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Research article

Traveling wave solutions for an integrodifference equation of higher order

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Abstract: This article is concerned with the minimal wave speed of traveling wave solutions for an integrodifference equation of higher order. Besides the operator may be nonmonotone, the kernel functions may be not Lebesgue measurable and integrable such that the equation has lower regularity. By constructing a proper set of potential wave profiles, we obtain the existence of smooth traveling wave solutions when the wave speed is larger than a threshold. Here, the profile set is obtained by giving a pair of upper and lower solutions. When the wave speed is the threshold, the existence of nontrivial traveling wave solutions is proved by passing to a limit function. Moreover, we obtain the nonexistence of nontrivial traveling wave solutions when the wave speed is smaller than the threshold.

Keywords: nonmonotone equation; upper-lower solutions; fixed point theorem; weaker regularity; minimal wave speed
Mathematics Subject Classification: 39A70, 47G10, 92D25

1. Introduction

Traveling wave solutions of intergrodifference equations have been widely utilized to model the spatial expansion process in population dynamics, see earlier classical works by Kot [7], Lui [13, 14], Weinberger [23] and a very recent monograph by Lutscher [15]. One widely studied integrodifference equation is

$$w_{n+1}(x) = \int_{\mathbb{R}} b(w_n(y))k(x-y)dy,\ \ x \in \mathbb{R},\ \ n = 0, 1, 2, \cdots, \tag{1.1}$$

in which $w_n(x)$ denotes the population density of the $n-$th generation at location x, k is the movement law or a kernel function describing the spatial dispersal of individuals, b depends on the reproductive behavior and is called the birth function. Such an equation could model the evolution of some species with non-overlapping generations. Mathematically, if k is Lebesgue measurable and integrable, then the corresponding difference operator may have nice regularity properties, of which the propagation dynamics has been widely studied even if b is not monotone increasing, see some results by Hsu and Zhao [4], Li et al. [8], Lin [10], Wang and Castillo-Chavez [21].

On the one hand, the corresponding difference equation of (1.1) is

$$u_{n+1} = b(u_n),\ n = 0, 1, 2, \cdots,$$

which shows that the $(n + 1)-$th generation only depends on the $n-$th generation. In population dynamics, this equation is not sufficiently precise to model some cases. For examples, although some species reproduce at most once in a period, individuals of different ages may have different birth behavior. When the phenomenon is concerned, many difference equations of higher order have been proposed and studied, see Kocic and Ladas [6]. For example,

$$u_{n+1} = u_n e^{r(1-u_n-au_{n-1})}, r > 0, a > 0,\ n = 0, 1, 2, \cdots,$$

is a difference equation of second order, which could be obtained by the study of long time behavior of a logistic equation with piecewise constant arguments [17].

On the other hand, the movement law in (1.1) may be not Lebesgue measurable and integrable. In Li [9] and Lutscher [15, Section 12.4], we may find some reasons why the law may depend on the Dirac function. Evidently, such a kernel function may lead to the deficiency of nice regularity of difference operator. To study its dynamics, different techniques were developed, see, e.g., Pan [20]. In a recent article by Pan and Lin [18], the authors further studied the existence of traveling wave solutions to the following model

$$u_{n+1}(x) = g(u_n(x)) + \int_{\mathbb{R}} f(u_n(y))k(x-y)dy,\ x \in \mathbb{R},\ n = 0, 1, 2, \cdots, \tag{1.2}$$

in which g, f are given birth functions.

The purpose of this article is to study the existence and nonexistence of nontrivial traveling wave solutions in a more general intrgrodifference equation that considers the above two factors. Our model is

$$u_{n+1}(x) = \sum_{j=0}^{m} g_j(u_{n-j}(x)) + \sum_{j=0}^{m} \int_{\mathbb{R}} f_j(u_{n-j}(y))k_j(x-y)dy,\ x \in \mathbb{R},\ n = 0, 1, 2, \cdots, \tag{1.3}$$

in which $m \in \{0, 1, 2, \cdots\}$ is a finite constant, g_j, f_j, k_j are given functions for each $j \in \{0, 1, 2, \cdots, m\} := J$. By developing the idea in Pan and Lin [18] and Lin [10], we shall define a threshold that determines the existence and nonexistence of nontrivial traveling wave solutions even if g_i, f_j are not monotone increasing. To prove our main conclusion when the wave speed is larger than the threshold, we use the fixed point theorem as well as the technique of upper and lower solutions. When the wave speed is the threshold, the existence of nontrivial traveling wave solutions is confirmed by passing to a limit function. Comparing with Pan and Lin [18] for $m = 0$, we can deal with the case that kernel functions do not have compact supports when the wave speed is the threshold and study the convergence of traveling wave solutions when (1.3) is nonmonotone, which can be applied to extend the conclusions in Pan and Lin [18]. Moreover, the nonexistence of nontrivial traveling wave solutions is proved when the wave speed is smaller than the threshold, which can complete the conclusions in Pan and Lin [18].

2. Main results

We first introduce some assumptions utilized in what follows.

(A1) For each $j \in J$, k_j is Lebesgue measurable and integrable such that $\int_{\mathbb{R}} k_j(y)dy = 1$ and $\int_{\mathbb{R}} k_j(y)e^{\lambda y}dy < \infty$ for all $\lambda > 0$, it is nonnegative and even in the sense that

$$\int_{-b}^{-a} k_j(y)dy = \int_{a}^{b} k_j(y)dy \geq 0 \text{ for any } b > a,$$

there also exists $\mathcal{K} > 0$ such that $\int_{\mathbb{R}} |k_j(x-t) - k_j(y-t)|dt \leq \mathcal{K}|x-y|$;

(A2) there exists $K > 0$ such that

$$g_j(0) = f_j(0) = 0, \ \sum_{j=0}^{m} g_j(K) + \sum_{j=0}^{m} f_j(K) = K, \ j \in J,$$

for any $x \in (0, K)$, we have

$$\sum_{j=0}^{m} g_j(x) + \sum_{j=0}^{m} f_j(x) > x;$$

(A3) there exists $\overline{K} \geq K$ such that for each $j \in J$, $g_j, f_j : [0, \overline{K}] \to [0, \overline{K}]$ are continuous differentiable such that either $g_j(x)(f_j(x)) \equiv 0$ or

$$\max\{0, g_j'(0)x - Lx^2\} < g_j(x) \leq g_j'(0)x \ \ (\max\{0, f_j'(0)x - Lx^2\} < f_j(x) \leq f_j'(0)x)$$

for any $x \in (0, \overline{K}]$ and

$$\sum_{j=0}^{m} \max_{x\in[0,\overline{K}]} \{g_j(x)\} + \sum_{j=0}^{m} \max_{x\in[0,\overline{K}]} \{f_j(x)\} \leq \overline{K},$$

we also fixed $\overline{K}$ as the smallest one;

(A4) $\sum_{j=0}^{m} g_j'(0) < 1, \ \ \sum_{j=0}^{m} (g_j'(0) + f_j'(0)) > 1$;

(A5) for each $j \in J$, we have

$$0 \leq g_j'(x) \leq g_j'(0), \ 0 \leq f_j'(x) \leq f_j'(0), \ x \in [0, K].$$

There are many functions satisfying the above assumptions in the known works, for example, we may refer to [4, Section 4]. In particular, we take

$$g_j(x) = \frac{\alpha_j \beta_j x}{1 + \beta_j x}, j \in J \text{ with } \alpha_j \geq 0, \beta_j > 0, \sum_{j\in J} \alpha_j \beta_j < 1$$

and

$$f_j(x) = \gamma_j \delta_j x e^{\delta_j(1-\eta_j x)}, j \in J \text{ with } \gamma_j \geq 0, \delta_j > 0, \eta_j > 0$$

such that

$$\sum_{j=0}^{m}\alpha_j\beta_j+\sum_{j=0}^{m}\gamma_j\delta_j>1,$$

then we may verify that (A2)–(A4) hold. Moreover, there exists $\delta'>0$ such that (A2)–(A5) hold when $\delta_j\in(0,\delta')$, $j\in J$.

In this article, $C(\mathbb{R},\mathbb{R})$ denotes the space of all uniformly continuous and bounded functions equipped with supremum norm. If $b>a$, then

$$C_{[a,b]}=\{u\in C: a\leq u(x)\leq b, x\in\mathbb{R}\}.$$

A traveling wave solution of (1.3) is a special entire solution taking the following form

$$u_n(x)=\phi(t),\ t=x+cn\in\mathbb{R}$$

or

$$\phi(t)=\sum_{j=0}^{m}g_j(\phi(t-(j+1)c))+\sum_{j=0}^{m}\int_{\mathbb{R}}f_j(\phi(y))k_j(t-(j+1)c-y)dy,\ t=x+cn\in\mathbb{R} \tag{2.1}$$

and satisfying the following asymptotic boundary conditions

$$\lim_{t\to-\infty}\phi(t)=0,\ \liminf_{t\to\infty}\phi(t)>0. \tag{2.2}$$

For $\phi\in C_{[0,\overline{K}]}$, define an operator

$$F(\phi)(t)=\sum_{j=0}^{m}g_j(\phi(t-(j+1)c))+\sum_{j=0}^{m}\int_{\mathbb{R}}f_j(\phi(y))k_j(t-(j+1)c-y)dy.$$

Then, in the set $C_{[0,\overline{K}]}$, a positive solution of (2.1) is a fixed point of F while a fixed point of F also satisfies (2.1).

To state our main results, we further consider the following eigenvalue problem

$$\Lambda(\lambda,c)=\sum_{j=0}^{m}g_j(0)e^{-\lambda(j+1)c}+\sum_{j=0}^{m}f_j'(0)\int_{\mathbb{R}}e^{\lambda y-\lambda(j+1)c}k_j(y)dy$$

for $\lambda>0$, which is obtained by the linearized equation of (2.1) at 0. Evidently, $\Lambda(\lambda,c)$ is bounded for all $\lambda>0, c>0$.

Lemma 2.1. *Assume that (A1) and (A4) hold. Then there exists a positive constant $c^*>0$ satisfying the following facts.*

(B1) *If $c\in(0,c^*)$, then $\Lambda(\lambda,c)>1$ for all $\lambda>0$.*

(B2) *If $c=c^*$, then $\Lambda(\lambda,c)=1$ has a unique real root $\lambda^*>0$ such that*

$$\sum_{j=0}^{m}-(j+1)cg_j(0)e^{-\lambda(j+1)c}+\sum_{j=0}^{m}f_j'(0)\int_{\mathbb{R}}(y-(j+1)c)e^{\lambda y-\lambda(j+1)c}k_j(y)dy=0$$

if $\lambda=\lambda^*$.

(B3) *If $c > c^*$, then $\Lambda(\lambda, c) = 1$ has a positive root λ_1^c such that $\Lambda(\lambda, c) < 1$ when $\lambda - \lambda_1^c > 0$ is small.*

Proof. Firstly, for each $j \in J$, since k_j is even and nonnegative, we see that $\int_{\mathbb{R}} e^{\lambda y - \lambda(j+1)c} k_j(y)dy$ is strictly convex in $\lambda \geq 0$. Moreover, $\Lambda(\lambda, c)$ is strictly decreasing in $c \geq 0$. Finally, $\Lambda(\lambda, c)$ is continuous in both $\lambda \geq 0, c \geq 0$. When $c > 0$ is fixed and small, we see that

$$\lim_{\lambda \to \infty} \Lambda(\lambda, c) = \infty,$$

which is uniform if c belongs to any bounded interval. When $c = 0$ or $c > 0$ is small enough, (A4) implies that $\Lambda(\lambda, c) > 1$ for all $\lambda > 0$. Fix $\lambda > 0$, let c be large, we see that $\Lambda(\lambda, c) < 1$. Then the conclusion is clear by the convex and monotonicity. We complete the proof. □

By these constants, we state our main results as follows.

Theorem 2.2. *Assume that (A1)–(A5) hold with $\overline{K} = K$. Then (2.1)–(2.2) has a monotone solution $\phi(t)$ such that $\lim_{t\to\infty} \phi(t) = K$ if and only if $c \geq c^*$. Moreover, if $c > c^*$, then $\lim_{t\to-\infty} \phi(t)e^{-\lambda_1^c t} > 0$.*

The above conclusion answers our question if all of g_j, f_j are monotone increasing. When (A5) does not hold, we have the following result.

Theorem 2.3. *Assume that (A1)–(A4) hold. Then (2.1)–(2.2) has a nontrivial positive solution $\phi(t)$ if and only if $c \geq c^*$. Moreover, if $c > c^*$, then $\lim_{t\to-\infty} \phi(t)e^{-\lambda_1^c t} > 0$.*

Remark 2.4. If each k_j has compact support, then we can obtain the precise limit behavior when $t \to -\infty$ similar to that in Pan and Lin [18].

Even if (A5) does not hold, we can give the following convergence result based on the contracting idea in Lin [10].

Theorem 2.5. *Assume that $a(s), b(s), s \in [a', b']$ are continuous functions such that*

(C1) *$0 \leq a(a') < a(s) \leq a(b') = K = b(b') \leq b(s) \leq b(b') \leq \overline{K}$ such that $a(s)$ is strictly increasing while $b(s)$ is strictly decreasing;*

(C2) *for each $s \in [a', b')$ we have*

$$b(s) > \sum_{j=0}^{m} g_j(x_j) + \sum_{j=0}^{m} f_j(y_j) > a(s)$$

if $x_j, y_j \in [a(s), b(s)], j \in J$.

If $\phi(t)$ satisfies (2.1)–(2.2) such that

$$a(s_0) \leq \liminf_{t\to\infty} \phi(t) \leq \limsup_{t\to\infty} \phi(t) \leq b(s_0)$$

for some $s_0 \in (a', b']$, then $\lim_{t\to\infty} \phi(t) = K$.

Example 2.6. Let

$$g_j(x) = \theta_j x e^{r_j(1-x)}, \quad f_j(x) = \vartheta_j x e^{R_j(1-x)}, \quad j \in J,$$

where r_j, R_j are positive and θ_j, ϑ_j are nonnegative such that

$$0 \leq \sum_{j=0}^{m} \theta_j e^{r_j} < 1 < \sum_{j=0}^{m} \theta_j e^{r_j} + \sum_{j=0}^{m} \vartheta_j e^{R_j}.$$

By direct calculation, we see that Theorems 2.2, 2.3 and 2.5 hold when $\max_{j\in J}\{0, r_j - 1\}, \max_{j\in J}\{0, R_j - 1\}$ are small.

3. Proof of main results

In this part, we shall prove our main results by dividing the discussion into two cases.

3.1. Monotone case

In this part, we study the existence of traveling wave solutions under (A1)–(A5) without further illustration. We first give the following definition.

Definition 3.1. Assume that $\overline{\phi}, \underline{\phi} \in C_{[0,K]}$ such that $\overline{\phi} \geq \underline{\phi}$ and

$$F(\overline{\phi})(t) \geq \overline{\phi}(t),\ F(\underline{\phi})(t) \leq \underline{\phi}(t),\ t \in \mathbb{R}. \tag{3.1}$$

Then $\overline{\phi}$ is an upper solution while $\underline{\phi}$ is a lower solution of (2.1).

Lemma 3.2. *For each fixed $c > c^*$, (2.1) has a pair of upper and lower solutions $\overline{\phi}, \underline{\phi}$ that are Lipschitz continuous. Moreover, the upper solution is monotone.*

Proof. Define continuous functions

$$\overline{\phi}(t) = \min\{e^{\lambda_1^c t}, K\}, \quad \underline{\phi}(t) = \max\{0, e^{\lambda_1^c t} - qe^{(\lambda_1^c+\epsilon)t}\}, \quad t \in \mathbb{R},$$

in which $\epsilon > 0$ is small while $q > 1$ is large. We now verify these functions are upper and lower solutions of (2.1).

If $\overline{\phi}(t) = K$, then the monotonicity implies that

$$\begin{aligned}
&\sum_{j=0}^{m} g_j(\overline{\phi}(t-(j+1)c)) + \sum_{j=0}^{m} \int_{\mathbb{R}} f_j(\overline{\phi}(y))k_j(t-(j+1)c-y)dy \\
\leq\ & \sum_{j=0}^{m} g_j(K) + \sum_{j=0}^{m} \int_{\mathbb{R}} f_j(K)k_j(t-(j+1)c-y)dy \\
=\ & \sum_{j=0}^{m} g_j(K) + \sum_{j=0}^{m} f_j(K) \\
=\ & K.
\end{aligned}$$

If $\overline{\phi}(t) = e^{\lambda_1^c t}$, then the monotonicity implies that

$$\begin{aligned}
&\sum_{j=0}^{m} g_j(\overline{\phi}(t-(j+1)c)) + \sum_{j=0}^{m} \int_{\mathbb{R}} f_j(\overline{\phi}(y))k_j(t-(j+1)c-y)dy \\
\leq\ & \sum_{j=0}^{m} g_j'(0)(\overline{\phi}(t-(j+1)c)) + \sum_{j=0}^{m} f_j'(0) \int_{\mathbb{R}} \overline{\phi}(y)k_j(t-(j+1)c-y)dy \\
\leq\ & \sum_{j=0}^{m} g_j'(0)e^{\lambda_1^c(t-(j+1)c)} + \sum_{j=0}^{m} f_j'(0) \int_{\mathbb{R}} e^{\lambda_1^c y}k_j(t-(j+1)c-y)dy \\
=\ & e^{\lambda_1^c t}\Lambda(\lambda_1^c, c)
\end{aligned}$$

$$= \quad e^{\lambda_1^c t}.$$

If $\underline{\phi}(t) = 0$, then the result is clear by the positivity (A3). When $\underline{\phi}(t) = e^{\lambda_1^c t} - qe^{(\lambda_1^c+\epsilon)t}$, then $q > K + 1$ implies that $t < 0$ and

$$\begin{aligned}
& \sum_{j=0}^{m} g_j(\underline{\phi}(t-(j+1)c)) + \sum_{j=0}^{m} \int_{\mathbb{R}} f_j(\underline{\phi}(y))k_j(t-(j+1)c-y)dy \\
\geq\ & \sum_{j=0}^{m} g_j'(0)(\underline{\phi}(t-(j+1)c)) + \sum_{j=0}^{m} f_j'(0) \int_{\mathbb{R}} \underline{\phi}(y)k_j(t-(j+1)c-y)dy \\
& -L\sum_{j=0}^{m} \underline{\phi}^2(t-(j+1)c) - L\sum_{j=0}^{m} \int_{\mathbb{R}} \underline{\phi}^2(y)k_j(t-(j+1)c-y)dy \\
\geq\ & \sum_{j=0}^{m} g_j'(0)e^{\lambda_1^c(t-(j+1)c)} + \sum_{j=0}^{m} f_j'(0) \int_{\mathbb{R}} qe^{\lambda_1^c y}k_j(t-(j+1)c-y)dy \\
& -L\sum_{j=0}^{m} \underline{\phi}^2(t-(j+1)c) - L\sum_{j=0}^{m} \int_{\mathbb{R}} \underline{\phi}^2(y)k_j(t-(j+1)c-y)dy \\
\geq\ & e^{\lambda_1^c t} - L\sum_{j=0}^{m} \overline{\phi}^2(t-(j+1)c) - L\sum_{j=0}^{m} \int_{\mathbb{R}} e^{(\lambda_1^c+\epsilon)y}k_j(t-(j+1)c-y)dy \\
\geq\ & e^{\lambda_1^c t} - Lme^{2\lambda_1^c t} - Le^{(\lambda_1^c+\epsilon)t}\sum_{j=0}^{m} \int_{\mathbb{R}} e^{(\lambda_1^c+\epsilon)(y+(m+1)c)}k_j(y)dy \\
\geq\ & e^{\lambda_1^c t} - e^{(\lambda_1^c+\epsilon)t}\left[Lm + L\sum_{j=0}^{m} \int_{\mathbb{R}} e^{(\lambda_1^c+\epsilon)(y+(m+1)c)}k_j(y)dy\right] \\
\geq\ & e^{\lambda_1^c t} - qe^{(\lambda_1^c+\epsilon)t}
\end{aligned}$$

provided that $\epsilon \in (0, \lambda_1^c)$ is small such that $\Lambda(\lambda_1^c + \epsilon, c) < 1$ and

$$q = K + 1 + LM + L\sum_{i=0}^{m} \int_{\mathbb{R}} e^{(\lambda_1^c+\epsilon)(y+(m+1)c)}k_j(y)dy.$$

The proof is complete. □

Lemma 3.3. *For each fixed $c > c^*$,* (2.1) *has a monotone fixed point satisfying* (2.2) *and* $\lim_{t\to\infty} \phi(t) = K, \lim_{t\to-\infty} \phi(t)e^{-\lambda_1^c t} > 0$.

Proof. Let $\overline{\phi}, \underline{\phi}$ be defined as those in the proof of Lemma 3.2, which are Lipscitz continuous with constant $L_1 > 0$. Let $\mathcal{L} > L$ be a constant such that

$$\mathcal{L} \geq \frac{K\mathcal{K}}{1 - \sum\limits_{j=0}^{m} g_j'(0)} + L_1.$$

Define the potential profile set

$$\Gamma\left(\underline{\phi},\overline{\phi}\right)=\left\{\phi\in C:\begin{array}{l}(i)\ \underline{\phi}(t)\leq\phi(t)\leq\overline{\phi}(t);\\ (ii)\ |\phi(t_1)-\phi(t_2)|\leq\mathcal{L}|t_1-t_2|,t_1,t_2\in\mathbb{R};\\ (iii)\ \phi(t)\text{ is nondecreasing in }t\in\mathbb{R}\end{array}\right\}.$$

Then it is convex, nonempty. We now prove that $F:\Gamma\to\Gamma$. By Definition 3.1 and monotonicity of F, (i) and (ii) are evident. Select $\phi\in\Gamma$, then

$$\begin{aligned}
&|F(\phi)(t_1)-F(\phi)(t_2)|\\
=&\left|\sum_{j=0}^{m}g_j(\phi(t_1-(j+1)c))+\sum_{j=0}^{m}\int_{\mathbb{R}}f_j(\phi(y))k_j(t_1-(j+1)c-y)dy\right.\\
&\left.-\sum_{j=0}^{m}g_j(\phi(t_2-(j+1)c))+\sum_{j=0}^{m}\int_{\mathbb{R}}f_j(\phi(y))k_j(t_2-(j+1)c-y)dy\right|\\
\leq&\sum_{j=0}^{m}\left|g_j(\phi(t_1-(j+1)c))-g_j(\phi(t_2-(j+1)c))\right|\\
&+\sum_{j=0}^{m}\int_{\mathbb{R}}f_j(\phi(y))\left|k_j(t_2-(j+1)c-y)-k_j(t_1-(j+1)c-y)\right|dy\\
\leq&\sum_{j=0}^{m}g_j'(0)\left|\phi(t_1-(j+1)c)-\phi(t_2-(j+1)c)\right|\\
&+\sum_{j=0}^{m}\int_{\mathbb{R}}f_j(K)\left|k_j(t_2-(j+1)c-y)-k_j(t_1-(j+1)c-y)\right|dy\\
\leq&\sum_{j=0}^{m}g_j'(0)\mathcal{L}|t_1-t_2|+\sum_{j=0}^{m}f_j(K)\mathcal{K}|t_1-t_2|\\
\leq&\sum_{j=0}^{m}g_j'(0)\mathcal{L}|t_1-t_2|+K\mathcal{K}|t_1-t_2|\\
\leq&\mathcal{L}|t_1-t_2|.
\end{aligned}$$

Let $\mu>0$ be a small constant and we define

$$B_\mu=\left\{\phi\in C:\sup_{t\in\mathbb{R}}\{|\phi(t)|e^{-\mu|t|}\}<\infty\right\},\quad |\phi|_\mu=\sup_{t\in\mathbb{R}}\{|\phi(t)|e^{-\mu|t|}\},$$

then $\left(B_\mu,|\cdot|_\mu\right)$ is a Banach space. Moreover, Γ is closed and bounded in the sense of $|\cdot|_\mu$. By the above continuous property of Γ, we see that $F:\Gamma\to\Gamma$ is completely continuous in the sense of $|\cdot|_\mu$.

Using Schauder's fixed point theorem, F has a fixed point $\phi\in\Gamma$, which is a monotone solution of (2.1). By the definition of upper and lower solutions, we see that

$$\lim_{t\to-\infty}\phi(t)=0,\quad\lim_{t\to\infty}\phi(t)=:\Phi\in(0,K].$$

Applying the dominated convergence theorem, we see that

$$\Phi = \sum_{j=0}^{m} g_j(\Phi) + \sum_{j=0}^{m} f_j(\Phi),$$

which implies that $\Phi = K$ by (A2). The proof is complete. □

Remark 3.4. Evidently, $\mathcal{L}$ can be fixed for any $c \in (c^*, c^* + 1)$.

Lemma 3.5. *If $c = c^*$, then* (2.1)–(2.2) *has a monotone solution.*

Proof. We prove the result by passing to a limit function similar to that in Hsu and Zhao [4]. Let $\{c_n\}$ be a monotone decreasing sequence such that

$$c_n \to c^*, \quad n \to \infty.$$

Then for each c_n satisfying $c_n > c^*$, (2.1)–(2.2) has a monotone solution $\phi_n(t)$ such that

$$\phi_n(0) = K/2, \quad n \in \mathbb{N}$$

after phase shift. Note that ϕ_n may have the same Lipschitz constant for all $n \in \mathbb{N}$. Then $\{\phi_n\}$ is equicontinuous and bounded. Using the Ascoli-Arzela lemma, it has a subsequence, still denoted by $\{\phi_n\}$, such that it converges to a function ϕ^*, in which the convergence is uniform in any bounded interval and pointwise in whole space. This ϕ^* is the fixed point of F with $c = c^*$. Note that every ϕ_n is monotone, so for ϕ^*. Therefore, we have

$$\lim_{t\to-\infty} \phi^*(t) = \Phi_* \in [0, K/2], \ \lim_{t\to\infty} \phi^*(t) = \Phi^* \in [K/2, K].$$

Using the dominated convergence theorem, we have

$$\Phi = \sum_{j=0}^{m} g_j(\Phi) + \sum_{j=0}^{m} f_j(\Phi)$$

for $\Phi = \Phi_*$ or $\Phi = \Phi^*$. By (A2), we see that $\Phi_* = 0, \Phi^* = K$. The proof is complete. □

To study the nonexistence of traveling wave solutions, we further consider the following initial value problem

$$\begin{cases} u_{n+1}(x) = \sum_{j=0}^{m} g_j(u_{n-j}(x)) + \sum_{j=0}^{m} \int_{\mathbb{R}} f_j(u_{n-j}(y))k_j(x-y)dy, \\ u_{-m}, u_{-m+1}, \cdots, u_0 \in C_{[0,\overline{K}]}, \end{cases} \tag{3.2}$$

in which $x \in \mathbb{R}$, $n = 0, 1, 2, \cdots$, each of $u_{-m}, u_{-m+1}, \cdots, u_0$ has nonempty support. Similar to Li [9, Proposition 2.1], we have the following conclusion.

Proposition 3.6. *Assume that $u_n(x)$ is defined by* (3.2)*. For any given $c \in (0, c^*)$,*

$$\lim_{n\to\infty, |x|<cn} u_n(x) = K.$$

Moreover, if $w_n(x) \in C_{[0,K]}$ *satisfies*

$$\begin{cases} w_{n+1}(x) \geq \sum_{j=0}^{m} g_j(w_{n-j}(x)) + \sum_{j=0}^{m} \int_{\mathbb{R}} f_j(w_{n-j}(y))k_j(x-y)dy, \ x \in \mathbb{R}, n \in \mathbb{N}, \\ w_{-m}, w_{-m+1}, \cdots, w_0 \in C_{[0,K]}, \\ w_{-m}(x) \geq u_{-m}(x), \cdots, w_0(x) \geq u_0(x), x \in \mathbb{R}, \end{cases}$$

then $w_n(x) \geq u_n(x), x \in \mathbb{R}, n \in \mathbb{N}$.

Lemma 3.7. *If* $c < c^*$, *then* (2.1)–(2.2) *does not have a positive solution.*

Proof. Were the statement false such that (2.1)–(2.2) has a positive solution $\phi(t)$ with some $c_1 < c^*$. By the limit behavior when $t \to \infty$, there exists a closed interval $I = [a, b]$ with $b - a > c^*(m+2)$ such that

$$\phi(t) > 0, \ t \in [a, b].$$

Since a traveling wave solution is a special entire solution, we see that

$$\phi(t + a) = \phi(x + c_1 n + a) = u_n(x)$$

satisfies (3.2) by letting

$$u_{-m}(x) = \phi(x + a), \ u_{-m+1}(x) = \phi(x + a + c), \cdots, u_0(x) = \phi(x + a + mc),$$

then Proposition 3.6 implies that

$$\lim_{n\to\infty, -2x(n)=(c_1+c^*)n} u_n(x) = K.$$

Let $n \to \infty$, then $x(n) + c_1 n \to -\infty$ such that $\liminf_{t\to-\infty} \phi(t) \geq K$. A contradiction occurs. The proof is complete. $\square$

3.2. Nonmonotone case

For this case, we define

$$g_j^+(x) = \max_{u\in[0,x]} g_j(u), \ f_j^+(x) = \max_{u\in[0,x]} f_j(u), \ x \in [0, \overline{K}],$$

and

$$g_j^-(x) = \min_{u\in[x,\overline{K}]} g_j(u), \ f_j^-(x) = \min_{u\in[x,\overline{K}]} f_j(u), \ x \in [0, \overline{K}].$$

Evidently, there exist $0 < K^- \leq K \leq K^+ \leq \overline{K}$ such that

$$\sum_{j=0}^{m} g_j^\pm(K^\pm) + \sum_{j=0}^{m} f_j^\pm(K^\pm) = K^\pm.$$

For fixed $c \geq c^*$, define $F^\pm$ by

$$F^\pm(\phi)(t) = \sum_{j=0}^{m} g_j^\pm(\phi(t-(j+1)c)) + \sum_{j=0}^{m} \int_{\mathbb{R}} f_j^\pm(\phi(y))k_j(t-(j+1)c-y)dy$$

for any $\phi \in C_{[0,\overline{K}]}$. Then the previous subsection implies that for each fixed $c > c^*$, $F^\pm$ has a fixed point $\phi^\pm \in C_{[0,\overline{K}]}$ such that

$$\lim_{t\to\infty} \phi^\pm(t) = 0,\ \lim_{t\to\infty} \phi^\pm(t) = K^\pm$$

and

$$\lim_{t\to-\infty} \{\phi^\pm(t)e^{-\lambda_1^c t}\} = 1,\ \phi^\pm(t) \le \min\{e^{\lambda_1^c t}, K^\pm\},\ t \in \mathbb{R}.$$

Proposition 3.8. *Assume that $\phi \in C_{[0,\overline{K}]}$. Then*

$$F^-(\phi)(t) \le F(\phi)(t) \le F^+(\phi)(t),\ t \in \mathbb{R}.$$

Lemma 3.9. *Assume that $c > c^*$. Then* (2.1)–(2.2) *has a positive solution such that*

$$K^- \le \liminf_{t\to\infty} \phi(t) \le \limsup_{t\to\infty} \phi(t) \le K^+.$$

Proof. The proof is similar to that in Subsection 3.1. Let

$$\overline{\phi}(t) = \min\{e^{\lambda_1^c t}, K^+\},\ \underline{\phi}(t) = \phi^-(t),\ t \in \mathbb{R}.$$

For a large $\mathcal{L} > 0$, we define

$$\Gamma\left(\underline{\phi}, \overline{\phi}\right) = \left\{\phi \in C :\ \begin{array}{l} (i)\ \underline{\phi}(t) \le \phi(t) \le \overline{\phi}(t); \\ (ii)\ |\phi(t_1) - \phi(t_2)| \le \mathcal{L}|t_1 - t_2|, t_1, t_2 \in \mathbb{R} \end{array}\right\}.$$

Then we can verify that Γ is nonempty and convex. Using Proposition 3.8, we see that $F : \Gamma \to \Gamma$ is also completely continuous in the sense of $|\cdot|_\mu$ with $\mu \in (0, \lambda_1^c)$. Applying the fixed point theorem, we complete the proof. □

Using (A3), we also have the following conclusion.

Proposition 3.10. *There exists $\delta \in (0, K^-)$ such that $g_j(x), f_j(x)$ are nondecreasing if $x \in [0, \delta]$. Moreover, we have*

$$\sum_{j=0}^{m} g_j(x) + \sum_{j=0}^{m} f_j(x) > x \text{ if } x \in (0, \delta].$$

Using the above results, we give the conclusion when $c = c^*$.

Lemma 3.11. *Theorem 2.3 holds when $c = c^*$.*

Proof. Let $\{c_n\}$ be a monotone decreasing sequence such that

$$c_n \to c^*,\quad n \to \infty.$$

Then for each c_n satisfying $c_n > c^*$, (2.1)–(2.2) has a positive solution $\phi_n(t)$ such that

$$\phi_n(0) = \delta/2,\ \phi_n(t) < \delta/2,\ n \in \mathbb{N}, t < 0$$

after phase shift. Similar to the proof of Lemma 3.5 by passing to a limit function, (2.1) has a positive solution ϕ^* such that

$$\phi^*(0) = \delta/2,\quad \phi^*(t) \le \delta/2,\quad t < 0.$$

We now consider the limit behavior when $t \to \pm\infty$. Define

$$\underline{\Phi} = \limsup_{t\to-\infty} \phi^*(t),$$

then $\underline{\Phi} \in [0, \delta/2]$. Using the dominated convergence theorem, we see that

$$\underline{\Phi} \le \sum_{j=0}^{m} g_j(\underline{\Phi}) + \sum_{j=0}^{m} f_j(\underline{\Phi}),$$

which implies that $\underline{\Phi} = 0$ by Proposition 3.10.

By (A1) and (A4), we see that a nonconstant positive traveling wave solution is strictly positive. That is, there exists $\eta > 0$ such that

$$\phi^*(t) > \eta, \quad |t| < (2m^* + 2)c^*.$$

Since a traveling wave solution is a special entire solution, then

$$u_n(x) = \phi^*(x + c^*n) = \phi^*(t)$$

satisfies

$$u_{n+1}(x) \ge \sum_{j=0}^{m} g_j^-(u_{n-j}(x)) + \sum_{j=0}^{m} \int_{\mathbb{R}} f_j^-(u_{n-j}(y))k_j(x-y)dy, \quad x \in \mathbb{R}, n \in \mathbb{N},$$

with

$$u_0(x) = \phi^*(x), \; u_{-1}(x) = \phi^*(x - c^*), \cdots, u_{-m}(x) = \phi^*(x - mc^*),$$

which have nonempty supports. Then the limit behavior when $t \to \infty$ is evident by Proposition 3.6. The proof is complete. □

Lemma 3.12. *If $c < c^*$, then* (2.1)–(2.2) *does not have a positive solution.*

Proof. Similar to the proof of Lemma 3.7, if (2.1)–(2.2) has a positive solution $\phi(t)$ with some $c_1 < c^*$. Then for some $a \in \mathbb{R}$,

$$\phi(t + a) = \phi(x + c_n n + a) = u_n(x)$$

satisfies

$$u_{n+1}(x) \ge \sum_{j=0}^{m} g_j^-(u_{n-j}(x)) + \sum_{j=0}^{m} \int_{\mathbb{R}} f_j^-(u_{n-j}(y))k_j(x-y)dy, \quad x \in \mathbb{R}, n \in \mathbb{N},$$

and

$$u_{-m}(x) = \phi(x + a), \; u_{-m+1}(x) = \phi(x + a + c), \cdots, u_0(x) = \phi(x + a + mc).$$

Using Proposition 3.6, we have

$$\lim_{n\to\infty, -2x(n)=(c_1+c^*)n} u_n(x) \ge K^-.$$

Let $n \to \infty$, then $x(n) + c_1 n \to -\infty$ such that $\limsup_{t\to-\infty} \phi(t) \ge K^-$. A contradiction occurs. The proof is complete. □

Lemma 3.13. *Theorem 2.5 holds.*

Proof. Let

$$\underline{\Phi} = \liminf_{t\to\infty} \phi(t), \ \overline{\Phi} = \limsup_{t\to\infty} \phi(t).$$

If $\underline{\Phi} = \overline{\Phi}$, then the dominated convergence theorem implies that $\underline{\Phi} = \overline{\Phi} = K$ and the proof is complete. We now prove that $\underline{\Phi} < \overline{\Phi}$ is impossible. Without loss of generality, when $\underline{\Phi} < \overline{\Phi}$, we may assume that

$$\underline{\Phi} = a(s_0), \ \overline{\Phi} \geq b(s_0) \text{ for some } s_0 \in (a', b').$$

By the definition of lim inf, there exists a sequence $\{t_n\}$ such that

$$t_n \to \infty, \ \phi(t_n) \to \underline{\Phi}, \ n \to \infty.$$

Using the dominated convergence theorem, we see that

$$\sum_{j=0}^{m} \inf_{x\in[\underline{\Phi},\overline{\Phi}]} g_j(x) + \sum_{j=0}^{m} \inf_{x\in[\underline{\Phi},\overline{\Phi}]} f_j(x) \leq \underline{\Phi}.$$

Since g_j, f_j are continuous, then

$$\sum_{j=0}^{m} g_j(x_j) + \sum_{j=0}^{m} f_j(y_j) \leq \underline{\Phi}$$

for some $x_j, y_j \in [\underline{\Phi}, \overline{\Phi}]$, $j \in J$. From (C1) and (C2), a contradiction occurs and $\underline{\Phi} = \overline{\Phi}$. The proof is complete. □

4. Conclusions and Discussion

In this article, a special integrodifference equation is investigated. Since the kernel functions may depend on the Dirac function, the difference operator often has lower regularity effect. Moreover, the birth function may be nonmonotonic, well known results on monotone semiflows do not work directly. We constructed a special potential wave profile set that satisfies proper regularity property. With this set, we obtained the existence of nontrivial traveling wave solutions by fixed point theorem, which also implies the precisely asymptotic behavior of traveling wave solutions. If the wave speed is the threshold, we studied the existence of nontrivial traveling wave solutions by passing to a limit function. Moreover, the nonexistence of nontrivial traveling wave solutions was shown and the minimal wave speed was confirmed.

In this article, we studied an integrodifference equation of high order. In the case of continuous temporal models, the corresponding effect is called the time delay. In the past decades, the traveling wave solutions of different delayed models have been widely studied, see [12, 16, 22, 24] for delayed reaction-diffusion systems and [2, 3, 25] for delayed cellular neural networks. In this article, our nonmonotonic property is similar to that in Hsu and Zhao [4] and is different from that in Lin and Su [11]. Motivated by monotone conditions in the works mentioned above, we shall further investigate the traveling wave solutions of integrodifference systems satisfying different non-monotonic properties.

Comparing with the well studied models with continuous spatial and temporal variables, the models with discrete spatial and temporal variables often has their distinct dynamical behavior. For the integrodifference equations, we see that many simple systems do not satisfy the comparison principle such that the dynamics may be complex [4, Section 4]. When the spatial variable is discrete, the model could better describe some special phenomena including propagation failure [5]. Recently, the traveling wave solutions in some non-cooperative systems with discrete spatial variables have been widely studied, see [19, 26, 27]. When the spatial is discrete, it is possible that the diffusion mode is degenerate since some diffusive parameters are zero [1]. With the degeneracy, the smoothness of solutions often encounters some difficulties. We hope our methods can be applied to study the propagation dynamics of these models.

Conflict of interest

The author declares that he has no competing interests.

References

1. A. R. A. Anderson, B. D. Sleeman, Wave front propagation and its failure in coupled systems of discrete bistable cells modeled by FitzHugh-Nagumo dynamics, *Int. J. Bifurcat. Chaos*, **5** (1995), 63–74. https://doi.org/10.1142/S0218127495000053
2. A. Arbi, Novel traveling waves solutions for nonlinear delayed dynamical neural networks with leakage term, *Chaos Soliton. Fract.*, **152** (2021), 111436. http://doi.org/10.1016/j.chaos.2021.111436
3. Y. Guo, S. S. Ge, A. Arbi, Stability of traveling waves solutions for nonlinear cellular neural networks with distributed delays, *J. Syst. Sci. Complex*, **35** (2022), 18–31. http://doi.org/10.1007/s11424-021-0180-7
4. S.-B. Hsu, X.-Q. Zhao, Spreading speeds and traveling waves for nonmonotone integrodifference equations, *SIAM J. Math. Anal.*, **40** (2008), 776–789. http://doi.org/10.1137/070703016
5. J. P. Keener, Propagation and its failure to coupled systems of discrete excitable cells, *SIAM J. Appl. Math.*, **47** (1987), 556–572. http://doi.org/10.1137/0147038
6. V. L. Kocic, G. Ladas, *Global behavior of nonlinear difference equations of higher order with applications*, Dordrecht: Springer, 1993. https://doi.org/10.1007/978-94-017-1703-8
7. M. Kot, Discrete-time travelling waves: Ecological examples, *J. Math. Biol.*, **30** (1992), 413–436. http://doi.org/10.1007/BF00173295
8. B. Li, M. A. Lewis, H. F. Weinberger, Existence of traveling waves for integral recursions with nonmonotone growth functions, *J. Math. Biol.*, **58** (2009), 323–338. http://doi.org/10.1007/s00285-008-0175-1
9. B. Li, Traveling wave solutions in a plant population model with a seed bank, *J. Math. Biol.*, **65** (2012), 855–873. http://doi.org/10.1007/s00285-011-0481-x
10. G. Lin, Travelling wave solutions for integro-difference systems, *J. Differ. Equations*, **258** (2015), 2908–2940. http://doi.org/10.1016/j.jde.2014.12.030
11. G. Lin, T. Su, Asymptotic speeds of spread and traveling wave solutions of a second order integrodifference equation without monotonicity, *J. Differ. Equ. Appl.*, **22** (2016), 542–557. http://doi.org/10.1080/10236198.2015.1112383

12. G. Lin, S. Ruan, Traveling wave solutions for delayed reaction-diffusion systems and applications to Lotka-Volterra competition-diffusion models with distributed delays, *J. Dyn. Differ. Equ.*, **26** (2014), 583–605. http://doi.org/10.1007/s10884-014-9355-4

13. R. Lui, Biological growth and spread modeled by systems of recursions. I. Mathematical theory, *Math. Biosci.*, **93** (1989), 269–295. http://doi.org/10.1016/0025-5564(89)90026-6

14. R. Lui, Biological growth and spread modeled by systems of recursions. II. Biological theory, *Math. Biosci.*, **107** (1991), 255–287. http://doi.org/10.1016/0025-5564(89)90027-8

15. F. Lutscher, *Integrodifference equations in spatial ecology*, Cham: Springer, 2019. http://doi.org/10.1007/978-3-030-29294-2

16. S. Ma, Traveling wavefronts for delayed reaction-diffusion systems via a fixed point theorem, *J. Differ. Equations*, **171** (2001), 294–314. http://doi.org/10.1006/jdeq.2000.3846

17. I. *Öztü*rk, F. Bozkurt, F. Gurcan, Stability analysis of a mathematical model in a microcosm with piecewise constant arguments, *Math. Biosci.*, **240** (2012), 85–91. http://doi.org/10.1016/j.mbs.2012.08.003

18. S. Pan, G. Lin, Traveling wave solutions in an integrodifference equation with weak compactness, *J. Nonl. Mod. Anal.*, **3** (2021), 465–475. http://doi.org/10.12150/jnma.2021.465

19. L.-Y. Pang, S.-L. Wu, Propagation dynamics for lattice differential equations in a time-periodic shifting habitat, *Z. Angew. Math. Phys.*, **72** (2021), 93. http://doi.org/10.1007/s00033-021-01522-w

20. Y. Pan, New methods for the existence and uniqueness of traveling waves of non-monotone integro-difference equations with applications, *J. Differ. Equations*, **268** (2020), 6319–6349. http://doi.org/10.1016/j.jde.2019.11.030

21. H. Wang, C. Castillo-Chavez, Spreading speeds and traveling waves for non-cooperative integro-difference systems, *Discrete Contin. Dyn. Syst. B*, **17** (2012), 2243–2266. http://doi.org/10.3934/dcdsb.2012.17.2243

22. Z. C. Wang, W. T. Li, S. Ruan, Traveling wave fronts of reaction-diffusion systems with spatio-temporal delays, *J. Differ. Equations*, **222** (2006), 185–232. http://doi.org/10.1016/j.jde.2005.08.010

23. H. F. Weinberger, Long-time behavior of a class of biological model, *SIAM J. Math. Anal.*, **13** (1982), 353–396. http://doi.org/10.1137/0513028

24. J. Wu, X. Zou, Traveling wave fronts of reaction-diffusion systems with delay, *J. Dyn. Differ. Equ.*, **13** (2001), 651–687. http://doi.org/10.1023/A:1016690424892

25. Z.-X. Yu, R. Yuan, C.-H. Hsu, Q. Jiang, Traveling waves for nonlinear cellular neural networks with distributed delays, *J. Differ. Equations*, **251** (2011), 630–650. http://doi.org/10.1016/j.jde.2011.05.008

26. R. Zhang, J. Wang, S. Liu, Traveling wave solutions for a class of discrete diffusive SIR epidemic model, *J. Nonlinear Sci.*, **31** (2021), 10. http://doi.org/10.1007/s00332-020-09656-3

27. J. Zhou, L. Song, J. Wei, Mixed types of waves in a discrete diffusive epidemic model with nonlinear incidence and time delay, *J. Differ. Equations*, **268** (2020), 4491–4524. http://doi.org/10.1016/j.jde.2019.10.034