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Research article

Global stability solution of the 2D incompressible anisotropic magneto-micropolar fluid equations

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Abstract: In this paper, we consider the two dimensional incompressible anisotropic magneto-micropolar fluid equations with partial mixed velocity dissipations, magnetic diffusion and horizontal vortex viscosity, and analyze the stability near a background magnetic field. At present, major works on the equations of magneto-micropolar fluid mainly focus on the global regularity of the solutions. While the stability of the solutions remains an open problem. This paper concentrates on establishing the stability for the linear and nonlinear system respectively. Two goals have been achieved. The first is to obtain the explicit decay rates for the solution of the linear system in $H^s(\mathbb{R}^2)$ Sobolev space. The second assesses the nonlinear stability by establishing the *a priori estimate* and employing bootstrapping arguments. Our results reveal that any perturbations near a background magnetic field is globally stable in Sobolev space $H^2(\mathbb{R}^2)$.

Keywords: 2D magneto-micropolar fluid; partial dissipation; stability; decay rates
Mathematics Subject Classification: 35A05,35Q35,76D03

1. Introduction

The incompressible magneto-micropolar fluid equations describes the motion of an incompressible conducting micropolar fluid in an arbitrary magnetic field. In this paper, we consider the 2D incompressible anisotropic magneto-micropolar fluid equations,

$$
\begin{cases}
\partial_t u_1 + u\cdot\nabla u_1 + \partial_1 P = (\mu+\chi)\partial_2^2 u_1 + B\cdot\nabla B_1 - 2\chi\partial_2 m, \qquad x\in\mathbb{R}^2,\ t>0\\
\partial_t u_2 + u\cdot\nabla u_2 + \partial_2 P = (\mu+\chi)\partial_1^2 u_2 + B\cdot\nabla B_2 + 2\chi\partial_1 m,\\
\partial_t B + u\cdot\nabla B = \nu\partial_1^2 B + B\cdot\nabla u,\\
\partial_t m + u\cdot\nabla m + 4\chi m = \kappa\partial_1^2 m + 2\chi\nabla\times u,\\
\nabla\cdot u = 0, \quad \nabla\cdot B = 0,
\end{cases}
\tag{1.1}
$$

where $u=(u_1,u_2)$, $b=(b_1,b_2)$, $\nabla^{\perp}m=(-\partial_2 m,\partial_1 m)$. Also m and P are the scalars. The nonnegative parameters $\mu>0$, $\nu>0$ and $\chi>0$ denote the kinematic viscosity, magnetic diffusion coefficient and the dynamic micro-rotation viscosity. Besides, γ and κ are the angular viscosities. The operators ∂_1, ∂_2 represent the horizontal and vertical direction respectively.

Our goal is to investigate the stability problem on the perturbation (u,b,m) near the steady solution (u^0,B^0,m^0) with $b=B-B^0$. Here

$$u^0=(0,0), B^0=(0,1), m^0=0.$$

Without loss of generality, set $\mu=\chi=\frac{1}{2}$ and $\nu=\kappa=1$. It is easy to verify that (u,b,m) satisfies

$$\begin{cases} \partial_t u_1+u\cdot\nabla u_1+\partial_1 P=\partial_2^2 u_1+b\cdot\nabla b_1-\partial_2 m+\partial_2 b_1,\\ \partial_t u_2+u\cdot\nabla u_2+\partial_2 P=\partial_1^2 u_2+b\cdot\nabla b_2+\partial_1 m+\partial_2 b_2,\\ \partial_t b+u\cdot\nabla b=\partial_1^2 b+b\cdot\nabla u+\partial_2 u,\\ \partial_t m+u\cdot\nabla m+2m=\partial_1^2 m+\nabla\times u,\\ \nabla\cdot u=\nabla\cdot b=0. \end{cases} \tag{1.2}$$

The standard magneto-micropolar fluid equations with full velocity field dissipation, magnetic diffusion and angular viscosities can be written as

$$\begin{cases} \partial_t u+(u\cdot\nabla)u+\nabla P=(\mu+\chi)\Delta u+(B\cdot\nabla)B+2\chi\nabla\times m,\\ \partial_t B+(u\cdot\nabla)B=\nu\Delta B+(B\cdot\nabla)u,\\ \partial_t m+(u\cdot\nabla)m+4\chi m=\gamma\Delta m+2\chi\nabla\times m+\kappa\nabla\mathrm{div}m,\\ \nabla\cdot u=\nabla\cdot B=0. \end{cases} \tag{1.3}$$

Because of the mathematically significant, the magneto-micropolar fluid equations and closely related equations have attracted considerable attentions for mathematical scholars and many important results have been achieved. Major works mainly concentrated on the global well-posedness and global regularity of the solution. Let's recall some of these results.

For the 2D incompressible magneto-micropolar equations, Yuan and Qiao [1] established the global smooth solution for the equations with zero angular viscosity and zero magnetic diffusion or with only angular viscosity and magnetic diffusion. In a two dimensional bounded domain with Navier type boundary condition for the velocity, Fan and Zhou [2] proved the existence and uniqueness of global strong solutions to the incompressible magneto-micropolar system. Ma in [3] obtained the global existence and regularity of classical solutions to the equations with mixed partial dissipation, magnetic diffusion and angular viscosity. In addition, some conditional regularity of strong solutions also be obtained. Guo and Shang in [4] showed the global regularity of solutions to the 2D incompressible magneto-micropolar equations with partial dissipation. For $2\frac{1}{2}$ dimensional system, the results of the global well-posedness for the incompressible magneto-micropolar fluid equations with mixed partial dissipation have been obtained (see e.g., [5, 6]). Besides, for 3D case, the global existence results for the Cauchy problem in $\mathbb{R}^3$ are obtained by Tan and Wu in [7]. The global well-posedness and global regularity of the incompressible magneto-micropolar system have been studied in [8, 9]. For more results, we refer to [7, 10–14] and references therein.

However, there are few results to our knowledge on the large-time behavior of the magneto-micropolar fluids. Shang and Gu [15] obtained the L^2-decay estimates of solutions for the two-dimensional incompressible magneto-micropolar fluid equations, which is $\|u(t)\|_{L^2}+\|w(t)\|_{L^2}\leq C(1+$

$t)^{-\frac{4}{3}}$ and $\|b\|_{L^2} \le C(1+t)^{-\frac{1}{2}}$. Moreover, by proving the optimal decay for $\|b(t)\|_{L^\infty}$, the authors optimized the decay rates to $\|u(t)\|_{L^2} + \|w(t)\|_{L^2} \le Ct^{-2}$ in [15]. In this paper, we will show decay rates for the linear system (1.2) in the next section.

If the magnetic field $B = 0$, the equations (1.3) become the micropolar fluid equations. For the 2D incompressible micropolar equations, Liu in [16] studied the global well-posedness to the Cauchy problem of 2D micropolar equations with large initial data and vacuum, and showed that the problem admits a unique global strong solution. Ye [17] studied the global regularity for the system of the 2D incompressible micropolar equations with vertical dissipation in the horizontal velocity equation, horizontal dissipation in the vertical velocity equation. Dong and Li [18] studied the global regularity in time and large time behavior of solutions to the 2D micropolar equations with only angular viscosity dissipation. The more results of the well-posedness, regularity and large time decay problems on the micropolar fluid equations can be shown in [19–22]. On the other hand, if $\chi = 0$ and $m = 0$, the equations in (1.3) reduce to the magneto-hydrodynamic equations (MHD). The case of full dissipation and magnetic diffusion, the classical solution is global (see e.g., [23]). There are numerous works on the global regularity and the stability. One of the significant works on the global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion is completed by Wu and Cao in [24]. One can also refer to [25–29] and so on.

Motivated by the results of the magneto-micropolar fluid equations and closely related equations, this paper investigates the stability of the solution of the system (1.2). We attempt to achieve two main goals. The first is to give the linear asymptotic stability, which is equivalent to assessing the small data global well-posedness. We need to consider the corresponding linear system of (1.2) to illustrate it,

$$\begin{cases} \partial_t u_1 = \partial_2^2 u_1 - \partial_2 m + \partial_2 b_1, \\ \partial_t u_2 = \partial_1^2 u_2 + \partial_1 m + \partial_2 b_2, \\ \partial_t b = \partial_1^2 b + \partial_2 u, \\ \partial_t m + 2m = \partial_1^2 m + \nabla \times u, \\ \nabla \cdot u = \nabla \cdot b = 0. \end{cases} \tag{1.4}$$

With some assumptions on the initial data, we establish explicit decay rates of the solutions for the linear system (1.4). To give decay rates, we define the fractional operator $\Lambda^\alpha f$ via the Fourier transform,

$$\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \widehat{f}(\xi).$$

Our first result is as follows.

Theorem 1.1. *For any $s \ge 0$, let the initial data $(u_0, b_0, m_0) \in H^s(\mathbb{R}^2)$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Suppose that (u, b, m) is the solution of the linear system (1.4).*

1. *Assume that $(\nabla u_0, \nabla b_0, \nabla m_0) \in H^s(\mathbb{R}^2)$. Then the decay rates holds*

$$\|\nabla u\|_{H^s(\mathbb{R}^2)} + \|\nabla b\|_{H^s(\mathbb{R}^2)} + \|\nabla m\|_{H^s(\mathbb{R}^2)} \le C(1+t)^{-\frac{1}{2}}. \tag{1.5}$$

2. *Suppose $(\Lambda_1^{-\sigma} u_0, \Lambda_1^{-\sigma} b_0, \Lambda_1^{-\sigma} m_0), (\Lambda_2^{-\sigma} u_0, \Lambda_2^{-\sigma} b_0, \Lambda_2^{-\sigma} m_0) \in H^s(\mathbb{R}^2)$, where $\sigma > 0$ is a real number. Then (u, b, m) satisfies*

$$\|u(t)\|_{H^s(\mathbb{R}^2)} + \|b(t)\|_{H^s(\mathbb{R}^2)} + \|m(t)\|_{H^s(\mathbb{R}^2)} \le C(1+t)^{-\frac{\sigma}{2}}. \tag{1.6}$$

The second goal is to prove the stability of the nonlinear system in (1.2).

Theorem 1.2. *Suppose that* $(u_0, b_0, m_0) \in H^2(\mathbb{R}^2)$ *with* $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. *Let* (u, b, m) *be the solution of the nonlinear system (1.2). Then there exists* $\delta > 0$, *such that if*

$$\|(u_0, b_0, m_0)\|_{H^2(\mathbb{R}^2)} \leq \delta,$$

then (1.2) possesses a unique global solution $(u, b, m) \in C(0, \infty; H^2(\mathbb{R}^2))$ *satisfying*

$$\begin{aligned}
&\|(u_1(t), u_2(t), b(t), m(t))\|^2_{H^2(\mathbb{R}^2)} + \int_0^t (\|\partial_2 u_1\|^2_{H^2} + \|\partial_1 u_2\|^2_{H^2} + \|\partial_1 b\|^2_{H^2} \\
&+ \|\partial_1 m\|^2_{H^2} + 2\|m\|^2_{H^2}) d\tau \leq C\delta^2,
\end{aligned} \tag{1.7}$$

for all $t \geq 0$, *where C is a pure constant.*

By the technology of bootstrapping argument (see [30, p.21]) and the energy method, we are able to obtain the nonlinear stability. To prove Theorem 1.1, we first introduce the energy $E(t)$ as follows

$$\begin{aligned}
E(t) = &\sup_{0\leq\tau\leq t} \|(u(\tau), b(\tau), m(\tau))\|^2_{H^2(\mathbb{R}^2)} + 2\int_0^t (\|\partial_2 u_1(\tau)\|^2_{H^2(\mathbb{R}^2)} + \|\partial_1 u_2(\tau)\|^2_{H^2(\mathbb{R}^2)} \\
&+\|\partial_1 b(\tau)\|^2_{H^2(\mathbb{R}^2)} + \|\partial_1 m(\tau)\|^2_{H^2(\mathbb{R}^2)} + 2\|m(\tau)d\|^2_{H^2(\mathbb{R}^2)}) d\tau,
\end{aligned}$$

for any $t \geq 0$. Our efforts concentrate on establishing the *a priori* estimate of $E(t)$ in Section 3,

$$E(t) \leq E(0) + C\, E(t)^{\frac{3}{2}}. \tag{1.8}$$

Then the bootstrapping argument implies the global bound and also the stability.

This paper is organized as follows. In section 2, we give the proof of decay rates in Theorem 1.1. In section 3, by employing the energy method and using the bootstrapping argument, we establish the H^2-estimate and then complete the proof of Theorem 1.2.

2. Proof of Theorem 1.1

In this section, we will show the decay rates in H^s for the solutions based on the linearized system (1.2). Under the different assumptions on the initial data, we establish the asymptotic stability for the linear system. Before stating our results in (1.5), we first give a tool which will be used in the proof of (1.5).

Lemma 2.1. *Let* $f = f(t)$ *be a nonnegative continuous function satisfying, for two constants* $a_0 > 0$ *and* $a_1 > 0$,

$$f(t) \leq a_0 f(s) \quad \textit{and} \quad \int_0^\infty f(\tau)\, d\tau \leq a_1 < \infty \quad \textit{for any } 0 \leq s < t. \tag{2.1}$$

Then, for any $t > 0$, *for* $a_2 = \max\{2a_0 f(0), 2a_1 a_0\}$,

$$f(t) \leq a_2(1+t)^{-1}. \tag{2.2}$$

The tool of Lemma 2.1 (see [25]) will be used to establish the decay rate (1.5). It indicates that generalized monotone nonnegative integrable functions have a precise decay rate.

2.1. Proof of (1.5)

Proof of (1.5). We show the proof of $s = 0$, then by the iterate, we can derive the case of $s > 0$. First of all, we consider the first condition of monotonous. Taking the L^2-inner product of (1.4) with $(\Delta u, \Delta b, \Delta m)$,we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2 + \|\nabla m(t)\|_{L^2}^2 + 2(\|\partial_2\nabla u_1(t)\|_{L^2}^2 + \|\partial_1\nabla u_2(t)\|_{L^2}^2)\\&+ \|\partial_1\nabla b(t)\|_{L^2}^2 + 2\|\nabla m(t)\|_{L^2}^2 + \|\partial_1\nabla m(t)\|_{L^2}^2 = 0.\end{aligned} \tag{2.3}$$

where we used the fact that

$$\int \Delta u \cdot \nabla^\perp m \mathrm{d}x + \int \nabla \times u \cdot \Delta m \mathrm{d}x = 0.$$

We denote the $f(t)$ as

$$f(t) = \|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2 + \|\nabla m(t)\|_{L^2}^2. \tag{2.4}$$

Thus (2.3) implies that

$$f(t) \le f(s), \tag{2.5}$$

for any $s < t$.

As a result, we prove the first condition. Next we verify the second condition that is $\int_0^\infty f(t)dt \le C$. First, we have the H^1-estimates,

$$\begin{aligned}&\|(u, b, m)\|_{H^1}^2 + 2\int_0^t (\|\partial_2 u_1\|_{H^1}^2 + \|\partial_1 u_2\|_{H^1}^2 + \|\partial_1 b\|_{H^1}^2 + 2\|m\|_{H^1}^2 + \|\partial_1 m\|_{H^1}^2)\mathrm{d}\tau\\=&\|u_0\|_{H^1}^2 + \|b_0\|_{H^1}^2 + \|m_0\|_{H^1}^2.\end{aligned} \tag{2.6}$$

By integration by parts and Hölder's inequality, we infer

$$\begin{aligned}\|\partial_1 u_1\|_{L^2}^2 &= -\int \partial_2 u_2 \cdot \partial_1 u_1 \mathrm{d}x = -\int \partial_1 u_2 \cdot \partial_2 u_1 \mathrm{d}x\\&\le \frac{1}{2}(\|\partial_1 u_2\|_{L^2}^2 + \|\partial_2 u_1\|_{L^2}^2),\end{aligned} \tag{2.7}$$

thus, it implies

$$\|\nabla u\|_{L^2}^2 \le 2(\|\partial_1 u_2\|_{L^2}^2 + \|\partial_2 u_1\|_{L^2}^2). \tag{2.8}$$

Combining (2.6), we can deduce

$$\int_0^\infty \|\nabla u(t)\|_{L^2}^2 \mathrm{d}t \le C. \tag{2.9}$$

Besides, from (2.6), we can infer

$$\int_0^\infty \|\partial_1 b(t)\|_{L^2}^2 \mathrm{d}t \le C \qquad \textit{and} \qquad \int_0^\infty \|\nabla m(t)\|_{L^2}^2 \mathrm{d}t \le C. \tag{2.10}$$

Now it suffices to prove the integrability for $\|\partial_2 b\|_{L^2}^2$ on the time. Dotting $\partial_2 b$ to the velocity equation of (1.4), integrating over $\mathbb{R}^2$, then replacing $\partial_t b$ by the other terms of the magnetic equation, we obtain

$$\begin{aligned}\|\partial_2 b\|_{L^2}^2 =& \int \partial_t u\cdot\partial_2 b\,\mathrm{d}x - \int \partial_2^2 u_1\cdot\partial_2 b_1 \mathrm{d}x - \int \partial_1^2 u_2\cdot\partial_2 b_2\mathrm{d}x - \int \nabla^\perp m\cdot\partial_2 b\,\mathrm{d}x\\ =&\frac{\mathrm{d}}{\mathrm{d}t}\int u\cdot\partial_2 b\,\mathrm{d}x - \int u\cdot\partial_2\partial_t b\,\mathrm{d}x - \int \partial_2^2 u_1\cdot\partial_2 b_1\mathrm{d}x\\ &-\int \partial_1^2 u_2\cdot\partial_2 b_2\mathrm{d}x - \int \nabla^\perp m\cdot\partial_2 b\,\mathrm{d}x\\ =&\frac{\mathrm{d}}{\mathrm{d}t}\int u\cdot\partial_2 b\,\mathrm{d}x + \int \partial_2 u\cdot\partial_1^2 b\,\mathrm{d}x + \|\partial_2 u\|_{L^2}^2 - \int \partial_2^2 u_1\cdot\partial_2 b_1\mathrm{d}x\\ &-\int \partial_1^2 u_2\cdot\partial_2 b_2\mathrm{d}x - \int \nabla^\perp m\cdot\partial_2 b\,\mathrm{d}x.\end{aligned}\tag{2.11}$$

The four integral terms above can be estimated as

$$\begin{aligned}&\int \partial_2 u\cdot\partial_1^2 b\,\mathrm{d}x - \int \partial_2^2 u_1\cdot\partial_2 b_1\mathrm{d}x - \int \partial_1^2 u_2\cdot\partial_2 b_2\mathrm{d}x - \int \nabla^\perp m\cdot\partial_2 b\,\mathrm{d}x\\ \le&\frac{1}{2}(\|\partial_2 u\|_{L^2}^2 + \|\partial_1^2 b\|_{L^2}^2 + \|\partial_2^2 u_1\|_{L^2}^2 + \|\partial_2 b_1\|_{L^2}^2 + \|\partial_1^2 u_2\|_{L^2}^2 + \|\partial_2 b_2\|_{L^2}^2 + \|\nabla^\perp m\|_{L^2}^2 + \|\partial_2 b\|_{L^2}^2).\end{aligned}\tag{2.12}$$

where we used the Hölder's inequality and Young's inequality. Inserting (2.12) into (2.11) and integrating it on $[0,t]$, we have

$$\begin{aligned}&\int_0^t \|\partial_2 b(\tau)\|_{L^2}^2\mathrm{d}\tau \le (\|u(t)\|_{L^2}^2 + \|\partial_2 b(t)\|_{L^2}^2 + \|u_0\|_{L^2}^2 + \|\partial_2 b_0\|_{L^2}^2)\\ &+\int_0^t (3\|\partial_2 u\|_{L^2}^2 + \|\partial_1^2 b\|_{L^2}^2 + \|\partial_2^2 u_1\|_{L^2}^2 + \|\partial_1^2 u_2\|_{L^2}^2 + \|\nabla^\perp m\|_{L^2}^2 + \|\partial_2 b\|_{L^2}^2)\mathrm{d}\tau.\end{aligned}\tag{2.13}$$

Then, adding $\lambda\times$(2.13) into (2.6), where $\lambda>0$ is a small number, we have

$$\begin{aligned}&\|(u,b,m)\|_{H^1}^2 + 2\int_0^t (\|\partial_2 u_1\|_{H^1}^2 + \|\partial_1 u_2\|_{H^1}^2 + \|\nabla b\|_{H^1}^2 + 2\|m\|_{H^1}^2 + \|\partial_1 m\|_{H^1}^2)\mathrm{d}\tau\\ \le& C(\|u_0\|_{H^1}^2 + \|b_0\|_{H^1}^2 + \|m_0\|_{H^1}^2).\end{aligned}\tag{2.14}$$

From (2.14), we infer

$$\int_0^t \|\nabla b\|_{L^2}^2\mathrm{d}t \le C.\tag{2.15}$$

Collecting (2.9), (2.10) and (2.15), it suffices to verify $f(t)$ satisfying the second condition which is nonnegative integrable. As a consequence, by Lemma 2.1 we conclude,

$$f(t)\le C(1+t)^{-1},\tag{2.16}$$

where C is a constant and (2.16) gives the desired result (2.2). Thus, we conclude the proof of (1.5). □

2.2. Proof of (1.6)

Proof of (1.6). Using the consideration of the itera again, we only prove the case of $s = 0$ similarly to the (1.5). We have the H^1-estimate as follows,

$$\frac{1}{2}\frac{d}{dt}\|(u_1, u_2, b, m)\|_{H^1}^2 + \|\partial_1 u_2\|_{H^1}^2 + \|\partial_2 u_1\|_{H^1}^2 + \|\partial_1 b\|_{H^1}^2 + \|\partial_1 m\|_{H^1}^2 + 2\|m\|_{H^1}^2 = 0.$$

where

$$F(t) = \|u_1(t)\|_{H^1}^2 + \|u_2(t)\|_{H^1}^2 + \|b(t)\|_{H^1}^2 + \|m(t)\|_{H^1}^2,$$

and

$$G(t) = 2(\|\partial_1 u_2\|_{H^1}^2 + \|\partial_2 u_1\|_{H^1}^2 + \|\partial_1 b\|_{H^1}^2 + \|\partial_1 m\|_{H^1}^2 + 2\|m\|_{H^1}^2).$$

Applying $\Lambda_1^{-\sigma}$ and $\Lambda_2^{-\sigma}$ to the linear system (1.4) respectively, and dotting the corresponding equations with $(\Lambda_1^{-\sigma}u_1, \Lambda_1^{-\sigma}u_2, \Lambda_1^{-\sigma}b, \Lambda_1^{-\sigma}m)$ and $(\Lambda_2^{-\sigma}u_1, \Lambda_2^{-\sigma}u_2, \Lambda_2^{-\sigma}b, \Lambda_2^{-\sigma}m)$,then integrating over $\mathbb{R}^2$, we have

$$\begin{aligned}&\frac{d}{dt}\|(\Lambda_1^{-\sigma}u_1, \Lambda_1^{-\sigma}u_2, \Lambda_1^{-\sigma}b, \Lambda_1^{-\sigma}m)(t)\|_{L^2}^2 + 2\|\Lambda_1^{-\sigma}\partial_2 u_1\|_{L^2}^2\\ &+2\|\Lambda_1^{-\sigma}\partial_1 u_2\|_{L^2}^2 + 2\|\Lambda_1^{-\sigma}\partial_1 b\|_{L^2}^2 + 2\|\Lambda_1^{-\sigma}\partial_1 m\|_{L^2}^2 + 4\|\Lambda_1^{-\sigma}m\|_{L^2}^2\\ &= 0.\end{aligned} \tag{2.17}$$

and,

$$\begin{aligned}&\frac{d}{dt}\|(\Lambda_2^{-\sigma}u_1, \Lambda_2^{-\sigma}u_2, \Lambda_2^{-\sigma}b, \Lambda_2^{-\sigma}m)(t)\|_{L^2}^2 + 2\|\Lambda_2^{-\sigma}\partial_2 u_1\|_{L^2}^2\\ &+2\|\Lambda_2^{-\sigma}\partial_1 u_2\|_{L^2}^2 + 2\|\Lambda_2^{-\sigma}\partial_1 b\|_{L^2}^2 + 2\|\Lambda_2^{-\sigma}\partial_1 m\|_{L^2}^2 + 4\|\Lambda_2^{-\sigma}m\|_{L^2}^2\\ &= 0.\end{aligned} \tag{2.18}$$

Combining (2.17) and (2.18), then integrating by parts, it infers that

$$\begin{aligned}&\|(\Lambda_1^{-\sigma}(u_1, u_2, b, m), \Lambda_2^{-\sigma}(u_1, u_2, b, m))\|_{L^2}^2\\ &\leq \|(\Lambda_1^{-\sigma}(u_{1,0}, u_{2,0}, b, m), \Lambda_2^{-\sigma}(u_{1,0}, u_{2,0}, b, m))\|_{L^2}^2,\end{aligned} \tag{2.19}$$

which indicates that

$$H(t) \leq H(0).$$

Then the estimate of $\|u_1\|_{L^2}$ follows from the Plancherel's identity and Hölder's inequality, which can be written as

$$\begin{aligned}\|u_1(t)\|_{L^2}^2 &= \int |\widehat{u_1}(\xi, t)|^2 d\xi\\ &= \int (|\xi_2|^2|\widehat{u_1}(\xi, t)|^2)^{\frac{\sigma}{\sigma+1}}(|\xi_2|^{-2\sigma}|\widehat{u_1}(\xi, t)|^2)^{\frac{1}{\sigma+1}} d\xi\\ &\leq \|\partial_2 u_1(t)\|_{L^2}^{\frac{2\sigma}{\sigma+1}}\|\Lambda_2^{-\sigma}u_1(t)\|_{L^2}^{\frac{2}{\sigma+1}}.\end{aligned} \tag{2.20}$$

Similarly,

$$\|u_2(t)\|_{L^2}^2 \leq \|\partial_1 u_2(t)\|_{L^2}^{\frac{2\sigma}{\sigma+1}}\|\Lambda_1^{-\sigma}u_2(t)\|_{L^2}^{\frac{2}{\sigma+1}}, \tag{2.21}$$

$$\|b(t)\|_{L^2}^2 \le \|\partial_1 b\|_{L^2}^{\frac{2\sigma}{\sigma+1}} \|\Lambda_1^{-\sigma} b(t)\|_{L^2}^{\frac{2}{\sigma+1}}, \tag{2.22}$$

and

$$\|m(t)\|_{L^2}^2 \le \|\partial_1 m\|_{L^2}^{\frac{2\sigma}{\sigma+1}} \|\Lambda_1^{-\sigma} m(t)\|_{L^2}^{\frac{2}{\sigma+1}}. \tag{2.23}$$

Collecting (2.20), (2.22), (2.23) and (2.19), we have

$$F(t) \le CG(t)^{\frac{\sigma}{1+\sigma}} H(t)^{\frac{1}{1+\sigma}} \le CG(t)^{\frac{\sigma}{1+\sigma}} H(0)^{\frac{1}{1+\sigma}}.$$

Then we infer that

$$G(t) \ge C\, F(t)^{1+\frac{1}{\sigma}}\, H(0)^{-\frac{1}{\sigma}}. \tag{2.24}$$

Which immediately leads to

$$F(t) \le \Big(C_1(\sigma, \|(u_0, b_0, m_0)\|_{L^2}) + C_2(\sigma, \|(\Lambda_1^{-\sigma}(u_0, b_0, m_0), \Lambda_2^{-\sigma}(u_0, b_0, m_0))\|_{L^2})t\Big)^{-\sigma}.$$

Thus, we finish the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

This section proves the stability of the nonlinear system (1.2). By exploiting the methods of bootstrapping and energy method, we establish the H^2-estimate. Before our proof, we give two useful tools. The first provides an anisotropic inequality for the integral of triple product and the proof can be found in [24]. The second shows a basic fact.

Lemma 3.1. *Suppose that f, g, $\partial_2 g$, h and $\partial_1 h$ are all in $L^2(\mathbb{R}^2)$. Then, for some constant $C > 0$,*

$$\int\int |fgh| dx \le C\|f\|_{L^2} \|g\|_{L^2}^{\frac{1}{2}} \|\partial_2 g\|_{L^2}^{\frac{1}{2}} \|h\|_{L^2}^{\frac{1}{2}} \|\partial_1 h\|_{L^2}^{\frac{1}{2}}. \tag{3.1}$$

Lemma 3.2. *Due to $\nabla \cdot u = 0$, we have the fact that*

$$\|\nabla^2 u\|_{H^1}^2 \le 3(\|\partial_2 \nabla u_1\|_{H^1}^2 + \|\partial_1 \nabla u_2\|_{H^1}^2). \tag{3.2}$$

The key step in the proof is to deal with the nonlinear and coupled terms. Therefore, we will take full use of the Lemma 3.1. Combining Sobolev inequality, Hölder's inequality and Young's inequality, the closed priori estimate of the energy $E(t)$ can be established.

Proof. This section aims to obtain the H^2-estimate. Since the equivalent norms,

$$\|\, (u_1, u_2, b, m)\, \|_{H^2}^2 \sim \|\, (u_1, u_2, b, m)\, \|_{L^2}^2 + \sum_{i=1}^{2} \|\, (\partial_i^2 u_1, \partial_i^2 u_2, \partial_i^2 b, \partial_i^2 m)\, \|_{L^2}^2, \tag{3.3}$$

it suffices to make the estimate on $\|\, (u_1, u_2, b, m)\, \|_{L^2}^2$ and $\|\, (\partial_i^2 u_1, \partial_i^2 u_2, \partial_i^2 b, \partial_i^2 m)\, \|_{L^2}^2$ respectively.

First of all, by taking L^2-inner product of (1.2) with (u, b, m) and using integration by parts, it easily infers

$$\frac{1}{2}\frac{d}{dt}\parallel (u_1,u_2,b,m)\parallel_{L^2}^2+\parallel \partial_2 u_1\parallel_{L^2}^2+\parallel \partial_1 u_2\parallel_{L^2}^2+\parallel \partial_1 b\parallel_{L^2}^2+\parallel \partial_1 m\parallel_{L^2}^2+2\parallel m\parallel_{L^2}^2 \\ =0. \tag{3.4}$$

Next, it suffices to estimate $\|(u_1,u_2,b,m)\|_{\dot{H}^2}^2$. Applying ∂_1^2 and ∂_2^2 to every equation in (1.2) respectively, then taking the L^2-inner product with $(\partial_i^2 u_1,\partial_i^2 u_2,\partial_i^2 b,\partial_i^2 m)$, and integrating them on $[0,t]$, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{2}(\parallel \partial_i^2 u_1\parallel_{L^2}^2+\parallel \partial_i^2 u_2\parallel_{L^2}^2+\parallel \partial_i^2 b\parallel_{L^2}^2+\parallel \partial_i^2 m\parallel_{L^2}^2)\\&+\sum_{i=1}^{2}(\parallel \partial_i^2\partial_2 u_1\parallel^2+\parallel \partial_i^2\partial_1 u_2\parallel_{L^2}^2+\parallel \partial_i^2\partial_1 b\parallel_{L^2}^2+2\parallel \partial_i^2 m\parallel_{L^2}^2+\parallel \partial_i^2\partial_1 m\parallel_{L^2}^2)\\&=I_1+I_2\cdots+I_7.\end{aligned} \tag{3.5}$$

They can be written as follows respectively,

$$\begin{aligned}I_1&=-\sum_{i=1}^{2}\int \partial_i^2(u\cdot\nabla u_1)\partial_i^2 u_1\mathrm{d}x,\\I_2&=\sum_{i=1}^{2}\int(\partial_i^2(b\cdot\nabla b_1)\partial_i^2 u_1-b\cdot\partial_i^2\nabla b_1\cdot\partial_i^2 u_1)\mathrm{d}x,\\I_3&=-\sum_{i=1}^{2}\int \partial_i^2(u\cdot\nabla u_2)\partial_i^2 u_2\mathrm{d}x,\\I_4&=\sum_{i=1}^{2}\int(\partial_i^2(b\cdot\nabla b_2)\partial_i^2 u_2-b\cdot\partial_i^2\nabla b_2\cdot\partial_i^2 u_2)\mathrm{d}x,\\I_5&=-\sum_{i=1}^{2}\int(\partial_i^2(u\cdot\nabla b)\partial_i^2 b\mathrm{d}x,\\I_6&=\sum_{i=1}^{2}\int(\partial_i^2(b\cdot\nabla u)\partial_i^2 b-b\cdot\partial_i^2\nabla u\cdot\partial_i^2 b)\mathrm{d}x,\\I_7&=-\sum_{i=1}^{2}\int \partial_i^2(u\cdot\nabla m)\cdot\partial_i^2 m\mathrm{d}x,\end{aligned} \tag{3.6}$$

where we have used the facts that

$$\sum_{i=1}^{2}\int(\partial_i^2\partial_2 b\cdot\partial_i^2 u+\partial_i^2\partial_2 u\cdot\partial_i^2 b)dx=0, \tag{3.7}$$

$$\sum_{i=1}^{2}\int(b\cdot\partial_i^2\nabla b\cdot\partial_i^2 u+b\cdot\partial_i^2\nabla u\cdot\partial_i^2 b)dx=0, \tag{3.8}$$

$$\sum_{i=1}^{2}\int \partial_i^2 u\cdot\partial_i^2\nabla P dx, \tag{3.9}$$

and

$$\sum_{i=1}^{2}\int(\partial_i^2\nabla^{\perp}m\cdot\partial_i^2u+\partial_i^2\nabla\times u\cdot\partial_i^2m)dx=0. \tag{3.10}$$

Then we estimate $I_1+I_2\cdots+I_7$ one by one. In the following calculations, the Hölder's inequality, Young's inequality and Sobolev embedding inequality will be applied frequently. By using the Lemma (3.2), I_1 and I_3 can be estimated together,

$$\begin{aligned}
I_1+I_3=&-\sum_{i=1}^{2}\int\partial_i^2(u\cdot\nabla u)\cdot\partial_i^2u\mathrm{d}x\\
=&-\int\partial_1^2(u\cdot\nabla u)\cdot\partial_1^2u\mathrm{d}x-\int\partial_2^2(u\cdot\nabla u)\cdot\partial_2^2u\mathrm{d}x\\
=&-\sum_{k=1}^{2}C_2^k\int\partial_1^ku\cdot\partial_1^{2-k}\nabla u\cdot\partial_1^2u\mathrm{d}x-\sum_{k=1}^{2}C_2^k\int\partial_2^ku\cdot\partial_2^{2-k}\nabla u\cdot\partial_2^2u\mathrm{d}x\\
=&-2\int\partial_1u\cdot\partial_1\nabla u\cdot\partial_1^2u\mathrm{d}x-\int\partial_1^2u\cdot\nabla u\cdot\partial_1^2u\mathrm{d}x\\
&-2\int\partial_2u\cdot\partial_2\nabla u\cdot\partial_2^2u\mathrm{d}x-\int\partial_2^2u\cdot\nabla u\cdot\partial_2^2u\mathrm{d}x\\
\leq&6\|\nabla u\|_{L^2}\|\nabla^2u\|_{L^4}^2\\
\leq&C\|u\|_{H^2}(\|\partial_1u_2\|_{H^2}^2+\|\partial_2u_1\|_{H^2}^2).
\end{aligned} \tag{3.11}$$

The term I_2 can be transformed into four terms,

$$\begin{aligned}
I_2=&\int\partial_1^2(b\cdot\nabla b_1)\cdot\partial_1^2u_1\mathrm{d}x+\int\partial_2^2(b\cdot\nabla b_1)\cdot\partial_2^2u_1\mathrm{d}x\\
=&\sum_{k=1}^{2}C_2^k\int\partial_1^kb\cdot\partial_1^{2-k}\nabla b_1\cdot\partial_1^2u_1\mathrm{d}x+\sum_{k=1}^{2}C_2^k\int\partial_2^kb\cdot\partial_2^{2-k}\nabla b_1\cdot\partial_2^2u_1\mathrm{d}x\\
=&2\int\partial_1b\cdot\partial_1\nabla b_1\cdot\partial_1^2u_1\mathrm{d}x+\int\partial_1^2b\cdot\nabla b_1\cdot\partial_1^2u_1\mathrm{d}x\\
&+2\int\partial_2b\cdot\partial_2\nabla b_1\cdot\partial_2^2u_1\mathrm{d}x+\int\partial_2^2b\cdot\nabla b_1\cdot\partial_2^2u_1\mathrm{d}x\\
=&I_{21}+I_{22}+I_{23}+I_{24}.
\end{aligned} \tag{3.12}$$

For I_{21} and I_{22}, we have

$$\begin{aligned}
I_{21}+I_{22}=&2\int\partial_1b\cdot\partial_1\nabla b_1\cdot\partial_1^2u_1\mathrm{d}x-\int\partial_1^2b\cdot\nabla b_1\cdot\partial_{12}u_2\mathrm{d}x\\
\leq&2\|\partial_1^2u_1\|_{L^2}\|\partial_1b\|_{L^4}\|\partial_1\nabla b_1\|_{L^4}+\|\nabla b_1\|_{L^2}\|\partial_1^2b\|_{L^4}\|\partial_{12}u_2\|_{L^2}\\
\leq&C(\|u\|_{H^2}\|+\|b\|_{H^2})(\|\partial_1b\|_{H^2}^2+\|\partial_1u_2\|_{H^2}^2).
\end{aligned} \tag{3.13}$$

When we estimate I_{23} and I_{24}, the incompressible condition $\nabla \cdot b = 0$ will be used,

$$\begin{aligned}
I_{23} + I_{24} =& 2\int \partial_2 b_1 \cdot \partial_{12} b_1 \cdot \partial_2^2 u_1 \mathrm{d}x + 2\int \partial_2 b_2 \cdot \partial_2^2 b_1 \cdot \partial_2^2 u_1 \mathrm{d}x \\
&+ \int \partial_2^2 b_1 \cdot \partial_1 b_1 \cdot \partial_2^2 u_1 \mathrm{d}x + \int \partial_2^2 b_2 \cdot \partial_2 b_1 \cdot \partial_2^2 u_1 \mathrm{d}x \\
=& 2\int \partial_2 b_1 \cdot \partial_{12} b_1 \cdot \partial_2^2 u_1 \mathrm{d}x - 2\int \partial_1 b_1 \cdot \partial_2^2 b_1 \cdot \partial_2^2 u_1 \mathrm{d}x \\
&+ \int \partial_2^2 b_1 \cdot \partial_1 b_1 \cdot \partial_2^2 u_1 \mathrm{d}x - \int \partial_{12} b_1 \cdot \partial_2 b_1 \cdot \partial_2^2 u_1 \mathrm{d}x \\
\leq& C(\|\partial_2 b_1\|_{L^2} \|\partial_{12} b_1\|_{L^4} \|\partial_2^2 u_1\|_{L^4} + \|\partial_2^2 b_1\|_{L^2} \|\partial_1 b_1\|_{L^4} \|\partial_2^2 u_1\|_{L^4}) \\
\leq& C\|b\|_{H^2}\|(\|\partial_1 b\|_{H^2}^2 + \|\partial_2 u_1\|_{H^2}^2).
\end{aligned} \tag{3.14}$$

Collecting the bounds for I_2, we have

$$I_2 \leq C(\| u \|_{H^2} + \| b \|_{H^2})(\| \partial_1 u_2 \|_{H^2}^2 + \| \partial_2 u_1 \|_{H^2}^2 + \| \partial_1 b \|_{H^2}^2). \tag{3.15}$$

For I_4, due to $\nabla \cdot u = \nabla \cdot b = 0$, it can be estimated as follows,

$$\begin{aligned}
I_4 =& \sum_{k=1}^{2} \int \partial_1^2 (b \cdot \nabla b_2) \cdot \partial_1^2 u_2 \mathrm{d}x + \sum_{k=1}^{2} \int \partial_2^2 (b \cdot \nabla b_2) \cdot \partial_2^2 u_2 \mathrm{d}x \\
=& \sum_{k=1}^{2} C_2^k \int \partial_1^k b \cdot \partial_1^{2-k} \nabla b_2 \cdot \partial_1^2 u_2 \mathrm{d}x + \sum_{k=1}^{2} C_2^k \int \partial_2^k b \cdot \partial_2^{2-k} \nabla b_2 \cdot \partial_2^2 u_2 \mathrm{d}x \\
=& 2\int \partial_1 b \cdot \partial_1 \nabla b_2 \cdot \partial_1^2 u_2 \mathrm{d}x + \int \partial_1^2 b \cdot \nabla b_2 \cdot \partial_1^2 u_2 \mathrm{d}x + 2\int \partial_2 b \cdot \partial_1 \nabla b_1 \cdot \partial_2 \partial_1 u_1 \mathrm{d}x \\
&- \int \partial_2^2 b_1 \cdot \partial_1 b_2 \cdot \partial_2 \partial_1 u_2 \mathrm{d}x + \int \partial_2^2 b_2 \cdot \partial_1 b_1 \cdot \partial_2 \partial_1 u_1 \mathrm{d}x \\
\leq& C\|\partial_1^2 u_2\|_{L^4}(\|\partial_1 b\|_{L^4}\|\partial_1 \nabla b_2\|_{L^2} + \|\partial_1^2 b\|_{L^4}\|\nabla b_2\|_{L^2}) + C\|\partial_1 b\|_{L^4}\|\partial_2 \partial_1 u_2\|_{L^4}\|\partial_2^2 b_1\|_{L^2} \\
&+ C\|\partial_2 \partial_1 u_1\|_{L^4}(\|\partial_1 \nabla b_1\|_{L^4}\|\partial_2 b\|_{L^2} + \|\partial_1 b_1\|_{L^4}\|\partial_2^2 b_2\|_{L^2}) \\
\leq& C\|b\|_{H^2}(\|\partial_1 u_2\|_{H^2}^2 + \|\partial_2 u_1\|_{H^2}^2 + \|\partial_1 b\|_{H^2}^2).
\end{aligned} \tag{3.16}$$

Now we think about I_5, which can be rewritten as following four parts,

$$\begin{aligned}
I_5 =& -\int \partial_1^2 (u \cdot \nabla b) \cdot \partial_1^2 b \, \mathrm{d}x - \int \partial_2^2 (u \cdot \nabla b) \cdot \partial_2^2 b \, \mathrm{d}x \\
=& -\sum_{k=1}^{2} C_2^k \int \partial_1^k u \cdot \partial_1^{2-k} \nabla b \cdot \partial_1^2 b \, \mathrm{d}x - \sum_{k=1}^{2} C_2^k \int \partial_2^k u \cdot \partial_2^{2-k} \nabla b \cdot \partial_2^2 b \, \mathrm{d}x \\
=& -2\int \partial_1 u \cdot \partial_1 \nabla b \cdot \partial_1^2 b \, \mathrm{d}x - \int \partial_1^2 u \cdot \nabla b \cdot \partial_1^2 b \, \mathrm{d}x - 2\int \partial_2 u \cdot \partial_2 \nabla b \cdot \partial_2^2 b \, \mathrm{d}x \\
&- \int \partial_2^2 u \cdot \nabla b \cdot \partial_2^2 b \, \mathrm{d}x \\
=& I_{51} + I_{52} + I_{53} + I_{54}.
\end{aligned} \tag{3.17}$$

I_{51} and I_{52} can be estimated easily. By Sobolev embedding inequality, we have

$$
\begin{aligned}
I_{51}+I_{52}=&-2\int \partial_1 u\cdot\partial_1\nabla b\cdot\partial_1^2 b\,\mathrm{d}x-\int \partial_1^2 u_1\cdot\partial_1 b\cdot\partial_1^2 b\,\mathrm{d}x-\int \partial_1^2 u_2\cdot\partial_2 b\cdot\partial_1^2 b\,\mathrm{d}x\\
\le& C(\|\partial_1 u\|_{L^2}\|\partial_1\nabla b\|_{L^4}\|\partial_1^2 b\|_{L^4}+\|\partial_1 b\|_{L^4}\|\partial_1^2 b\|_{L^4}\|\partial_1^2 u_1\|_{L^2}\\
&+\|\partial_1^2 b\|_{L^4}\|\partial_1^2 u_2\|_{L^4}\|\partial_2 b\|_{L^2})\\
\le& C\|u\|_{H^2}\|\partial_1 b\|_{H^2}^2.
\end{aligned}
\tag{3.18}
$$

To deal with I_{53} and I_{54}, writing them into four terms,

$$
\begin{aligned}
I_{53}+I_{54}=&-2\int \partial_2 u_1\cdot\partial_{12} b\cdot\partial_2^2 b\,\mathrm{d}x-2\int \partial_2 u_2\cdot\partial_2^2 b\cdot\partial_2^2 b\,\mathrm{d}x\\
&-\int \partial_2^2 u_1\cdot\partial_1 b\cdot\partial_2^2 b\,\mathrm{d}x-\int \partial_2^2 u_2\cdot\partial_2 b\cdot\partial_2^2 b\,\mathrm{d}x\\
=&J_1+J_2+J_3+J_4.
\end{aligned}
\tag{3.19}
$$

Using the Hölder's inequality and Young's inequality, J_1 and J_3 can be estimated as follows,

$$
\begin{aligned}
J_1+J_3\le& C(\|\partial_2 u_1\|_{L^4}\|\partial_{12} b\|_{L^4}\|\partial_2^2 b\|_{L^2}+\|\partial_2^2 u_1\|_{L^4}\|\partial_1 b\|_{L^4}\|\partial_2^2 b\|_{L^2})\\
\le& C\|b\|_{H^2}(\|\partial_2 u_1\|_{H^2}^2+\|\partial_1 b\|_{H^2}^2).
\end{aligned}
\tag{3.20}
$$

Integrating by parts and using Lemma 3.1, we obtain

$$
\begin{aligned}
J_2+J_4=&-2\int \partial_1 u_1\cdot\partial_2^2 b\cdot\partial_2^2 b\,\mathrm{d}x+\int \partial_2\partial_1 u_1\cdot\partial_2 b\cdot\partial_2^2 b\,\mathrm{d}x\\
=&-4\int u_1\cdot\partial_1\partial_2^2 b\cdot\partial_2^2 b\,\mathrm{d}x-\int \partial_2 u_1\cdot\partial_1\partial_2 b\cdot\partial_2^2 b\,\mathrm{d}x+\int \partial_2 u_1\cdot\partial_2 b\cdot\partial_1\partial_2^2 b\,\mathrm{d}x\\
\le& C(\|u_1\|_{L^2}^{\frac12}\|\partial_2 u_1\|_{L^2}^{\frac12}\|\partial_2^2 b\|_{L^2}^{\frac12}\|\partial_1\partial_2^2 b\|_{L^2}^{\frac12}\|\partial_1\partial_2^2 b\|_{L^2}\\
&+\|\partial_2 u_1\|_{L^2}^{\frac12}\|\partial_2\partial_2 u_1\|_{L^2}^{\frac12}\|\partial_2 b\|_{L^2}^{\frac12}\|\partial_2\partial_1 b\|_{L^2}^{\frac12}\|\partial_1\partial_2^2 b\|_{L^2}\\
&+\|\partial_2 u_1\|_{L^4}\|\partial_1\partial_2 b\|_{L^4}\|\partial_2^2 b\|_{L^2})\\
\le& C(\|u\|_{H^2}+\|b\|_{H^2})(\|\partial_2 u_1\|_{H^2}^2+\|\partial_1 b\|_{H^2}^2).
\end{aligned}
\tag{3.21}
$$

Inserting (3.20) and (3.21) into (3.19),

$$
I_{53}+I_{54}\le C(\|u\|_{H^2}+\|b\|_{H^2})(\|\partial_2 u_1\|_{H^2}^2+\|\partial_1 b\|_{H^2}^2).
\tag{3.22}
$$

Collecting (3.18) and (3.22), we infer

$$
I_5\le C(\| u \|_{H^2}+\| b \|_{H^2})(\| \partial_2 u_1 \|_{H^2}^2+\| \partial_1 b \|_{H^2}^2).
\tag{3.23}
$$

Similarly to I_5, I_6 can be written as

$$
\begin{aligned}
I_6 =& \int \partial_1^2(b \cdot \nabla u) \cdot \partial_1^2 b \, \mathrm{d}x + \int \partial_2^2(b \cdot \nabla u) \cdot \partial_2^2 b \, \mathrm{d}x \\
=& \sum_{k=1}^{2} C_2^k \int \partial_1^k b \cdot \partial_1^{2-k} \nabla u \cdot \partial_1^2 b \, \mathrm{d}x + \sum_{k=1}^{2} C_2^k \int \partial_2^k b \cdot \partial_2^{2-k} \nabla u \cdot \partial_2^2 b \, \mathrm{d}x \\
=& 2 \int \partial_1 b \cdot \partial_1 \nabla u \cdot \partial_1^2 b \, \mathrm{d}x + \int \partial_1^2 b \cdot \nabla u \cdot \partial_1^2 b \, \mathrm{d}x + 2 \int \partial_2 b \cdot \partial_2 \nabla u \cdot \partial_2^2 b \, \mathrm{d}x \\
&+ \int \partial_2^2 b \cdot \nabla u \cdot \partial_2^2 b \, \mathrm{d}x \\
=& I_{61} + I_{62} + I_{63} + I_{64}.
\end{aligned}
\tag{3.24}
$$

For I_{61} and I_{62}, by the Sobolev embedding inequality, we obtain

$$
\begin{aligned}
I_{61} + I_{62} =& 2 \int \partial_1 b \cdot \partial_1 \nabla u \cdot \partial_1^2 b \, \mathrm{d}x + \int \partial_1^2 b \cdot \nabla u \cdot \partial_1^2 b \, \mathrm{d}x \\
\leq& C \|\partial_1^2 b\|_{L^4} (\|\partial_1 b\|_{L^4} \|\partial_1 \nabla u\|_{L^2} + \|\nabla u\|_{L^2} \|\partial_1^2 b\|_{L^4}) \\
\leq& C \|u\|_{H^2} \|\partial_1 b\|_{H^2}^2.
\end{aligned}
\tag{3.25}
$$

I_{63} and I_{64} can be divided into six terms,

$$
\begin{aligned}
I_{63} + I_{64} =& 2 \int \partial_2 b \cdot \partial_2 \nabla u \cdot \partial_2^2 b \, \mathrm{d}x + \int \partial_2^2 b \cdot \nabla u \cdot \partial_2^2 b \, \mathrm{d}x \\
=& 2 \int \partial_2 b_1 \cdot \partial_{12} u \cdot \partial_2^2 b \, \mathrm{d}x + 2 \int \partial_2 b_2 \cdot \partial_2^2 u_1 \cdot \partial_2^2 b_1 \, \mathrm{d}x + 2 \int \partial_2 b_2 \cdot \partial_2^2 u_2 \cdot \partial_2^2 b_2 \, \mathrm{d}x \\
&+ \int \partial_2^2 b_1 \cdot \partial_1 u \cdot \partial_2^2 b \, \mathrm{d}x + \int \partial_2^2 b_2 \cdot \partial_2 u_1 \cdot \partial_2^2 b_1 \mathrm{d}x + \int \partial_2^2 b_2 \cdot \partial_2 u_2 \cdot \partial_2^2 b_2 \mathrm{d}x \\
=& K_1 + K_2 + K_3 + K_4 + K_5 + K_6.
\end{aligned}
\tag{3.26}
$$

Estimating K_2, K_3, K_5 and K_6 together,

$$
\begin{aligned}
K_2 + K_3 + K_5 + K_6 =& -2 \int \partial_1 b_1 \cdot \partial_2^2 u_1 \cdot \partial_2^2 b_1 \mathrm{d}x + 2 \int \partial_1 b_1 \cdot \partial_2^2 u_2 \cdot \partial_2 \partial_1 b_1 \mathrm{d}x \\
&- \int \partial_2 \partial_1 b_1 \cdot \partial_2 u_1 \cdot \partial_2^2 b_1 \mathrm{d}x + \int \partial_2 \partial_1 b_1 \cdot \partial_2 u_2 \cdot \partial_2 \partial_1 b_1 \mathrm{d}x \\
\leq& C \|\partial_1 b\|_{L^4} (\|\partial_2^2 u_1\|_{L^4} \|\partial_2^2 b_1\|_{L^2} + \|\partial_2 \partial_1 b_1\|_{L^4} \|\partial_2^2 u_2\|_{L^2}) \\
&+ C \|\partial_1 \partial_2 b_1\|_{L^4} (\|\partial_2 u_1\|_{L^4} \|\partial_2^2 b_1\|_{L^2} + \|\partial_1 \partial_2 b_1\|_{L^4} + \|\partial_2 u_2\|_{L^2}) \\
\leq& C (\|u\|_{H^2} + \|b\|_{H^2}) (\|\partial_1 b\|_{H^2}^2 + \|\partial_2 u_1\|_{H^2}^2).
\end{aligned}
\tag{3.27}
$$

Considering K_1, integrating it by parts,

$$\begin{aligned}K_1 =&2\int \partial_2 b_1\cdot\partial_{12}u\cdot\partial_2^2 b\mathrm{d}x\\ =&2\int \partial_2 b_1\cdot\partial_{12}u_1\cdot\partial_2^2 b_1\mathrm{d}x+2\int \partial_2 b_1\cdot\partial_{12}u_2\cdot\partial_2^2 b_2\mathrm{d}x\\ =&-2\int \partial_{12} b_1\cdot\partial_{2}u_1\cdot\partial_2^2 b_1\mathrm{d}x-2\int \partial_2 b_1\cdot\partial_{2}u_1\cdot\partial_1\partial_2^2 b_1\mathrm{d}x-2\int \partial_2 b_1\cdot\partial_{12}u_2\cdot\partial_{12} b_1\mathrm{d}x\\ \le&C\|\partial_{12}b_1\|_{L^4}(\|\partial_2u_1\|_{L^4}\|\partial_2^2 b_1\|_{L^2}+\|\partial_2 b_1\|_{L^2}\|\partial_{12}u_2\|_{L^4})+C\|\partial_2 b_1\|_{L^4}\|\partial_2 u_1\|_{L^4}\|\partial_1\partial_2^2 b_1\|_{L^2}\\ \le&C(\|u\|_{H^2}+\|b\|_{H^2})(\|\partial_1 b\|_{H^2}^2+\|\partial_1 u_2\|_{H^2}^2+\|\partial_2 u_1\|_{H^2}^2).\end{aligned}\tag{3.28}$$

Similarly to K_1, and using the Lemma 3.1 again,

$$\begin{aligned}K_4 =&\int \partial_2^2 b_1\cdot\partial_1u_1\cdot\partial_2^2 b_1\mathrm{d}x+\int \partial_2^2 b_1\cdot\partial_1u_2\cdot\partial_2^2 b_2\mathrm{d}x\\ =&-2\int u_1\cdot\partial_1\partial_2^2 b_1\cdot\partial_2^2 b_1\mathrm{d}x-\int \partial_2^2 b_1\cdot\partial_1u_2\cdot\partial_2\partial_1 b_1\mathrm{d}x\\ \le&C(\|u_1\|_{L^2}^{\frac12}\|\partial_2u_1\|_{L^2}^{\frac12}\|\partial_2^2b_1\|_{L^2}^{\frac12}\|\partial_1\partial_2^2b_1\|_{L^2}^{\frac12}\|\partial_1\partial_2^2b_1\|_{L^2}\\ &+\|\partial_1b_1\|_{L^4}\|\partial_1u_2\|_{L^4}\|\partial_2^2b_1\|_{L^2}\\ \le&C(\|u\|_{H^2}+\|b\|_{H^2})(\|\partial_2u_1\|_{H^2}^2+\|\partial_1b\|_{H^2}^2).\end{aligned}\tag{3.29}$$

Hence,

$$I_{63}+I_{64}\le C(\|u\|_{H^2}+\|b\|_{H^2})(\|\partial_1u_2\|_{H^2}^2+\|\partial_2u_1\|_{H^2}^2+\|\partial_1b\|_{H^2}^2).\tag{3.30}$$

Consequently, combining (3.25) and (3.30), we have

$$I_6\le C(\| u \|_{H^2}+\| b \|_{H^2})(\|\partial_1u_2\|_{H^2}^2+\| \partial_2u_1 \|_{H^2}^2+\| \partial_1b \|_{H^2}^2).\tag{3.31}$$

Now we concerning the last term I_7, which can be rewritten as follows,

$$\begin{aligned}I_7 =&-\int \partial_1^2(u\cdot\nabla m)\cdot\partial_1^2m\mathrm{d}x-\int \partial_2^2(u\cdot\nabla m)\cdot\partial_2^2m\mathrm{d}x\\ =&-\int \partial_1^2(u_1\cdot\partial_1m+u_2\cdot\partial_2m)\cdot\partial_1^2m\mathrm{d}x-\int \partial_2^2(u_1\cdot\partial_1m+u_2\cdot\partial_2m)\cdot\partial_2^2m\mathrm{d}x\\ =&I_{71}+I_{72}.\end{aligned}\tag{3.32}$$

First, by combining the Hölder's inequality and Young's inequality, I_{71} becomes

$$\begin{aligned}I_{71} =&-\sum_{k=1}^{2}C_2^k\int \partial_1^ku_1\cdot\partial_1^{2-k}\partial_1m\cdot\partial_1^2m\mathrm{d}x-\sum_{k=1}^{2}C_2^k\int \partial_1^ku_2\cdot\partial_1^{2-k}\partial_2m\cdot\partial_1^2m\mathrm{d}x\\ =&-2\int \partial_1u_1\cdot\partial_1^2m\cdot\partial_1^2m\mathrm{d}x-\int \partial_1^2u_1\cdot\partial_1m\cdot\partial_1^2m\mathrm{d}x\\ &-2\int \partial_1u_2\cdot\partial_{12}m\cdot\partial_1^2m\mathrm{d}x-\int \partial_1^2u_2\cdot\partial_2m\cdot\partial_1^2m\mathrm{d}x\\ \le&C\| \partial_1^2m \|_{L^4}(\| \partial_1u_1 \|_{L^2}\| \partial_1^2m \|_{L^4}+\| \partial_1^2u_1 \|_{L^2}\| \partial_1m \|_{L^4}\\ &+\| \partial_1u_2 \|_{L^4}\| \partial_2m \|_{L^2}+\| \partial_1^2u_2 \|_{L^4}\| \partial_2m \|_{L^2})\\ \le&C\| \partial_1m \|_{H^2}(\| u_1 \|_{H^2}\| \partial_1m \|_{H^2}+\| \partial_1u_2 \|_{H^2}\| m \|_{H^2})\\ \le&C(\| u \|_{H^2}+\| m \|_{H^2})(\| \partial_1m \|_{H^2}^2+\| \partial_1u_2 \|_{H^2}^2).\end{aligned}\tag{3.33}$$

We can infer I_{72},

$$\begin{aligned}
I_{72} =& -\sum_{k=1}^{2} C_2^k \int \partial_2^k u_1 \cdot \partial_2^{2-k}\partial_1 m \cdot \partial_2^2 m \mathrm{d}x - \sum_{k=1}^{2} C_2^k \int \partial_2^k u_2 \cdot \partial_2^{2-k}\partial_2 m \cdot \partial_2^2 m \mathrm{d}x \\
=& -2\int \partial_2 u_1 \cdot \partial_2\partial_1 m \cdot \partial_2^2 m \mathrm{d}x - \int \partial_2^2 u_1 \cdot \partial_1 m \cdot \partial_2^2 m \mathrm{d}x \\
&+ 3\int \partial_1 u_1 \cdot \partial_2\partial_2 m \cdot \partial_2^2 m \mathrm{d}x \\
\leq& C \parallel \partial_2^2 m \parallel_{L^2} (\parallel \partial_2 u_1 \parallel_{L^4} \parallel \partial_2\partial_1 m \parallel_{L^4} + \parallel \partial_2^2 u_1 \parallel_{L^4} \parallel \partial_1 m \parallel_{L^4} \\
&+ \parallel \partial_2 u_1 \parallel_{L^4} \parallel \partial_2\partial_1 m \parallel_{L^4}) \\
\leq& C \parallel m \parallel_{H^2} (\parallel \partial_2 u_1 \parallel_{H^2}^2 + \parallel \partial_1 m \parallel_{H^2}^2).
\end{aligned} \tag{3.34}$$

Therefore, I_7 is estimated as

$$I_7 \leq C(\parallel u \parallel_{H^2} + \parallel m \parallel_{H^2})(\parallel \partial_1 u_2 \parallel_{H^2}^2 + \parallel \partial_2 u_1 \parallel_{H^2}^2 + \parallel \partial_1 m \parallel_{H^2}^2). \tag{3.35}$$

Collecting all the estimate $I_1 + I_2 \cdots + I_7$ and inserting them into (3.5), we deduce

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{2}(\parallel \partial_i^2 u_1 \parallel_{L^2}^2 + \parallel \partial_i^2 u_2 \parallel_{L^2}^2 + \parallel \partial_i^2 b \parallel_{L^2}^2 + \parallel \partial_i^2 m \parallel_{L^2}^2) \\
&+ \sum_{i=1}^{2}(\parallel \partial_i^2\partial_2 u_1 \parallel^2 + \parallel \partial_i^2\partial_1 u_2 \parallel_{L^2}^2 + \parallel \partial_i^2\partial_1 b \parallel_{L^2}^2 + 2 \parallel \partial_i^2 m \parallel_{L^2}^2 + \parallel \partial_i^2\partial_1 m \parallel_{L^2}^2) \\
&\leq C(\|(u,b,m)(t)\|_{H^2})(\parallel \partial_1 u_2 \parallel_{H^2}^2 + \parallel \partial_2 u_1 \parallel_{H^2}^2 + \parallel \partial_1 b \parallel_{H^2}^2 + \parallel \partial_1 m \parallel_{H^2}^2 + 2 \parallel m \parallel_{H^2}^2).
\end{aligned} \tag{3.36}$$

Then we finished the estimate of $\parallel (\partial_i^2 u_1, \partial_i^2 u_2, \partial_i^2 b, \partial_i^2 m) \parallel_{L^2}^2$. Combining (3.4) and (3.36), then integrating the resulted equation on $[0, t]$, we conclude

$$|(u,b,m)(t)\|_{H^2}^2 + 2\int_0^t(\|\partial_2 u_1\|_{H^2}^2 + \|\partial_1 u_2\|_{H^2}^2 + \|\partial_1 b\|_{H^2}^2 + \|\partial_1 m\|_{H^2}^2 + 2\|m\|_{H^2}^2)d\tau$$

$$\leq \|(u_0, b_0, m_0)(t)\|_{H^2}^2 + C\int_0^t(\|u\|_{H^2} + \|b\|_{H^2} + \parallel m \parallel_{H^2}) \tag{3.37}$$

$$\times(\parallel \partial_1 u_2 \parallel_{H^2}^2 + \parallel \partial_2 u_1 \parallel_{H^2}^2 + \parallel \partial_1 b \parallel_{H^2}^2 + \parallel \partial_1 m \parallel_{H^2}^2)d\tau, \tag{3.38}$$

which indicates the desired estimate

$$E(t) \leq E(0) + C\,E(t)^{\frac{3}{2}}. \tag{3.39}$$

Thus this completes the proof of (1.8).

□

Proof of Theorem 1.2. We have the energy inequality, namely

$$E(t) \leq E(0) + CE(t)^{\frac{3}{2}}, \tag{3.40}$$

where C is a pure constant. Due to the assumption that $\|(u_0, b_0, m_0)\|_{H^2(\mathbb{R}^2)} \le \delta$ is sufficiently small, such that

$$E(0) \le \delta^2 := \frac{M}{2C}. \tag{3.41}$$

To initiate the bootstrapping argument to the energy inequality, we make the ansatz

$$E(t) \le \frac{1}{4C^2} = M. \tag{3.42}$$

It then implies that

$$CE(t)^{\frac{1}{2}} \le \frac{1}{2}. \tag{3.43}$$

Substituting (3.43) into (3.40) and combining with (3.41), we obtain

$$E(t) \le 2E(0) \le C\delta^2 := \frac{M}{2}. \tag{3.44}$$

Then we have obtained that $E(t)$ actually admits an smaller upper bound, which is

$$E(t) \le 2E(0) \le C\delta^2 \le \frac{M}{2}. \tag{3.45}$$

By the bootstrapping argument, this completes the proof of Theorem 1.2. □

4. Conclusions

In this paper, the stability of the 2D incompressible anisotropic magneto-micropolar fluid equations near a background magnetic field with partial mixed velocity dissipations, magnetic diffusion and horizontal vortex viscosity is considered. We obtained the explicit decay rates for the solution of the linear system in $H^s(\mathbb{R}^2)$ Sobolev space and the stability of nonlinear system. And the results reveal that the background magnetic field can stabilize the electrically conducting fluids.

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Conflict of interest

The authors declare that there are no conflicts of interest.

References

1. B. Yuan, Y. Qiao, Global regularity of 2D Leray-alpha regularized incompressible magneto-micropolar equations, *J. Math. Anal. Appl.,* **474** (2019), 492–512. https://doi.org/10.1016/j.jmaa.2019.01.057

2. J. Fan, Y. Zhou, Global solutions to the incompressible magneto-micropolar system in a bounded domain in 2D, *Appl. Math. Lett.,* **118** (2021), 107125. https://doi.org/10.1016/j.aml.2021.107125
3. L. Ma, On two-dimensional incompressible magneto-micropolar system with mixed partial viscosity, *Nonlinear Anal-Real,* **40** (2018), 95–129. https://doi.org/10.1016/j.nonrwa.2017.08.014
4. Y. Guo, H. Shang, Global well-posedness of two-dimensional magneto-micropolar equations with partial dissipation, *Appl. Math. Comput.,* **313** (2017), 392–407. https://doi.org/10.1016/j.amc.2017.06.017
5. H. Lin, S. Li, Global well-posedness for the $2\frac{1}{2}$D incompressible magneto-micropolar fluid equations with mixed partial viscosity, *Comput. Math. Appl.,* **72** (2016), 1066–1075. https://doi.org/10.1016/j.camwa.2016.06.028
6. Y. Lin, S. Li, Global well-posedness for magneto-micropolar system in $2\frac{1}{2}$ dimensions, *Appl. Math. Comput.,* **280** (2016), 72–85. https://doi.org/10.1016/j.amc.2016.01.002
7. Z. Tan, W. Wu, Global existence and decay estimate of solutions to magneto-micropolar fluid equations, *J. Differ. Equations,* **7** (2019), 4137–4169. https://doi.org/10.1016/j.jde.2018.09.027
8. Y. Wang, K. Wang, Global well-posedness of 3D magneto-micropolar fluid equations with mixed partial viscosity, *Nonlinear Anal-Real,* **33** (2017), 348–362. https://doi.org/10.1016/j.nonrwa.2016.07.003
9. P. Zhang, M. Zhu, Global regularity of 3D nonhomogeneous incompressible magneto-micropolar system with the density-dependent viscosity, *Comput. Math. Appl.,* **76** (2018), 2304–2314. https://doi.org/10.1016/j.camwa.2018.08.041
10. B. Yuan, Y. Qiao, Global regularity for the 2D magneto-micropolar equations with partial and fractional dissipation, *Comput. Math. Appl.,* **76** (2018), 2345–2359. https://doi.org/10.1016/j.camwa.2018.08.029
11. K. Yamazaki, Global regularity of the two-dimensional magneto-micropolar fluid system with zero angular viscosity, *Nonlinear Anal-Real,* **35** (2015), 21932207. https://doi.org/10.3934/dcds.2015.35.2193
12. H. Shang, C. Gu, Global regularity and decay estimates for 2D magneto-micropolar equations with partial dissipation, *Z. Angew. Math. Phys.,* **70** (2019), 22, Art. https://doi.org/85.10.1007/s00033-019-1129-8
13. H. Shang, J. Zhao, Global regularity for 2D magneto-micropolar equations with only micro-rotational velocity dissipation and magnetic diffusion, *Nonlinear Anal-Theor.,* **150** (2017), 194–209. https://doi.org/10.1016/j.na.2016.11.011
14. Li. Deng, H. Shang, Global well-posedness for n-dimensional magneto-micropolar equations with hyperdissipation, *Appl. Math. Lett.,* **111** (2021), 106610. https://doi.org/10.1016/j.aml.2020.106610
15. H. Shang, C. Gu, Large time behavior for two-dimensional magneto-micropolar equations with only micro-rotational dissipation and magnetic diffusion, *Appl. Math. Lett.,* **99** (2020), 105977. https://doi.org/10.1016/j.aml.2019.07.008
16. Y. Liu, Global well-posedness to the Cauchy problem of 2D density-dependent micropolar equations with large initial data and vacuum, *J. Math. Anal. Appl.,* **132** (2020), 124294. https://doi.org/10.1016/j.jmaa.2020.124294

17. Z. Ye, Global regularity results for the 2D Boussinesq equations and micropolar equations with partial dissipation, *J. Differ. Equations,* **268** (2020), 910–944. https://doi.org/10.1016/j.jde.2019.08.037
18. B. Dong, J. Li, J. Wu, Global well-posedness and large-time decay for the 2D micropolar equations, *J. Differ. Equations,* **262** (2017), 3488–3523. https://doi.org/10.1016/j.jde.2016.11.029
19. R. Guterres, W. Melo, J. Nunes, C. Perusato, On the large time decay of asymmetric flows in homogeneous Sobolev space, *J. Math. Anal. Appl.,* **471** (2019), 88–101. https://doi.org/10.1016/j.jmaa.2018.10.065
20. B. Dong, J. Wu, X. Xu, Z. Ye, Global regularity for the 2D micropolar equations with fractional dissipation, *Discrete Contin. Dyn. Syst.,* **38** (2018), 41334162.
21. Q. Chen, C. Miao, Global well-posedness for the micropolar fluid system in critical Besov spaces, *J. Differ. Equations,* **252** (2012), 2698–2724. https://doi.org/10.1016/j.jde.2011.09.035
22. W. Tan, B. Dong, Z. Chen, Regularity criterion for a critical fractional diffusion model of two-dimensional micropolar flows, *J. Math. Anal. Appl.,* **470** (2019), 500–514. https://doi.org/10.1016/j.jmaa.2018.10.017
23. M. Sermange, R. Temam, Some mathematical questions related to the MHD equations, *Comm. Pure Appl. Math.,* **36** (1983), 635–664. https://doi.org/10.1016/0167-7136(83)90286-X
24. C. Cao, J. Wu, Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion, *Adv. Math.,* **226** (2011), 1803–1822. https://doi.org/10.1016/j.aim.2010.08.017
25. N. Boardman, H. Lin, J. Wu, Stabilization of a background magnetic field on a 2 dimensional magnetohydrodynamic flow, *SIAM J. Math. Anal.,* **52** (2020), 5001–5035. https://doi.org/10.1137/20M1324776
26. B. Dong, Y. Jia, J. Li, J. Wu, Global regularity and time decay for the 2D magnetohydrodynamic equations with fractional dissipation and partial magnetic diffusion, *J. Math. Fluid Mech.,* **20** (2018), 1541–1565. https://doi.org/10.1007/s00021-018-0376-3
27. H. Lin, R. Ji, J. Wu, L. Yan, Stability of perturbations near a background magnetic field of the 2D incompressible MHD equations with mixed partial dissipation, *J. Funct. Anal.,* **279** (2020), 108519. https://doi.org/10.1016/j.jfa.2020.108519
28. S. Lai, J. Wu, J. Zhang, Stabilizing phenomenon for 2D anisotropic magnetohydrodynamic system near a background magnetic field, *SIAM J. Math. Anal.,* **53** (2021), https://doi.org/6073-6093. 10.1137/21M139791X
29. W. Feng, F. Hafeez, J. Wu, Influence of a background magnetic field on a 2D magnetohydrodynamic flow, *Nonlinerity,* **34** (2021), 2527–2562. https://doi.org/10.1088/1361-6544/abb928
30. T. Tao, *Nonlinear Dispersive Equations: Local and Global Analysis*, CBMS Regional Conference Series in Mathematics, Providence, RI: American Mathematical Society, 2006.

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