AIMS Mathematics, 7(10): 19553–19561.
DOI: 10.3934/math.20221073
Received: 03 May 2022
Revised: 10 August 2022
Accepted: 18 August 2022
Published: 05 September 2022

Mathematics

http://www.aimspress.com/journal/Math

Research article

A related problem on s-Hamiltonian line graphs

Xia Liu[1,2,*]

[1] Department of Mathematics, Northwest Normal University, Lanzhou 730070, China
[2] Key Laboratory of Discrete Mathematics with Applications of Ministry of Education, Center for Applied Mathematics of Fujian Province, Key Laboratory of Operations Research and Cybernetics of Fujian Universities, Fuzhou University, Fuzhou 350116, China

* **Correspondence:** Email: liuxia_90@163.com.

Abstract: A graph G is said to be *claw-free* if G does not contain $K_{1,3}$ as an induced subgraph. For an integer $s \geq 0$, G is *s-Hamiltonian* if for any vertex subset $S \subset V(G)$ with $|S| \leq s$, $G - S$ is Hamiltonian. Lai et al. in [On s-Hamiltonian line graphs of claw-free graphs, Discrete Math., 342 (2019)] proved that for a connected claw-free graph G and any integer $s \geq 2$, its line graph $L(G)$ is s-Hamiltonian if and only if $L(G)$ is $(s + 2)$-connected.
Motivated by above result, we in this paper propose the following conjecture. Let G be a claw-free connected graph such that $L(G)$ is 3-connected and let $s \geq 1$ be an integer. If one of the following holds:

(i) $s \in \{1, 2, 3, 4\}$ and $L(G)$ is essentially $(s + 3)$-connected,
(ii) $s \geq 5$ and $L(G)$ is essentially $(s + 2)$-connected,

then for any subset $S \subseteq V(L(G))$ with $|S| \leq s$, $|D_{\leq 1}(L(G) - S)| \leq \left\lfloor \frac{s}{2} \right\rfloor$ and $L(G) - S - D_{\leq 1}(L(G) - S)$ is Hamiltonian. Here, $D_{\leq 1}(L(G) - S)$ denotes the set of vertices of degree at most 1 in $L(G) - S$. Furthermore, we in this paper deal with the cases $s \in \{1, 2, 3, 4\}$ and $L(G)$ is essentially $(s+3)$-connected about this conjecture.

Keywords: essentially; s-Hamiltonian; supereulerian; collapsible; dominating
Mathematics Subject Classification: 05C45

1. Introduction

For the notation or terminology not defined here, see [1]. A graph is called *trivial* if it has only one vertex, *nontrivial* otherwise. Let $\kappa'(G)$ represent the *edge-connectivity* of a graph G. An edge (vertex) cut X is *essential* if $G - X$ has at least two non-trivial components. A graph G is *essentially k-edge-*

connected (or *essentially k-connected*) if G does not have an essential edge cut X (or an essential vertex cut X) with $|X| < k$. For a connected graph, define $ess(G) = \max\{k : G \text{ is essentially } k\text{-connected}\}$. For any $u \in V(G)$, we use $N_G(u)$ to denote the set of vertices which are adjacent to u in the graph G and define $d_G(u) = |N_G(u)|$, $N_G[u] = N_G(u) \cup \{u\}$. For an integer $i \geq 0$, define $D_{\geq i}(G) = \{v \in V(G) : d_G(v) \geq i\}$, $D_{\leq i}(G) = \{v \in V(G) : d_G(v) \leq i\}$, $D_i(G) = \{v \in V(G) : d_G(v) = i\}$ and $d_i(G) = |D_i(G)|$. We use $H \subseteq G$ ($H \cong G$) to denote the fact that H is a subgraph of G (H and G are isomorphic). Define $G[S]$ is the subgraph induced in G by S for $S \subseteq V(G)$ or $S \subseteq E(G)$. For $H_1, H_2 \subseteq G$, two disjoint sets $S_1, S_2 \subseteq V(G)$ and $X \subseteq E(G)$, define $G - S_1 = G[V(G) - S_1]$, $G - X = G[E(G) - X]$, $[S_1, S_2]_G = \{uv \in E(G) : u \in S_1, v \in S_2\}$, $[H_1, H_2]_G = [V(H_1), V(H_2)]_G$. We use v for $\{v\}$ and e for $\{e\}$. Throughout this paper, for an integer $n \geq 1$, P_n denotes a path of order n, C_n denotes a cycle on n vertices, W_n denotes the graph obtained from an n-cycle by adding a new vertex and connecting it to every vertex of the n-cycle, and $K_5 - e$ denotes the graph obtained from K_5 by deleting an edge. We call a bipartite graph $K_{1,n}$ a *star*.

A graph is *k-triangular* if each edge is in at least k triangles. The *line graph* of a given graph G, denoted by $L(G)$, is a graph with vertex set $E(G)$ such that two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are incident to a common vertex in G. For an integer $s \geq 0$, a graph G is *s-Hamiltonian* if for any vertex subset $S \subset V(G)$ such that $|S| \leq s$, $G - S$ is Hamiltonian. Broersma and Veldman in [2] raised the following question.

Problem 1. *(Broersma and Veldman, [2]) For an integer $k \geq 0$, determine the value s such that the line graph $L(G)$ of a k-triangular graph G is s-Hamiltonian if and only if $L(G)$ is $(s+2)$-connected.*

They commented in [2] that Problem 1 holds for $0 \leq s \leq k$ and conjectured that it holds if $0 \leq s \leq 2k$. Chen, Lai, Shiu and Li in [7] confirmed it holds when $0 \leq s \leq \max\{2k, 6k-16\}$. Then Lai et al. gave some attempt to characterize s-Hamiltonian line graph. A graph G is *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph.

Theorem 2. *Let G be a graph and $s \geq 2$ be an integer.*

(1) *(Lai and Shao, [9]) For $s \geq 5$, $L(G)$ is s-Hamiltonian if and only if $L(G)$ is $(s+2)$-connected.*
(2) *(Lai, Zhan, Zhang and Zhou, [11]) For $s \geq 2$, if G is claw-free, then $L(G)$ is s-Hamiltonian if and only if $L(G)$ is $(s+2)$-connected.*

In fact, the authors mainly proved the cases $s \in \{2, 3, 4\}$ of Theorem 2(2) in [11]. Motivated by Theorem 2(2), we propose the following conjecture.

Conjecture 3. *Let G be a claw-free connected graph such that $L(G)$ is 3-connected and let $s \geq 1$ be an integer. If one of the following holds:*

(i) $s \in \{1, 2, 3, 4\}$ *and $L(G)$ is essentially $(s+3)$-connected, or*
(ii) $s \geq 5$ *and $L(G)$ is essentially $(s+2)$-connected,*

then for any subset $S \subseteq V(L(G))$ with $|S| \leq s$, $|D_{\leq 1}(L(G) - S)| \leq \left\lfloor \frac{s}{2} \right\rfloor$ and $L(G) - S - D_{\leq 1}(L(G) - S)$ is Hamiltonian.

Define the *core* of G, denoted by G_0, to be the graph obtained from G by deleting all the vertices of degree 1, and replacing each path xyz with $y \in D_2(G)$ by an edge xz. It is easy to see that if G is claw-free, then the core G_0 is claw-free. Our main result of this paper is as follows, which settles Conjecture 3(i).

Theorem 4. *Let $s \in \{1, 2, 3, 4\}$ and G be a connected graph such that $L(G)$ is 3-connected and essentially $(s + 3)$-connected and the core G_0 is claw-free. Then for any $S \subseteq V(L(G))$ with $|S| \leq s$, $|D_{\leq 1}(L(G) - S)| \leq \left\lfloor \frac{s}{2} \right\rfloor$ and $L(G) - S - D_{\leq 1}(L(G) - S)$ is Hamiltonian.*

A *dominating closed trail* (abbreviated DCT) in a graph G is a closed trail (or, equivalently, an Eulerian subgraph) T in G such that every edge of G has at least one vertex on T. The following result by Harary and Nash-Williams relates the existence of a DCT in a graph G and the existence of a Hamiltonian cycle in its line graph $L(G)$.

Theorem 5. *(Harary and Nash-Williams, [8]) Let G be a graph with at least three edges. Then $L(G)$ is Hamiltonian if and only if G has a DCT.*

Remark 1. *For integer $i \in \{0, 1, 2, 3, 4\}$ and the graph H_i depicted in Figure 1, let H_i' be the graph obtained from H_i by deleting the bold lines. Then H_i' has no DCT and by Theorem 5, $L(H_i')$ is non Hamiltonian. Since $\kappa(L(H_0)) = 2$ and $ess(L(H_0)) = s + 5$, the condition "$L(G)$ is 3-connected" in Theorem 4 is sharp. Furthermore, for $i \in \{1, 2, 3, 4\}$, $ess(L(H_i)) = i + 2$ and $L(H_i)$ is not i-Hamiltonian, then the condition "$L(G)$ is essentially $(s + 3)$-connected" in Theorem 4 is sharp.*

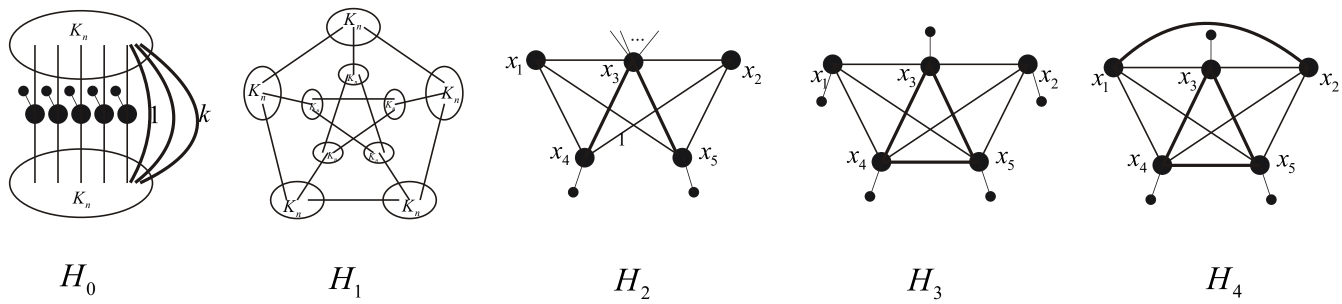

Figure 1. Some special graphs.

2. Proof of Theorem 4

Before starting the proof, we need some definitions and additional results. For $X \subseteq E(G)$, define the *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. If H is a subgraph of G, we write G/H for $G/E(H)$. If v_H is the contraction image of H in G/H, then H is called the preimage of v, and denoted by $PI(v)$. Call v is *non-trivial* if $|V(PI(v))| \geq 2$; *trivial*, otherwise. Let $O(G)$ denote the set of odd degree vertices in G. A graph G is *eulerian* if $O(G) = \emptyset$ and G is connected. A graph G is *supereulerian*if G has a spanning Eulerian subgraph. Catlin in [3] defined collapsible graphs. A graph G is *collapsible* if for any even subset R of $V(G)$, G has a connected spanning subgraph Γ_R with $O(\Gamma_R) = R$. The *reduction* of G is obtained from G by contracting all maximal collapsible subgraphs of G. Let $\tau(G)$ denote the maximum number of edge-disjoint spanning trees of G. Let $F(G)$ be the minimum number of additional edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees. We summarize some results on Catlin's reduction method and other related facts below.

Theorem 6. *Let G be a connected graph and H, G' be a collapsible subgraph and the reduction of G, respectively. Then each of the following holds.*

(1) *(Catlin, [3]) G is collapsible if and only if G/H is collapsible. And G is collapsible if and only if G' is K_1.*
(2) *(Catlin, [4]) $F(G') = 2|V(G')| - 2 - |E(G')|$.*
(3) *(Catlin, Han and Lai, [5]) If $F(G) \leq 2$, then $G' \in \{K_1, K_2, K_{2,t}\}$ for some $t \geq 1$.*
(4) *(Catlin, Lai and Shao, [6]) Let $k \geq 1$ be an integer. Then $\kappa'(G) \geq 2k$ if and only if for any edge subset $X \subseteq E(G)$ with $|X| < k$, $\tau(G - X) \geq k$.*

An edge cut X of G is a P_2-*edge-cut* of G if at least two components of $G - X$ contain P_3. Define $\kappa'_2(G) = \min\{|X| : X \text{ is a } P_2\text{-edge-cut of } G\}$.

Lemma 7. *If G is 3-edge-connected with $\kappa'_2(G) \geq 4$, then G is essentially 4-edge-connected.*

Proof. For any edge cut X of G such that $G - X$ has two non-trivial components G_1, G_2, if both G_1 and G_2 contain P_3, then X is a P_2-edge-cut and hence $|X| \geq 4$; otherwise, at least one of G_1, G_2 is isomorphic to K_2 and then $|X| \geq 4$. □

The graphs $P_{i,j,k}, K_{i,j,k} \subseteq G$ are two subgraphs isomorphic to a P_3 and a K_3 such that three vertices have degree i, j, k in G, respectively.

Lemma 8. *Let G be a 3-edge-connected graph and $G \notin \{K_4, W_4, K_5 - e, K_5\}$. Then*

(1) *G has no $K_{3,3,3}$ if $\kappa'_2(G) \geq 4$ and G has no $K_{3,3,4}$ if $\kappa'_2(G) \geq 4$,*
(2) *G has no $P_{3,3,3}$, $K_{3,4,4}$ and $K_{3,3,k}$ for $k \leq 5$ if $\kappa'_2(G) \geq 6$,*
(3) *G has no $P_{3,3,3}$, $P_{3,3,4}$, $K_{3,4,l}$ and $K_{3,3,k}$ for $l \leq 5$, $k \leq 6$ if $\kappa'_2(G) \geq 7$.*

Proof. Let $x_1x_2x_3 \subseteq G$ and $X = [\{x_1, x_2, x_3\}, V(G) - \{x_1, x_2, x_3\}]_G$. We assume that $|X| < \min\{\kappa'_2(G), 7\}$. Let $\mathcal{D}$ be the set of components of $G - X$. Then each component of $G - \{x_1, x_2, x_3\}$ belongs to $\{K_1, K_2\}$ and hence $|\mathcal{D}| \leq 2$ and $|\mathcal{D}| = 1$ if $P_2 \in \mathcal{D}$. Then $|V(G)| \leq 5$ and hence $G \in \{K_4, W_4, K_5, K_5 - e\}$, a contradiction. So $|X| \geq \kappa'_2(G)$ and lemma holds. □

Lemma 9. *Let G be a claw-free graph of order at least 6 such that $\kappa'(G) \geq 3$ and $\kappa'_2(G) \geq 4$. Then there is a set of edge-disjoint triangles $\triangle(G)$ such that $D_3(G) \subseteq V(\triangle(G))$ and $D_3(G) \cap V(K) \neq \emptyset$ for each $K \in \triangle(G)$.*

Proof. Since G is claw-free with $\kappa'(G) \geq 3$, each vertex with degree 3 is in a triangle. Then we can choose a set of triangles $\triangle(G)$ such that $D_3(G) \subseteq V(\triangle(G))$, $D_3(G) \cap V(K) \neq \emptyset$ for each $K \in \triangle(G)$, and then

$$|\bigcup_{K_1, K_2 \in \triangle(G)} E(K_1) \cap E(K_2)| \text{ is as small as possible.}$$

Suppose that there are two triangles $w_1u_1u_2w_1$, $w_2u_1u_2w_2 \in \triangle(G)$, then $d_G(w_1) = d_G(w_2) = 3$; for otherwise, delete the triangle $w_iu_1u_2w_i$ in $\triangle(G)$ if $d_G(w_i) \geq 4$ for any $i \in \{1, 2\}$. Besides, $w_1w_2 \notin E(G)$; for otherwise, replace $w_1u_1u_2w_1$, $w_2u_1u_2w_2$ by $w_1w_2u_2w_1$ in $\triangle(G)$. By Lemma 8, $\max\{d_G(u_1), d_G(u_2)\} \geq 4$. Without loss of generality, assume that $d_G(u_2) \geq 4$ and there is a vertex $x_1 \in N_G(u_2)$. Since $G[\{u_2, w_1, w_2, x_1\}] \not\cong K_{1,3}$, $[x_1, \{w_1, w_2\}]_G \neq \emptyset$. By symmetry, assume that $x_1w_1 \in E(G)$.

If there is a vertex $x_2 \in N_G(u_2)\backslash\{w_1, w_2, x_1\}$, then, by symmetry, $x_2w_2 \in E(G)$. So $4 \le d_G(u_2) \le 5$ and we can delete the triangle $w_1u_1u_2w_1$ and add the triangle $x_1w_1u_2x_1$ in $\triangle(G)$ if $x_1w_1u_2x_1 \notin \triangle(G)$, a contradiction. Hence any two triangles of $\triangle(G)$ are edge-disjoint. □

Theorem 10. *Let G be a connected graph such that $L(G)$ is 3-connected, essentially k-connected for some integer $k \ge 1$. Then*

(1) (Shao, [12]) the core G_0 of G is uniquely defined and $\kappa'(G_0) \ge 3$,
(2) (Lai, Shao, Wu and Zhou, [10]) $\kappa_2'(G_0) \ge \kappa_2'(G) \ge k$.

For a connected graph G and an Eulerian subgraph T, define $D[T] = \{uv : \{u, v\} \cap V(T) \ne \emptyset\}$.
Proof of Theorem 4. We first have $|D_{\le 1}(L(G) - S)| \le \left\lfloor \frac{s}{2} \right\rfloor$ as $L(G)$ is 3-connected. For any $X = \{e_1, \cdots, e_s\} \subseteq E(G)$, let $E_1 = N_{G-X}[D_1(G - X)]$, $S_2 = V(E_1) \cap D_2(G - X)$. Then $E_1 = (G - X)[V(E_1)]$ and $|E_1| \le \left\lfloor \frac{s}{2} \right\rfloor$. By Theorem 5, it suffices to prove that

$$G - X \text{ has an Eulerian subgraph } T \text{ such that } E(G)\backslash D[T] \subseteq E_1. \tag{2.1}$$

Let G_0 be the core of G. By Theorem 10, G_0 is 3-edge-connected with $\kappa_2'(G_0) \ge s + 3$ and $D_3(G_0) \subseteq D_3(G)$. It suffices to prove that for any $X = \{e_1, \cdots, e_s\} \subseteq E(G_0)$

$$G_0 - X \text{ has a spanning Eulerian subgraph } T \text{ such that } D_{\ge 2}(G_0 - X) \subseteq V(T). \tag{2.2}$$

(Since then for any $u \notin V(T)$, either $u \in S_2$ or $u \in V(G)$ has no neighbor in $D_1(G)$ and hence T can be extended to an Eulerian subgraph T' of G satisfying (2.1).)

By Lemma 7, G_0 is essentially 4-edge-connected. By Lemma 9, G_0 has a set of edge-disjoint triangles $\triangle(G)$ such that $D_3(G) \subseteq V(\triangle(G))$ and $D_3(G) \cap V(K) \ne \emptyset$ for each $K \in \triangle(G)$. Let $\triangle'(G) = \{K \in \triangle(G) : E(K) \cap X = \emptyset\}$ and $G_1 = G_0/\triangle'(G)$. Then G_1 is 3-edge-connected, essentially 4-edge-connected and $\kappa_2'(G_1) \ge s + 3$. Besides, $D_3(G_1) \subseteq D_3(G_0)$ since G_0 is essentially 4-edge-connected.

We first assume that $|V(G_1)| \le 5$. If $G_1 - X$ has no cycle, then it is isomorphic to the graph obtained from a star and some isolated vertices by subdividing some edges of star exactly once, respectively, and then the preimage of the center of the star is an Eulerian subgraph of G satisfying (2.1). Then we assume that $G_1 - X$ contains a longest cycle C. If for any 1-component x_0 of $G_1 - X - C$, $|[x, V(C)]_{G_1}| \le 1$, then (2.3) holds. We then assume that $|[x_0, V(C)]_{G_1}| \ge 2$ for some 1-component x_0 of $G_1 - X - C$, then $|V(C)| = 4$, $|V(G_1)| = 5$. Let $C = ux_1vx_2u$. Then $E(G_1) = E(C) \cup \{x_0u, x_0v\} \cup X$. Then at least one vertex $u_0 \in \{x_0, x_1, x_2\}$ is non-adjacent to one vertex of degree at most 2. Suppose otherwise. Then $\{x_0, x_1, x_2\} \subseteq D_4(G_1)$ and $s \in \{3, 4\}$. If $s = 3$, then $X = \{x_0x_1, x_0x_2, x_1x_2\}$ and $G_1 \cong K_5 - e$. If $s = 4$, then $G_1 \cong K_5$. However, there is a P_2-edge-cut with order at most $s + 2$, a contradiction. By symmetry, assume $u_0 = x_0$. Then C is a dominating trail of $G - X$.

Thus we assume that $|V(G_1)| \ge 6$ in the proof below. Note that a triangle is collapsible. Thus it suffices to prove that

$$G_1 - X \text{ has an Eulerian subgraph } T \text{ such that } D_{\ge 2}(G_1 - X) \subseteq V(T). \tag{2.3}$$

Case 1. $G_1 - X$ is disconnected.

In this case, if edges $e_1, \cdots, e_s$ have same end vertices u, v for any $u, v \in V(G_1)$ and $s \geq 2$, then $e_1, \cdots e_s$ are actually parallel edges. Since G_1 is 3-edge-connected, there are vertices $v, x_1, \cdots x_s$ such that either $d_{G_1}(v) = s$ and $\{vx_1, \cdots, vx_s\} = \{e_1, \cdots, e_s\}$ or $s = 4, d_{G_1}(v) = 3$ and $\{vx_1, vx_2, vx_3\} = \{e_1, e_2, e_3\}$. Then $D_3(G_1) \subseteq \{v\} \subseteq V(G)$ and hence $G_1[N_{G_1}[v]]$ is claw-free.

Subcase 1.1. $d_{G_1}(v) = s$.

Let $(d_1, \cdots, d_s) = (d_{G_1}(x_1), \cdots, d_{G_1}(x_s))$ and $(d'_1, \cdots, d'_s) = (d_{G_1-v}(x_1), \cdots, d_{G_1-v}(x_s))$. If $s = 3$, then $\kappa'_2(G_1) \geq 6$. By symmetry, assume that $x_1x_2 \in E(G_1)$ and $d_{G_1}(x_1) \leq d_{G_1}(x_2)$. By Lemma 8(2), $(d_1, d_2, d_3) \in \{(3, m, n) : m \geq 6, n \geq 4\} \cup \{(4, m, n) : m \geq 5, n \geq 3\} \cup \{(m, n, t) : m \geq 5, n \geq 5, t \geq 3\}$ and then $(d'_1, d'_2, d'_3) \in \{(2, m, n) : m \geq 5, n \geq 3\} \cup \{(3, m, n) : m \geq 4, n \geq 2\} \cup \{(m, n, t) : m \geq 4, n \geq 4, t \geq 2\}$. Let

$$E' = \begin{cases} \{x_1x_2, x_1x_3\}, & \text{if } (d'_1, d'_2, d'_3) \in \{(2, m, n) : m \geq 5, n \geq 3\}, \\ \{x_1x_3, x_2x_3\}, & \text{if } (d'_1, d'_2, d'_3) \in \{(m, n, l) : m \geq 3, n \geq 4, l \geq 2\}. \end{cases}$$

If $d_{G_1}(x_1) = 4$, then $\kappa'_2(G_1) \geq 7$. By symmetry, either $\{x_1x_2, x_2x_3\} \subseteq E(G_1)$ and $d_{G_1}(x_1) \leq d_{G_1}(x_2) \leq d_{G_1}(x_3)$ or $\{x_1x_2, x_3x_4\} \subseteq E(G_1)$, $d_{G_1}(x_1) \leq d_{G_1}(x_2)$ and $d_{G_1}(x_3) \leq d_{G_1}(x_4)$. We firstly assume that $\{x_1x_2, x_2x_3\} \subseteq E(G_1)$. By Lemma 8(3), $(d_1, d_2, d_3, d_4) \in \{(3, m, n, l) : m \geq 6, n \geq 6, l \geq 4\} \cup \{(m, n, l, p) : m \geq 4, n \geq 4, l \geq 4, p \geq 4\} \cup \{(4, m, n, 3) : m \geq 5, n \geq 5\} \cup \{(m, n, l, 3) : m \geq 5, n \geq 5, l \geq 5\}$. Let

$$E' = \begin{cases} \{x_1x_2, x_1x_4\}, & \text{if } (d'_1, d'_2, d'_3, d'_4) \in \{(2, m, n, l) : m \geq 5, n \geq 5, l \geq 3\}, \\ \{x_1x_2, x_3x_4\}, & \text{if } (d'_1, d'_2, d'_3, d'_4) \in \{(m, n, l, p) : m \geq 3, n \geq 3, l \geq 3, p \geq 3\}, \\ \{x_2x_4, x_3x_4\}, & \text{if } (d'_1, d'_2, d'_3, d'_4) \in \{(m, n, l, 2) : m \geq 3, n \geq 4, l \geq 4\}. \end{cases}$$

We then assume that $\{x_1x_2, x_3x_4\} \subseteq E(G_1)$. By Lemma 8(3), $(d_1, d_2, d_3, d_4) \in \{(3, m, 3, n) : m \geq 6, n \geq 6\} \cup \{(3, m, 4, n) : m \geq 6, n \geq 5\} \cup \{(4, m, 4, n) : m \geq 5, n \geq 5\} \cup \{(m, n, l, p) : m \geq 5, n \geq 5, l \geq 5, p \geq 5\}$. Let

$$E' = \begin{cases} \{x_1x_2, x_1x_2\}, & \text{if } (d'_1, d'_2, d'_3, d'_4) \in \{(2, m, n, l) : m \geq 5, n \geq 2, l \geq 4\}, \\ \{x_1x_3\}, & \text{if } (d'_1, d'_2, d'_3, d'_4) \in \{(m, n, l, p) : m \geq 3, n \geq 4, l \geq 3, p \geq 4\}. \end{cases}$$

Note that $D_{\leq 3}(G_1 - v) \subseteq \{x_1, \cdots, x_s\}$. Let Q_1 be the graph obtained from $G_1 - v$ by adding the edge set E'. Then Q_1 is 4-edge-connected. By Theorem 6(4), $\tau(Q_1 - E') = \tau(G_1 - v) \geq 2$. Therefore, $G_1 - v$ is collapsible and then is supereulerian. Hence $G_1 - X$ has a dominating Eulerian subgraph T_1 such that $V(G_1) \backslash V(T_1) = \{v\}$. Hence (2.3) holds.

Subcase 1.2. $s = 4$ and $d_{G_1}(v) = 3$.

Then $\kappa'_2(G_1) \geq 7$. By Subcase 1.1, $\tau(G_1 - v) \geq 2$. Then $F(G_1 - v - X) \leq 1$. Note that $\kappa'(G_1 - v - X) \geq 2$ since G_1 is essentially 4-edge-connected. By Theorem 6(3), $G_1 - v - X$ is collapsible and hence it has a dominating Eulerian subgraph T_2 such that $V(G_1) \backslash V(T_2) = \{v\}$. Hence (2.3) holds.

Case 2. $G_1 - X$ is connected.

Let G' be the reduction of $G_1 - X$.

Claim 1. *If $F(G') \leq 2$, then (2.3) holds.*

Proof. By Theorem 6(3), $G' \in \{K_1, K_2, K_{2,t}\}$ for some integer $t \geq 2$. If $G' \cong K_{2,t}$ for some odd integer $t \geq 3$, then each vertex of degree 2 in G' is trivial; for otherwise, assume that $|PI(u)| \geq 3$, then $PI(u)$ has a P_3 and there is a P_2-edge-cut $X' = [V(PI(u)), V(G_1) - V(PI(u))]_{G_1}$ with $|X'| \leq |X| + 2$, a contradiction.

If one vertex u of degree 3 in G' is non-trivial, then $X \subseteq [V(PI(u)), V(G_1) - V(PI(u))]_{G_1}$. If $s \leq 2$, then there is a vertex of degree 2, a contradiction. If $s = 3$, then there is a $P_{3,3,3}$, a contradiction. If $s = 4$, there is a $P_{3,3,4}$, a contradiction. Hence $G' \subseteq G_1 - X$. If $s = 3$, then either $G' \cong K_{2,5}$ and G' has a $K_{3,3,5}$ or $G' \cong K_{2,3}$ and $G_1 \cong K_5 - e$, a contradiction. If $s = 4$, then G' either has a $P_{3,3,4}$ (if $t \geq 7$) or has a $K_{3,3,5}$ (if $t \leq 5$), a contradiction.

If $G' \in \{K_1, K_{2,t}\}$ for some even integer $t \geq 2$, then G' is supereulerian and then $G_1 - X$ is supereulerian by Theorem 6(1). If $G' \cong K_2 = uv$, then at least one of u, v is trivial; for otherwise, $X \cup \{uv\}$ is a P_2-edge-cut of G_1 with $|X \cup \{uv\}| \leq s + 1$, a contradiction. By symmetry, assume that u is trivial and $PI(v)$ is collapsible. Then $u \in D_1(G_1 - X)$ and $G_1 - X$ has a dominating Eulerian subgraph T_3 such that $V(G_1)\backslash V(T_3) = \{u\}$ and (2.3) holds.

If $G' \cong v_1uv_2$, then v_1, v_2 are trivial and $PI(u)$ is collapsible. Then $v_1, v_2 \in D_1(G_1 - X)$ and $G_1 - X$ has a dominating Eulerian subgraph T_4 such that $V(G_1)\backslash V(T_4) = \{v_1, v_2\}$ and (2.3) holds. □

Let $G_2 = G' \cup X$ and define $\phi(G_2) = 2|V(G_2)| - |E(G_2)| - 2$. Then G_2 is 3-edge-connected, essentially 4-edge-connected with $\kappa'_2(G_2) \geq s + 3$. By Theorem 6(2), $F(G') \leq \phi(G_2) + s = \frac{1}{2}(d_3(G_2) - \sum_{i\geq 5}(i - 4)d_i(G_2)) + (s - 2)$. By Claim 1, it suffices to prove that $F(G') \leq 2$, that is,

$$d_3(G_2) - \sum_{i\geq 5}(i - 4)d_i(G_2) \leq 8 - 2s. \tag{2.4}$$

If $|D_3(G_2)| \leq 2$ when $s = 3$, then add at most one edge e such that $D_3(G_2) \subseteq V(e)$ and the resulting graph, say G'_2, is 4-edge-connected. Then $\tau(G_2) = \tau(G'_2 - \{e, f\}) \geq 2$ for any edge $f \in E(G'_2)$ by Theorem 6(4) and hence $F(G') \leq F(G_2 - X) \leq 2$. By the same argument, $F(G') \leq 2$ if $|D_3(G_2)| \leq 6$ when $s = 1$, $|D_3(G_2)| \leq 4$ when $s = 2$ and $|D_3(G_2)| = 0$ when $s = 4$. Hence we only consider the cases $|D_3(G_2)| \geq 3$ when $s = 3$ and $|D_3(G_2)| \geq 1$ when $s = 4$.

Note that $D_3(G_2) \subseteq V(G)$. Then $G_2[N_{G_2}[u]]$ is claw-free for any $u \in D_3(G_2)$ and then u is in a triangle of G_2. Recall G_2 is obtained from G' by adding X, then each vertex of degree 3 is in a triangle of G_2 which contains at least one edge of X. Then G_2 has at most s edge-disjoint triangles containing all vertices of degree 3 and each of them must contain at least one edge of X. Since $\kappa'_2(G_2) \geq 5$, G_2 has no $K_{3,3,3}$. Then $|D_3(G_2)| \leq 2s$.

Besides, for any vertex $u \in V(G_2)$ with degree less than $s + 2$,

$$\text{if } G_2 - u \text{ contains } P_3\text{, then } u \in V(G_0) \text{ and } G_2[N_{G_2}[u]] \text{ is claw-free.} \tag{2.5}$$

(For otherwise, $[V(PI(u)), V(G_1) - V(PI(u))]_{G_1}$ is a P_2-edge-cut of G_1, a contradiction.) We then consider the following two subcases to finish our proof.

Subcase 2.1. $s = 3$ and $3 \leq |D_3(G_2)| \leq 6$.

Then $\kappa'_2(G_2) \geq 6$ and it suffices to prove that

$$d_3(G_2) - \sum_{i\geq 5}(i - 4)d_i(G_2) \leq 2. \tag{2.6}$$

By Lemma 8(2), there is a triangle $x_1x_2x_3x_1$ such that $\max\{d_{G_2}(x_1), d_{G_2}(x_2), d_{G_2}(x_3)\} \geq 5$ if $|D_3(G_2)| = 3$ and $d_{G_2}(x_1) = d_{G_2}(x_2) = 3$, $d_{G_2}(x_3) \geq 6$ if $|D_3(G_2)| = 4$, and hence (2.6) holds.

If $|D_3(G_2)| = 5$, then there are three edge-disjoint triangles $u_1x_1x_2u_1, u_2y_1y_2u_2, u_3z_1z_2u_3$ such that $\{x_1, x_2, y_1, y_2, z_1\} = D_3(G_2)$ and $d_{G_2}(u_1) \geq 6$, $d_{G_2}(u_2) \geq 6$ and $d_{G_2}(u_3) \geq 5$. So (2.6) holds if at least two of u_1, u_2, u_3 are distinct or $d_{G_2}(z_2) \geq 5$. Otherwise, $G_2[N_{G_2}[z_2]]$ is claw-free and hence either both

x_1 and x_2 or both y_1 and y_2 are nonadjacent to z_1. By symmetry, say x_1, x_2. If x_1, x_2 have a common neighbor x_{12} outside $\{u_1\}$ with $d_{G_2}(x_{12}) \geq 6$ or $x_1' \in N_{G_2}(x_1)$, $x_2' \in N_{G_2}(x_2)$ with $\max\{d_{G_2}(x_1'), d_{G_2(x_2')}\} \geq 5$, then (2.6) holds. If $d_{G_2}(x_1') = d_{G_2(x_2')} = 4$, then $\{x_1'u_1, x_1'u_2\} \subseteq E(G_2)$, $d_{G_2}(u_1) \geq 8$ and (2.6) holds.

If $|D_3(G_2)| = 6$, the discussion is similar to the case when $|D_3(G_2)| = 5$, then we omit it here.

Subcase 2.2. $s = 4$ and $1 \leq |D_3(G_2)| \leq 8$.

Then $\kappa_2'(G_2) \geq 7$. For a vertex v of degree 5 or 6, $G_2[N_{G_2}[v]]$ is claw-free by (2.5). Then there are at most two vertices of degree 3 in $N_{G_2}(v)$; for otherwise, there is a $K_{3,3,5}$ or $K_{3,3,6}$, contradicting Lemma 8(3).

For a vertex w of degree 7, if $\{x_1, \cdots, x_7\} = N_{G_2}(w) \subseteq D_3(G_2)$ and $\{x_1x_2, x_3x_4, x_5x_6\} \subseteq E(G_2)$, then x_7 has two neighbors y_1, y_2 such that $y_1y_2 \in E(G_2)$ and $d_{G_2}(y_1) \leq d_{G_2}(y_2)$ since G_2 has no $P_{3,3,3}$. Assume that $|D_3(G_2)| = 8$. Then $d_{G_2}(y_1) = 3$, $d_{G_2}(y_2) \geq 7$ and y_1 has a neighbor y_3 with $d_{G_2}(y_3) \geq 5$. If $d_{G_2}(y_3) \geq 7$ or $N_{G_2}\{x_1, \cdots, x_6\} \nsubseteq \{y_2, y_3\}$, then (2.4) holds. Otherwise, note that $5 \leq d_{G_2}(y_3) \leq 6$, then $y_2y_3 \in E(G_2)$ and at least 5 vertices of $\{x_1, \cdots, x_6\}$ are adjacent to y_2. Then $d_{G_2}(y_2) \geq 8$ and (2.4) holds. Assume that $|D_3(G_2)| = 7$. If $d_{G_2}(y_1) \geq 7$, then (2.4) holds. Otherwise, $G_2[N_{G_2}[y_1]]$ is claw-free and then $|N_{G_2}(y_1) \cap \{x_1, \cdots, x_7\}| = 1$ and hence there are at least two vertices $u_1, u_2 \in N_{G_2}(y_1)$ with $u_1u_2 \in E(G_2)$. Then there are 5 edge-disjoint triangles, a contradiction.

Thus we consider the case when w has at most six neighbors of degree 3. Define a function $l(u) = \begin{cases} \frac{1}{2}, & \text{if } d_{G_2}(u) \geq 5; \\ 0, & \text{otherwise.} \end{cases}$ For a vertex u of degree 3 and its neighbors x_1, x_2, x_3, at least two of them have degree at least 5 by the argument in Subcase 1.1. Then $l(x_1) + l(x_2) + l(x_3) \geq 1$. So

$$\begin{aligned} d_3(G_2) &= \sum_{u \in D_3(G_2)} 1 \leq \sum_{u \in D_3(G_2)} \sum_{v \in N_{G_2}(u)} l(v) \\ &\leq \sum_{v \in D_5(G_2) \cup D_6(G_2)} 2 \times \frac{1}{2} + \sum_{v \in D_7(G_2)} 6 \times \frac{1}{2} + \sum_{v \in D_{\geq 8}(G_2)} d_{G_2}(v) \times \frac{1}{2} \\ &\leq \sum_{i \geq 5} (i-4) d_i(G_2). \end{aligned}$$

Then (2.4) holds. This completes the proof of Theorem 4. □

3. Conclusions

Note that Conjecture 3(ii) is a generalization of Theorem 2 when $s \geq 5$. By dealing with Conjecture 3(i), we know that how the connectivity effects the s-hamiltonicity of claw-free graphs. By comparing them, for a 3-connected line graph $L(G)$ with $ess(L(G)) \geq s$ condition other than $L(G)$ is s-connected, graph G may have essential l-edge-cut for some $3 \leq l \leq s$, which leads to $L(G) - X$ is disconnected for some vertex set $X \subseteq V(L(G))$ with $|X| \leq s$. There are still many properties for us to further explore.

Acknowledgments

The work is supported by the Open Project Program of Key Laboratory of Discrete Mathematics with Applications of Ministry of Education, Center for Applied Mathematics of Fujian Province, Key Laboratory of Operations Research and Cybernetics of Fujian Universities, Fuzhou University.

Conflict of interest

The author declares no conflict of interest.

References

1. J. A. Bondy, U. S. R. Murty, *Graph theory*, Springer, 2008. http://dx.doi.org/10.1007/978-1-84628-970-5
2. H. J. Broersma, H. J. Veldman, 3-connected line graphs of triangular graphs are pan-connected and 1-Hamiltonian, *J. Graph Theor.*, **11** (1987), 399–407. https://doi.org/10.1002/jgt.3190110314
3. P. A. Catlin, A reduction method to find spanning eulerian subgraphs, *J. Graph Theor.*, **12** (1988), 29–44. https://doi.org/10.1002/jgt.3190120105
4. P. A. Catlin, Supereulerian graphs, collapsible graphs, and four-cycles, *Congressus Numerantium*, **58** (1988), 233–246.
5. P. A. Catlin, Z. Y. Han, H. J. Lai, Graphs without spanning closed trals, *Discrete Math.*, **160** (1996), 81–91. https://doi.org/10.1016/S0012-365X(95)00149-Q
6. P. A. Catlin, H. J. Lai, Y. Shao, Edge-connectivity and edge-disjoint spanning trees, *Discrete Math.*, **309** (2009), 1033–1040. https://doi.org/10.1016/j.disc.2007.11.056
7. Z. H. Chen, H. J. Lai, W. Shiu, D. Li, An s-Hamiltonian line graph problem, *Graph. Combinator.*, **23** (2007), 241–248. https://doi.org/10.1007/s00373-007-0727-y
8. F. Harary, C. S. J. A. Nash-Williams, On eulerian and Hamiltonian graphs and line graphs, *Can. Math. Bull.*, **8** (1965), 701–710. https://doi.org/10.4153/CMB-1965-051-3
9. H. J. Lai, Y. Shao, On s-Hamiltonian line graphs, *J. Graph Theor.*, **74** (2013), 344–358. https://doi.org/10.1002/jgt.21713
10. H. J. Lai, Y. Shao, H. Wu, J. Zhou, Every 3-connected, essentially 11-connected line graph is Hamiltonian, *J. Comb. Theory B*, **96** (2006), 571–576. https://doi.org/10.1016/j.jctb.2005.11.002
11. H. J. Lai, M. Zhan, T. Zhang, J. Zhou, On s-Hamiltonian line graphs of claw-free graphs, *Discrete Math.*, **342** (2019), 3006–3016. https://doi.org/10.1016/j.disc.2019.06.006
12. Y. Shao, *Claw-free graphs and line graphs*, Ph.D. Dissertation, West Virginia University, 2005.