AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(1): 194–212.
DOI: 10.3934/math.2023009
Received: 21 May 2022
Revised: 19 August 2022
Accepted: 19 August 2022
Published: 28 September 2022

*Research article*

# Computational analysis of fractional modified Degasperis-Procesi equation with Caputo-Katugampola derivative

**Jagdev Singh*** **and Arpita Gupta**

Department of Mathematics, JECRC University, Jaipur, India

* **Correspondence:** Email: jagdevsinghrathore@gmail.com; Tel: +919460905224.

**Abstract:** Main aim of the current study is to examine the outcomes of nonlinear partial modified Degasperis-Procesi equation of arbitrary order by using two analytical methods. Both methods are based on homotopy and a novel adjustment with generalized Laplace transform operator. Nonlinear terms are handled by using He's polynomials. The fractional order modified Degasperis-Procesi (FMDP) equation, is capable to describe the nonlinear aspects of dispersive waves. The Katugampola derivative of fractional order in the caputo type is employed to model this problem. The numerical results and graphical representation demonstrate the efficiency and accuracy of applied techniques.

**Keywords:** Caputo-Katugampola fractional derivative; generalized Laplace transform; Degasperis-Procesi equation
**Mathematics Subject Classification:** 34A08, 35A20, 35A22, 35C05

## 1. Introduction

The nonlinear partial Degasperis-Procesi equation is a very important differential equation, that arises in the modeling of dispersive water wave propagation. In mathematical physics, the modified Degasperis-Procesi differential equation is written as

$$v_t - v_{xxt} + 4v^2 v_x = 3v_x v_{xx} + v v_{xxx}. \tag{1.1}$$

The Degasperis-Procesi [1] equation was discovered by, 'Antonio Degasperis' and 'Michela Procesi', while researching asymptotically integrable partial differential equations. In addition, this third order nonlinear modified dispersive Degasperis-Procesi equation, is also considered for the modeling of shallow water dynamics. Because of these properties, this equation is centre of attraction for many researchers.

Due to its local nature, model (1.1) can not describe the entire memory effect of the system. Thus, in order to involve whole memory of the system, we modify the model (1.1) by changing ordinary time derivative to the Katugampola fractional derivative in the Caputo sense.

In this research work, we are considering the nonlinear time fractional modified Degasperis-Procesi (FMDP) equation that models the unidirectional propagation of two-dimensional shallow water waves over a flat plate. Hence, FMDP equation associated with the Caputo-Katugampola fractional derivative is given as

$$ {}_{a}^{KC}D_{t}^{\alpha,\rho}v(x,t) - v_{xxt} + 4v^{2}v_{x} = 3v_{x}v_{xx} + vv_{xxx}, \tag{1.2}$$

with initial condition

$$ v_{0}(x,t) = v(x,0) = \wp(x) = -\frac{15}{8}\operatorname{sech}^{2}\left(\frac{x}{2}\right). \tag{1.3}$$

In last few decades, many researchers and mathematicians have taken a great interest in the study of fractional calculus and its application areas, like physical sciences, chemistry, engineering, life sciences, etc. The main reason behind their interest in fractional calculus is that, these fractional order models give more accurate results in comparison to the integer order models. Many definitions of fractional calculus, fractional order derivatives (integrals) and their various properties are available [2–6]. Usually, it is tough to obtain the exact solutions of fractional order differential equations. So, several numerical and analytical methods are available to obtain the approximate solution of a differential equation of fractional order. Many analytical and numerical methods are given to obtain the more efiicient and approximate results of FMDP equation. Some of them are: homotopy perturbation technique applied by Zhang et al. [7] to get the solitary wave solution of modified Degasperis-Procesi and Camassa-Holm equations, variational homotopy perturbation method (VHPM) given by Yousif et al. [8] is a coupling of variational iteration method and the homotopy perturbation approach. Gupta et al. [9] obtained the approximate analytical solution of modified fractional Degasperis-Procesi equation by using the homotopy perturbation technique. Abourabia et al. [10] gave the analytical solutions of Camassa-Holm and Degasperis-Procesi equations by employing three different methods, which are the Cole-Hopf method, the Schwarzian derivatives method and the factorization method, the $q$-homotopy analysis sumudu transform method ($q$-HASTM) applied by Dubey et al. [11] to examine the results of fractional modified Degasperis-Procesi equation. Singh et al. [12] employed homotopy analysis transform method (HATM) to determine the solution of fractional fish farm model. An efficient computational approach, namely $q$-homotopy analysis transform method ($q$-HATM), implemented by Singh et al. [13] to analyze the local fractional Poisson equation. The homotopy perturbation sumudu transform method (HPSTM) is used by Goswami et al. [14] to obtain the solution of time-fractional Kersten-Krasil'shchik coupled KdV-mKdV nonlinear system.

In this research work, we are studying the FMDP equation by employing two techniques. One is $q$-homotopy analysis generalized transform method ($q$-HAGTM), which is a graceful coupling of $q$-homotopy analysis method [15], generalized Laplace transform (LT) [16] and homotopy polynomials, and the another is homotopy perturbation generalized transform method (HPGTM). The HPGTM is a mixture of the homotopy perturbation method [17], generalized LT and He's polynomials [18]. Reason behind the application of these techniques is the potential of combining two powerful computational methods for analysing nonlinear fractional order equations.

The current article is organized as follows: Section 2 contains some preliminary definitions. In section 3, the elementary procedure of both analytic methods ($q$-HAGTM and HPGTM) is mentioned. In section 4, FMDP equation is analyzed by using $q$-HAGTM and HPGTM. Section 5 is devoted to numerical results. Lastly, section 6 presents the conclusion of this research work.

## 2. Some preliminary definitions

Here, we present some definitions and results related to fractional operators and generalized LT [4,16,19–24].

**Definition 2.1.** The Caputo derivative [4] of non-integer order $\alpha$ of function $g(t)$ is given as follows

$$\left({}_a^C D_t^\alpha g\right)(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-w)^{-\alpha} g'(w)dw, \text{ where } 0 < \alpha \leq 1. \tag{2.1}$$

**Definition 2.2.** The Caputo-Hadamard derivative [20] of fractional order $\alpha$ of function $g(t)$ is defined as follows

$$\left({}_a^{CH} D_t^\alpha g\right)(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \left(\log \frac{t}{w}\right)^{-\alpha} \delta g(w)dw, \tag{2.2}$$

where $\delta$ represents differential operator and is defined as $\delta = t\dfrac{d}{dt}$.

**Definition 2.3.** The Katugampola derivative of fractional order $\alpha$ of function $g(t)$ in Caputo type [22] is given as

$$\left({}_a^{KC} D_t^{\alpha,\rho} g\right)(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \left(\frac{t^\rho - w^\rho}{\rho}\right)^{-\alpha} \gamma g(w) \frac{du}{w^{1-\rho}}, \tag{2.3}$$

where $\gamma$ represents differential operator and is given as $\gamma = t^{1-\rho}\dfrac{d}{dt}$.

As, we can observe that if $\rho = 1$, then the Katugampola fractional derivative given by Eq (2.3) reduces to the Caputo derivative of fractional order $\alpha$ and if $\rho$ approaches to 0, then the fractional derivative given by Eq (2.3) reduces to the Caputo-Hadamard derivative of fractional order $\alpha$.

**Definition 2.4.** Let $g, h : [a, \infty) \to \mathbb{R}$ be two real-valued functions in such a way that $h(t)$ is continuous and $h'(t) > 0$ on $[a, \infty)$, then generalized LT [16] of function $g(t)$ is defined as

$$\mathcal{L}_h\{g(t)\}(s) = \int_a^\infty e^{-s(h(t)-h(a))} g(t) h'(t) dt, \tag{2.4}$$

here, $s$ is used as generalized LT parameter.

On putting $h(t) = t$ and $a = 0$ in Eq (2.4), the generalized LT reduces to the classical LT and if we put $h(t) = \frac{t^\rho}{\rho}$ and $a = 0$ then the generalized LT becomes the $\rho$-LT and is defined as

$$\mathcal{L}_{\frac{t^\rho}{\rho}}\{g(t)\}(s) = \int_0^\infty e^{-s\frac{t^\rho}{\rho}} g(t) \frac{dt}{t^{1-\rho}}. \tag{2.5}$$

In this paper, we are using generalized LT given by Eq (2.5).

**Definition 2.5.** Generalized LT of Katugampola derivative of fractional order in Caputo type [22] is given as follows

$$\mathcal{L}_{\frac{t^\rho}{\rho}}\left\{\left({}_a^{KC} D_t^{\alpha,\rho} g\right)(t)\right\}(s) = s^\alpha \mathcal{L}_{\frac{t^\rho}{\rho}}\{g(t)\}(s) - s^{\alpha-1} g(0). \tag{2.6}$$

## 3. Elementary description of analytical methods

*3.1. q-Homotopy analysis generalized transform method (q-HAGTM)*

To illustrate basic working plan of the first implemented analytical scheme, consider a non-homogeneous nonlinear fractional differential equation

$$ {}_{a}^{KC}D_t^{\alpha,\rho} v(x,t) + Rv(x,t) + Nv(x,t) = \phi(x,t), \quad m-1<\alpha\le m, \quad m\in\mathbb{N}, \tag{3.1}$$

with initial condition $v(x,0) = \wp(x)$ for any $x \in \mathbb{R}$, here, $v(x,t)$ is a function of $x$ and $t$, ${}_{a}^{KC}D_t^{\alpha,\rho}$ is the Katugampola fractional derivative of order $\alpha$, $R$ is bounded linear operator of $x$ and $t$. The general nonlinear operator is presented by $N$, which is Lipschitz continuous and $\phi(x,t)$ is a source term.

Using generalized LT on Eq (3.1), we get

$$\mathcal{L}_{\frac{\wp}{\rho}}\left[{}_{a}^{KC}D_t^{\alpha,\rho} v(x,t)\right] + L_{\frac{\wp}{\rho}}\left[Rv(x,t)+Nv(x,t)\right] = L_{\frac{\wp}{\rho}}[\phi(x,t)]. \tag{3.2}$$

Now, on utilizing the generalized LT of Katugampola derivative of fractional order in Caputo type, we get

$$s^{\alpha}\mathcal{L}_{\frac{\wp}{\rho}}[v(x,t)](s) - s^{\alpha-1}v(x,0) + \mathcal{L}_{\frac{\wp}{\rho}}[Rv(x,t)+Nv(x,t)] - \mathcal{L}_{\frac{\wp}{\rho}}[\phi(x,t)] = 0. \tag{3.3}$$

On simplifying the Eq (3.3), we get

$$\mathcal{L}_{\frac{\wp}{\rho}}[v(x,t)](s) - s^{-1}v(x,0) + s^{-\alpha}\left[\mathcal{L}_{\frac{\wp}{\rho}}[Rv(x,t)+N(x,t)]\right] - s^{-\alpha}\mathcal{L}_{\frac{\wp}{\rho}}[\phi(x,t)] = 0. \tag{3.4}$$

The nonlinear operator can be written in the following manner

$$N[\psi(x,t;q)] = \mathcal{L}_{\frac{\wp}{\rho}}[\psi(x,t;q)] - s^{-1}\psi(x,0;q) + s^{-\alpha}\left[\mathcal{L}_{\frac{\wp}{\rho}}[R\psi(x,t;q)+N\psi(x,t;q)] - \mathcal{L}_{\frac{\wp}{\rho}}[\phi(x,t)]\right]. \tag{3.5}$$

In Eq (3.5) $\psi(x,t;q)$ represents a function of $x$, $t$ and $q$, also $q$ is an embedding parameter s.t. $q \in \left[0, \frac{1}{n}\right]$, where $n \ge 1$. Now the homotopy can be developed in this way

$$(1-nq)\mathcal{L}_{\frac{\wp}{\rho}}[\psi(x,t;q) - v_0(x,t)] = \hbar q N[\psi(x,t;q)], \tag{3.6}$$

where, $\mathcal{L}_{\frac{\wp}{\rho}}$ represents the generalized LT operator, $v_0(x,t)$ is an initial approximation of $v(x,t)$, $\psi(x,t;q)$ is an unknown function and $\hbar$ is a nonzero auxiliary parameter. Moreover, it may be clarified that, by substituting the values of embedding parameter $q = 0$ as well as $q = \frac{1}{n}$, it gives

$$\psi(x,t;0) = v_0(x,t), \qquad \psi\left(x,t;\frac{1}{n}\right) = v(x,t), \tag{3.7}$$

respectively. Thus, we can note that as the value of $q$ varies from 0 to $\frac{1}{n}$, the solution of $\psi(x,t;q)$ changes from initial approximation $v_0(x,t)$ to the solution $v(x,t)$. The Taylor's series extension of function $\psi(x,t;q)$ is given as follows

$$\psi(x,t;q) = v_0(x,t) + \sum_{k=1}^{\infty} v_k(x,t)q^k, \tag{3.8}$$

where,

$$v_k(x,t) = \frac{1}{k!}\frac{\partial^k}{\partial q^k}\left\{\psi(x,t;q)\right\}|_{q=0}. \tag{3.9}$$

If the initial guess $v_0(x,t)$, the convergence control parameter $\hbar$ and asymptotic parameter $n$ are described appropriately, then Eq (3.8) converges at $q=\frac{1}{n}$. Then we get the following equation

$$v(x,t) = v_0(x,t) + \sum_{k=1}^{\infty} v_k(x,t)\left(\frac{1}{n}\right)^k. \tag{3.10}$$

Result given by Eq (3.10) must be one of the solutions of studied nonlinear fractional differential equation. With the aid of Eqs (3.10) and (3.6), the governing equation can be obtained as

$$\vec{v}_k = \{v_1(x,t),\ v_2(x,t),\ v_3(x,t), ..., v_k(x,t)\}. \tag{3.11}$$

On differentiating Eq (3.6) k-times w.r.t. $q$ and then dividing by $k!$, after that putting $q=0$, it gives the subsequent equation

$$\mathcal{L}_{\frac{\wp}{\rho}}[v_k(x,t) - \chi_k v_{k-1}(x,t)] = \hbar\mathcal{R}_k(\vec{v}_{k-1}). \tag{3.12}$$

Employing the inverse generalized LT operator on Eq (3.12), we attain the subsequent result

$$v_k(x,t) = \chi_k v_{k-1}(x,t) + \hbar\mathcal{L}^{-1}_{\frac{\wp}{\rho}}\left[\mathcal{R}_k(\vec{v}_{k-1})\right], \tag{3.13}$$

where $\chi_k$ is defined as

$$\chi_k = \begin{cases} 0, & k \le 1, \\ n, & k > 1, \end{cases} \tag{3.14}$$

and we express the value of $\mathcal{R}_k(\vec{v}_{k-1})$ in an enhanced manner as

$$\mathcal{R}_k(\vec{v}_{k-1}) = \mathcal{L}_{\frac{\wp}{\rho}}[v_{k-1}(x,t)] - \left(1-\frac{\chi_k}{n}\right)\left[s^{-1}v(x,0) + s^{-\alpha}\mathcal{L}_{\frac{\wp}{\rho}}[\phi(x,t)]\right] + s^{-\alpha}\mathcal{L}_{\frac{\wp}{\rho}}[Rv_{k-1} + A_{k-1}]. \tag{3.15}$$

In Eq (3.15), $A_k$ represents the homotopy polynomial [25] and is given as

$$A_k = \frac{1}{\Gamma(k)}\left[\frac{\partial^k}{\partial q^k}N\psi(x,t;q)\right]_{q=0}, \tag{3.16}$$

and

$$\psi(x,t;q) = \psi_0 + q\psi_1 + q^2\psi_2 + ... \tag{3.17}$$

Using Eq (3.15) in Eq (3.13), we get

$$v_k(x,t) = (\chi_k+\hbar)v_{k-1}(x,t) - \hbar\left(1-\frac{\chi_k}{n}\right)\mathcal{L}^{-1}_{\frac{\wp}{\rho}}\left[s^{-1}v(x,0) + s^{-\alpha}\mathcal{L}_{\frac{\wp}{\rho}}[\phi(x,t)]\right] + \hbar\mathcal{L}^{-1}_{\frac{\wp}{\rho}}\left[s^{-\alpha}\mathcal{L}_{\frac{\wp}{\rho}}[Rv_{k-1} + A_{k-1}]\right]. \tag{3.18}$$

Hence, by using Eq (3.18), the various components of $v_k(x,t)$ for $k \ge 1$ can be determined and we obtain $q$-HAGTM solution given by the subsequent equation as

$$v(x,t) = \sum_{k=0}^{\infty} v_k(x,t)\left(\frac{1}{n}\right)^k. \tag{3.19}$$

### 3.2. Homotopy perturbation generalized transform method (HPGTM)

To demonstrate fundamental working plan of the next implemented analytical scheme, take a non-homogeneous nonlinear fractional differential equation

$$ {}^{KC}_{a}D_t^{\alpha,\rho} v(x,t) + Rv(x,t) + Nv(x,t) = \phi(x,t), \quad m-1 < \alpha \le m, \quad m \in \mathbb{N}, \tag{3.20} $$

with initial condition $v(x,0) = \wp(x)$ for any $x \in \mathbb{R}$, here, $v(x,t)$ is a function of $x$ and $t$, ${}^{KC}_{a}D_t^{\alpha,\rho}$ is the Katugampola fractional derivative of order $\alpha$, $R$ is bounded linear operator of $x$ and $t$. The general nonlinear operator is presented by $N$, which is Lipschitz continuous and $\phi(x,t)$ is a source term.

Employing generalized LT on Eq (3.20) ,we get

$$ \mathcal{L}_{\frac{t^\rho}{\rho}}\left[{}^{KC}_{a}D_t^{\alpha,\rho} v(x,t)\right] + L_{\frac{t^\rho}{\rho}}[Rv(x,t) + Nv(x,t)] = L_{\frac{t^\rho}{\rho}}[\phi(x,t)]. \tag{3.21} $$

Now, on utilizing the generalized LT of Katugampola derivative of fractional order in Caputo type, we get

$$ s^\alpha \mathcal{L}_{\frac{t^\rho}{\rho}}[v(x,t)](s) - s^{\alpha-1} v(x,0) + \mathcal{L}_{\frac{t^\rho}{\rho}}[Rv(x,t) + Nv(x,t)] - \mathcal{L}_{\frac{t^\rho}{\rho}}[\phi(x,t)] = 0. \tag{3.22} $$

On simplifying the Eq (3.22), we get

$$ \mathcal{L}_{\frac{t^\rho}{\rho}}[v(x,t)](s) = \frac{\wp(x)}{s} + \frac{1}{s^\alpha}\mathcal{L}_{\frac{t^\rho}{\rho}}[\phi(x,t)] - \frac{1}{s^\alpha}\left[\mathcal{L}_{\frac{t^\rho}{\rho}}\left[Rv(x,t) + Nv(x,t)\right]\right]. \tag{3.23} $$

Now, operating the inverse generalized LT on Eq (3.23), we obtain the following equation

$$ v(x,t) = F(x,t) - \mathcal{L}^{-1}_{\frac{t^\rho}{\rho}}\left[\frac{1}{s^\alpha}\left\{\mathcal{L}_{\frac{t^\rho}{\rho}}[Rv(x,t) + Nv(x,t)]\right\}\right], \tag{3.24} $$

where, $F(x,t)$ stands for the term arising from the prescribed initial condition and the source term. Next, we implement the HPM

$$ v(x,t) = \sum_{k=0}^{\infty} p^k v_k(x,t). \tag{3.25} $$

The nonlinear terms can be decomposed as

$$ Nv(x,t) = \sum_{k=0}^{\infty} p^k H_k(v), \tag{3.26} $$

using the He's polynomials $H_k(v)$ that are given as

$$ H_k(v_0,\ v_1,\ ,...,v_k) = \frac{1}{k!}\frac{\partial^k}{\partial p^k}\left[N\left(\sum_{j=0}^{k} p^j v_j\right)\right]_{p=0}, \quad k = 0,1,2,3,... \tag{3.27} $$

On using Eqs (3.25) and (3.26) in Eq (3.24), we get

$$ \sum_{k=0}^{\infty} p^k v_k(x,t) = F(x,t) - p\left[\mathcal{L}^{-1}_{\frac{t^\rho}{\rho}}\left\{\frac{1}{s^\alpha}\mathcal{L}_{\frac{t^\rho}{\rho}}\left(R\sum_{k=0}^{\infty} p^k v_k(x,t) + \sum_{k=0}^{\infty} p^k H_k(v)\right)\right\}\right], \tag{3.28} $$

that is a combination of generalized LT and HPM utilizing He's polynomials. Next, on equating the coefficients of like powers of $p$, we obtain the subsequent approximations

$$p^0 \; : \; v_0(x,t) = F(x,t), \tag{3.29}$$

$$p^1 \; : \; v_1(x,t) = \mathcal{L}^{-1}_{\frac{t^\rho}{\rho}}\left[\frac{1}{s^\alpha}\mathcal{L}_{\frac{t^\rho}{\rho}}[Rv_0(x,t) + H_0(v)]\right], \tag{3.30}$$

$$p^2 \; : \; v_2(x,t) = \mathcal{L}^{-1}_{\frac{t^\rho}{\rho}}\left[\frac{1}{s^\alpha}\mathcal{L}_{\frac{t^\rho}{\rho}}[Rv_1(x,t) + H_1(v)]\right], \tag{3.31}$$

$$p^3 \; : \; v_3(x,t) = \mathcal{L}^{-1}_{\frac{t^\rho}{\rho}}\left[\frac{1}{s^\alpha}\mathcal{L}_{\frac{t^\rho}{\rho}}[Rv_2(x,t) + H_2(v)]\right]. \tag{3.32}$$

Hence, we can find remaining components $v_k(x,t)$ completely by proceeding in the same way, and we get the series solution. Finally, the approximate solution of the problem using this technique is presented as

$$v(x,t) = \lim_{K\to\infty}\sum_{k=0}^{K} v_k(x,t). \tag{3.33}$$

## 4. Analysis of uniqueness and convergence of the solution

In this part, we check the uniqueness and convergence of the obtained solution.

**Theorem 4.1.** (Uniqueness Theorem). The solution of FMDP Eq (1.2) is unique, while $0 < \lambda < 1$, where, $\lambda = (n+\hbar) + \hbar[\delta_3' + 4((A+B)\delta_1 A + B^2\delta_1) + 3(\delta_2 A\delta_1 + \delta_1 B\delta_2) + (\delta_3 A + B\delta_3)]T$.

*Proof.* Here, the solution of FMDP Eq (1.2) is given as

$$v(x,t) = \sum_{k=0}^{\infty} v_k(x,t)\left(\frac{1}{n}\right)^k, \tag{4.1}$$

where,

$$v_k(x,t) = (\chi_k + \hbar)v_{k-1}(x,t) - \hbar\left(1 - \frac{\chi_k}{n}\right)v_0(x,t) - \hbar\mathcal{L}^{-1}_{\frac{t^\rho}{\rho}}\left[\frac{1}{s^\alpha}\mathcal{L}_{\frac{t^\rho}{\rho}}\left(v_{(k-1)xxt} - 4A_{k-1} + 3B_{k-1} + C_{k-1}\right)\right]. \tag{4.2}$$

Let, $v$ and $v^*$ be two different solutions of FMDP Eq (1.2) s.t. $|v| \le A,\ |v^*| \le B$, then using the Eq (4.2), we have

$$|v - v^*| = \Bigg|(n+\hbar)(v - v^*) - \hbar\mathcal{L}^{-1}_{\frac{t^\rho}{\rho}}\left[\frac{1}{s^\alpha}\mathcal{L}_{\frac{t^\rho}{\rho}}\{(v_{xxt} - v^*_{xxt}) - 4(v^2 v_x - v^{*2}v^*_x) + 3(v_x v_{xx} - v^*_x v^*_{xx}) + (v v_{xxx} - v^* v^*_{xxx})\}\right]\Bigg|. \tag{4.3}$$

Now, on applying convolution theorem [22] for generalized LT, we obtain

$$|v - v^*| \le (n+\hbar)|v - v^*| + \hbar\int_0^t \Big(|v_{xxt} - v^*_{xxt}| + 4|v^2 v_x - v^{*2}v^*_x| + 3|v_x v_{xx} - v^*_x v^*_{xx}|$$

$$+|vv_{xxx} - v^* v^*_{xxx}|) \frac{1}{\Gamma(\alpha)} \left[\frac{(t^\rho - w^\rho)}{\rho}\right]^{\alpha-1} w^{\rho-1} dw.$$

$$|v - v^*| \le (n + \hbar)|v - v^*| + \hbar \int_0^t \left( \left| \frac{\partial^3}{\partial x^2 \partial t}(v - v^*) \right| + 4 \left| \frac{\partial v}{\partial x}(v - v^*)(v + v^*) + v^{*2} \frac{\partial}{\partial x}(v - v^*) \right| \right.$$
$$\left. +3 \left| \frac{\partial}{\partial x}(v - v^*) \frac{\partial^2 v}{\partial x^2} + \frac{\partial v^*}{\partial x} \frac{\partial^2}{\partial x^2}(v - v^*) \right| + \left| (v - v^*) \frac{\partial^3 v}{\partial x^3} + v^* \frac{\partial^3}{\partial x^3}(v - v^*) \right| \right) \frac{1}{\Gamma(\alpha)} \left[\frac{(t^\rho - w^\rho)}{\rho}\right]^{\alpha-1} w^{\rho-1} dw.$$

$$|v - v^*| \le (n + \hbar)|v - v^*| + \hbar \int_0^t \left( \delta_3' |(v - v^*)| + 4((A + B)\delta_1 A + B^2 \delta_1)|(v - v^*)| \right.$$
$$\left. + 3(\delta_2 A \delta_1 + \delta_1 B \delta_2)|(v - v^*)| + (\delta_3 A + B\delta_3)|(v - v^*)| \right) \frac{1}{\Gamma(\alpha)} \left[\frac{(t^\rho - w^\rho)}{\rho}\right]^{\alpha-1} w^{\rho-1} dw. \quad (4.4)$$

Now, on implementing mean value theorem [26, 27], we get

$$|v - v^*| \le (n + \hbar)|v - v^*| + \hbar \left( \delta_3' |(v - v^*)| + 4((A + B)\delta_1 A + B^2 \delta_1)|(v - v^*)| + 3(\delta_2 A \delta_1 + \delta_1 B \delta_2)|(v - v^*)| \right.$$
$$\left. +(\delta_3 A + B\delta_3)|(v - v^*)| \right) T. \quad (4.5)$$

On simplifying Eq (4.5), we obtain the subsequent relation as

$$|v - v^*| \le \lambda |v - v^*|, \quad (4.6)$$

where, $\lambda = (n + \hbar) + \hbar[\delta_3' + 4((A + B)\delta_1 A + B^2 \delta_1) + 3(\delta_2 A \delta_1 + \delta_1 B \delta_2) + (\delta_3 A + B\delta_3)]T$.

It yields $(1 - \lambda)|v - v^*|$ and here $0 < \lambda < 1$, hence $|v - v^*| = 0$ which confers that $v = v^*$.

Therefore, we can say that the obtained solution is unique.

**Theorem 4.2.** (Convergence Theorem). Let $F : B \to B$ be a nonlinear mapping, where $B$ is a Banach space, also assume that

$$\|F(v) - F(u)\| \le |v - u|, \ \forall \ v, u \in B. \quad (4.7)$$

Then by the fixed point theory [26, 27] of Banach space, we know that $F$ has a fixed point. Also, the sequence formed by using $q$-HAGTM solution having an arbitrary solution of $v_0, u_0 \in B$, converges to the fixed point of $F$ and

$$\|v_k - v_j\| \le \frac{\lambda^j}{1 - \lambda} \|v_1 - v_0\|, \ \forall \ v, u \in B. \quad (4.8)$$

*Proof.* Let $(C[I], \|.\|)$ be the Banach space of all continuous functions on $I$ associated with the norm, given as $\|f(t)\| = \max_{t \in I} |f(t)|$.

Now, to prove the convergence of this solution, we will show that $\{v_j\}$ is a Cauchy sequence in the Banach space.

$$\|v_k - v_j\| = \max_{t \in I} |(v_k - v_j)|,$$

$$\|v_k - v_j\| = \max_{t\in I}\left|(n+\hbar)\left(v_{(k-1)} - v_{(j-1)}\right) - \hbar\mathcal{L}^{-1}_{\frac{t^\rho}{\rho}}\left[\frac{1}{s^\alpha}\mathcal{L}_{\frac{t^\rho}{\rho}}\left\{\left(v_{(k-1)xxt} - v_{(j-1)xxt}\right)\right.\right.\right.$$
$$\left.\left.\left.- \ 4\left(v^2_{(k-1)}v_{(k-1)x} - v^2_{(j-1)}v_{(j-1)x}\right) + 3\left(v_{(k-1)x}v_{(k-1)xx} - v_{(j-1)x}v_{(j-1)xx}\right) + \left(v_{(k-1)}v_{(k-1)xxx} - v_{(j-1)}v_{(j-1)xxx}\right)\right\}\right]\right|.$$

$$\|v_k - v_j\| \le \max_{t\in I}\left[(n+\hbar)|v_{(k-1)} - v_{(j-1)}| + \hbar\mathcal{L}^{-1}_{\frac{t^\rho}{\rho}}\left\{\frac{1}{s^\alpha}\mathcal{L}_{\frac{t^\rho}{\rho}}\left(\left|v_{(k-1)xxt} - v_{(j-1)xxt}\right|\right.\right.\right.$$
$$\left.\left.\left.+ \ 4\left|v^2_{(k-1)}v_{(k-1)x} - v^2_{(j-1)}v_{(j-1)x}\right| + 3\left|v_{(k-1)x}v_{(k-1)xx} - v_{(j-1)x}v_{(j-1)xx}\right| + \left|v_{(k-1)}v_{(k-1)xxx} - v_{(j-1)}v_{(j-1)xxx}\right|\right)\right\}\right].$$

Now, employing convolution theorem for generalized LT, we obtain

$$\|v_k - v_j\| \le \max_{t\in I}\left[(n+\hbar)\left|v_{(k-1)} - v_{(j-1)}\right| + \hbar\int_0^t\left(\left|v_{(k-1)xxt} - v_{(j-1)xxt}\right| + 4\left|v^2_{(k-1)}v_{(k-1)x} - v^2_{(j-1)}v_{(j-1)x}\right|\right.\right.$$
$$\left.\left.+ \ 3\left|v_{(k-1)x}v_{(k-1)xx} - v_{(j-1)x}v_{(j-1)xx}\right| + \left|v_{(k-1)}v_{(k-1)xxx} - v_{(j-1)}v_{(j-1)xxx}\right|\right)\frac{1}{\Gamma(\alpha)}\left[\frac{(t^\rho - w^\rho)}{\rho}\right]^{\alpha-1} w^{\rho-1}dw\right].$$

$$\|v_k - v_j\| \le \max_{t\in I}\left[(n+\hbar)\left|v_{(k-1)} - v_{(j-1)}\right| + \hbar\int_0^t\left(\delta_3'\left|v_{(k-1)} - v_{(j-1)}\right| + 4((A+B)\delta_1 A + B^2\delta_1)\left|v_{(k-1)} - v_{(j-1)}\right|\right.\right.$$
$$\left.\left.+3(\delta_2 A\delta_1 + \delta_1 B\delta_2)\left|v_{(k-1)} - v_{(j-1)}\right| + (\delta_3 A + B\delta_3)\left|v_{(k-1)} - v_{(j-1)}\right|\right)\frac{1}{\Gamma(\alpha)}\left[\frac{(t^\rho - w^\rho)}{\rho}\right]^{\alpha-1} w^{\rho-1}dw\right].$$

Now, applying mean value theorem, we obtain

$$\|v_k - v_j\| \le \max_{t\in I}\left[(n+\hbar)\left|v_{(k-1)} - v_{(j-1)}\right| + \hbar\left(\delta_3'\left|v_{(k-1)} - v_{(j-1)}\right| + 4\left((A+B)\delta_1 A + B^2\delta_1\right)\left|v_{(k-1)} - v_{(j-1)}\right|\right.\right.$$
$$\left.\left.+3(\delta_2 A\delta_1 + \delta_1 B\delta_2)\left|v_{(k-1)} - v_{(j-1)}\right| + (\delta_3 A + B\delta_3)\left|v_{(k-1)} - v_{(j-1)}\right|\right)T\right],$$

then we have

$$\|v_k - v_j\| \le \lambda\|v_{k-1} - v_{j-1}\|.$$

Setting $k = j+1$, it gives

$$\|v_{j+1} - v_j\| \le \lambda\|v_j - v_{j-1}\| \le \lambda^2\|v_{j-1} - v_{j-2}\| \le ... \le \lambda^j\|v_1 - v_0\|.$$

Using triangular inequality, we have

$$\begin{aligned}\|v_k - v_j\| &\le \|v_{j+1} - v_j\| + \|v_{j+2} - v_{j+1}\| + ... + \|v_k - v_{k-1}\|\\ &\le [\lambda^j + \lambda^{j+1} + ... + \lambda^{k-1}]\|v_1 - v_0\|\\ &\le \lambda^j[1 + \lambda + \lambda^2 + ... + \lambda^{k-j-1}]\|v_1 - v_0\|\\ &\le \lambda^j\left(\frac{1-\lambda^{k-j-1}}{1-\lambda}\right)\|v_1 - v_0\|.\end{aligned}$$

As $0 < \lambda < 1$, so $1 - \lambda^{k-j-1} < 1$, then we have

$$\|v_k - v_j\| \leq \frac{\lambda^j}{1-\lambda}\|v_1 - v_0\|.$$

Since $\|v_1 - v_0\| < \infty$, so as $k \to \infty$ then $\|v_k - v_j\| \to 0$.

Hence, the sequence $\{v_j\}$ is convergent as it is a Cauchy sequence in $C[I]$.

## 5. Solution of fractional Degasperis-Procesi equation

### 5.1. Solution by applying q-HAGTM

The FMDP equation associated with Katugampola fractional derivative is given as

$$ {}_{a}^{KC}D_t^{\alpha,\rho} v(x,t) - v_{xxt} + 4v^2 v_x = 3v_x v_{xx} + v v_{xxx}, \tag{5.1}$$

with initial condition

$$v_0(x,t) = v(x,0) = \wp(x) = -\frac{15}{8}\operatorname{sech}^2\left(\frac{x}{2}\right). \tag{5.2}$$

The exact solution [11] of standard modified Degasperis-Procesi equation obtained by substituting $\alpha = 1$ in Eq (5.1) is given as

$$v(x,t) = -\frac{15}{8}\operatorname{sech}^2\left(\frac{x}{2} - \frac{5t}{4}\right). \tag{5.3}$$

Now, by employing generalized LT on Eq (5.1) and using initial approximation given by Eq (5.2), we obtain

$$\mathcal{L}_{\frac{t^\rho}{\rho}}[v(x,t)] - s^{-1}\wp(x) - s^{-\alpha}\mathcal{L}_{\frac{t^\rho}{\rho}}[v_{xxt}] + 4s^{-\alpha}\mathcal{L}_{\frac{t^\rho}{\rho}}[v^2 v_x] - 3s^{-\alpha}\mathcal{L}_{\frac{t^\rho}{\rho}}[v_x v_{xx}] - s^{-\alpha}\mathcal{L}_{\frac{t^\rho}{\rho}}[v v_{xxx}] = 0. \tag{5.4}$$

Now, the nonlinear operator is given as follows

$$\begin{aligned} N[\psi(x,t;q)] = {} & \mathcal{L}_{\frac{t^\rho}{\rho}}[\psi(x,t;q)] - \frac{1}{s}\wp(x) - s^{-\alpha}\mathcal{L}_{\frac{t^\rho}{\rho}}[\psi_{xxt}(x,t;q)] + 4s^{-\alpha}\mathcal{L}_{\frac{t^\rho}{\rho}}\left[\psi^2(x,t;q)\psi_x(x,t;q)\right] \\ & - 3s^{-\alpha}\mathcal{L}_{\frac{t^\rho}{\rho}}[\psi_x(x,t;q)\psi_{xx}(x,t;q)] - s^{-\alpha}\mathcal{L}_{\frac{t^\rho}{\rho}}[\psi(x,t;q)\psi_{xxx}(x,t;q)], \end{aligned} \tag{5.5}$$

and the value of $\mathcal{R}_k(\vec{v}_{k-1})$ is given as

$$\mathcal{R}_k(\vec{v}_{k-1}(x,t)) = \mathcal{L}_{\frac{t^\rho}{\rho}}[v_{k-1}(x,t)] - \left(1 - \frac{\chi_k}{n}\right)\left[\frac{1}{s}\wp(x)\right] - s^{-\alpha}\mathcal{L}_{\frac{t^\rho}{\rho}}\left[v_{(k-1)xxt} - 4A_{(k-1)} + 3B_{(k-1)} + C_{(k-1)}\right]. \tag{5.6}$$

Now, making use of the initial approximation $\wp(x) = -\frac{15}{8}\operatorname{sech}^2\left(\frac{x}{2}\right)$ and iterative formula given by Eq (3.13), we attain the subsequent iterative terms of the approximate solution

$$v_1(x,t) = \hbar\left[4\wp^2(x)\wp'(x) - 3\wp'(x)\wp''(x) - \wp(x)\wp'''(x)\right]\frac{1}{\Gamma(\alpha+1)}\left(\frac{t^\rho}{\rho}\right)^\alpha. \tag{5.7}$$

Hence, performing in the similar way, we can find rest of the components $v_k$, $k \geq 2$, and approximate solution using $q$HAGTM is obtained.

Consequently, $q$-HAGTM solution is given as

$$v(x,t) = \lim_{K\to\infty}\sum_{k=0}^{K} v_k(x,t)\left(\frac{1}{n}\right)^k. \tag{5.8}$$

*5.2. Solution by applying HPGTM*

In this part, we are finding the approximate solution of FMDP Eq (5.1) with an initial guess given by Eq (5.2) using HPGTM.

Employing generalized LT on both sides of Eq (5.1) and utilizing initial guess (5.2), we get

$$\mathcal{L}_{\frac{t^\rho}{\rho}}[v(x,t)] = \frac{1}{s}\wp(x) + \frac{1}{s^\alpha}\mathcal{L}_{\frac{t^\rho}{\rho}}[v_{xxt} - 4v^2v_x + 3v_xv_{xx} + vv_{xxx}]. \tag{5.9}$$

Now, operating the inverse generalized LT on Eq (5.9), we obtain

$$v(x,t) = \wp(x) + \mathcal{L}^{-1}_{\frac{t^\rho}{\rho}}\left[\frac{1}{s^\alpha}\mathcal{L}_{\frac{t^\rho}{\rho}}[v_{xxt} - 4v^2v_x + 3v_xv_{xx} + vv_{xxx}]\right]. \tag{5.10}$$

Employing HPM, we get subsequent equation

$$\sum_{k=0}^{\infty} p^k v_k(x,t) = \wp(x) + p\left[\mathcal{L}^{-1}_{\frac{t^\rho}{\rho}}\left\{\frac{1}{s^\alpha}\mathcal{L}_{\frac{t^\rho}{\rho}}\left(\sum_{k=0}^{\infty} p^k (v_k)_{xxt} - 4\sum_{k=0}^{\infty} p^k A_k(v) + 3\sum_{k=0}^{\infty} p^k B_k(v) + \sum_{k=0}^{\infty} p^k C_k(v)\right)\right\}\right], \tag{5.11}$$

where $A_k(v)$, $B_k(v)$ and $C_k(v)$ are He's polynomials, which represent the nonlinear terms.

On equating the coefficients of like powers of $p$, we get

$$p^0 \;:\; v_0(x,t) = \wp(x), \tag{5.12}$$

$$\begin{aligned} p^1 \;:\; v_1(x,t) &= \mathcal{L}^{-1}_{\frac{t^\rho}{\rho}}\left[\frac{1}{s^\alpha}\mathcal{L}_{\frac{t^\rho}{\rho}}\{(v_0)_{xxt} - 4A_0(v) + 3B_0(v) + C_0(v)\}\right] \\ &= -\left[4\wp^2(x)\wp'(x) - 3\wp'(x)\wp''(x) - \wp(x)\wp'''(x)\right]\frac{1}{\Gamma(\alpha+1)}\left(\frac{t^\rho}{\rho}\right)^\alpha. \end{aligned}$$

Hence, performing in the similar way, we can find rest of the components $v_k,\ k \geq 2$ and approximate solution using HPGTM is obtained.

Consequently, HPGTM solution is given as

$$v(x,t) = \lim_{K\to\infty}\sum_{k=0}^{K} v_k(x,t). \tag{5.13}$$

## 6. Numerical results

In this part, we analyze the solutions of FMDP equation obtained by $q$-HAGTM and HPGTM. The numerical simulation of the discussed problem is performed for numerous values of the time variable $t$, space variable $x$ and fractional order $\alpha$. The given table shows the comparative study of solutions attained by implemented techniques versus exact solution. Table 1 shows that approximate solutions obtained by implemented techniques are quite close to their exact solution. The outcomes of this numerical simulation are presented in the form of Figures 1–14. Figures 1–4 represent the behaviour of solution $v(x,t)$ obtained by $q$-HAGTM and Figures 5–8 represent the behaviour of solution $v(x,t)$ obtained by HPGTM with respect to $x, t$ and for distinct values of $\alpha$. Figure 9 is plotted for the exact solution of the classical modified Degasperis-Procesi equation. Figure 10 (for $q$-HAGTM) and Figure 11 (for HPGTM) represent the response of $v(x,t)$ w.r.t. time variable for various values of $\alpha$. Figure 12 (for $q$-HAGTM) and Figure 13 (for HPGTM) depict the aspect of $v(x,t)$ w.r.t. space variable. Figure 14 expresses the $n$-curves for $q$-HAGTM solution at various values of $\alpha$.

**Table 1.** Comparative study of exact solution and obtained solutions for $v(x,t)$ when $\alpha = 1, \hbar = -1$ and $n = 1$.

| $(x,t)$ | Exact Solution | Approximate ($q$-HAGTM) | Approximate (HPGTM) | Absolute Error |
|---|---|---|---|---|
| (8,0.05) | -0.002848800949 | -0.002515546006 | -0.002515546006 | 0.000333254943 |
| (9,0.05) | -0.001048518941 | -0.0009255163340 | -0.0009255163340 | 0.0001230026070 |
| (10,0.05) | -0.0003857967557 | -0.0003404917396 | -.0003404917396 | 0.0000453050161 |
| (8,0.1) | -0.003227787763 | -0.002516809482 | -0.002516809482 | 0.000710978281 |
| (9,0.1) | -0.001188083378 | -0.0009256875444 | -0.0009256875444 | 0.0002623958336 |
| (10,0.1) | -0.0004371590085 | -0.0003405149212 | -0.0003405149212 | 0.0000966440873 |
| (8,0.2) | -0.004143548046 | -0.002519336432 | -0.002519336432 | 0.001624211614 |
| (9,0.2) | -0.001525391967 | -0.0009260299653 | -0.0009260299653 | 0.0005993620017 |
| (10,0.2) | -0.0005613046914 | -0.0003405612846 | -0.0003405612846 | 0.0002207434068 |

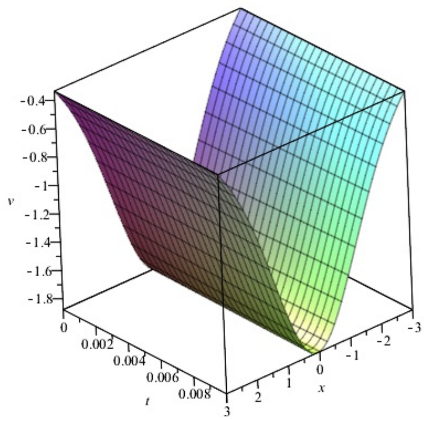


**Figure 1.** The surface of $v(x,t)$ for $q$-HAGTM solution w.r.t. $x$ and $t$ for $\alpha = 1$.

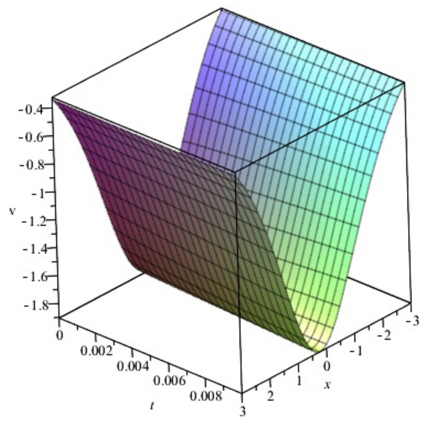


**Figure 2.** The surface of $v(x,t)$ for $q$-HAGTM solution w.r.t. $x$ and $t$ for $\alpha = 0.75$.

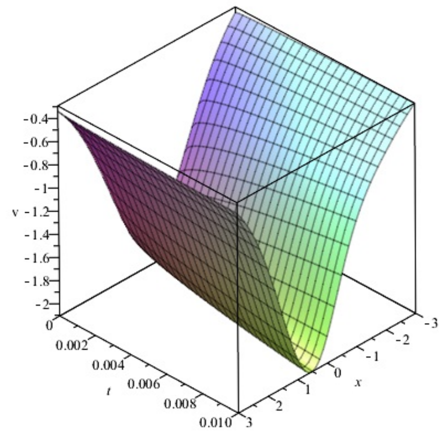


**Figure 3.** The surface of $v(x,t)$ for $q$-HAGTM solution w.r.t. $x$ and $t$ for $\alpha = 0.50$.

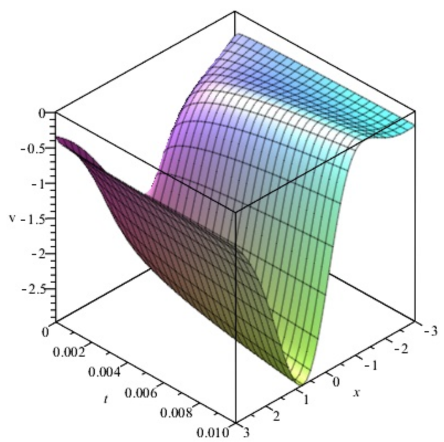


**Figure 4.** The surface of $v(x,t)$ for $q$-HAGTM solution w.r.t. $x$ and $t$ for $\alpha = 0.25$.

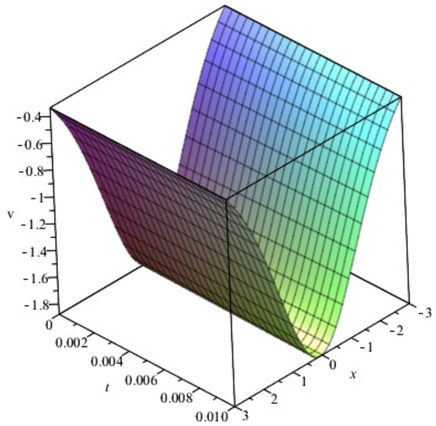


**Figure 5.** The surface of $v(x,t)$ for HPGTM solution w.r.t. $x$ and $t$ for $\alpha = 1$.

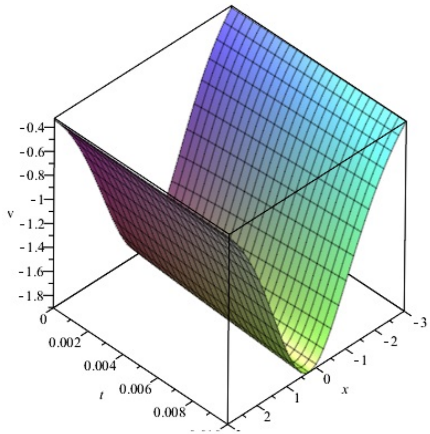


**Figure 6.** The surface of $v(x,t)$ for HPGTM solution w.r.t. $x$ and $t$ for $\alpha = 0.75$.

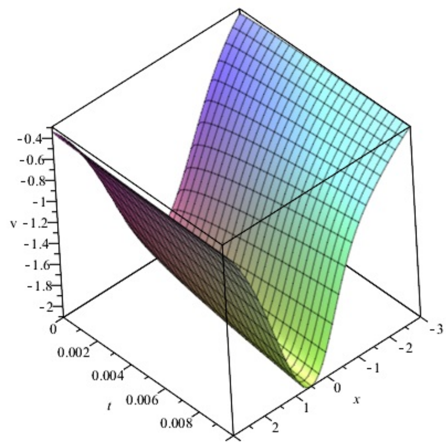


**Figure 7.** The surface of $v(x,t)$ for HPGTM solution w.r.t. $x$ and $t$ for $\alpha = 0.50$.

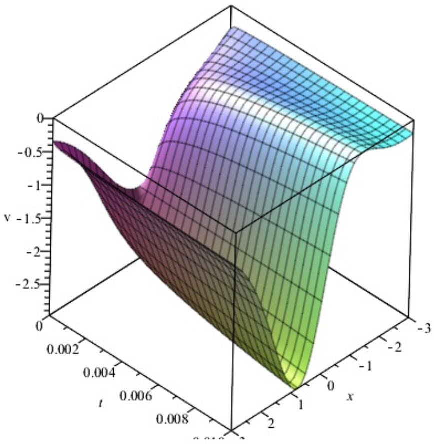


**Figure 8.** The surface of $v(x,t)$ for HPGTM solution w.r.t. $x$ and $t$ for $\alpha = 0.25$.

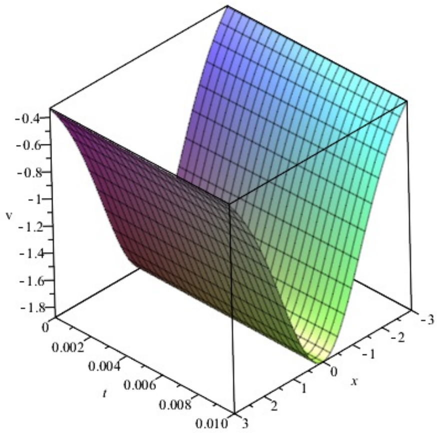


**Figure 9.** The surface of Exact solution $v(x,t)$ w.r.t. $x$ and $t$.

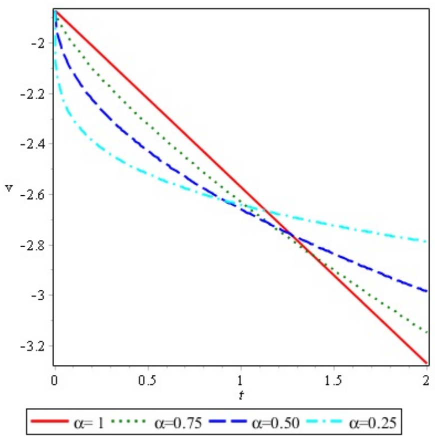


**Figure 10.** Response of $q$-HAGTM solution $v(x,t)$ w.r.t. $t$ for distinct values of $\alpha$.

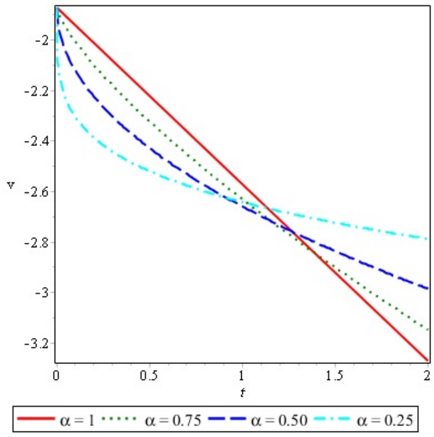


**Figure 11.** Response of HPGTM solution $v(x, t)$ w.r.t. $t$ for distinct values of $\alpha$.

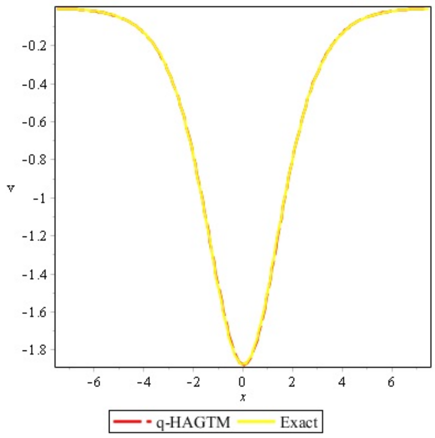


**Figure 12.** Nature of $v(x, t)$ w.r.t. $x$ at $t = 0.005$.

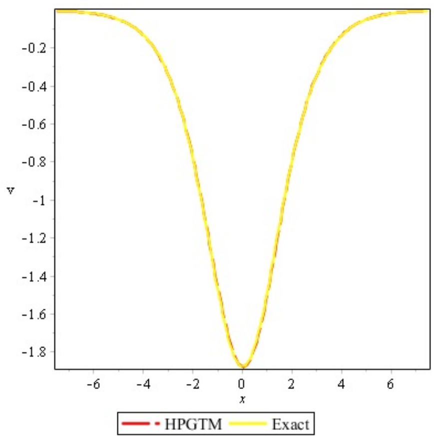


**Figure 13.** Nature of $v(x, t)$ w.r.t. $x$ at $t = 0.005$.

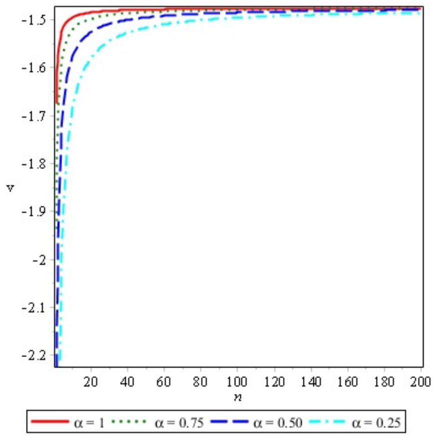


**Figure 14.** $n$-curves for distinct values of $\alpha$ at $x = 1$, $t = 0.05$ and $\hbar = -1$.

## 7. Conclusions

In this current work, we successfully implemented two techniques, namely $q$-HAGTM and HPGTM, to analyze the approximate series solution of FMDP equation. Graphical representation of the obtained results indicates that the implemented techniques are powerful and efficient for solving FMDP equation. The comparative study of approximate solutions with exact solution shows the accuracy and applicability of the applied techniques. Hence, we can conclude that the applied methods are efficient to solve such types of problems arising in physical sciences.

## Conflict of interest

The authors declare that there is no conflict of interests.

## References


1. A. Degasperis, M. Procesi, Asymptotic integrability, In: *Symmetry and pertubation theory*, Singapore: World Scientific, 1999, 23–37.
2. J. Singh, Analysis of fractional blood alcohol model with composite fractional derivative, *Chaos Soliton. Fract.*, **140** (2020), 110127. https://doi.org/10.1016/j.chaos.2020.110127
3. J. Singh, H. K. Jassim, D. Kumar, An efficient computational technique for local fractional Fokker Planck equation, *Physica A*, **555** (2020), 124525. https://doi.org/10.1016/j.physa.2020.124525
4. I. Podlubny, *Fractional differential equations*, New York: Academic Press, 1998.
5. K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, Wiley, 1993.
6. K. B. Oldham, J. Spanier, *The fractional calculus: Theory and applications of differentiation and integration to arbitrary order*, Elsevier, 1974.

7. B. G. Zhang, S. Y. Li, Z. R. Liu, Homotopy perturbation method for modified Camassa-Holm and Degasperis-Procesi equations, *Phys. Lett. A*, **372** (2008), 1867–1872. https://doi.org/10.1016/j.physleta.2007.10.072

8. M. A. Yousif, B. A. Mahmood, F. H. Easif, A new analytical study of modified Camassa-Holm and Degasperis-procesi equations, *Am. J. Comput. Math.*, **5** (2015), 267-273. https://doi.org/10.4236/ajcm.2015.53024

9. P. K. Gupta, M. Singh, A. Yildirim, Approximate analytical solution of the time-fractional Camassa-Holm, modified Camassa-Holm and Degasperis-Procesi equations by homotopy perturbation method , *Sci. Iran. A*, **23** (2016), 155–165.

10. A. M. Abourabia, I. M. Soliman, Analytical solutions of the Camassa-Holm, Degasperis-Procesi equation and phase plane analysis, *AJMS*, **5** (2021), 9–19. https://doi.org/10.22377/ajms.v5i30379

11. V. P. Dubey, R. Kumar, J. Singh, D. Kumar, An efficient computational technique for time-fractional modofied Degasperis-Procesi equation arising in propagation of nonlinear dispersive waves, *J. Ocean Eng. Sci.*, **6** (2021), 30–39. https://doi.org/10.1016/j.joes.2020.04.006

12. J. Singh, D. Kumar, D. Baleanu, A new analysis of fractional fish farm model associated with Mittag-Leffler type kernel, *Int. J. Biomath.*, **13** (2020), 2050010. https://doi.org/10.1142/S1793524520500102

13. J. Singh, A. Ahmadian, S. Rathore, D. Kumar, D. Baleanu, M. Salimi, et al., An efficient computational approach for local fractional Poisson equation in fractal media, *Numer. Meth. Part. D. E.*, **37** (2021), 1439–1448. https://doi.org/10.1002/num.22589

14. A. Goswami, Sushila, J. Singh, D. Kumar, Numerical computation of fractional Kersten-Krasilshchik coupled KdV-mKdV system occuring in multi-component plasmas, *AIMS Math.*, **5** (2020), 2346–2368. https://doi.org/10.3934/math.2020155

15. M. A. El Tawil, S. N. Huseen, The q-homotopy analysis method (q-HAM), *Int. J. Appl. Math. Mech.*, **8** (2012), 51–75.

16. F. Jarad, T. Abdeljawad, Generalized fractional derivatives and Laplace transform, *Discrete Cont. Dyn. S*, **13** (2020), 709–722. https://doi.org/10.3934/dcdss.2020039

17. J. H. He, Homotopy perturbation technique, *Comput. Method. Appl. M.*, **178** (1999), 257–262. https://doi.org/10.1016/S0045-7825(99)00018-3

18. A. Ghorbani, J. Saberi-Nadjafi, He's homotopy perturbation method for calculating Adomian polynomials, *Int. J. Nonlin. Sci. Num.*, **8** (2007), 229–232. https://doi.org/10.1515/IJNSNS.2007.8.2.229

19. S. Thanompolkrang, W. Sawangtong, P. Sawangtong, Application of the generalized laplace homotopy perturbation method to time-fractional Black-Scholes equations based on the Katugampola fractional derivative in Caputo type, *Computation*, **9** (2021), 33. https://doi.org/10.3390/computation9030033

20. R. Zafar, M. Ur-Rehman, M. Shams, On caputo modification of Hadamard type fractional derivative and fractional Taylor series, *Adv. Differ. Equ.*, **2020** (2020), 219. https://doi.org/10.1186/s13662-020-02658-1

21. R. Almeida, A. B. Malinowska, T. Odzijewicz, Frartional differential equations with dependence on the Caputo-Katigampola derivative, *J. Comput. Nonlinear Dynam.*, **11** (2016), 061017. https://doi.org/10.1115/1.4034432

22. F. Jarad, T. Abdeljawad, A modified Laplace transform for certain generalized fractional operators, *Res. Nonlinear Anal.*, **1** (2018), 88–98.

23. U. N. Katugampola, New approach to a generalized fractional integral, *Appl. Math. Comput.*, **218** (2011), 860–865. https://doi.org/10.1016/j.amc.2011.03.062

24. U. N. Katugampola, A new approach to generalized fractional derivatives, *Bull. Math. Anal. Appl.*, **6** (2014), 1–15.

25. Z. Odibat, S. A. Bataineh, An adaptation of homotopy analysis method for reliable treatment of strongly nonlinear problems, *Math. Method. Appl. Sci.*, **38** (2015), 991–1000. https://doi.org/10.1002/mma.3136

26. I. K. Argyros, *Convergence and applications of Newton-type iterations*, New York: Springer-Verlag, 2008. https://doi.org/10.1007/978-0-387-72743-1

27. A. A. Magrenan, A new tool to study real dynamics: The convergence plane, *Appl. Math. Comput.*, **248** (2014), 215–224. https://doi.org/10.1016/j.amc.2014.09.061


AIMS Press