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Research article

The existence of entire solutions of some systems of the Fermat type differential-difference equations

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Abstract: In this paper, we investigate some systems of the Fermat type differential-difference equations with polynomial coefficients and obtain the condition for the existence of finite order transcendental entire solutions and the expression for the entire solutions. We also give some corresponding examples.

Keywords: existence; Fermat type; differential-difference equation; finite order; entire solution
Mathematics Subject Classification: 30D35, 39A45

1. Introduction

It is assumed that the reader is familiar with the standard notations and basic results of Nevanlinna's value distribution theory (see [2,10]). Especially, we use the notions $\sigma(f)$ to denote the order of growth of the meromorphic function $f(z)$ and $S(r, f)$ to denote any quantity that satisfies $S(r, f) = o(T(r, f))$, where $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure.

The celebrated Fermat's last theorem [17] elaborates that it do not exist nonzero rational numbers x, y and an integer $n \geq 3$ such that $x^n + y^n = 1$. The equation $x^2 + y^2 = 1$ does admit nontrivial rational solutions. Replacing x, y in it by entire or meromorphic functions f, g, Fermat type functional equations were studied by Gross [8] and many others thereafter. In 2004, Yang and Li [15] did some related research and they obtained the following result.

Theorem A. [15] Let n be a positive integer, $a(z), b_0(z), b_1(z), \cdots, b_{n-1}(z)$ are polynomials, and $b_n(z) \equiv b_n$ be a nonzero constant. Let $L(f) = \sum_{k=0}^{n} b_k(z) f^{(k)}$. If $a(z) \not\equiv 0$, then a transcendental meromorphic solution of the equation

$$f^2 + (L(f))^2 = a(z),$$

must have the form $f(z) = \frac{1}{2}\left(P(z)e^{R(z)} + Q(z)e^{-R(z)}\right)$, where $P(z), Q(z), R(z)$ are polynomials, and $P(z)Q(z) = a(z)$.

Recently, as the difference analogues of Nevanlinna's theory are being investigated [2], there are many interests in the complex analytic properties of meromorphic solutions of complex difference equations, and many results on the complex linear or nonlinear difference equations are got rapidly, such as [2, 9, 11]. In particular, some results on the solutions of the Fermat type functional equation are obtained [11–13].

In 2012, Liu et al. [11] investigated entire solutions with finite order of the Fermat type differential-difference equation and obtained the following result.

Theorem B. [11] The transcendental entire solutions with finite order of the differential-difference equation

$$f'(z)^2 + f(z+c)^2 = 1,$$

must satisfy $f(z) = \sin(z + iB)$, where B is a constant and $c = 2k\pi$ or $c = 2k\pi + \pi, k$ is an integer.

In 2012, Gao had discussed the existence or growth of some types of systems of complex difference equations, and obtained some results [3–7]. Especially, Gao [7] investigated the existence of transcendental entire functions for the system of the nonlinear differential-difference equations

$$\begin{cases} w_1'(z)^2 + P_1(z)^2 w_2(z+c)^2 = Q_1(z), \\ w_2'(z)^2 + P_2(z)^2 w_1(z+c)^2 = Q_2(z), \end{cases}$$

and obtained the following interesting result.

Theorem C. [7] Suppose that $(w_1(z), w_2(z))$ are the transcendental entire solutions for the above differential-difference equations, $\sigma(w_1, w_2) < \infty$, then $P_1(z) = A, P_2(z) = B, AB \neq 0$, and

$$w_1(z) = \frac{c_{11}e^{az+b_1} - c_{12}e^{-az-b_1}}{2a}, \qquad w_2(z) = \frac{c_{21}e^{az+b_2} - c_{22}e^{-az-b_2}}{2a},$$

where $a^4 = A^2B^2, A, B, b_j, c_{jk}, k, j = 1, 2$ are all constants.

The order of growth of meromorphic solutions (f, g) of the system of the nonlinear differential-difference equations is defined by

$$\sigma(f, g) = \max\{\sigma(f), \sigma(g)\}, \qquad \sigma(f) = \limsup_{r\to\infty} \frac{\log^+ T(r, f)}{\log r}.$$

In 2019, Liu and Gao [13] investigated the existence of entire functions for the nonlinear differential-difference equation

$$\omega''^2(z) + \omega(z+c)^2 = Q(z), \tag{1.1}$$

where $Q(z)$ is a nonzero polynomials and they obtained the following result.

Theorem D. [13] There is a transcendental entire solution $w(z)$ with finite order for the differential-difference equation (1.1), then $Q(z) = c_1c_2$ is a constant, and $w(z)$ must satisfy

$$w(z) = \frac{c_1e^{az+b} + c_2e^{-az-b}}{2a^2},$$

where a is a constant such that $a^4 = 1$, b and b' are arbitrary constants, $c = \frac{ln(-ia^2)+2k\pi i}{a}$, k is an integer.

Inspired by the above theorems, we [14] consider the existence of entire functions for the nonlinear differential-difference

$$\omega''(z)^2 + [P(z)(d_1\omega(z+c) + d_0\omega(z))]^2 = Q(z), \tag{1.2}$$

where $P(z)$ and $Q(z)$ be nonzero polynomials, d_0, d_1 are nonzero constants and we obtained the following result.

Theorem E. [14] Let $P(z)$ and $Q(z)$ be nonzero polynomials, d_0, d_1 are nonzero constants such that $d_0 = \pm d_1$, then there is no finite order entire solution $\omega(z)$ satisfying the nonlinear differential-difference equation (1.2).

More interestingly, we find that for $d_1 = 1, d_0 = 0$, the Eq (1.2) has a transcendental entire solution with the following form

$$\omega(z) = \frac{q_1 e^{az+b} - q_2 e^{-(az+b)}}{2a^2},$$

where $P(z) \equiv A(\neq 0)$ and $Q(z) = q_1 q_2(\neq 0)$, $a^4 = A^2, ac = \frac{ik\pi}{2}, b \in \mathbb{C}, q_1, q_2, c$ are nonzero complex constants, $k \in \mathbb{Z}$.

A natural and interesting question is, what we can say about the system of the differential-difference equation

$$\begin{cases} f''(z)^2 + [P_1(z)(d_1 g(z+c) + d_0 g(z))]^2 = Q_1(z), \\ g''(z)^2 + [P_2(z)(d_1 f(z+c) + d_0 f(z))]^2 = Q_2(z), \end{cases} \tag{1.3}$$

where $P_j(z)(j = 1, 2)$, $Q_i(z)(i = 1, 2)$ are nonzero polynomials and d_0, d_1 are nonzero constants?

In this paper, we investigate the the existence of finite order transcendental entire solutions for the Fermart type of the system of differential-difference equations (1.3), and we obtain the following result.

Theorem 1.1. *Let $P_j(z)(j = 1, 2)$, $Q_i(z)(i = 1, 2)$ be nonzero polynomials, d_0, d_1 be constants such that $d_0 \neq \pm d_1$. If the Fermart type of systems of differential-difference equation (1.3) has transcendental entire solutions $(f(z), g(z))$ such that $\sigma(f, g) < \infty$, then $P_j(z) \equiv P_j(j = 1, 2)$ are nonzero constants and $(f(z), g(z))$ can be expressed as*

$$\begin{cases} f(z) = \dfrac{S_1^*(z)e^{az+b_1} + S_2^*(z)e^{-az-b_1}}{2i}, \\ g(z) = \dfrac{T_1^*(z)e^{az+b_2} + T_2^*(z)e^{-az-b_2}}{2}, \end{cases}$$

where $a^4 = P_1 P_2 (d_0 + d_1 e^{ac})^2, d_0 + d_1 e^{ac} \neq 0$, either $ac = k\pi i$, $k \in \mathbb{Z}$ or $-2\frac{d_0}{d_1} = e^{-ac} + e^{ac}$, and $S_i^*(z), T_i^*(z)(i = 1, 2)$ are polynomials related to $Q_i(z)(i = 1, 2), b_1$ and b_2 are arbitrary constants.

Example 1.1. Let $a = 2i, P_j = \frac{1}{2}, S_1 = S_2 = -4, T_1 = 4i, T_2 = -4i, d_0 = -d_1 = 4$, then the solutions of the systems of differential-difference equation

$$\begin{cases} f''(z)^2 + \left[\dfrac{1}{2}\left(4g(z - \dfrac{\pi}{2}) - 4g(z)\right)\right]^2 = 16 \\ g''(z)^2 + \left[\dfrac{1}{2}\left(4f(z - \dfrac{\pi}{2}) - 4f(z)\right)\right]^2 = 16 \end{cases}$$

has the transcendental entire solutions

$$\begin{cases} f(z) = \dfrac{-4e^{i(2z-\frac{\pi}{2})} - 4e^{-i(2z-\frac{\pi}{2})}}{-8} = \dfrac{e^{i(2z-\frac{\pi}{2})} + e^{-i(2z-\frac{\pi}{2})}}{2} = \sin 2z, \\ g(z) = \dfrac{4ie^{i(2z+\frac{\pi}{2})} - 4ie^{-i(2z+\frac{\pi}{2})}}{8i^2} = \dfrac{e^{i(2z+\frac{\pi}{2})} - e^{-i(2z+\frac{\pi}{2})}}{2i} = \cos 2z. \end{cases}$$

Corollary 1.1. Let $P_j(z)(j = 1, 2)$ be nonconstant polynomials, then the Fermart type of systems of differential-difference equation (1.3) has no finite order transcendental entire solutions $(f(z), g(z))$.
Remark 1.1. We want to prove a stronger conclusion that $S_i(z), T_i(z)(i = 1, 2)$ are constants too, unfortunately we fail. In some special cases, we obtain the ideal result, such as $d_1 = 1, d_0 = 0$.

In fact, we investigate the differential-difference equation

$$\begin{cases} f''(z)^2 + [P_1(z)g(z+c)]^2 = Q_1(z), \\ g''(z)^2 + [P_2(z)f(z+c)]^2 = Q_2(z), \end{cases} \tag{1.4}$$

and obtain the following result.

Theorem 1.2. *Let $P_j(z)(j = 1, 2)$ and $Q_i(z)(i = 1, 2)$ be nonzero polynomials. If the Fermart type of systems of differential-difference equation (1.5) has transcendental entire solutions $(f(z), g(z))$ such that $\sigma(f, g) < \infty$, then $P_j(z) \equiv P_j(j = 1, 2), Q_1(z) = \beta_1\beta_2$ and $Q_2(z) = \alpha_1\alpha_2$, where $P_i, \alpha_i, \beta_i(i = 1, 2)$ are nonzero constants. Furthermore, $(f(z), g(z))$ can be expressed as*

$$\begin{cases} f(z) = \dfrac{\beta_1 e^{az+b_1} + \beta_2 e^{-az-b_1}}{2a^2}, \\ g(z) = \dfrac{\alpha_1 e^{az+b_2} + \alpha_2 e^{-az-b_2}}{2a^2}, \end{cases} \tag{1.5}$$

where $a^8 = P_1^2 P_2^2$ and $ac = \frac{1}{2}k\pi i$, $k \in \mathbb{Z}, b_1$ and b_2 are arbitrary constants.

Example 1.2. Let $a = 2i, P_j = \pm 4, \alpha_{11} = \alpha_{12} = \beta_{11} = \beta_{12} = 1$, then the solutions of the systems of differential-difference equation

$$\begin{cases} f''(z)^2 + 16g(z+c)^2 = 1 \\ g''(z)^2 + 16f(z+c)^2 = 1 \end{cases} \tag{1.6}$$

must have the form

$$\begin{cases} f(z) = \dfrac{e^{2iz+b_1} + e^{-2iz-b_1}}{-8} = \dfrac{e^{i(2z-ib_1)} + e^{-i(2z-ib_1)}}{-8} = -\dfrac{1}{4}\cos(2z - ib_1), \\ g(z) = \dfrac{e^{2iz+b_2} + e^{-2iz-b_2}}{-8} = \dfrac{e^{i(2z-ib_2)} + e^{-i(2z-ib_2)}}{-8} = -\dfrac{1}{4}\cos(2z - ib_2). \end{cases}$$

Furthermore, we take $c = \frac{1}{4}\pi$, then $(-\frac{1}{4}\cos(2z - \frac{\pi}{2}), -\frac{1}{4}\cos(2z + \frac{\pi}{2}))$ is a pair of entire solutions of the Eq (1.6).
Remark 1.2. Let $a = i, P_j = \pm i, \alpha_{11} = \alpha_{12} = \beta_{11} = \beta_{12} = 1, c = \frac{1}{2}\pi$ then

$$\begin{cases} f(z) = \dfrac{e^{iz-\frac{\pi}{2}i} + e^{-iz+\frac{\pi}{2}i}}{-2} = \dfrac{e^{i(z-\frac{\pi}{2})} + e^{-i(z-\frac{\pi}{2})}}{-2} = -\cos(z - \dfrac{\pi}{2}) \\ g(z) = \dfrac{e^{iz+\frac{\pi}{2}i} + e^{-iz-\frac{\pi}{2}i}}{-2} = \dfrac{e^{i(z+\frac{\pi}{2})} + e^{-i(z+\frac{\pi}{2})}}{-2} = -\cos(z + \dfrac{\pi}{2}) \end{cases}$$

are the solutions of the systems of the differential-difference equation

$$\begin{cases} f''(z)^2 - g(z+c)^2 = 0, \\ g''(z)^2 - f(z+c)^2 = 0. \end{cases} \tag{1.7}$$

This example indicates that the Eq (1.4) maybe also have the solutions of the form as (1.5) when $Q_k(z) \equiv 0 \quad (k = 1, 2)$.

Corollary 1.2. Let $P_j(z)(j = 1, 2)$ and $Q_i(z)(i = 1, 2)$ be nonconstant polynomials, then the Fermart type of systems of differential-difference equation (1.4) has no finite order transcendental entire solutions $(f(z), g(z))$.

Next, the last question is what happens when $d_0 = \pm d_1$ in the Eq (1.3)? When $d_0 = \pm d_1$, we can rewrite the Eq (1.3) as the following equations

$$\begin{cases} f''(z)^2 + P_1(z)^2[g(z+c) + qg(z)]^2 = Q_1(z), \\ g''(z)^2 + P_2(z)^2[f(z+c) + qf(z)]^2 = Q_2(z), \end{cases} \tag{1.8}$$

where $q = \pm 1$, and obtain the following theorem.

Theorem 1.3. *Let $P_j(z)(j = 1, 2)$ and $Q_i(z)(i = 1, 2)$ be nonzero polynomials. If the Fermart type of systems of differential-difference equation (1.8) has transcendental entire solutions $(f(z), g(z))$ such that $\sigma(f, g) < \infty$, then $P_j(z) \equiv P_j(j = 1, 2), Q_1(z) = \beta_1\beta_2$ and $Q_2(z) = \alpha_1\alpha_2$, where $P_i, \alpha_i, \beta_i(i = 1, 2)$ are nonzero constants. Furthermore, $(f(z), g(z))$ can be expressed as*

$$\begin{cases} f(z) = \dfrac{\beta_1 e^{az+b_1} + \beta_2 e^{-az-b_1}}{2a^2}, \\ g(z) = \dfrac{\alpha_1 e^{az+b_2} + \alpha_2 e^{-az-b_2}}{2a^2}, \end{cases} \tag{1.9}$$

where $a^4 = 4P_1P_2, b_1$ and b_2 are arbitrary constants. Especially, for $\forall k \in \mathbb{Z}$,

(i) $ac = 2k\pi i$, when $q = 1$;

(ii) $ac = (2k + 1)\pi i$, when $q = -1$.

2. Lemmas for proof of Theorems

To prove our theorems, the following lemmas are used play the key roles in proving our main theorems.

Lemma 2.1. *[16] Suppose that $n \geq 2$, and let $f_j(z)(j = 1, \cdots, n)$ be meromorphic functions and $g_j(z)(j = 1, \cdots, n)$ be entire functions such that*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;

(ii) when $1 \leq j < k \leq n, g_j(z) - g_k(z)$ is not a constant;

(iii) when $1 \leq j \leq n, 1 \leq h < k \leq n$,

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\}(r \to \infty, r \notin E),$$

where $E \subset (1, \infty)$ is of finite logarithmic measure.

Then $f_j(z) \equiv 0(j = 1, \cdots, n)$.

Lemma 2.2. *[1] Suppose $Q(z)$ is nonzero entire function, $P(z)$ is nonzero polynomial, $h(z)$ is not constant polynomial and satisfy*

$$(Q'(z) \pm Q(z)h'(z))P(z) - Q(z)P'(z) \equiv 0 \tag{2.1}$$

then $Q(z)$ is a transcendent entire function.

Lemma 2.3. *[16] Let $n \geq 3$ and $f_j(z)(j = 1, \cdots, n)$ be meromorphic functions satisfying $\sum_{j=1}^{n} f_j(z) = 1$ such that $f_k(z)(k = 1, \cdots, n-1)$ being nonconstant. If $f_n(z) \not\equiv 0$ and*

$$\sum_{j=1}^{n} N(r, \frac{1}{f_j}) + (n-1) \sum_{j=1}^{n} \overline{N}(r, f_j) < (\lambda + o(1))T(r, f_k),$$

where $\lambda < 1$ and $k = 1, \cdots, n-1$, then $f_n(z) \equiv 1$.

3. Proof of Theorem

3.1. Proof of Theorem 1.1

Proof. Suppose that $(f(z), g(z))$ is a transcendental entire solution with $\sigma(f, g) < \infty$ satisfying (1.3).

Equation (1.3) can be rewritten as follows

$$\begin{cases} P_1(z)[d_1 g(z+c) + d_0 g(z)] + i f''(z) = S_1(z)e^{h_1(z)}, \\ P_1(z)[d_1 g(z+c) + d_0 g(z)] - i f''(z) = S_2(z)e^{-h_1(z)}, \\ g''(z) + iP_2(z)[d_1 f(z+c) + d_0 f(z)] = T_1(z)e^{h_2(z)}, \\ g''(z) - iP_2(z)[d_1 f(z+c) + d_0 f(z)] = T_2(z)e^{-h_2(z)}, \end{cases} \tag{3.1}$$

where $h_1(z)$ and $h_2(z)$ are nonconstant polynomials, $Q_1(z) = S_1(z)S_2(z), Q_2(z) = T_1(z)T_2(z), S_i(z)(i = 1, 2)$ and $T_j(z)(j = 1, 2)$ are nonzero polynomials. Then we obtain that

$$\begin{cases} f''(z) = \dfrac{S_1(z)e^{h_1(z)} - S_2(z)e^{-h_1(z)}}{2i}, \\ d_1 f(z+c) + d_0 f(z) = \dfrac{T_1(z)e^{h_2(z)} - T_2(z)e^{-h_2(z)}}{2iP_2(z)}, \end{cases} \tag{3.2}$$

and

$$\begin{cases} g''(z) = \dfrac{T_1(z)e^{h_2(z)} + T_2(z)e^{-h_2(z)}}{2}, \\ d_1 g(z+c) + d_0 g(z) = \dfrac{S_1(z)e^{h_1(z)} + S_2(z)e^{-h_1(z)}}{2P_1(z)}. \end{cases} \tag{3.3}$$

By the second equation of (3.2), we see that

$$d_1 f''(z+c) + d_0 f''(z) = \frac{A_1(z)e^{h_2(z)} + A_2(z)e^{-h_2(z)}}{2iP_2(z)^3}, \tag{3.4}$$

where

$$A_1(z) = (H_1' + H_1 h_2')P_2 - 2P_2' H_1, \quad A_2(z) = -(H_2' - H_2 h_2')P_2 + 2P_2' H_2,$$

and $H_1 = T_1' P_2 + T_1 h_2' P_2 - T_1 P_2', H_2 = T_2' P_2 - T_2 h_2' P_2 - T_2 P_2'$.

Combining the above several equations, we have that

$$a_1(z)e^{h_2(z) - h_1(z+c)} + a_2(z)e^{-h_1(z) - h_2(z)} - \frac{d_1 S_1(z+c)}{d_1 S_1(z)} e^{h_1(z+c) - h_1(z)}$$

$$+\frac{d_1S_2(z+c)}{d_0S_1(z)}e^{-h_1(z+c)-h_1(z)}+\frac{S_2(z)}{S_1(z)}e^{-2h_1(z)}\equiv 1, \tag{3.5}$$

where

$$a_1(z)=\frac{A_1(z)}{d_0S_1(z)P_2(z)^3},\quad a_2(z)=\frac{A_2(z)}{d_0S_1(z)P_2(z)^3}.$$

To facilitate discussion, we rewrite the Eq (3.5) as

$$\begin{aligned}A_1(z)e^{h_2(z)-h_1(z+c)}+A_2(z)e^{-h_1(z)-h_2(z)}+A_3(z)e^{h_1(z+c)-h_1(z)}+A_4(z)e^{-h_1(z+c)-h_1(z)}\\+A_5(z)e^{-2h_1(z)}+A_6(z)e^{h_0(z)}\equiv 0,\end{aligned} \tag{3.6}$$

where $h_0(z)\equiv 0$ and $A_3(z)=d_1S_1(z+c)P_2(z)^3$, $A_4(z)=d_1S_2(z+c)P_2(z)^3$, $A_5(z)=d_0S_2(z)P_2(z)^3$, $A_6(z)=d_0S_1(z)P_2(z)^3$.

Similar discussion to the Eq (3.3), we obtain that

$$\begin{aligned}b_1(z)e^{h_1(z)+h_2(z)}+b_2(z)e^{h_2(z)-h_1(z)}-\frac{d_1T_1(z+c)}{d_0T_2(z)}e^{h_2(z+c)+h_2(z)}\\-\frac{d_1T_2(z+c)}{d_0T_2(z)}e^{h_2(z)-h_2(z+c)}-\frac{T_1(z)}{T_2(z)}e^{2h_2(z)}\equiv 1,\end{aligned} \tag{3.7}$$

where

$$b_1(z)=\frac{B_1(z)}{d_0T_2(z)P_1(z)^3},\quad b_2(z)=\frac{B_2(z)}{d_0T_2(z)P_1(z)^3}.$$

and

$$\begin{aligned}B_1(z)e^{h_1(z)+h_2(z)}+B_2(z)e^{h_2(z)-h_1(z)}+B_3(z)e^{h_2(z+c)+h_2(z)}+\\B_4(z)e^{h_2(z)-h_2(z+c)}+B_5(z)e^{2h_2(z)}+B_6(z)e^{h_0(z)}\equiv 0,\end{aligned} \tag{3.8}$$

where $h_0(z)\equiv 0$ and

$$\begin{aligned}B_1(z)=&(U_1'+U_1h_1')P_1-2P_1'U_1,\quad B_2(z)=(U_2'-U_2h_1')P_1-2P_1'U_2.\\B_3(z)=&-d_1T_1(z+c)P_1(z)^3,\quad B_4(z)=-d_1T_2(z+c)P_1(z)^3,\\B_5(z)=&-d_0T_1(z)P_1(z)^3,\quad B_6(z)=-d_0T_2(z)P_1(z)^3,\end{aligned}$$

denoting $U_1=S_1'P_1+S_1h_1'P_1-S_1P_1'$ and $U_2=S_2'P_1-S_2h_1'P_1-S_2P_1'$.

From the formula (3.2), using the fact that $f''(z)$ and $d_1f(z+c)+d_0f(z)$ having the same finite order of growth and $f(z)$ transcendental, we can deduce easily that $\deg h_1=\deg h_2\geq 1$. We claim that $\deg h_1=\deg h_2=1$. Conversely, we assume that $\deg h_1\geq 2$ and $\deg h_2\geq 2$. Since $P_i(z),S_i(z),T_i(z),h_i(z)(i=1,2)$ are polynomials, $A_j(z)(j=1,2,3,4,5,6)$ and $B_k(z)(k=1,2,3,4,5,6)$ are polynomials, too. Thus,

$$\begin{aligned}T(r,A_i)=o\left(e^{h_2(z)+h_1(z+c)-h_0(z)}\right),\quad T(r,A_i)=o\left(e^{h_2(z)+h_1(z+c)-2h_1(z+c)}\right),\\\cdots\\T(r,A_i)=o\left(e^{2h_1(z+c)-h_0(z)}\right),\quad T(r,B_i)=o\left(e^{h_1(z)+h_2(z+c)-h_0(z)}\right),\\\cdots\end{aligned}$$

$$T(r,B_i)=o\left(e^{2h_2(z)-h_0(z)}\right),\quad i=1,2,3,4,5,6.$$

By Lemma 2.1, we see that

$$A_1(z)\equiv 0,\quad A_2(z)\equiv 0,\quad B_1(z)\equiv 0,\quad B_2(z)\equiv 0.$$

i.e.,

$$\begin{aligned}&(H_1'+H_1h_2')P_2-2P_2'H_1\equiv 0,\quad (H_2'-H_2h_2')P_2+2P_2'H_2\equiv 0,\\&(U_1'+U_1h_1')P_1-2P_1'U_1\equiv 0,\quad (U_2'-U_2h_1')P_1+2P_1'U_2\equiv 0.\end{aligned}\tag{3.9}$$

Noting that $S_i(z)(i=1,2)$ and $T_j(z)(j=1,2)$ are polynomials, by Lemma 2.2, we see that $T_i'P_2\pm T_ih_2'P_2-T_iP_2'\not\equiv 0(i=1,2)$ and $S_i'P_1\pm S_ih_1'P_1-S_iP_1'\not\equiv 0(i=1,2)$. Hence $H_i\not\equiv 0(i=1,2)$ and $U_i\not\equiv 0(i=1,2)$. Thus there is only a item that it has the largest degree in $A_i(z),B_i(z)(i=1,2)$. For example, $T_1(z)h_2'(z)^2P_2(z)^2$ is the only item that it has the largest degree in $A_1(z)$. Then $A_i(z)\not\equiv 0,B_i(z)\not\equiv 0(i=1,2)$. It contradicts the Eq (3.9).

Now we consider that $\deg h_1=1$ and $\deg h_2=1$. Let $h_1(z)=a_1z+b_1$ and $h_2(z)=a_2z+b_2$, by (3.2) or (3.3), we easily obtain that $a_1=a_2$. Hence, we assume that $h_1(z)=az+b_1$ and $h_2(z)=az+b_2$.

We rewrite the Eq (3.5) as

$$\begin{aligned}[a_2(z)e^{-b_1-b_2-ac}-\frac{d_1S_2(z+c)}{d_0S_1(z)}e^{-2b_1-ac}-\frac{S_2(z)}{S_1(z)}e^{-2b_1}]e^{-2az}&\\+[a_1(z)e^{b_2-b_1}-\frac{d_1S_1(z+c)}{d_0S_1(z)}e^{ac}]\equiv 1,&\end{aligned}$$

hence

$$\begin{cases}a_1(z)e^{b_2-b_1}-\dfrac{d_1S_1(z+c)}{d_0S_1(z)}e^{ac}\equiv 1,\\ a_2(z)e^{-b_1-b_2-ac}-\dfrac{d_1S_2(z+c)}{d_0S_1(z)}e^{-2b_1-ac}-\dfrac{S_2(z)}{S_1(z)}e^{-2b_1}\equiv 0.\end{cases}\tag{3.10}$$

Noticing the expressions of $a_1(z)$ and $a_2(z)$, we see that

$$A_1(z)=e^{b_1-b_2}P_2(z)^3[d_0S_1(z)+d_1S_1(z+c)e^{ac}],\tag{3.11}$$

$$A_2(z)=e^{b_2-b_1}P_2(z)^3[d_0S_2(z)+d_1S_2(z+c)e^{-ac}].\tag{3.12}$$

Similar discussion to the Eq (3.7), we obtain that

$$B_1(z)=e^{b_2-b_1}P_1(z)^3[d_0T_1(z)+d_1T_1(z+c)e^{ac}],\tag{3.13}$$

$$B_2(z)=e^{b_1-b_2}P_1(z)^3[d_0T_2(z)+d_1T_2(z+c)e^{-ac}].\tag{3.14}$$

Firstly we claim that $d_0+d_1e^{ac}\neq 0$ and $d_0+d_1e^{-ac}\neq 0$. Conversely, either $d_0+d_1e^{ac}=0$ and $d_0+d_1e^{-ac}=0$, or one of $d_0+d_1e^{ac}$ and $d_0+d_1e^{-ac}$ is not equal 0. On the one hand, if $d_0+d_1e^{ac}=0$ and $d_0+d_1e^{-ac}=0$, it contradicts the condition that $d_0\neq\pm d_1$. On the other hand, without losing generality, we assume that $d_0+d_1e^{ac}\neq 0$ and $d_0+d_1e^{-ac}=0$. Let $\deg P_j(z)=p_j,\deg S_j(z)=s_j,\deg T_j(z)=t_j(j=1,2)$, then

$$\deg[d_0S_1(z)+d_1S_1(z+c)e^{ac}]=s_1-1,\quad \deg[d_0T_1(z)+d_1T_1(z+c)e^{ac}]=t_1-1,$$

$$\deg[d_0S_2(z)+d_1S_2(z+c)e^{-ac}]=s_2, \qquad \deg[d_0T_2(z)+d_1T_2(z+c)e^{-ac}]=t_2.$$

Comparing the degree of the polynomials on two sides of the Eqs (3.11) and (3.13), we have that $t_1+2p_2=s_1-1+3p_2$, and $s_1+2p_1=t_1-1+3p_1$, i.e., $p_1+p_2=2$. Comparing the degree of the polynomials on two sides of the Eq (3.12) and (3.14), we have that $t_2+2p_2=s_2+3p_2$, and $s_2+2p_1=t_2+3p_1$, i.e., $p_1+p_2=0$. It is a contradiction.

Secondly we claim that $P_1(z)$ and $P_2(z)$ are constant functions. We set $\deg P_j(z)=p_j, \deg S_j(z)=s_j, \deg T_j(z)=t_j (j=1,2)$. Comparing the degree of the polynomials on two sides of the Eqs (3.11) and (3.13), we obtain that $t_1+2p_2=s_1+3p_2$, and $s_1+2p_1=t_1+3p_1$, i.e., $p_1+p_2=0$. Since $\deg P_j(z)=p_j\geq 0$, then $p_1=p_2=0$. The claim is proved, i.e., $P_1(z)\equiv P_1$ and $P_2(z)\equiv P_2$ are constants. Furthermore, we see that $\deg S_1(z)=\deg T_1(z)$ and $\deg S_2(z)=\deg T_2(z)$. Rewriting (3.11)–(3.14) as follows:

$$T_1''(z)+2aT_1'(z)+a^2T_1(z)=P_2e^{b_1-b_2}[d_0S_1(z)+d_1S_1(z+c)e^{ac}], \tag{3.15}$$
$$2aT_2'(z)-T_2''(z)-a^2T_2(z)=P_2e^{b_2-b_1}[d_0S_2(z)+d_1S_2(z+c)e^{-ac}], \tag{3.16}$$
$$S_1''(z)+2aS_1'(z)+a^2S_1(z)=P_1e^{b_2-b_1}[d_0T_1(z)+d_1T_1(z+c)e^{ac}], \tag{3.17}$$
$$2aS_2'(z)-S_2''(z)-a^2S_2(z)=P_1e^{b_1-b_2}[d_0T_2(z)+d_1T_2(z+c)e^{-ac}]. \tag{3.18}$$

Let

$$\begin{aligned} T_1(z)&=\alpha_sz^s+\alpha_{s-1}z^{s-1}+\cdots+\alpha_1z^1+\alpha_0, \qquad \alpha_s\neq 0,\\ S_1(z)&=\beta_sz^s+\beta_{s-1}z^{s-1}+\cdots+\beta_1z^1+\beta_0, \qquad \beta_s\neq 0. \end{aligned} \tag{3.19}$$

Comparing the leading coefficients of polynomials on both sides of the Eqs (3.15) and (3.17), we obtain that $a^2\alpha_s=P_2e^{b_2-b_1}(d_0+d_1e^{ac})\beta_s$, $a^2\beta_s=P_1e^{b_1-b_2}(d_0+d_1e^{ac})\alpha_s$. Thus

$$a^4=P_1P_2(d_0+d_1e^{ac})^2. \tag{3.20}$$

Similarly, from (3.16) and (3.18), we have that

$$a^4=P_1P_2(d_0+d_1e^{-ac})^2. \tag{3.21}$$

By (3.20) and (3.21), we obtain that $(d_0+d_1e^{ac})^2=(d_0+d_1e^{-ac})^2$. Thus, either $ac=k\pi i$, $k\in\mathbb{Z}$ or $-2\frac{d_0}{d_1}=e^{-ac}+e^{ac}$.

Combining $h_1(z)=az+b_1$ and $h_2(z)=az+b_2$ and integrating the first equation of (3.2) and (3.3) twice, we have that

$$\begin{cases} f(z)=\dfrac{\int\int(S_1(z)e^{az+b_1}+S_2(z)e^{-az-b_1})dzdz}{2i}=\dfrac{S_1^*(z)e^{az+b_1}+S_2^*(z)e^{-az-b_1}}{2i},\\ g(z)=\dfrac{\int\int(T_1(z)e^{az+b_2}+T_2(z)e^{-az-b_2})dzdz}{2}=\dfrac{T_1^*(z)e^{az+b_2}+T_2^*(z)e^{-az-b_2}}{2}, \end{cases}$$

where $S_i^*(z), T_i^*(z)(i=1,2)$ are polynomials related to $Q_i(z)(i=1,2)$.

Thus, Theorem 1.1 is proved. □

3.2. Proof of Theorem 1.2

Proof. We do the same proof as in the proof in Theorem 1.1, and easily obtain that $P_1(z) \equiv P_1$ and $P_2(z) \equiv P_2$ are constants and $\deg S_1(z) = \deg T_1(z)$ and $\deg S_2(z) = \deg T_2(z)$. Noticing that $d_1 = 1, d_0 = 0$ and rewriting (3.15)–(3.18) as follows.

$$2aT_2'(z) - T_2''(z) - a^2T_2(z) = P_2e^{b_2-ac-b_1}S_2(z+c), \tag{3.22}$$
$$T_1''(z) + 2aT_1'(z) + a^2T_1(z) = P_2e^{b_1+ac-b_2}S_1(z+c), \tag{3.23}$$
$$2aS_2'(z) - S_2''(z) - a^2S_2(z) = P_1e^{b_1-ac-b_2}T_2(z+c), \tag{3.24}$$
$$S_1''(z) + 2aS_1'(z) + a^2S_1(z) = P_1e^{b_2+ac-b_1}T_1(z+c). \tag{3.25}$$

Let

$$T_2(z) = \alpha_s z^s + \alpha_{s-1}z^{s-1} + \cdots + \alpha_1 z^1 + \alpha_0, \quad \alpha_s \neq 0,$$
$$S_2(z) = \beta_s z^s + \beta_{s-1}z^{s-1} + \cdots + \beta_1 z^1 + \beta_0, \quad \beta_s \neq 0.$$

Comparing the leading coefficients of polynomials on both sides of the Eqs (3.22) and (3.24), we obtain that $-a^2\alpha_s = P_2e^{b_2-ac-b_1}\beta_s$, $-a^2\beta_s = P_1e^{b_1-ac-b_2}\alpha_s$. Hence

$$a^4 = P_1P_2e^{-2ac}. \tag{3.26}$$

Similarly, from (3.23) and (3.25), we have that

$$a^4 = P_1P_2e^{2ac}. \tag{3.27}$$

By (3.26) and (3.27), we see that

$$a^8 = P_1^2P_2^2, \quad ac = \frac{1}{2}k\pi i, \quad k \in \mathbb{Z}. \tag{3.28}$$

Multiplying (3.22) and (3.24), we get

$$[2aT_2'(z) - T_2''(z) - a^2T_2(z)][2aS_2'(z) - S_2''(z) - a^2S_2(z)] = -P_1P_2e^{-2ac}T_2(z+c)S_2(z+c).$$

Combining with (3.26), we rewrite the above equation as follow,

$$a^4T_2(z)S_2(z) - 2a^3[T_2'(z)S_2(z) + T_2(z)S_2'(z)] + H_{2s-2}(z) = a^4T_2(z+c)S_2(z+c), \tag{3.29}$$

where $H_{2s-2}(z) = 4a^2S_2'(z)T_2'(z) - S_2''(z)(2aT_2'(z) - a^2T_2(z)) - T_2''(z)(2aS_2'(z) - a^2S_2(z)) + T_2''(z)S_2''(z)$ is a polynomial with degree $2s-2$.

Next we prove that $T_2(z)$ and $S_2(z)$ are constants. Conversely, we assume that $\deg S_2(z) = \deg T_2(z) = s \geq 1$, then $2s-1 \geq 1$.

By the definition of $T_2(z)$ and $S_2(z)$, we easily have that

$$T_2'(z) = s\alpha_s z^{s-1} + (s-1)\alpha_{s-1}z^{s-2} + \cdots + \alpha_1,$$
$$S_2'(z) = s\beta_s z^{s-1} + (s-1)\beta_{s-1}z^{s-2} + \cdots + \beta_1,$$
$$T_2(z+c) = \alpha_s z^s + (sc\alpha_s + \alpha_{s-1})z^{s-1} + \cdots + c\alpha_1 + \alpha_0,$$

$$S_2(z+c) = \beta_s z^s + (sc\beta_s + \beta_{s-1})z^{s-1} + \cdots + c\beta_1 + \beta_0.$$

Comparing the coefficients of the terms of the degree equivalent to $2s-1$ of the two sides of the Eq (3.29), we see that

$$a^4(\alpha_s\beta_{s-1} + \alpha_{s-1}\beta_s) - 2a^3(s\alpha_s\beta_s + s\alpha_s\beta_s) = a^4[\alpha_s(sc\beta_s + \beta_{s-1}) + \beta_s(sc\alpha_s + \alpha_{s-1})].$$

This gives $ac = -2$, which contradicts with (3.28). Similarly, we obtain that $T_1(z)$ and $S_1(z)$ are constants. Setting

$$T_1(z) \equiv \alpha_1, \quad T_2(z) \equiv \alpha_2, \quad S_1(z) \equiv \beta_1, \quad S_2(z) \equiv \beta_2.$$

According $h_1(z) = az + b_1$ and $h_2(z) = az + b_2$, we have that

$$f(z) = \frac{\beta_1 e^{az+b_1} + \beta_2 e^{-az-b_1}}{2a^2},$$
$$g(z) = \frac{\alpha_1 e^{az+b_2} + \alpha_2 e^{-az-b_2}}{2a^2}.$$

Thus, Theorem 1.2 is proved. □

3.3. Proof of Theorem 1.3

Proof. Without losing generality, we only consider that $q = 1$.

We do the same proof as in the proof in Theorem 1.1, and easily obtain that $1 + e^{ac} \neq 0, P_1(z) \equiv P_1$ and $P_2(z) \equiv P_2$ are constants and $\deg S_1(z) = \deg T_1(z), \deg S_2(z) = \deg T_2(z)$. Noticing that $d_1 = 1, d_0 = 0$ and rewriting (3.15)–(3.18) as follows:

$$2aT_2'(z) - T_2''(z) - a^2T_2(z) = P_2e^{b_2-b_1}[S_2(z) + S_2(z+c)e^{-ac}], \tag{3.30}$$
$$T_1''(z) + 2aT_1'(z) + a^2T_1(z) = P_2e^{b_1-b_2}[S_1(z) + S_1(z+c)e^{ac}], \tag{3.31}$$
$$2aS_2'(z) - S_2''(z) - a^2S_2(z) = P_1e^{b_1-b_2}[T_2(z) + T_2(z+c)e^{-ac}], \tag{3.32}$$
$$S_1''(z) + 2aS_1'(z) + a^2S_1(z) = P_1e^{b_2-b_1}[T_1(z) + T_1(z+c)e^{ac}]. \tag{3.33}$$

Let

$$T_2(z) = \alpha_s z^s + \alpha_{s-1}z^{s-1} + \cdots + \alpha_1 z^1 + \alpha_0, \quad \alpha_s \neq 0,$$
$$S_2(z) = \beta_s z^s + \beta_{s-1}z^{s-1} + \cdots + \beta_1 z^1 + \beta_0, \quad \beta_s \neq 0.$$

Comparing the leading coefficients of polynomials on both sides of the Eqs (3.30) and (3.32), we obtain that $-a^2\alpha_s = P_2e^{b_2-b_1}\beta_s(1+e^{-ac})$, $-a^2\beta_s = P_1e^{b_1-b_2}\alpha_s(1+e^{-ac})$. Thus

$$a^4 = P_1P_2(1+e^{-ac})^2. \tag{3.34}$$

Similarly, from (3.31) and (3.33), we have that

$$a^4 = P_1P_2(1+e^{ac})^2. \tag{3.35}$$

By (3.34) and (3.35), we see that $(1+e^{ac})^2 = (1+e^{-ac})^2$, i.e., $1+e^{ac} = \pm(1+e^{-ac})$, hence $e^{2ac} = 1$ or $e^{ac} + e^{-ac} = 2$. By $e^{ac} + e^{-ac} = 2$, we obtain that $1 + e^{ac} = 0$, that contradicts $1 + e^{ac} \neq 0$. Thus $e^{2ac} = 1$, i.e., $ac = l\pi i, l \in \mathbb{Z}$. Noticing that $1 + e^{ac} \neq 0$, we see that l is a even number. Hence $ac = 2k\pi i, k \in \mathbb{Z}$.

Multiplying (3.30) and (3.32), noticing that $ac = 2k\pi i$, i.e., $e^{-ac} = 1$, we get

$$\begin{aligned}&[2aT_2'(z) - T_2''(z) - a^2T_2(z)][2aS_2'(z) - S_2''(z) - a^2S_2(z)]\\&= P_1P_2[T_2(z) + T_2(z + c)][S_2(z) + S_2(z + c)].\end{aligned}$$

Combining with (3.34), we rewrite the above equation as follow,

$$\begin{aligned}&a^4T_2(z)S_2(z) - 2a^3[T_2'(z)S_2(z) + T_2(z)S_2'(z)] + H_{2s-2}(z)\\&= P_1P_2[T_2(z) + T_2(z + c)][S_2(z) + S_2(z + c)],\end{aligned} \tag{3.36}$$

where $H_{2s-2}(z) = 4a^2S_2'(z)T_2'(z) - S_2''(z)(2aT_2'(z) - a^2T_2(z)) - T_2''(z)(2aS_2'(z) - a^2S_2(z)) + T_2''(z)S_2''(z)$ is a polynomial with degree not higher than $2s - 2$.

Next we prove that $T_2(z)$ and $S_2(z)$ are constants. Conversely, we assume that $\deg S_2(z) = \deg T_2(z) = s \geq 1$, then $2s - 1 \geq 1$. We easily have that

$$\begin{aligned}T_2'(z) &= s\alpha_s z^{s-1} + (s-1)\alpha_{s-1}z^{s-2} + \cdots + \alpha_1,\\S_2'(z) &= s\beta_s z^{s-1} + (s-1)\beta_{s-1}z^{s-2} + \cdots + \beta_1,\\T_2(z + c) &= \alpha_s z^s + (sc\alpha_s + \alpha_{s-1})z^{s-1} + \cdots + c\alpha_1 + \alpha_0,\\S_2(z + c) &= \beta_s z^s + (sc\beta_s + \beta_{s-1})z^{s-1} + \cdots + c\beta_1 + \beta_0.\end{aligned}$$

Comparing the coefficients of the terms of the degree equivalent to $2s - 1$ of the two sides of the Eq (3.36), we see that

$$a^4(\alpha_s\beta_{s-1} + \alpha_{s-1}\beta_s) - 2a^3(s\alpha_s\beta_s + s\alpha_s\beta_s) = 4P_1P_2[\alpha_s(sc\beta_s + \beta_{s-1}) + \beta_s(sc\alpha_s + \alpha_{s-1})].$$

This gives $ac = -2$, which contradicts with $ac = 2k\pi i$. Similarly, we obtain that $T_1(z)$ and $S_1(z)$ are constants. Setting

$$T_1(z) \equiv \alpha_1, \quad T_2(z) \equiv \alpha_2, \quad S_1(z) \equiv \beta_1, \quad S_2(z) \equiv \beta_2.$$

According with $h_1(z) = az + b_1$ and $h_2(z) = az + b_2$, we have that

$$\begin{aligned}f(z) &= \frac{\beta_1e^{az+b_1} + \beta_2e^{-az-b_1}}{2a^2},\\g(z) &= \frac{\alpha_1e^{az+b_2} + \alpha_2e^{-az-b_2}}{2a^2}.\end{aligned}$$

Thus, Theorem 1.3 is proved.

□

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Conflict of interest

The authors declare no conflict of interest.

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