F)x^*,\bar{q}_n-x^*\rangle\}\\&\le\beta_n\|x_n-x^*\|^2+(1-\beta_n)\{\alpha_n\delta\|x_n-x^*\|^2+(1-\alpha_n\tau+\theta_n)\|z_n-x^*\|^2\\&\quad+2\alpha_n\langle(f-\mu F)x^*,\bar{q}_n-x^*\rangle\}\\&\le[1-\alpha_n(1-\beta_n)(\tau-\delta)]\|x_n-x^*\|^2+\alpha_n(1-\beta_n)(\tau-\delta)\\&\quad\times[\frac{2\langle(f-\mu F)x^*,\bar{q}_n-x^*\rangle}{\tau-\delta}+\frac{\theta_n}{\alpha_n}\cdot\frac{M}{\tau-\delta}],\end{aligned}$$

where $\sup_{n\ge1}\|x_n-x^*\|^2\le M$ for some $M>0$.

Aspect 5. We assert that $x_n\to x^*\in\Omega$, which is only a solution of the HVI (3.16).

In fact, from (3.44), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2\le[1-\alpha_n(1-\beta_n)(\tau-\delta)]\|x_n-x^*\|^2+\alpha_n(1-\beta_n)(\tau-\delta)\\\times[\frac{2\langle(f-\mu F)x^*,\bar{q}_n-x^*\rangle}{\tau-\delta}+\frac{\theta_n}{\alpha_n}\cdot\frac{M}{\tau-\delta}].\end{aligned}\tag{3.45}$$

Setting $\Lambda_n=\|x_n-x^*\|^2$, we demonstrate the convergence of $\{\Lambda_n\}$ to zero by the following two situations.

Situation 1. $\exists$ (integer) $n_0 \ge 1$ s.t. $\{\Lambda_n\}$ is nonincreasing. It is clear that the limit $\lim_{n\to\infty}\Lambda_n = k < +\infty$ and $\lim_{n\to\infty}(\Lambda_n - \Lambda_{n+1}) = 0$. From Aspect 2 and $\{\beta_n\} \subset [a, b] \subset (0, 1)$ we obtain

$$
\begin{aligned}
&(1-b)\{[1-\frac{\alpha_n(\tau+\delta)}{2}]\|w_n - z_n\|^2 + \|\bar{q}_n - P_C(\bar{q}_n)\|^2\} + a(1-b)\|w_n - P_C(\bar{q}_n)\|^2 \\
&\le (1-\beta_n)\{[1-\frac{\alpha_n(\tau+\delta)}{2}]\|w_n - z_n\|^2 + \|\bar{q}_n - P_C(\bar{q}_n)\|^2\} + \beta_n(1-\beta_n)\|w_n - P_C(\bar{q}_n)\|^2 \\
&\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n M_1 = \Lambda_n - \Lambda_{n+1} + \alpha_n M_1.
\end{aligned}
$$

Owing to the facts that $\alpha_n \to 0$ and $\Lambda_n - \Lambda_{n+1} \to 0$, from $\frac{\tau+\delta}{2} \in (0, 1)$ one deduces that

$$
\lim_{n\to\infty}\|w_n - z_n\| = \lim_{n\to\infty}\|\bar{q}_n - P_C(\bar{q}_n)\| = \lim_{n\to\infty}\|w_n - P_C(\bar{q}_n)\| = 0. \tag{3.46}
$$

Hence it is readily known that

$$
\|w_n - \bar{q}_n\| \le \|w_n - P_C(\bar{q}_n)\| + \|P_C(\bar{q}_n) - \bar{q}_n\| \to 0 \quad (n\to\infty),
$$

$$
\|x_{n+1} - w_n\| = (1-\beta_n)\|P_C(\bar{q}_n) - w_n\| \le \|\bar{q}_n - w_n\| \to 0 \quad (n\to\infty), \tag{3.47}
$$

and

$$
\begin{aligned}
\|S^n z_n - w_n\| &= \|\bar{q}_n - w_n - \alpha_n(f(z_n) - \mu F S^n z_n)\| \\
&\le \|\bar{q}_n - w_n\| + \alpha_n(\|f(z_n)\| + \mu\|FS^n z_n\|) \to 0 \quad (n\to\infty).
\end{aligned}
$$

Next, we show that $\|x_n - u_n\| \to 0$ and $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$. Indeed, note that $y^* = T^{\Theta_2}_{\mu_2}(x^* - \mu_2 B_2 x^*)$, $v_n = T^{\Theta_2}_{\mu_2}(w_n - \mu_2 B_2 w_n)$ and $u_n = T^{\Theta_1}_{\mu_1}(v_n - \mu_1 B_1 v_n)$. Then $u_n = \mathcal{G} w_n$. Using Lemma 2.1, from (3.19) we have

$$
\|u_n - x^*\|^2 \le \|x_n - x^*\|^2 - \mu_2(2\beta - \mu_2)\|B_2 w_n - B_2 x^*\|^2 - \mu_1(2\alpha - \mu_1)\|B_1 v_n - B_1 y^*\|^2.
$$

This together with (3.41), implies that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n - x^*\|^2 + [(1-\alpha_n\tau) + \theta_n] \\
&\quad \times [\sigma_n\|x_n - x^*\|^2 + (1-\sigma_n)\|u_n - x^*\|^2] + 2\alpha_n\langle (f - \mu F)x^*, \bar{q}_n - x^*\rangle\} \\
&\le \|x_n - x^*\|^2 - (1-\beta_n)(1-\sigma_n)[1 - \frac{\alpha_n(\tau+\delta)}{2}]\{\mu_2(2\beta - \mu_2)\|B_2 w_n - B_2 x^*\|^2 \\
&\quad + \mu_1(2\alpha - \mu_1)\|B_1 v_n - B_1 y^*\|^2\} + \alpha_n M_1,
\end{aligned}
$$

which immediately arrives at

$$
\begin{aligned}
&(1-\beta_n)(1-\sigma_n)[1 - \frac{\alpha_n(\tau+\delta)}{2}]\{\mu_2(2\beta - \mu_2)\|B_2 w_n - B_2 x^*\|^2 + \mu_1(2\alpha - \mu_1)\|B_1 v_n - B_1 y^*\|^2\} \\
&\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n M_1 = \Lambda_n - \Lambda_{n+1} + \alpha_n M_1.
\end{aligned}
$$

This hence ensures that

$$
\lim_{n\to\infty}\|B_2 w_n - B_2 x^*\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|B_1 v_n - B_1 y^*\| = 0.
$$

On the other hand, using Lemma 1.1, from (3.19) we get

$$\begin{aligned}\|u_n - x^*\|^2 \le &\|x_n - x^*\|^2 - \|w_n - v_n + y^* - x^*\|^2 - \|v_n - u_n + x^* - y^*\|^2 \\ &+ 2\mu_1\|B_1y^* - B_1v_n\|\|u_n - x^*\| + 2\mu_2\|B_2x^* - B_2w_n\|\|v_n - y^*\|.\end{aligned}$$

This together with (3.41), implies that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \le &\beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n - x^*\|^2 + [(1-\alpha_n\tau) + \theta_n] \\ &\times [\sigma_n\|x_n - x^*\|^2 + (1-\sigma_n)\|u_n - x^*\|^2] + 2\alpha_n\langle (f - \mu F)x^*, \bar{q}_n - x^*\rangle\} \\ \le &\|x_n - x^*\|^2 - (1-\beta_n)(1-\sigma_n)[1 - \frac{\alpha_n(\tau+\delta)}{2}]\{\|w_n - v_n + y^* - x^*\|^2 \\ &+ \|v_n - u_n + x^* - y^*\|^2\} + 2\mu_1\|B_1y^* - B_1v_n\|\|u_n - x^*\| \\ &+ 2\mu_2\|B_2x^* - B_2w_n\|\|v_n - y^*\| + \alpha_nM_1,\end{aligned}$$

which immediately leads to

$$\begin{aligned}&(1-\beta_n)(1-\sigma_n)[1 - \frac{\alpha_n(\tau+\delta)}{2}]\{\|w_n - v_n + y^* - x^*\|^2 + \|v_n - u_n + x^* - y^*\|^2\} \\ &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\mu_1\|B_1y^* - B_1v_n\|\|u_n - x^*\| \\ &\quad + 2\mu_2\|B_2x^* - B_2w_n\|\|v_n - y^*\| + \alpha_nM_1 \\ &= \Lambda_n - \Lambda_{n+1} + 2\mu_1\|B_1y^* - B_1v_n\|\|u_n - x^*\| + 2\mu_2\|B_2x^* - B_2w_n\|\|v_n - y^*\| + \alpha_nM_1.\end{aligned}$$

This hence ensures that

$$\lim_{n\to\infty}\|w_n - v_n + y^* - x^*\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|v_n - u_n + x^* - y^*\| = 0.$$

Therefore,

$$\begin{aligned}\|w_n - \mathcal{G}w_n\| &= \|w_n - u_n\| \\ &\le \|w_n - v_n + y^* - x^*\| + \|v_n - u_n + x^* - y^*\| \to 0 \quad (n\to\infty).\end{aligned} \tag{3.48}$$

Noticing $w_n = \sigma_nx_n + (1-\sigma_n)u_n$, we get

$$\begin{aligned}\|w_n - x^*\|^2 &= \sigma_n\langle x_n - x^*, w_n - x^*\rangle + (1-\sigma_n)\langle u_n - x^*, w_n - x^*\rangle \\ &\le \sigma_n\langle x_n - x^*, w_n - x^*\rangle + (1-\sigma_n)\|w_n - x^*\|^2 \\ &= \frac{1}{2}\sigma_n[\|x_n - x^*\|^2 + \|w_n - x^*\|^2 - \|x_n - w_n\|^2] + (1-\sigma_n)\|w_n - x^*\|^2,\end{aligned}$$

which immediately yields

$$\|w_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|x_n - w_n\|^2.$$

This together with (3.41), arrives at

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \le &\beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n - x^*\|^2 + [(1-\alpha_n\tau) + \theta_n] \\ &\times [\sigma_n\|x_n - x^*\|^2 + (1-\sigma_n)\|w_n - x^*\|^2] + 2\alpha_n\langle (f - \mu F)x^*, \bar{q}_n - x^*\rangle\} \\ \le &\|x_n - x^*\|^2 - (1-\beta_n)(1-\sigma_n)[1 - \frac{\alpha_n(\tau+\delta)}{2}]\|x_n - w_n\|^2 + \alpha_nM_1.\end{aligned}$$

So it follows that

$$
\begin{aligned}
&(1-\beta_n)(1-\sigma_n)[1-\frac{\alpha_n(\tau+\delta)}{2}]\|x_n-w_n\|^2\\
&\le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_n M_1=\Lambda_n-\Lambda_{n+1}+\alpha_n M_1,
\end{aligned}
$$

which immediately yields

$$\lim_{n\to\infty}\|x_n-w_n\|=0.$$

So it follows from (3.46)–(3.48) that

$$\|x_n-z_n\|\le\|x_n-w_n\|+\|w_n-z_n\|\to 0 \quad (n\to\infty), \tag{3.49}$$

$$\|x_n-x_{n+1}\|\le\|x_n-w_n\|+\|w_n-x_{n+1}\|\to 0 \quad (n\to\infty), \tag{3.50}$$

and

$$\|x_n-\mathcal{G}w_n\|\le\|x_n-w_n\|+\|w_n-\mathcal{G}w_n\|\to 0 \quad (n\to\infty). \tag{3.51}$$

Also, using the similar arguments to those of (3.31) in the proof of Theorem 3.6, we can obtain that

$$\lim_{n\to\infty}\|w_n-y_n\|=0. \tag{3.52}$$

By the boundedness of $\{x_n\}$, we know that $\exists$ subsequence $\{x_{n_p}\}\subset\{x_n\}$ s.t.

$$\limsup_{n\to\infty}\langle(f-\mu F)x^*,x_n-x^*\rangle=\lim_{p\to\infty}\langle(f-\mu F)x^*,x_{n_p}-x^*\rangle. \tag{3.53}$$

Since H is reflexive and $\{x_n\}$ is bounded, we might assume that $x_{n_p}\rightharpoonup\bar{x}$. Thus, from (3.53) one has

$$
\begin{aligned}
\limsup_{n\to\infty}\langle(f-\mu F)x^*,x_n-x^*\rangle&=\lim_{p\to\infty}\langle(f-\mu F)x^*,x_{n_p}-x^*\rangle\\
&=\langle(f-\mu F)x^*,\bar{x}-x^*\rangle.
\end{aligned}
\tag{3.54}
$$

Since $x_n-x_{n+1}\to 0,\ x_n-\mathcal{G}w_n\to 0,\ w_n-y_n\to 0,\ x_n-z_n\to 0$ and $x_{n_p}\rightharpoonup\bar{x}$, by Lemma 3.4 we infer that $\bar{x}\in\Omega$. Thus, using (3.16) and (3.54) one has

$$\limsup_{n\to\infty}\langle(f-\mu F)x^*,x_n-x^*\rangle=\langle(f-\mu F)x^*,\bar{x}-x^*\rangle\le 0, \tag{3.55}$$

which together with (3.46), arrives at

$$
\begin{aligned}
&\limsup_{n\to\infty}\langle(f-\mu F)x^*,\bar{q}_n-x^*\rangle\\
&=\limsup_{n\to\infty}[\langle(f-\mu F)x^*,\bar{q}_n-P_C(\bar{q}_n)+P_C(\bar{q}_n)-w_n+w_n-x_n\rangle\\
&\quad+\langle(f-\mu F)x^*,x_n-x^*\rangle]\\
&\le\limsup_{n\to\infty}[\|(f-\mu F)x^*\|(\|\bar{q}_n-P_C(\bar{q}_n)\|+\|P_C(\bar{q}_n)-w_n\|+\|w_n-x_n\|)\\
&\quad+\langle(f-\mu F)x^*,x_n-x^*\rangle]\le 0.
\end{aligned}
\tag{3.56}
$$

Note that $\{\alpha_n(1-\beta_n)(\tau-\delta)\} \subset [0,1]$, $\sum_{n=1}^{\infty} \alpha_n(1-\beta_n)(\tau-\delta) = \infty$, and

$$\limsup_{n\to\infty}[\frac{2\langle(f-\mu F)x^*, \bar{q}_n - x^*\rangle}{\tau-\delta} + \frac{\theta_n}{\alpha_n}\cdot\frac{M}{\tau-\delta}] \leq 0.$$

Consequently, applying Lemma 2.5 to (3.45), one has $\lim_{n\to\infty}\|x_n - x^*\|^2 = 0$.

Situation 2. $\exists\{\Lambda_{n_p}\} \subset \{\Lambda_n\}$ s.t. $\Lambda_{n_p} < \Lambda_{n_p+1}$ $\forall p \in \mathcal{N}$, with $\mathcal{N}$ being the set of all natural numbers. Let $\phi : \mathcal{N} \to \mathcal{N}$ be formulated as

$$\phi(n) := \max\{p \leq n : \Lambda_p < \Lambda_{p+1}\}.$$

Using Lemma 2.9, we get

$$\Lambda_{\phi(n)} \leq \Lambda_{\phi(n)+1} \quad \text{and} \quad \Lambda_n \leq \Lambda_{\phi(n)+1}.$$

In the remainder of the proof, using the same reasonings as in Situation 2 of Aspect 5 in the proof of Theorem 3.6, we obtain the desired assertion. □

Theorem 3.10. *If $S : C \to C$ is nonexpansive and $\{x_n\}$ is the sequence constructed in the modified version of Algorithm 3.8, that is, for any initial $x_1 \in C$,*

$$\begin{cases} w_n = \sigma_n x_n + (1-\sigma_n)u_n, \\ v_n = T_{\mu_2}^{\Theta_2}(w_n - \mu_2 B_2 w_n), \\ u_n = T_{\mu_1}^{\Theta_1}(v_n - \mu_1 B_1 v_n), \\ y_n = P_C(w_n - \lambda A w_n), \\ t_n = (1-\tau_n)w_n + \tau_n y_n, \\ z_n = P_{C_n}(w_n), \\ x_{n+1} = \beta_n w_n + (1-\beta_n)P_C[\alpha_n f(z_n) + (I - \alpha_n\mu F)S z_n], \quad \forall n \geq 1, \end{cases} \tag{3.57}$$

where for each $n \geq 1$, C_n and τ_n are picked as in Algorithm 3.8, then $x_n \to x^* \in \Omega$ with $x^* \in \Omega$ being only a solution to the HVI: $\langle(\mu F - f)x^*, y - x^*\rangle \geq 0, \ \forall y \in \Omega$.

Proof. We divide the proof of the theorem into several aspects.

Aspect 1. We assert the boundedness of $\{x_n\}$. Indeed, using the same reasonings as in Aspect 1 of the proof of Theorem 3.9, one derives the desired assertion.

Aspect 2. We assert that

$$\begin{aligned}&(1-\beta_n)\{(1-\alpha_n\tau)\|w_n - z_n\|^2 + \|\bar{q}_n - P_C(\bar{q}_n)\|^2\} + \beta_n(1-\beta_n)\|w_n - P_C(\bar{q}_n)\|^2 \\ &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n M_1,\end{aligned}$$

for some $M_1 > 0$. In fact, putting $\theta_n = 0$, from (3.41) we get

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n - x^*\|^2 + (1-\alpha_n\tau)[\sigma_n\|x_n - x^*\|^2 \\ &\quad + (1-\sigma_n)\|u_n - x^*\|^2 - \|w_n - z_n\|^2] + 2\alpha_n\langle(f-\mu F)x^*, \bar{q}_n - x^*\rangle \\ &\quad - \|\bar{q}_n - P_C(\bar{q}_n)\|^2\} - \beta_n(1-\beta_n)\|w_n - P_C(\bar{q}_n)\|^2 \\ &\leq [1 - \alpha_n(1-\beta_n)(\tau-\delta)]\|x_n - x^*\|^2 - (1-\beta_n)\{(1-\alpha_n\tau)\|w_n - z_n\|^2 \\ &\quad + \|\bar{q}_n - P_C(\bar{q}_n)\|^2\} + \alpha_n M_1 - \beta_n(1-\beta_n)\|w_n - P_C(\bar{q}_n)\|^2 \\ &\leq \|x_n - x^*\|^2 - (1-\beta_n)\{(1-\alpha_n\tau)\|w_n - z_n\|^2 + \|\bar{q}_n - P_C(\bar{q}_n)\|^2\} \\ &\quad + \alpha_n M_1 - \beta_n(1-\beta_n)\|w_n - P_C(\bar{q}_n)\|^2,\end{aligned}$$

where $\sup_{n\geq 1} 2\|(f-\mu F)x^*\|\|\bar{q}_n - x^*\| \leq M_1$ for some $M_1 > 0$. This attains the desired assertion.

Aspect 3. We assert that

$$(1-\beta_n)(1-\alpha_n\tau)[\frac{\tau_n}{2\lambda\bar{L}}\|R_\lambda(w_n)\|^2]^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n M_1.$$

Indeed, using the same reasonings as in Aspect 3 of the proof of Theorem 3.9, one deduces the desired assertion.

Aspect 4. We assert that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \leq [1-\alpha_n(1-\beta_n)(\tau-\delta)]\|x_n - x^*\|^2 \\ + \alpha_n(1-\beta_n)(\tau-\delta)\cdot\frac{2\langle(f-\mu F)x^*, \bar{q}_n - x^*\rangle}{\tau-\delta}.\end{aligned} \tag{3.58}$$

Indeed, using the same reasonings as in Aspect 4 of the proof of Theorem 3.9, one obtains the desired assertion.

Aspect 5. We assert that $\{x_n\}$ converges strongly to the unique solution $x^* \in \Omega$ of the HVI (3.16). Indeed, using the same reasonings as in Aspect 5 of the proof of Theorem 3.9, one gets the desired assertion. □

Remark 3.11. Compared with the corresponding results in Cai, Shehu and Iyiola [13], Thong and Hieu [17] and Reich et al. [29], our results improve and extend them in the following aspects.

(i) The problem of finding an element of $\mathrm{Fix}(S) \cap \mathrm{Fix}(\mathcal{G})$ (with $\mathcal{G} = P_C(I-\mu_1 B_1)P_C(I-\mu_2 B_2)$) in [13] is extended to develop our problem of finding an element of $\mathrm{Fix}(S) \cap \mathrm{Fix}(\mathcal{G}) \cap \mathrm{VI}(C, A)$ where $\mathcal{G} = T_{\mu_1}^{\Theta_1}(I-\mu_1 B_1)T_{\mu_2}^{\Theta_2}(I-\mu_2 B_2)$ and S is asymptotically nonexpansive mapping. The modified viscosity implicit rule for finding an element of $\mathrm{Fix}(S) \cap \mathrm{Fix}(\mathcal{G})$ in [13] is extended to develop our modified Mann-like subgradient-like extragradient implicit rules with linear-search process for finding an element of $\mathrm{Fix}(S) \cap \mathrm{Fix}(\mathcal{G}) \cap \mathrm{VI}(C, A)$, which is on the basis of the subgradient extragradient rule with linear-search process, Mann implicit iteration approach, and hybrid deepest-descent technique.

(ii) The problem of finding an element of $\mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$ with quasi-nonexpansive mapping S in [17] is extended to develop our problem of finding an element of $\mathrm{Fix}(S) \cap \mathrm{Fix}(\mathcal{G}) \cap \mathrm{VI}(C, A)$ with asymptotically nonexpansive mapping S. The inertial subgradient extragradient method with linear-search process for finding an element of $\mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$ in [17] is extended to develop our modified Mann-like subgradient-like extragradient implicit rules with linear-search process for finding an element of $\mathrm{Fix}(S) \cap \mathrm{Fix}(\mathcal{G}) \cap \mathrm{VI}(C, A)$, which is on the basis of the subgradient extragradient rule with linear-search process, Mann implicit iteration approach, and hybrid deepest-descent technique.

(iii) The problem of finding an element of $\mathrm{VI}(C, A)$ with pseudomonotone uniform continuity mapping A is extended to develop our problem of finding an element of $\mathrm{Fix}(S) \cap \mathrm{Fix}(\mathcal{G}) \cap \mathrm{VI}(C, A)$ with both asymptotically nonexpansive mapping S and nonexpansive mapping $\mathcal{G}$. The modified projection-type method with linear-search process in [29] is extended to develop our modified Mann-like subgradient-like extragradient implicit rule with linear-search process, e.g., the original projection step $y_n = P_C(x_n - \lambda A x_n)$ in [29] is developed into the modified Mann-like implicit projection step $w_n = (1-\sigma_n)x_n + \sigma_n \mathcal{G} w_n$ and $y_n = P_C(w_n - \lambda A w_n)$; meantime, the original viscosity step $x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)P_{C_n}(x_n)$ is developed into the composite viscosity iterative step $x_{n+1} = P_C[\alpha_n f(x_n) + (I - \alpha_n \mu F)S^n P_{C_n}(w_n)]$.

4. Examples

In what follows, we provide an illustrated instance to show the feasibility and implementability of suggested rules. Put $\Theta_1 = \Theta_2 = 0, \mu = 2, \mu_1 = \mu_2 = \frac{1}{3},\ \nu = 1,\ \lambda = \ell = \frac{1}{2},\ \sigma_n = \beta_n = \frac{2}{3}$ and $\alpha_n = \frac{1}{3(n+1)}$. We first provide an example of two inverse-strongly monotone mappings $B_1, B_2 : C \to H$, Lipschitz continuous and pseudomonotone mapping A and asymptotically nonexpansive mapping S with $\Omega = \mathrm{Fix}(S) \cap \mathrm{Fix}(\mathcal{G}) \cap \mathrm{VI}(C, A) \neq \emptyset$, where $\mathcal{G} := T^{\Theta_1}_{\mu_1}(I - \mu_1 B_1) T^{\Theta_2}_{\mu_2}(I - \mu_2 B_2) = P_C(I - \mu_1 B_1) P_C(I - \mu_2 B_2)$. We set $H = \mathbf{R}$ and use the $\langle a, b\rangle = ab$ and $\|\cdot\| = |\cdot|$ to denote its inner product and induced norm, respectively. Moreover, we put $C = [-2, 3]$. The starting point x_1 is arbitrarily chosen in C. Let $f(x) = F(x) = \frac{1}{2}x\ \forall x \in C$ with

$$\delta = \frac{1}{2} < \tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} = 1 - \sqrt{1 - 2(2 \cdot \frac{1}{2} - 2(\frac{1}{2})^2)} = 1.$$

We let $B_1 x = B_2 x := Bx = x - \frac{1}{2}\sin x,\ \forall x \in C$. Let $A : H \to H$ and $S : C \to C$ be formulated as $Ax := \frac{1}{1+|\sin x|} - \frac{1}{1+|x|}$ and $Sx := \frac{5}{6}\sin x$. We now assert that B is $\frac{2}{9}$-inverse-strongly monotone. In fact, since B is $\frac{1}{2}$-strongly monotone and $\frac{3}{2}$-Lipschitz continuous, we know that B is $\frac{2}{9}$-inverse-strongly monotone with $\alpha = \beta = \frac{2}{9}$. Let us assert that A is pseudomonotone and Lipschitz continuous. In fact, for each $a, b \in H$ one has

$$\begin{aligned}
\|Aa - Ab\| &\le \left|\frac{\|b\| - \|a\|}{(1 + \|b\|)(1 + \|a\|)}\right| + \left|\frac{\|\sin b\| - \|\sin a\|}{(1 + \|\sin b\|)(1 + \|\sin a\|)}\right| \\
&\le \frac{\|a - b\|}{(1 + \|a\|)(1 + \|b\|)} + \frac{\|\sin a - \sin b\|}{(1 + \|\sin a\|)(1 + \|\sin b\|)} \\
&\le \|a - b\| + \|\sin a - \sin b\| \le 2\|a - b\|.
\end{aligned}$$

This means that A is Lipschitz continuous with $L = 2$. Next, we assert that A is pseudomonotone. For each $a, b \in H$, it is readily known that

$$\begin{aligned}
&\langle Aa, b - a\rangle = (\frac{1}{1 + |\sin a|} - \frac{1}{1 + |a|})(b - a) \ge 0 \\
\Rightarrow\ &\langle Ab, b - a\rangle = (\frac{1}{1 + |\sin b|} - \frac{1}{1 + |b|})(b - a) \ge 0.
\end{aligned}$$

Moreover, it is easy to check that S is asymptotically nonexpansive with $\theta_n = (\frac{5}{6})^n,\ \forall n \ge 1$, such that $\|S^{n+1}x_n - S^n x_n\| \to 0$ as $n \to \infty$. In fact, note that

$$\|S^n a - S^n b\| \le \frac{5}{6}\|S^{n-1}a - S^{n-1}b\| \le \cdots \le (\frac{5}{6})^n\|a - b\| \le (1 + \theta_n)\|a - b\|,$$

and

$$\|S^{n+1}x_n - S^n x_n\| \le (\frac{5}{6})^{n-1}\|S^2 x_n - S x_n\| = (\frac{5}{6})^{n-1}\|\frac{5}{6}\sin(S x_n) - \frac{5}{6}\sin x_n\| \le 2(\frac{5}{6})^n \to 0.$$

It is obvious that $\mathrm{Fix}(S) = \{0\}$ and

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = \lim_{n\to\infty} \frac{(5/6)^n}{1/3(n+1)} = 0.$$

Accordingly, $\Omega = \mathrm{Fix}(S) \cap \mathrm{Fix}(\mathcal{G}) \cap \mathrm{VI}(C, A) = \{0\} \neq \emptyset$. In this case, noticing

$$\mathcal{G} = P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2) = [P_C(I - \frac{1}{3}B)]^2,$$

we rewrite Algorithm 3.1 as follows:

$$\begin{cases} w_n = \frac{2}{3}x_n + \frac{1}{3}u_n, \\ v_n = P_C(I - \frac{1}{3}B)w_n, \\ u_n = P_C(I - \frac{1}{3}B)v_n, \\ y_n = P_C(w_n - \frac{1}{2}Aw_n), \\ t_n = (1 - \tau_n)w_n + \tau_n y_n, \\ z_n = P_{C_n}(w_n), \\ x_{n+1} = \frac{2}{3}x_n + \frac{1}{3}P_C[\frac{1}{3(n+1)} \cdot \frac{1}{2}x_n + (1 - \frac{1}{3(n+1)})S^n z_n], \quad \forall n \geq 1, \end{cases} \tag{4.1}$$

where for each $n \geq 1$, C_n and τ_n are chosen as in Algorithm 3.1. Then, by Theorem 3.6, we know that $\{x_n\}$ converges to $0 \in \Omega = \mathrm{Fix}(S) \cap \cap\mathrm{Fix}(\mathcal{G}) \cap \mathrm{VI}(C, A)$.

In particular, since $Sx := \frac{5}{6}\sin x$ is also nonexpansive, we consider the modified version of Algorithm 3.1, that is,

$$\begin{cases} w_n = \frac{2}{3}x_n + \frac{1}{3}u_n, \\ v_n = P_C(I - \frac{1}{3}B)w_n, \\ u_n = P_C(I - \frac{1}{3}B)v_n, \\ y_n = P_C(w_n - \frac{1}{2}Aw_n), \\ t_n = (1 - \tau_n)w_n + \tau_n y_n, \\ z_n = P_{C_n}(w_n), \\ x_{n+1} = \frac{2}{3}x_n + \frac{1}{3}P_C[\frac{1}{3(n+1)} \cdot \frac{1}{2}x_n + (1 - \frac{1}{3(n+1)})Sz_n], \quad \forall n \geq 1, \end{cases} \tag{4.2}$$

where for each $n \geq 1$, C_n and τ_n are chosen as above. Then, by Theorem 3.7, we know that $\{x_n\}$ converges to $0 \in \Omega = \mathrm{Fix}(S) \cap \mathrm{Fix}(\mathcal{G}) \cap \mathrm{VI}(C, A)$.

5. Conclusions

In this paper, we introduce the modified Mann-like subgradient-like extragradient implicit rules with linear-search process for finding a common solution of the SGEP, VIP and FPP. The proposed algorithms are on the basis of the subgradient extragradient rule with linear-search process, Mann implicit iteration approach, and hybrid deepest-descent technique. Under mild restrictions, we demonstrate the strong convergence of the suggested algorithms to a common solution of the SGEP, VIP and FPP, which is a unique solution of a certain HVI defined on their common solution set. In addition, an illustrated example is provided to show the feasibility and implementability of our proposed rule.

Acknowledgments

This research was supported by the 2020 Shanghai Leading Talents Program of the Shanghai Municipal Human Resources and Social Security Bureau (20LJ2006100), the Innovation Program of Shanghai Municipal Education Commission (15ZZ068) and the Program for Outstanding Academic Leaders in Shanghai City (15XD1503100). Li-Jun Zhu was supported by the National Natural Science Foundation of China [grant number 11861003], the Natural Science Foundation of Ningxia province [grant number NZ17015], the Major Research Projects of NingXia [grant numbers 2021BEG03049]and Major Scientific and Technological Innovation Projects of YinChuan [grant numbers 2022RKX03 and NXYLXK2017B09].

Conflict of interest

The authors declare no conflicts of interest.

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