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Research article

Fejér type inequalities for harmonically convex functions

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Abstract: In this study, some mappings related to the Fejér-type inequalities for harmonically convex functions are defined over $[0, 1]$. Some Fejér-type inequalities for harmonically convex functions are proved using these mappings. Properties of these mappings are considered and consequently, refinements are obtained of some known results.

Keywords: Hermite-Hadamard inequality; convex function; harmonic convex function; Fejér inequality
Mathematics Subject Classification: 26D15, 26D20, 26D07

1. Introduction

Hermite-Hadamard inequality is a double inequality for convex functions that has a lot of literary value (please see [16]).

Let $\zeta : I \longrightarrow \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, $\varsigma, \tau \in I$ with $\varsigma < \tau$, be a convex function. Then

$$\zeta\left(\frac{\varsigma+\tau}{2}\right) \leq \frac{1}{\tau-\varsigma}\int_{\varsigma}^{\tau} \zeta(\mathfrak{w})d\mathfrak{w} \leq \frac{\zeta(\varsigma)+\zeta(\tau)}{2}, \tag{1.1}$$

the inequality holds in reversed direction if ζ is concave.
Fejér [15] established the following double inequality as a weighted generalization of (1.1):

$$\zeta\left(\frac{\varsigma+\tau}{2}\right)\int_{\varsigma}^{\tau} \vartheta(\mathfrak{w})d\mathfrak{w} \leq \int_{\varsigma}^{\tau} \zeta(\mathfrak{w})\vartheta(\mathfrak{w})d\mathfrak{w} \leq \frac{\zeta(\varsigma)+\zeta(\tau)}{2}\int_{\varsigma}^{\tau} \vartheta(\mathfrak{w})d\mathfrak{w}, \tag{1.2}$$

where $\zeta : I \longrightarrow \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, $\varsigma, \tau \in I$ with $\varsigma < \tau$ is any convex function and $\vartheta : [\varsigma, \tau] \rightarrow \mathbb{R}$ is non-negative integrable and symmetric with respect to $\mathfrak{w} = \frac{\varsigma+\tau}{2}$.

These inequalities have many extensions and generalizations, see [2–18] and [19–31].
Consider the following mappings on $[0,1]$:

$$\breve{\mathcal{G}}(\iota)=\frac{1}{2}\int_{\varsigma}^{\tau}\left[\zeta\left(\iota\varsigma+(1-\iota)\frac{\varsigma+\tau}{2}\right)+\zeta\left(\iota\tau+(1-\iota)\frac{\varsigma+\tau}{2}\right)\right]d\mathfrak{w},$$

$$\mathbb{H}(\iota)=\frac{1}{\tau-\varsigma}\int_{\varsigma}^{\tau}\zeta\left(\iota\mathfrak{w}+(1-\iota)\frac{\varsigma+\tau}{2}\right)d\mathfrak{w},$$

$$\mathbb{H}_{\vartheta}(\iota)=\frac{1}{\tau-\varsigma}\int_{\varsigma}^{\tau}\zeta\left(\iota\mathfrak{w}+(1-\iota)\frac{\varsigma+\tau}{2}\right)\vartheta(\mathfrak{w})\,d\mathfrak{w},$$

$$\mathcal{L}(\iota)=\frac{1}{2(\tau-\varsigma)}\int_{\varsigma}^{\tau}\left[\zeta(\iota\varsigma+(1-\iota)\mathfrak{w})+\zeta(\iota\tau+(1-\iota)\mathfrak{w})\right]d\mathfrak{w}$$

and

$$\mathcal{L}_{\vartheta}(\iota)=\frac{1}{2}\int_{\varsigma}^{\tau}\left[\zeta(\iota\varsigma+(1-\iota)\mathfrak{w})+\zeta(\iota\tau+(1-\iota)\mathfrak{w})\right]\vartheta(\mathfrak{w})d\mathfrak{w},$$

where $\zeta:[\varsigma,\tau]\to\mathbb{R}$ is a convex function and $\vartheta:[\varsigma,\tau]\to\mathbb{R}$ is non-negative integrable and symmetric with respect to $\mathfrak{w}=\frac{\varsigma+\tau}{2}$.

The important results that characterize the properties of the above mappings and inequalities are discussed by a number of mathematicians.

Dragomir [2] established the theorem which refines the first inequality of (1.1).

Theorem 1. *[2] Let $\zeta:[\varsigma,\tau]\to\mathbb{R}$ be a convex function on $[\varsigma,\tau]$. Then $\mathbb{H}$ is monotonically increasing and convex on $[0,1]$. Moreover, one has the following inequalities*

$$\zeta\left(\frac{\varsigma+\tau}{2}\right)=\mathbb{H}(0)\leq\mathbb{H}(\iota)\leq\mathbb{H}(1)=\frac{1}{\tau-\varsigma}\int_{\varsigma}^{\tau}\zeta(\mathfrak{w})\,d\mathfrak{w}.$$

Dragomir et al. [7] obtained the refinements of (1.1).

Theorem 2. *[2] Let ζ, $\mathbb{H}$ be defined as above. Then*

(i) The following inequality holds

$$\begin{aligned}\zeta\left(\frac{\varsigma+\tau}{2}\right)&\leq\frac{2}{\tau-\varsigma}\int_{\frac{3\varsigma+\tau}{4}}^{\frac{\varsigma+3\tau}{4}}\zeta(\mathfrak{w})\,d\mathfrak{w}\leq\int_{0}^{1}\mathbb{H}(\iota)\,d\iota\\&\leq\frac{1}{2}\left[\zeta\left(\frac{\varsigma+\tau}{2}\right)+\frac{1}{\tau-\varsigma}\int_{\varsigma}^{\tau}\zeta(\mathfrak{w})\,d\mathfrak{w}\right].\end{aligned}$$

(ii) If ζ is differentiable on $[\varsigma,\tau]$, then for all $\iota\in[0,1]$, one has

$$\begin{aligned}0&\leq\frac{1}{\tau-\varsigma}\int_{\varsigma}^{\tau}\zeta(\mathfrak{w})\,d\mathfrak{w}-\mathbb{H}(\iota)\\&\leq(1-\iota)\left[\frac{\zeta(\varsigma)+\zeta(\tau)}{2}-\frac{1}{\tau-\varsigma}\int_{\varsigma}^{\tau}\zeta(\mathfrak{w})\,d\mathfrak{w}\right]\end{aligned}$$

and

$$0\leq\frac{\zeta(\varsigma)+\zeta(\tau)}{2}-\mathbb{H}(\iota)\leq\frac{\left(\zeta'(\tau)-\zeta'(\varsigma)\right)(\tau-\varsigma)}{4}.$$

Theorem 3. *[7] Let ζ, $\mathbb{H}$, $\breve{\mathcal{G}}$ be defined as above. We have*

(i) $\breve{\mathcal{G}}$ is convex and increasing on $[0,1]$.

(ii) The following hold

$$\inf_{\iota\in[0,1]}\breve{\mathcal{G}}(\iota)=\breve{\mathcal{G}}(0)=\zeta\left(\frac{\varsigma+\tau}{2}\right)$$

and

$$\sup_{\iota\in[0,1]}\breve{\mathcal{G}}(\iota)=\breve{\mathcal{G}}(1)=\frac{\zeta(\varsigma)+\zeta(\tau)}{2}.$$

(iii) The inequality

$$\mathbb{H}(\iota)\leq\breve{\mathcal{G}}(\iota)$$

holds for all $\iota\in[0,1]$.

(iv) Then

$$\begin{aligned}\frac{2}{\tau-\varsigma}\int_{\frac{3\varsigma+\tau}{4}}^{\frac{\varsigma+3\tau}{4}}\zeta(\mathfrak{w})\,d\mathfrak{w}&\leq\frac{1}{2}\left[\zeta\left(\frac{3\varsigma+\tau}{4}\right)+\zeta\left(\frac{\varsigma+3\tau}{4}\right)\right]\\&\leq\int_0^1\breve{\mathcal{G}}(\iota)\,d\iota\\&\leq\frac{1}{2}\left[\zeta\left(\frac{\varsigma+\tau}{2}\right)+\frac{\zeta(\varsigma)+\zeta(\tau)}{2}\right].\end{aligned}$$

(v) If ζ is differentiable on $[\varsigma,\tau]$, then for all $\iota\in[0,1]$, one has

$$0\leq\mathbb{H}(\iota)-\zeta\left(\frac{\varsigma+\tau}{2}\right)\leq\breve{\mathcal{G}}(\iota)-\mathbb{H}(\iota).$$

Theorem 4. *[7] Let ζ, $\mathbb{H}$, $\breve{\mathcal{G}}$, $\mathcal{L}$ be defined as above. Then*

(i) $\mathcal{L}$ is convex on $[0,1]$.

(ii) The inequalities

$$\breve{\mathcal{G}}(\iota)\leq\mathcal{L}(\iota)\leq\frac{1-\iota}{\tau-\varsigma}\int_{\varsigma}^{\tau}\zeta(\mathfrak{w})\,d\mathfrak{w}+\iota\frac{\zeta(\varsigma)+\zeta(\tau)}{2}\leq\frac{\zeta(\varsigma)+\zeta(\tau)}{2}$$

hold for all $\iota\in[0,1]$ and

$$\sup_{\iota\in[0,1]}\mathcal{L}(\iota)=\mathcal{L}(1)=\frac{\zeta(\varsigma)+\zeta(\tau)}{2}.$$

(iii) The inequalities

$$\mathbb{H}(1-\iota) \leq \mathcal{L}(\iota) \text{ and } \frac{\mathbb{H}(\iota)+\mathbb{H}(1-\iota)}{2} \leq \mathcal{L}(\iota)$$

hold for all $\iota \in [0,1]$.

Teseng et al. [24] proved the following result.

Lemma 1. *[24] Let* $\zeta : [\varsigma, \tau] \to \mathbb{R}$ *be a convex function and let* $\varsigma \leq \breve{\varkappa}_1 \leq \mathfrak{w}_1 \leq \mathfrak{w}_2 \leq \breve{\varkappa}_2 \leq \tau$ *with* $\mathfrak{w}_1 + \mathfrak{w}_2 = \breve{\varkappa}_1 + \breve{\varkappa}_2$. *Then*

$$\zeta(\mathfrak{w}_1) + \zeta(\mathfrak{w}_2) \leq \zeta(\breve{\varkappa}_1) + \zeta(\breve{\varkappa}_2).$$

Yang and Tseng [28] proven the theorem by using Lemma 1 which refines the first inequality of (1.2) and generalizes Theorem 1.

Theorem 5. *[28] Let* $\zeta : [\varsigma, \tau] \to \mathbb{R}$ *be a convex function and* $\vartheta : [\varsigma, \tau] \to \mathbb{R}$ *is non-negative integrable and symmetric with respect to* $\mathfrak{w} = \frac{\varsigma+\tau}{2}$. *Then* $\mathbb{H}_\vartheta$ *is convex, increasing on* $[0,1]$, *and for all* $\iota \in [0,1]$, *we have*

$$\zeta\left(\frac{\varsigma+\tau}{2}\right)\int_\varsigma^\tau \vartheta(\mathfrak{w})\,d\mathfrak{w} = \mathbb{H}_\vartheta(0) \leq \mathbb{H}_\vartheta(\iota) \leq \mathbb{H}_\vartheta(1) = \int_\varsigma^\tau \zeta(\mathfrak{w})\,\vartheta(\mathfrak{w})\,d\mathfrak{w}. \tag{1.3}$$

One of the generalizations of the convex functions is harmonic functions:

Definition 1. *[17] Define* $I \subseteq \mathbb{R}\setminus\{0\}$ *as an interval of real numbers. A function* ζ *from* I *to the real numbers is considered to be harmonically convex, if*

$$\zeta\left(\frac{\mathfrak{w}\breve{\varkappa}}{\iota\mathfrak{w} + (1-\iota)\breve{\varkappa}}\right) \leq \iota\zeta(\breve{\varkappa}) + (1-\iota)\zeta(\mathfrak{w}) \tag{1.4}$$

for all $\mathfrak{w}, \breve{\varkappa} \in I$ *and* $\iota \in [0,1]$. *Harmonically concave* ζ *is defined as the inequality in (1.4) reversed.*

İşcan [17] used harmonic-convexity to develop the inequalities of Hermite-Hadamard type.

Theorem 6. *[17] Let* $\zeta : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ *be a harmonically convex function and* $\varsigma, \tau \in I$ *with* $\varsigma < \tau$. *If* $\zeta \in L([\varsigma, \tau])$ *then the inequalities*

$$\zeta\left(\frac{2\varsigma\tau}{\varsigma+\tau}\right) \leq \frac{\varsigma\tau}{\tau-\varsigma}\int_\tau^\varsigma \frac{\zeta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w} \leq \frac{\zeta(\varsigma)+\zeta(\tau)}{2} \tag{1.5}$$

hold.

Let $\zeta : [\varsigma, \tau] \subset (0,\infty) \to \mathbb{R}$ be a harmonic convex mapping and let $\mathbb{S}, \mathbb{V} : [0,1] \to \mathbb{R}$ be defined by

$$\mathbb{S}(\iota) = \frac{\varsigma\tau}{\tau-\varsigma}\int_\varsigma^\tau \frac{1}{\mathfrak{w}^2}\zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{2\varsigma\tau\iota + (1-\iota)\mathfrak{w}(\varsigma+\tau)}\right)d\mathfrak{w} \tag{1.6}$$

and

$$\mathbb{V}(\iota) = \frac{\varsigma\tau}{2(\tau-\varsigma)}\int_\varsigma^\tau \frac{1}{\mathfrak{w}^2}\left[\zeta\left(\frac{2\tau\mathfrak{w}}{(1+\iota)\mathfrak{w} + (1-\iota)\tau}\right) + \zeta\left(\frac{2\varsigma\mathfrak{w}}{(1+\iota)\mathfrak{w} + (1-\iota)\varsigma}\right)\right]d\mathfrak{w}. \tag{1.7}$$

The author obtained the refinement inequalities for (1.5) related to the above mappings:

Theorem 7. *[21] Let $\zeta : [\varsigma, \tau] \subset (0, \infty) \to \mathbb{R}$ be a harmonic convex function on $[\varsigma, \tau]$. Then*

(i) $\mathbb{S}$ *is harmonic convex* $(0, 1]$ *and increases monotonically on* $[0, 1]$.
(ii) The following hold:

$$\zeta\left(\frac{2\varsigma\tau}{\varsigma+\tau}\right) = \mathbb{S}(0) \le \mathbb{S}(\iota) \le \mathbb{S}(1) = \frac{\varsigma\tau}{\tau-\varsigma}\int_{\varsigma}^{\tau} \frac{\zeta(\mathfrak{w})}{\mathfrak{w}^2} d\mathfrak{w}.$$

Theorem 8. *[21] Let $\zeta : [\varsigma, \tau] \subset (0, \infty) \to \mathbb{R}$ be a harmonic convex function on $[\varsigma, \tau]$. Then*

(i) $\mathbb{V}$ *is harmonic convex* $(0, 1]$ *and increases monotonically on* $[0, 1]$.
(ii) The following hold:

$$\frac{\varsigma\tau}{\tau-\varsigma}\int_{\varsigma}^{\tau} \frac{\zeta(\mathfrak{w})}{\mathfrak{w}^2} d\mathfrak{w} = \mathbb{V}(0) \le \mathbb{V}(\iota) \le \mathbb{V}(1) = \frac{\zeta(\varsigma)+\zeta(\tau)}{2}.$$

Harmonic symmetricity of a function is given in the definition below.

Definition 2. *[22] A function $\vartheta : [\varsigma, \tau] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is harmonically symmetric with respect to $\frac{2\varsigma\tau}{\varsigma+\tau}$ if*

$$\vartheta(\mathfrak{w}) = \vartheta\left(\frac{1}{\frac{1}{\varsigma}+\frac{1}{\tau}-\frac{1}{\mathfrak{w}}}\right)$$

holds for all $\mathfrak{w} \in [\varsigma, \tau]$.

Fejér type inequalities using harmonic convexity and the notion of harmonic symmetricity were presented in Chan and Wu [1].

Theorem 9. *[1] Let $\zeta : I \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and $\varsigma, \tau \in I$ with $\varsigma < \tau$. If $\zeta \in L([\varsigma, \tau])$ and $\vartheta : [\varsigma, \tau] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $\frac{2\varsigma\tau}{\varsigma+\tau}$, then*

$$\zeta\left(\frac{2\varsigma\tau}{\varsigma+\tau}\right)\int_{\tau}^{\varsigma} \frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2} d\mathfrak{w} \le \int_{\tau}^{\varsigma} \frac{\zeta(\mathfrak{w})\vartheta(\mathfrak{w})}{\mathfrak{w}^2} d\mathfrak{w} \le \frac{\zeta(\varsigma)+\zeta(\tau)}{2}\int_{\tau}^{\varsigma} \frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2} d\mathfrak{w}. \tag{1.8}$$

Chan and Wu [1] also defined some mappings related to (1.8) and discussed important properties of these mappings.

Motivated by the studies conducted in [2, 21, 24, 27], we define some new mappings in connection to (1.8) and to prove new Féjer type inequalities which indeed provide refinement inequalities as well.

2. Main results

To prove the major findings of this work, we employ the given important facts about harmonic convex and convex functions.

Theorem 10. *[8, 9] If $[\varsigma, \tau] \subset I \subset (0, \infty)$ and if we consider the function $\breve{h} : \left[\frac{1}{\tau}, \frac{1}{\varsigma}\right] \to \mathbb{R}$ defined by $\breve{h}(\iota) = \zeta\left(\frac{1}{\iota}\right)$, then ζ is harmonically convex on $[\varsigma, \tau]$ if and only if $\breve{h}$ is convex in the usual sense on $\left[\frac{1}{\tau}, \frac{1}{\varsigma}\right]$.*

Theorem 11. *[8, 9] If $I \subset (0,\infty)$ and ζ is convex and nondecreasing function then ζ is harmonic convex and if ζ is harmonic convex and nonincreasing function then ζ is convex.*

Let us now define some mappings on $[0,1]$ related to (1.8) and prove some refinement inequalities.

$$\breve{\mathcal{G}}_1(\iota) = \frac{1}{2}\left[\zeta\left(\frac{2\varsigma\tau}{2\varsigma\iota + (1-\iota)(\varsigma+\tau)}\right) + \zeta\left(\frac{2\varsigma\tau}{2\tau\iota + (1-\iota)(\varsigma+\tau)}\right)\right] d\mathfrak{w},$$

$$\mathbb{S}(\iota) = \frac{\varsigma\tau}{\tau-\varsigma}\int_{\varsigma}^{\tau}\frac{1}{\mathfrak{w}^2}\zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{2\varsigma\tau\iota + (1-\iota)(\varsigma+\tau)\mathfrak{w}}\right) d\mathfrak{w},$$

$$\mathbb{S}_{\vartheta}(\iota) = \int_{\varsigma}^{\tau}\zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{2\varsigma\tau\iota + (1-\iota)(\varsigma+\tau)\mathfrak{w}}\right)\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2} d\mathfrak{w},$$

$$\mathbb{T}(\iota) = \frac{\varsigma\tau}{2(\tau-\varsigma)}\int_{\varsigma}^{\tau}\frac{1}{\mathfrak{w}^2}\left[\zeta\left(\frac{\tau\mathfrak{w}}{\iota\mathfrak{w} + (1-\iota)\tau}\right) + \zeta\left(\frac{\varsigma\mathfrak{w}}{\iota\mathfrak{w} + (1-\iota)\varsigma}\right)\right] d\mathfrak{w}$$

and

$$\mathbb{T}_{\vartheta}(\iota) = \frac{1}{2}\int_{\varsigma}^{\tau}\left[\zeta\left(\frac{\tau\mathfrak{w}}{\iota\mathfrak{w} + (1-\iota)\tau}\right) + \zeta\left(\frac{\varsigma\mathfrak{w}}{\iota\mathfrak{w} + (1-\iota)\varsigma}\right)\right]\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2} d\mathfrak{w},$$

where $\zeta : [\varsigma,\tau] \to \mathbb{R}$ is a harmonic convex function and $\vartheta : [\varsigma,\tau] \to \mathbb{R}$ is non-negative integrable and symmetric with respect to $\mathfrak{w} = \frac{2\varsigma\tau}{\varsigma+\tau}$.

Lemma 2. *Let $\zeta : [\varsigma,\tau] \subset (0,\infty) \to \mathbb{R}$ be a harmonic convex function and let $\varsigma \le \breve{\varkappa}_1 \le \mathfrak{w}_1 \le \mathfrak{w}_2 \le \breve{\varkappa}_2 \le \tau$ with $\frac{\mathfrak{w}_1\mathfrak{w}_2}{\mathfrak{w}_1+\mathfrak{w}_2} = \frac{\breve{\varkappa}_1\breve{\varkappa}_2}{\breve{\varkappa}_1+\breve{\varkappa}_2}$. Then*

$$\zeta(\mathfrak{w}_1) + \zeta(\mathfrak{w}_2) \le \zeta(\breve{\varkappa}_1) + \zeta(\breve{\varkappa}_2).$$

Proof. For $\breve{\varkappa}_1 = \breve{\varkappa}_2$, the result is obvious. We observe that

$$\mathfrak{w}_1 = \frac{\breve{\varkappa}_1\breve{\varkappa}_2}{\left(\frac{\mathfrak{w}_1\breve{\varkappa}_1-\breve{\varkappa}_1\breve{\varkappa}_2}{\mathfrak{w}_1\breve{\varkappa}_1-\mathfrak{w}_1\breve{\varkappa}_2}\right)\breve{\varkappa}_2 + \left(\frac{\breve{\varkappa}_1\breve{\varkappa}_2-\mathfrak{w}_1\breve{\varkappa}_2}{\mathfrak{w}_1\breve{\varkappa}_1-\mathfrak{w}_1\breve{\varkappa}_2}\right)\breve{\varkappa}_1} \text{ and } \mathfrak{w}_2 = \frac{\breve{\varkappa}_1\breve{\varkappa}_2}{\left(\frac{\mathfrak{w}_2\breve{\varkappa}_1-\breve{\varkappa}_1\breve{\varkappa}_2}{\mathfrak{w}_2\breve{\varkappa}_1-\mathfrak{w}_2\breve{\varkappa}_2}\right)\breve{\varkappa}_2 + \left(\frac{\breve{\varkappa}_1\breve{\varkappa}_2-\mathfrak{w}_2\breve{\varkappa}_2}{\mathfrak{w}_2\breve{\varkappa}_1-\mathfrak{w}_2\breve{\varkappa}_2}\right)\breve{\varkappa}_1}.$$

By applying the harmonic convexity, we obtain

$$\begin{aligned}
\zeta(\mathfrak{w}_1) + \zeta(\mathfrak{w}_2) &\le \left(\frac{\mathfrak{w}_1\breve{\varkappa}_1 - \breve{\varkappa}_1\breve{\varkappa}_2}{\mathfrak{w}_1\breve{\varkappa}_1 - \mathfrak{w}_1\breve{\varkappa}_2}\right)\zeta(\breve{\varkappa}_1) + \left(\frac{\breve{\varkappa}_1\breve{\varkappa}_2 - \mathfrak{w}_1\breve{\varkappa}_2}{\mathfrak{w}_1\breve{\varkappa}_1 - \mathfrak{w}_1\breve{\varkappa}_2}\right)\zeta(\breve{\varkappa}_2) \\
&\quad + \left(\frac{\mathfrak{w}_2\breve{\varkappa}_1 - \breve{\varkappa}_1\breve{\varkappa}_2}{\mathfrak{w}_2\breve{\varkappa}_1 - \mathfrak{w}_2\breve{\varkappa}_2}\right)\zeta(\breve{\varkappa}_1) + \left(\frac{\breve{\varkappa}_1\breve{\varkappa}_2 - \mathfrak{w}_2\breve{\varkappa}_2}{\mathfrak{w}_2\breve{\varkappa}_1 - \mathfrak{w}_2\breve{\varkappa}_2}\right)\zeta(\breve{\varkappa}_2) \\
&= \left(\frac{\mathfrak{w}_1\breve{\varkappa}_1 - \breve{\varkappa}_1\breve{\varkappa}_2}{\mathfrak{w}_1\breve{\varkappa}_1 - \mathfrak{w}_1\breve{\varkappa}_2} + \frac{\mathfrak{w}_2\breve{\varkappa}_1 - \breve{\varkappa}_1\breve{\varkappa}_2}{\mathfrak{w}_2\breve{\varkappa}_1 - \mathfrak{w}_2\breve{\varkappa}_2}\right)\zeta(\breve{\varkappa}_1) \\
&\quad + \left(\frac{\breve{\varkappa}_1\breve{\varkappa}_2 - \mathfrak{w}_1\breve{\varkappa}_2}{\mathfrak{w}_1\breve{\varkappa}_1 - \mathfrak{w}_1\breve{\varkappa}_2} + \frac{\breve{\varkappa}_1\breve{\varkappa}_2 - \mathfrak{w}_2\breve{\varkappa}_2}{\mathfrak{w}_2\breve{\varkappa}_1 - \mathfrak{w}_2\breve{\varkappa}_2}\right)\zeta(\breve{\varkappa}_2)
\end{aligned}$$

$$= \frac{\breve{\varkappa}_1}{\breve{\varkappa}_2 - \breve{\varkappa}_1}\left(\left(\frac{\mathfrak{w}_1 + \mathfrak{w}_2}{\mathfrak{w}_1\mathfrak{w}_2}\right)\breve{\varkappa}_2 - 2\right)\zeta(\breve{\varkappa}_1)$$
$$+ \frac{\breve{\varkappa}_2}{\breve{\varkappa}_2 - \breve{\varkappa}_1}\left(2 - \left(\frac{\mathfrak{w}_1 + \mathfrak{w}_2}{\mathfrak{w}_1\mathfrak{w}_2}\right)\breve{\varkappa}_1\right)\zeta(\breve{\varkappa}_2) = \zeta(\breve{\varkappa}_1) + \zeta(\breve{\varkappa}_2).$$

□

We first prove a result similar to (1.3) for harmonically convex functions which provide refinement inequalities for (1.8).

Theorem 12. *Let* $\zeta : [\varsigma, \tau] \subset (0,\infty) \to \mathbb{R}$ *be a harmonic convex function,* $0 < \rho < 1$, $0 < \theta < 1$, $\lambda = \frac{\varsigma\tau}{\rho\varsigma + (1-\rho)\tau}$, $\tau_0 = \left(\frac{\varsigma\tau}{\tau - \varsigma}\right)\min\left\{\frac{\rho}{1-\theta}, \frac{1-\rho}{\theta}\right\}$ *and let* $\vartheta : [\varsigma, \tau] \to \mathbb{R}$ *be nonnegative and integrable and* $\vartheta\left(\frac{\lambda}{1-\theta\iota\lambda}\right) = \vartheta\left(\frac{\lambda}{1+(1-\theta)\iota\lambda}\right)$, $\iota \in [0, \tau_0]$. *Then*

$$\zeta\left(\frac{\varsigma\tau}{\rho\varsigma + (1-\rho)\tau}\right)\int_{\frac{\lambda}{1+(1-\theta)\iota\lambda}}^{\frac{\lambda}{1-\theta\iota\lambda}} \frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}$$
$$\leq \frac{1-\theta}{\theta}\int_{\frac{\lambda}{1-\theta\iota\lambda}}^{\lambda} \frac{\zeta(\mathfrak{w})\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w} + \frac{\theta}{1-\theta}\int_{\lambda}^{\frac{\lambda}{1+(1-\theta)\iota\lambda}} \frac{\zeta(\mathfrak{w})\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}$$
$$\leq \left[\rho\zeta(\tau) + (1-\rho)\zeta(\varsigma)\right]\int_{\frac{\lambda}{1+(1-\theta)\iota\lambda}}^{\frac{\lambda}{1-\theta\iota\lambda}} \frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}. \quad (2.1)$$

Proof. For every $\tau \in [0, \tau_0]$, we have the identity

$$\int_{\frac{\lambda}{1+(1-\theta)\iota\lambda}}^{\frac{\lambda}{1-\theta\iota\lambda}} \frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w} = \int_{\frac{\lambda}{1+(1-\theta)\iota\lambda}}^{\lambda} \frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w} + \int_{\lambda}^{\frac{\lambda}{1-\theta\iota\lambda}} \frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}$$
$$= \theta\int_0^{\iota} \frac{\vartheta\left(\frac{\lambda}{1-\theta\mathfrak{w}\lambda}\right)}{\mathfrak{w}^2}d\mathfrak{w} + (1-\theta)\int_0^{\iota} \frac{\vartheta\left(\frac{\lambda}{1-\theta\mathfrak{w}\lambda}\right)}{\mathfrak{w}^2}d\mathfrak{w} = \int_0^{\iota} \frac{\vartheta\left(\frac{\lambda}{1-\theta\mathfrak{w}\lambda}\right)}{\mathfrak{w}^2}d\mathfrak{w}. \quad (2.2)$$

We now prove that the mapping $\mathcal{W} : [0, \tau_0] \to \mathbb{R}$ defined by

$$\mathcal{W}(\iota) = (1-\theta)\zeta\left(\frac{\lambda}{1-\theta\iota\lambda}\right) + \theta\zeta\left(\frac{\lambda}{1+(1-\theta)\iota\lambda}\right)$$

is harmonic convex $(0, \tau_0]$ and monotonically increasing on $[0, \tau_0]$.
Since the sum of two harmonic convex functions is a harmonic convex, hence $\mathcal{W}$ is a harmonic convex on $(0, \tau_0]$. Let $\iota \in (0, \tau_0]$, it follows from the harmonic convexity of ζ that

$$\mathcal{W}(\iota) = (1-\theta)\zeta\left(\frac{\lambda}{1-\theta\iota\lambda}\right) + \theta\zeta\left(\frac{\lambda}{1+(1-\theta)\iota\lambda}\right)$$
$$\geq \zeta\left(\frac{\left(\frac{\lambda}{1-\theta\iota\lambda}\right)\left(\frac{\lambda}{1+(1-\theta)\iota\lambda}\right)}{\theta\left(\frac{\lambda}{1-\theta\iota\lambda}\right) + (1-\theta)\left(\frac{\lambda}{1+(1-\theta)\iota\lambda}\right)}\right)$$
$$= \zeta(\lambda) = \zeta\left(\frac{\varsigma\tau}{\rho\varsigma + (1-\rho)\tau}\right). \quad (2.3)$$

We observed that $0 < \rho \leq \frac{\rho(\tau-\varsigma)+\varsigma\tau\theta\iota}{\tau-\varsigma} \leq 1$, $0 \leq \frac{(1-\rho)(\tau-\varsigma)-\theta\iota\varsigma\tau}{\tau-\varsigma} \leq 1-\rho < 1$, $0 \leq \rho \leq \frac{\rho(\tau-\varsigma)-(1-\theta)\iota\varsigma\tau}{\tau-\varsigma} \leq \rho \leq 1$ and $0 < 1-\rho \leq \frac{(1-\rho)(\tau-\varsigma)+(1-\theta)\iota\varsigma\tau}{\tau-\varsigma} \leq 1$. Thus, by using the harmonic convexity, we obtain

$$\begin{aligned}
\mathcal{W}(\iota) &= (1-\theta)\zeta\left(\frac{\lambda}{1-\theta\iota\lambda}\right) + \theta\zeta\left(\frac{\lambda}{1+(1-\theta)\iota\lambda}\right) \\
&= (1-\theta)\zeta\left(\frac{\varsigma\tau}{\left(\frac{\rho(\tau-\varsigma)+\varsigma\tau\iota\theta}{\tau-\varsigma}\right)\varsigma + \left(\frac{(1-\rho)(\tau-\varsigma)-\theta\iota\varsigma\tau}{\tau-\varsigma}\right)\tau}\right) \\
&\quad + \theta\zeta\left(\frac{\varsigma\tau}{\left(\frac{\rho(\tau-\varsigma)-(1-\theta)\iota\varsigma\tau}{(\tau-\varsigma)}\right)\varsigma + \left(\frac{(1-\rho)(\tau-\varsigma)+(1-\theta)\iota\varsigma\tau}{\tau-\varsigma}\right)\tau}\right) \\
&\leq (1-\theta)\left(\frac{(1-\rho)(\tau-\varsigma)-\theta\iota\varsigma\tau}{\tau-\varsigma}\right)\zeta(\varsigma) + (1-\theta) \\
&\quad \times\left(\frac{\rho(\tau-\varsigma)+\iota\varsigma\tau\theta}{\tau-\varsigma}\right)\zeta(\tau) + \theta\left(\frac{(1-\rho)(\tau-\varsigma)+(1-\theta)\iota\varsigma\tau}{(\tau-\varsigma)}\right)\zeta(\varsigma) \\
&\quad + \theta\left(\frac{\rho(\tau-\varsigma)-(1-\theta)\iota\varsigma\tau}{\tau-\varsigma}\right)\zeta(\tau) = (1-\rho)\zeta(\varsigma) + \rho\zeta(\tau).
\end{aligned} \tag{2.4}$$

From (2.3) and (2.4), we obtain

$$\zeta\left(\frac{\varsigma\tau}{\rho\varsigma+(1-\rho)\tau}\right) \leq \mathcal{W}(\iota) \leq (1-\rho)\zeta(\varsigma) + \rho\zeta(\tau). \tag{2.5}$$

Finally, for ι_1, ι_2, such that $0 < \iota_1 < \iota_2 \leq \varsigma_0$, since $\mathcal{W}(\iota)$ is harmonic convex, it follows from (2.3) that

$$\frac{\mathcal{W}(\iota_2) - \mathcal{W}(\iota_1)}{\iota_2 - \iota_1} \geq 0.$$

This shows that $\mathcal{W}$ is increasing on $[0, \tau_0]$.

Since ϑ is nonnegative, multiplying (2.5) by $\frac{\vartheta\left(\frac{\lambda}{1-\theta\mathfrak{w}\lambda}\right)}{\mathfrak{w}^2}$, integrating the resulting inequalities over $[0, \iota]$ and using $\vartheta\left(\frac{\lambda}{1-\theta\mathfrak{w}\lambda}\right) = \vartheta\left(\frac{\lambda}{1+(1-\theta)\mathfrak{w}\lambda}\right)$, we have

$$\begin{aligned}
&\zeta\left(\frac{\varsigma\tau}{\rho\varsigma+(1-\rho)\tau}\right)\int_0^\iota \frac{\vartheta\left(\frac{\lambda}{1-\theta\mathfrak{w}\lambda}\right)}{\mathfrak{w}^2}d\mathfrak{w} \\
&\quad \leq (1-\theta)\int_0^\iota \frac{\zeta\left(\frac{\lambda}{1-\theta\mathfrak{w}\lambda}\right)\vartheta\left(\frac{\lambda}{1-\theta\mathfrak{w}\lambda}\right)}{\mathfrak{w}^2}d\mathfrak{w} + \theta\int_0^\iota \frac{\zeta\left(\frac{\lambda}{1+(1-\theta)\mathfrak{w}\lambda}\right)\vartheta\left(\frac{\lambda}{1+(1-\theta)\mathfrak{w}\lambda}\right)}{\mathfrak{w}^2}d\mathfrak{w} \\
&\quad \leq \left[(1-\rho)\zeta(\varsigma) + \rho\zeta(\tau)\right]\int_0^\iota \frac{\vartheta\left(\frac{\lambda}{1-\theta\mathfrak{w}\lambda}\right)}{\mathfrak{w}^2}d\mathfrak{w}.
\end{aligned} \tag{2.6}$$

By using the identity (2.2) in (2.6), we obtain (2.1). □

Remark 1. *If we choose $\rho = \frac{\vartheta}{\vartheta+q}$, $\theta = \frac{1}{2}$, $\iota = 2\varkappa$ in Theorem 12, then*

$$\zeta\left(\frac{\varsigma\tau(\vartheta+q)}{\vartheta\varsigma+q\tau}\right)\int_{\frac{\lambda}{1+\lambda\tau\aleph}}^{\frac{\lambda}{1-\lambda\tau\aleph}}\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}\leq\int_{\frac{\lambda}{1+\lambda\tau\aleph}}^{\frac{\lambda}{1-\lambda\tau\aleph}}\frac{\zeta(\mathfrak{w})\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}$$
$$\leq\left[\frac{\vartheta\zeta(\tau)+q\zeta(\varsigma)}{\vartheta+q}\right]\int_{\frac{\lambda}{1+\lambda\tau\aleph}}^{\frac{\lambda}{1-\lambda\tau\aleph}}\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}. \quad (2.7)$$

Remark 2. *If we choose $\rho=\theta=\frac{1}{2}$, $\iota=\tau_0=\frac{\varsigma\tau}{\tau-\varsigma}$ in Theorem 12, then we get (1.8).*

Remark 3. *If we choose $\rho=\theta=\frac{1}{2}$, $\iota=\tau_0=\frac{\varsigma\tau}{\tau-\varsigma}$ and $\vartheta(\mathfrak{w})=1$, $\mathfrak{w}\in[\varsigma,\tau]$ in Theorem 12, then we get (1.5).*

Theorem 13. *Let ζ, λ and τ_0 be defined as in Theorem 12, $0<\rho<1$, $0<\theta<1$, $\rho+\theta\leq 1$ and let $\mathbb{X}$ be defined on $[0,1]$ as*

$$\mathbb{X}(\iota)=\frac{\rho\varsigma\tau}{(1-\theta)(\tau-\varsigma)}\times\int_0^{\frac{\rho\varsigma\tau}{(1-\theta)(\tau-\varsigma)}}\left[(1-\theta)\zeta\left(\frac{\lambda}{1-\theta\iota\mathfrak{w}\lambda}\right)+\theta\zeta\left(\frac{\lambda}{1+(1-\theta)\iota\mathfrak{w}\lambda}\right)\right]d\mathfrak{w}. \quad (2.8)$$

Then, $\mathbb{X}$ is harmonically convex on $(0,1]$ and monotonically increasing on $[0,1]$, and

$$\zeta\left(\frac{\varsigma\tau}{\rho\varsigma+(1-\rho)\tau}\right)\leq\mathbb{X}(\iota)\leq\mathbb{X}(1)=\frac{\rho\varsigma\tau}{(1-\theta)(\tau-\varsigma)}\times\int_0^{\frac{\rho\varsigma\tau}{(1-\theta)(\tau-\varsigma)}}\left[(1-\theta)\zeta\left(\frac{\lambda}{1-\theta\mathfrak{w}\lambda}\right)+\theta\zeta\left(\frac{\lambda}{1+(1-\theta)\mathfrak{w}\lambda}\right)\right]d\mathfrak{w}$$
$$\leq(1-\rho)\zeta(\varsigma)+\rho\zeta(\tau).$$

Proof. Since ζ is harmonically convex on $[\varsigma,\tau]$ this prove the harmonic convexity of $\mathbb{X}$ on $(0,1]$. By using the condition $\rho+\theta\leq 1$ implies that $\tau_0=\frac{\rho\varsigma\tau}{(1-\theta)(\tau-\varsigma)}$. Since the mapping $\mathcal{W}:[0,\tau_0]\to\mathbb{R}$ defined by

$$\mathcal{W}(\iota)=(1-\theta)\zeta\left(\frac{\lambda}{1-\theta\iota\lambda}\right)+\theta\zeta\left(\frac{\lambda}{1+(1-\theta)\iota\lambda}\right) \quad (2.9)$$

has been proved to be monotonically increasing on $[0,\tau_0]$, thus the mapping $\mathbb{X}$ is also monotonically increasing on $[0,1]$.
Because X is monotonically increasing on $[0,1]$, it follows that the inequalities (2.8) can be deduced from these inequalities (2.5). The proof of the theorem was completed as a result of this. □

The next theorem can be proved similarly:

Theorem 14. *Let ζ, λ, τ_0, ρ and θ be defined as in Theorem 13. Let $\mathbb{X}_1$ be defined on $[0,1]$ as*

$$\mathbb{X}_1(\iota)=\frac{\rho\varsigma\tau}{(1-\theta)(\tau-\varsigma)}\times\int_0^{\frac{\rho\varsigma\tau}{(1-\theta)(\tau-\varsigma)}}\left[(1-\theta)\zeta\left(\frac{\lambda(1-\theta)\varsigma\tau}{(1-\theta)\varsigma\tau-\theta(\rho(\tau-\varsigma)-\mathfrak{w}(1-\iota)(1-\theta)\varsigma\tau)\lambda}\right)+\theta\zeta\left(\frac{\lambda(1-\theta)\varsigma\tau}{(1-\theta)\varsigma\tau+(1-\theta)(\rho(\tau-\varsigma)-\mathfrak{w}(1-\iota)(1-\theta)\varsigma\tau)\lambda}\right)\right]d\mathfrak{w}. \quad (2.10)$$

Then, $\mathbb{X}_1$ *is harmonically convex monotonically increasing on* $[0,1]$*, and*

$$\frac{(1-\theta)^2\varsigma\tau}{\rho\theta(\tau-\varsigma)}\int_{\lambda}^{\frac{(1-\theta)\varsigma\tau}{(1-\theta)\varsigma-\rho(\tau-\varsigma)}}\frac{\zeta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}+\frac{\theta\varsigma\tau}{\rho(\tau-\varsigma)}\int_{\varsigma}^{\lambda}\frac{\zeta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}$$
$$\leq\mathbb{X}_1(\iota)\leq\mathbb{X}_1(1)=(1-\theta)\zeta\left(\frac{(1-\theta)\varsigma\tau}{(\varsigma-\tau)\rho+(1-\theta)\tau}\right)+\theta\zeta(\varsigma)$$
$$\leq(1-\rho)\zeta(\varsigma)+\rho\zeta(\tau). \quad (2.11)$$

Remark 4. *Taking* $\rho=\theta=\frac{1}{2}$ *in the inequality (2.8) reduces to*

$$\mathbb{S}(\iota)=\frac{\varsigma\tau}{\tau-\varsigma}\int_{\varsigma}^{\tau}\zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{2\varsigma\tau\iota+(1-\iota)\mathfrak{w}(\varsigma+\tau)}\right)\frac{d\mathfrak{w}}{\mathfrak{w}^2}.$$

Remark 5. *Taking* $\rho=\theta=\frac{1}{2}$ *in the inequality (2.10) reduces to*

$$\mathbb{V}(\iota)=\frac{\varsigma\tau}{2(\tau-\varsigma)}\int_{\varsigma}^{\tau}\frac{1}{\mathfrak{w}^2}\left[\zeta\left(\frac{2\tau\mathfrak{w}}{(1+\iota)\mathfrak{w}+(1-\iota)\tau}\right)\right.$$
$$\left.+\zeta\left(\frac{2\varsigma\mathfrak{w}}{(1+\iota)\mathfrak{w}+(1-\iota)\varsigma}\right)\right]d\mathfrak{w}. \quad (2.12)$$

Theorem 15. *Let* ζ*,* ρ*,* θ*,* λ*,* τ_0 *be defined as in Theorem 13 and let* ϑ *be defined as in Theorem 12. Let* $\mathbb{Y}$ *be a function defined on* $[0,1]$ *by*

$$\mathbb{Y}(\iota)=\int_0^s\left[(1-\theta)\zeta\left(\frac{\lambda}{1-\theta\lambda\iota\mathfrak{w}}\right)\vartheta\left(\frac{\lambda}{1-\theta\lambda\mathfrak{w}}\right)\right.$$
$$\left.+\theta\zeta\left(\frac{\lambda}{1+(1-\theta)\lambda\iota\mathfrak{w}}\right)\vartheta\left(\frac{\lambda}{1+(1-\theta)\lambda\mathfrak{w}}\right)\right]d\mathfrak{w} \quad (2.13)$$

for some $s\in[0,\tau_0]$*. Then* $\mathbb{Y}$ *is harmonic convex and monotonically increasing on* $(0,1]$ *and*

$$\zeta\left(\frac{\varsigma\tau}{\rho\varsigma+(1-\rho)\tau}\right)\int_{\frac{\lambda}{1+(1-\theta)\lambda s}}^{\frac{\lambda}{1-\theta\lambda s}}\vartheta(\mathfrak{w})d\mathfrak{w}\leq\mathbb{Y}(\iota)\leq\mathbb{Y}(1)$$
$$=\frac{1-\theta}{\theta}\int_{\frac{\lambda}{1-\theta\lambda s}}^{\lambda}\frac{\zeta(\mathfrak{w})\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}+\frac{\theta}{1-\theta}\int_{\lambda}^{\frac{\lambda}{1+(1-\theta)\lambda s}}\frac{\zeta(\mathfrak{w})\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}. \quad (2.14)$$

Proof. Since ζ is harmonic convex and ϑ is nonnegative, we see that $\mathbb{Y}$ is harmonic convex on $(0,1]$. Let $\mathfrak{w}\in[0,s]$, where $s\in[0,\tau_0]$, from Theorem 12 we get $\breve{h}(\iota\mathfrak{w})=(1-\theta)\zeta\left(\frac{\lambda}{1-\theta\lambda\iota\mathfrak{w}}\right)+\theta\zeta\left(\frac{\lambda}{1+(1-\theta)\lambda\iota\mathfrak{w}}\right)$ is increasing for $\iota\in[0,1]$. Therefore the inequalities (2.14) are achieved immediately. $\square$

Theorem 16. *Let* ζ*,* ρ*,* θ*,* λ*,* τ_0 *be defined as in Theorem 15 and let* ϑ *be defined as in Theorem 12. Let* $\mathbb{Y}_1$ *be a function defined on* $[0,1]$ *by*

$$\mathbb{Y}_1(\iota)=\int_0^s\left[(1-\theta)\zeta\left(\frac{\lambda}{1-\theta s+\theta\mathfrak{w}(1-\iota)\lambda}\right)\vartheta\left(\frac{\lambda}{1-\theta(s-\mathfrak{w})\lambda}\right)\right.$$
$$\left.+\theta\zeta\left(\frac{\lambda}{1+(1-\theta)s-(1-\theta)\mathfrak{w}(1-\iota)\lambda}\right)\vartheta\left(\frac{\lambda}{1+(1-\theta)(s-\mathfrak{w})\lambda}\right)\right]d\mathfrak{w} \quad (2.15)$$

for some $s \in [0, \tau_0]$. *Then* $\mathbb{Y}_1$ *is harmonic convex* $(0, 1]$ *and monotonically increasing on* $[0, 1]$, *and*

$$\frac{1-\theta}{\theta}\int_{\frac{\lambda}{1-\theta\lambda s}}^{\lambda}\frac{\zeta(\mathfrak{w})\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}+\frac{\theta}{1-\theta}\int_{\lambda}^{\frac{\lambda}{1+(1-\theta)\lambda s}}\frac{\zeta(\mathfrak{w})\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}$$
$$\leq \mathbb{Y}(\iota)\leq \mathbb{Y}(1)=\left[(1-\theta)\zeta\left(\frac{\lambda}{1-\theta s\lambda}\right)+(1-\theta)\zeta\left(\frac{\lambda}{1+(1-\theta)s\lambda}\right)\right]$$
$$\times\int_{\frac{\lambda}{1-\theta\lambda s}}^{\frac{\lambda}{1+(1-\theta)\lambda s}}\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}\leq\left[(1-\rho)\zeta(\varsigma)+\rho\zeta(\tau)\right]\int_{\frac{\lambda}{1-\theta\lambda s}}^{\frac{\lambda}{1+(1-\theta)\lambda s}}\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}. \quad (2.16)$$

Proof. Since ζ is harmonic convex and ϑ is nonnegative, we see that $\mathbb{Y}_1$ is harmonic convex on $(0, 1]$. Next, for each $\mathfrak{w} \in [0, \iota]$, where $\iota \in [0, \tau_0]$, it follows from Theorem 12 that $\breve{h}(\iota) = (1-\theta)\zeta\left(\frac{\lambda}{1-\theta\lambda\iota}\right) + \theta\zeta\left(\frac{\lambda}{1+(1-\theta)\lambda\iota}\right)$ and $k(\iota) = s - (1-\iota)\mathfrak{w}$ are increasing on $[0, \tau_0]$ and $[0, 1]$ respectively. Hence

$$\breve{h}(k(\iota)) = (1-\theta)\zeta\left(\frac{\lambda}{1-\theta s+\theta\mathfrak{w}(1-\iota)\lambda}\right)\vartheta\left(\frac{\lambda}{1-\theta(s-\mathfrak{w})\lambda}\right)$$
$$+\theta\zeta\left(\frac{\lambda}{1+(1-\theta)s-(1-\theta)\mathfrak{w}(1-\iota)\lambda}\right)\vartheta\left(\frac{\lambda}{1+(1-\theta)(s-\mathfrak{w})\lambda}\right)$$

is increasing on $[0, 1]$. Using the identity $\vartheta\left(\frac{\lambda}{1-\theta\lambda\iota}\right) = \vartheta\left(\frac{\lambda}{1+(1-\theta)\lambda\iota}\right)$ we see that $\mathbb{Y}(\iota)$ is increasing on $[0, 1]$. Therefore the inequalities (2.16) follows from

$$\zeta\left(\frac{\varsigma\tau}{\rho\varsigma+(1-\rho)\tau}\right)\leq\mathcal{W}(k(\iota))\leq(1-\rho)\zeta(\varsigma)+\rho\zeta(\tau)$$

and (2.16). □

Remark 6. *Choose* $\rho = \theta = \frac{1}{2}$, $s = \tau_0 = \frac{\varsigma\tau}{\tau-\varsigma}$ *in Theorems 15 and 16. Then the inequalities (2.14) and (2.16) reduce to*

$$\zeta\left(\frac{2\varsigma\tau}{\varsigma+\tau}\right)\int_{\varsigma}^{\tau}\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}\leq\mathbb{Y}(\iota)\leq\mathbb{Y}(1)=\int_{\varsigma}^{\tau}\frac{\zeta(\mathfrak{w})\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}$$
$$\leq\mathbb{Y}_1(\iota)\leq\mathbb{Y}_1(1)=\frac{\zeta(\varsigma)+\zeta(\tau)}{2}\int_{\varsigma}^{\tau}\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}, \quad (2.17)$$

where

$$\mathbb{Y}(\iota)=\frac{\varsigma\tau}{\tau-\varsigma}\int_{\varsigma}^{\tau}\zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{2\varsigma\tau\iota+(1-\iota)\mathfrak{w}(\varsigma+\tau)}\right)\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}$$

and

$$\mathbb{Y}_1(\iota)=\frac{1}{2}\int_{\varsigma}^{\tau}\frac{1}{\mathfrak{w}^2}\left[\zeta\left(\frac{2\tau\mathfrak{w}}{(1+\iota)\mathfrak{w}+(1-\iota)\tau}\right)\vartheta\left(\frac{2\varsigma\mathfrak{w}}{\varsigma+\mathfrak{w}}\right)\right.$$
$$\left.+\zeta\left(\frac{2\varsigma\mathfrak{w}}{(1+\iota)\mathfrak{w}+(1-\iota)\varsigma}\right)\vartheta\left(\frac{2\mathfrak{w}\tau}{\mathfrak{w}+\tau}\right)\right]d\mathfrak{w}. \quad (2.18)$$

Remark 7. *The inequalities (2.17) provide weighted generalizations of Theorems 9 and 15.*

In the coming results we provide weighted generalizations of Theorems 2–4 for harmonic convex functions by using Lemma 2.

Theorem 17. *Let ζ, ϑ ,$\mathbb{S}_\vartheta$ be defined as above. Then*

(i) The inequality

$$\zeta\left(\frac{2\varsigma\tau}{\varsigma+\tau}\right)\int_\varsigma^\tau \frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w} \le 2\int_{\frac{4\varsigma\tau}{\varsigma+3\tau}}^{\frac{4\varsigma\tau}{3\varsigma+\tau}} \zeta(\mathfrak{w})\,\vartheta\left(\frac{2\varsigma\tau\mathfrak{w}}{4\varsigma\tau-(\varsigma+\tau)\mathfrak{w}}\right)\frac{d\mathfrak{w}}{\mathfrak{w}^2}$$
$$\le \int_0^1 \mathbb{S}_\vartheta(\iota)\,d\iota \le \frac{1}{2}\left[\zeta\left(\frac{2\varsigma\tau}{\varsigma+\tau}\right)\int_\varsigma^\tau \frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w} + \int_\varsigma^\tau \frac{\zeta(\mathfrak{w})\,\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}\right] \quad (2.19)$$

holds.

(ii) If ζ is differentiable on $[\varsigma,\tau]$ and ϑ is bounded on $[\varsigma,\tau]$, then the inequalities

$$0 \le \int_\varsigma^\tau \frac{\zeta(\mathfrak{w})\,\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w} - \mathbb{S}_\vartheta(\iota)$$
$$\le (1-\iota)\left[\left(\frac{\tau-\varsigma}{\varsigma\tau}\right)\left[\frac{\zeta(\varsigma)+\zeta(\tau)}{2}\right] - \int_\varsigma^\tau \frac{\zeta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}\right]\|\vartheta\|_\infty, \quad (2.20)$$

hold for all $\iota \in [0,1]$, where $\|\vartheta\|_\infty = \sup_{\mathfrak{w}\in[\varsigma,\tau]} \vartheta(\mathfrak{w})$.

(iii) If ζ is differentiable on $[\varsigma,\tau]$, then, for all $\iota \in [0,1]$, then

$$0 \le \frac{\zeta(\varsigma)+\zeta(\tau)}{2}\int_\varsigma^\tau \frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w} - \mathbb{S}_\vartheta(\iota)$$
$$\le \frac{(\tau-\varsigma)\left(\tau^2\zeta'(\tau)-\varsigma^2\zeta'(\varsigma)\right)}{4\varsigma\tau}\int_\varsigma^\tau \frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}. \quad (2.21)$$

Proof. (i) Using techniques of integration and the hypothesis of ϑ, we have the following identities:

$$\zeta\left(\frac{2\varsigma\tau}{\varsigma+\tau}\right)\int_\varsigma^\tau \frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w} = 4\int_\varsigma^{\frac{2\varsigma\tau}{\varsigma+\tau}}\int_0^{\frac{1}{2}} \zeta\left(\frac{2\varsigma\tau}{\varsigma+\tau}\right)\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\iota d\mathfrak{w}, \quad (2.22)$$

$$2\int_{\frac{4\varsigma\tau}{\varsigma+3\tau}}^{\frac{4\varsigma\tau}{3\varsigma+\tau}} \frac{\zeta(\mathfrak{w})}{\mathfrak{w}^2}\vartheta\left(\frac{2\varsigma\tau\mathfrak{w}}{4\varsigma\tau-(\varsigma+\tau)\mathfrak{w}}\right)d\mathfrak{w}$$
$$= 2\int_\varsigma^{\frac{2\varsigma\tau}{\varsigma+\tau}}\int_0^{\frac{1}{2}}\left[\zeta\left(\frac{4\varsigma\tau\mathfrak{w}}{2\varsigma\tau+(\varsigma+\tau)\mathfrak{w}}\right) + \zeta\left(\frac{4\varsigma\tau\mathfrak{w}}{3(\varsigma+\tau)\mathfrak{w}-2\varsigma\tau}\right)\right]\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\iota d\mathfrak{w}, \quad (2.23)$$

$$\int_0^1 \mathbb{S}_\vartheta(\iota)\,d\iota$$
$$= \int_\varsigma^{\frac{2\varsigma\tau}{\varsigma+\tau}}\int_0^{\frac{1}{2}}\left[\zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{\iota(\varsigma+\tau)\mathfrak{w}+2(1-\iota)\varsigma\tau}\right) + \zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{2\varsigma\tau\iota+(1-\iota)(\varsigma+\tau)\mathfrak{w}}\right)\right]$$

$$\times \frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2} d\iota d\mathfrak{w} + \int_{\varsigma}^{\frac{2\varsigma\tau}{\varsigma+\tau}} \int_0^{\frac{1}{2}} \left[\zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{2\iota(\varsigma\mathfrak{w}+\tau\mathfrak{w}-\varsigma\tau)+(1-\iota)(\varsigma+\tau)\mathfrak{w}} \right) \right.$$
$$\left. +\zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{2(1-\iota)(\varsigma\mathfrak{w}+\tau\mathfrak{w}-\varsigma\tau)+\iota(\varsigma+\tau)\mathfrak{w}} \right) \right] \frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2} d\iota d\mathfrak{w} \quad (2.24)$$

and

$$\frac{1}{2}\left[\zeta\left(\frac{2\varsigma\tau}{\varsigma+\tau}\right) \int_{\varsigma}^{\tau} \frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2} d\mathfrak{w} + \int_{\varsigma}^{\tau} \frac{\zeta(\mathfrak{w})\vartheta(\mathfrak{w})}{\mathfrak{w}^2} d\mathfrak{w} \right]$$
$$= \int_{\varsigma}^{\frac{2\varsigma\tau}{\varsigma+\tau}} \int_0^{\frac{1}{2}} \left[\zeta(\mathfrak{w}) + \zeta\left(\frac{2\varsigma\tau}{\varsigma+\tau}\right) \right] \frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2} d\iota d\mathfrak{w}$$
$$+ \int_{\varsigma}^{\frac{2\varsigma\tau}{\varsigma+\tau}} \int_0^{\frac{1}{2}} \left[\zeta\left(\frac{2\varsigma\tau}{\varsigma+\tau}\right) + \zeta\left(\frac{\varsigma\tau\mathfrak{w}}{\varsigma+\tau-\mathfrak{w}}\right) \right] \frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2} d\iota d\mathfrak{w}. \quad (2.25)$$

By using Lemma 2, we observe that the following inequalities hold for all $\iota \in \left[0, \frac{1}{2}\right]$ and $\mathfrak{w} \in \left[\varsigma, \frac{2\varsigma\tau}{\varsigma+\tau}\right]$:

$$4\zeta\left(\frac{2\varsigma\tau}{\varsigma+\tau}\right) \leq 2\left[\zeta\left(\frac{4\varsigma\tau\mathfrak{w}}{2\varsigma\tau+(\varsigma+\tau)\mathfrak{w}}\right) + \zeta\left(\frac{4\varsigma\tau\mathfrak{w}}{3(\varsigma+\tau)\mathfrak{w}-2\varsigma\tau}\right) \right], \quad (2.26)$$

$$2\zeta\left(\frac{4\varsigma\tau\mathfrak{w}}{2\varsigma\tau+(\varsigma+\tau)\mathfrak{w}}\right)$$
$$\leq \zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{\iota(\varsigma+\tau)\mathfrak{w}+2(1-\iota)\varsigma\tau}\right) + \zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{2\varsigma\tau\iota+(1-\iota)(\varsigma+\tau)\mathfrak{w}}\right), \quad (2.27)$$

$$2\zeta\left(\frac{4\varsigma\tau\mathfrak{w}}{3(\varsigma+\tau)\mathfrak{w}-2\varsigma\tau}\right) \leq \zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{2\iota(\varsigma\mathfrak{w}+\tau\mathfrak{w}-\varsigma\tau)+(1-\iota)(\varsigma+\tau)\mathfrak{w}}\right)$$
$$+ \zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{2(1-\iota)(\varsigma\mathfrak{w}+\tau\mathfrak{w}-\varsigma\tau)+\iota(\varsigma+\tau)\mathfrak{w}}\right), \quad (2.28)$$

$$\zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{\iota(\varsigma+\tau)\mathfrak{w}+2(1-\iota)\varsigma\tau}\right) + \zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{2\varsigma\tau\iota+(1-\iota)(\varsigma+\tau)\mathfrak{w}}\right)$$
$$\leq \zeta(\mathfrak{w}) + \zeta\left(\frac{2\varsigma\tau}{\varsigma+\tau}\right) \quad (2.29)$$

and

$$\zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{2\iota(\varsigma\mathfrak{w}+\tau\mathfrak{w}-\varsigma\tau)+(1-\iota)(\varsigma+\tau)\mathfrak{w}}\right)$$
$$+ \zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{2(1-\iota)(\varsigma\mathfrak{w}+\tau\mathfrak{w}-\varsigma\tau)+\iota(\varsigma+\tau)\mathfrak{w}}\right)$$
$$\leq \zeta\left(\frac{2\varsigma\tau}{\varsigma+\tau}\right) + \zeta\left(\frac{\varsigma\tau\mathfrak{w}}{\varsigma+\tau-\mathfrak{w}}\right). \quad (2.30)$$

Multiplying the inequalities (2.26)–(2.30) by $\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}$ and integrating them over ι on $\left[0, \frac{1}{2}\right]$, over $\mathfrak{w}$ on $\left[\varsigma, \frac{2\varsigma\tau}{\varsigma+\tau}\right]$ and using identities (2.22)–(2.25), we derive (2.19).
(ii) Since $\zeta : [\varsigma, \tau] \to \mathbb{R}$ is harmonic convex on $[\varsigma, \tau]$, hence $\breve{h} : \left[\frac{1}{\tau}, \frac{1}{\varsigma}\right] \to \mathbb{R}$ defined by $\breve{h}(\mathfrak{w}) = \zeta\left(\frac{1}{\mathfrak{w}}\right)$ is convex on $\left[\frac{1}{\tau}, \frac{1}{\varsigma}\right]$. Thus, by integration by parts, we get that following identity holds:

$$\begin{aligned}\int_{\frac{1}{\tau}}^{\frac{\varsigma+\tau}{2\varsigma\tau}} \left(\frac{\varsigma+\tau}{2\varsigma\tau} - \mathfrak{w}\right)\left[\breve{h}'\left(\frac{1}{\varsigma} + \frac{1}{\tau} - \mathfrak{w}\right) - \breve{h}'(\mathfrak{w})\right] d\mathfrak{w} \\ = \left(\frac{\tau-\varsigma}{2\varsigma\tau}\right)\left[\breve{h}\left(\frac{1}{\varsigma}\right) + \breve{h}\left(\frac{1}{\tau}\right)\right] - \int_{\frac{1}{\tau}}^{\frac{\varsigma+\tau}{2\varsigma\tau}} \left[\breve{h}\left(\frac{1}{\varsigma} + \frac{1}{\tau} - \mathfrak{w}\right) + \breve{h}(\mathfrak{w})\right] d\mathfrak{w}. \quad (2.31)\end{aligned}$$

The equality (2.31) is equivalent to the equality:

$$\begin{aligned}\int_{\varsigma}^{\frac{2\varsigma\tau}{\varsigma+\tau}} \frac{1}{\mathfrak{w}^2}\left(\frac{1}{\mathfrak{w}} - \frac{\varsigma+\tau}{2\varsigma\tau}\right)\left[\frac{\zeta'\left(\frac{1}{\frac{1}{\varsigma}+\frac{1}{\tau}-\frac{1}{\mathfrak{w}}}\right)}{\left(\frac{1}{\varsigma} + \frac{1}{\tau} - \frac{1}{\mathfrak{w}}\right)^2} - \mathfrak{w}^2\zeta'(\mathfrak{w})\right] d\mathfrak{w} \\ = \left(\frac{\tau-\varsigma}{\varsigma\tau}\right)\left[\frac{\zeta(\varsigma) + \zeta(\tau)}{2}\right] - \int_{\varsigma}^{\tau} \frac{\zeta(\mathfrak{w})}{\mathfrak{w}^2} d\mathfrak{w}. \quad (2.32)\end{aligned}$$

Using substitution rules for integration and the hypothesis of ϑ, we have the following identities:

$$\int_{\varsigma}^{\tau} \frac{\zeta(\mathfrak{w})\vartheta(\mathfrak{w})}{\mathfrak{w}^2} d\mathfrak{w} = \int_{\varsigma}^{\frac{2\varsigma\tau}{\varsigma+\tau}} \left[\zeta(\mathfrak{w}) + \zeta\left(\frac{1}{\frac{1}{\varsigma} + \frac{1}{\tau} - \frac{1}{\mathfrak{w}}}\right)\right] \frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2} d\mathfrak{w} \quad (2.33)$$

and

$$\begin{aligned}\mathbb{S}_{\vartheta}(\iota) = \int_{\varsigma}^{\frac{2\varsigma\tau}{\varsigma+\tau}} \Bigg[\zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{2\varsigma\tau\iota + (1-\iota)(\varsigma+\tau)\mathfrak{w}}\right) \\ +\zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{2\iota(\varsigma\mathfrak{w} + \tau\mathfrak{w} - \varsigma\tau) + (1-\iota)(\varsigma+\tau)\mathfrak{w}}\right)\Bigg] \frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2} d\mathfrak{w}. \quad (2.34)\end{aligned}$$

Now, using the convexity of $\breve{h}(\mathfrak{w}) = \zeta\left(\frac{1}{\mathfrak{w}}\right)$ on $\left[\frac{1}{\tau}, \frac{1}{\varsigma}\right]$ and the hypothesis of ϑ, the following inequality holds for all $\iota \in [0, 1]$ and $\mathfrak{w} \in \left[\frac{1}{\tau}, \frac{\varsigma+\tau}{2\varsigma\tau}\right]$:

$$\begin{aligned}&\left[\breve{h}(\mathfrak{w}) - \breve{h}\left(\iota\mathfrak{w} + (1-\iota)\left(\frac{\varsigma+\tau}{2\varsigma\tau}\right)\right)\right]\vartheta\left(\frac{1}{\mathfrak{w}}\right) \\ &\quad + \left[\breve{h}\left(\frac{1}{\varsigma} + \frac{1}{\tau} - \mathfrak{w}\right) - \breve{h}\left(\iota\left(\frac{1}{\varsigma} + \frac{1}{\tau} - \mathfrak{w}\right) + (1-\iota)\left(\frac{\varsigma+\tau}{2\varsigma\tau}\right)\right)\right]\vartheta\left(\frac{1}{\mathfrak{w}}\right) \\ &\leq (1-\iota)\left(\mathfrak{w} - \frac{\varsigma+\tau}{2\varsigma\tau}\right)\breve{h}'(\mathfrak{w})\vartheta\left(\frac{1}{\mathfrak{w}}\right) \\ &\quad + (1-\iota)\left(\frac{\varsigma+\tau}{2\varsigma\tau} - \mathfrak{w}\right)\breve{h}'\left(\frac{1}{\varsigma} + \frac{1}{\tau} - \mathfrak{w}\right)\vartheta\left(\frac{1}{\mathfrak{w}}\right) \\ &= (1-\iota)\left(\frac{\varsigma+\tau}{2\varsigma\tau} - \mathfrak{w}\right)\left[\breve{h}'\left(\frac{1}{\varsigma} + \frac{1}{\tau} - \mathfrak{w}\right) - \breve{h}'(\mathfrak{w})\right]\vartheta\left(\frac{1}{\mathfrak{w}}\right) \quad (2.35)\end{aligned}$$

which is equivalent to

$$\left[\zeta\left(\frac{1}{\mathfrak{w}}\right)-\zeta\left(\frac{1}{\iota\mathfrak{w}+(1-\iota)\left(\frac{\varsigma+\tau}{2\varsigma\tau}\right)}\right)\right]\vartheta\left(\frac{1}{\mathfrak{w}}\right)$$
$$+\left[\zeta\left(\frac{1}{\frac{1}{\varsigma}+\frac{1}{\tau}-\mathfrak{w}}\right)-\zeta\left(\frac{1}{\iota\left(\frac{1}{\varsigma}+\frac{1}{\tau}-\mathfrak{w}\right)+(1-\iota)\left(\frac{\varsigma+\tau}{2\varsigma\tau}\right)}\right)\right]\vartheta\left(\frac{1}{\mathfrak{w}}\right)$$
$$\leq(1-\iota)\left(\frac{\varsigma+\tau}{2\varsigma\tau}-\mathfrak{w}\right)\left[\frac{\zeta'\left(\frac{1}{\mathfrak{w}}\right)}{\mathfrak{w}^2}-\frac{\zeta'\left(\frac{1}{\frac{1}{\varsigma}+\frac{1}{\tau}-\mathfrak{w}}\right)}{\left(\frac{1}{\varsigma}+\frac{1}{\tau}-\mathfrak{w}\right)^2}\right]\vartheta\left(\frac{1}{\mathfrak{w}}\right). \quad (2.36)$$

Integrating the above inequalities over $\mathfrak{w}$ on $\left[\frac{1}{\tau},\frac{\varsigma+\tau}{2\varsigma\tau}\right]$, we get

$$\int_{\frac{1}{\tau}}^{\frac{\varsigma+\tau}{2\varsigma\tau}}\left[\zeta\left(\frac{1}{\mathfrak{w}}\right)-\zeta\left(\frac{1}{\iota\mathfrak{w}+(1-\iota)\left(\frac{\varsigma+\tau}{2\varsigma\tau}\right)}\right)\right]\vartheta\left(\frac{1}{\mathfrak{w}}\right)d\mathfrak{w}$$
$$+\int_{\frac{1}{\tau}}^{\frac{\varsigma+\tau}{2\varsigma\tau}}\left[\zeta\left(\frac{1}{\frac{1}{\varsigma}+\frac{1}{\tau}-\mathfrak{w}}\right)-\zeta\left(\frac{1}{\iota\left(\frac{1}{\varsigma}+\frac{1}{\tau}-\mathfrak{w}\right)+(1-\iota)\left(\frac{\varsigma+\tau}{2\varsigma\tau}\right)}\right)\right]\vartheta\left(\frac{1}{\mathfrak{w}}\right)d\mathfrak{w}$$
$$\leq(1-\iota)\int_{\frac{1}{\tau}}^{\frac{\varsigma+\tau}{2\varsigma\tau}}\left(\frac{\varsigma+\tau}{2\varsigma\tau}-\mathfrak{w}\right)\left[\frac{\zeta'\left(\frac{1}{\mathfrak{w}}\right)}{\mathfrak{w}^2}-\frac{\zeta'\left(\frac{1}{\frac{1}{\varsigma}+\frac{1}{\tau}-\mathfrak{w}}\right)}{\left(\frac{1}{\varsigma}+\frac{1}{\tau}-\mathfrak{w}\right)^2}\right]\vartheta\left(\frac{1}{\mathfrak{w}}\right)d\mathfrak{w}. \quad (2.37)$$

After making use of suitable substitution, the inequality (2.37) takes the form:

$$\int_{\varsigma}^{\frac{2\varsigma\tau}{\varsigma+\tau}}\frac{1}{\mathfrak{w}^2}\left[\zeta\left(\frac{1}{\iota\frac{1}{\mathfrak{w}}+(1-\iota)\left(\frac{\varsigma+\tau}{2\varsigma\tau}\right)}\right)-\zeta(\mathfrak{w})\right]\vartheta(\mathfrak{w})\,d\mathfrak{w}$$
$$+\int_{\varsigma}^{\frac{2\varsigma\tau}{\varsigma+\tau}}\frac{1}{\mathfrak{w}^2}\left[\zeta\left(\frac{1}{\iota\left(\frac{1}{\varsigma}+\frac{1}{\tau}-\frac{1}{\mathfrak{w}}\right)+(1-\iota)\left(\frac{\varsigma+\tau}{2\varsigma\tau}\right)}\right)-\zeta\left(\frac{1}{\frac{1}{\varsigma}+\frac{1}{\tau}-\frac{1}{\mathfrak{w}}}\right)\right]\vartheta(\mathfrak{w})\,d\mathfrak{w}$$
$$\leq\|\vartheta\|_\infty(1-\iota)\int_{\varsigma}^{\frac{2\varsigma\tau}{\varsigma+\tau}}\frac{1}{\mathfrak{w}^2}\left(\frac{1}{\mathfrak{w}}-\frac{\varsigma+\tau}{2\varsigma\tau}\right)\left[\frac{\zeta'\left(\frac{1}{\frac{1}{\varsigma}+\frac{1}{\tau}-\frac{1}{\mathfrak{w}}}\right)}{\left(\frac{1}{\varsigma}+\frac{1}{\tau}-\frac{1}{\mathfrak{w}}\right)^2}-\mathfrak{w}^2\zeta'(\mathfrak{w})\right]d\mathfrak{w}. \quad (2.38)$$

Inequality (2.20) follows from (2.31)–(2.34) and (2.38).
(iii) We use the fact that $\zeta:[\varsigma,\tau]\to\mathbb{R}$ is harmonic convex on $[\varsigma,\tau]$, hence $\breve{h}:\left[\frac{1}{\tau},\frac{1}{\varsigma}\right]\to\mathbb{R}$ defined by $\breve{h}(\mathfrak{w})=\zeta\left(\frac{1}{\mathfrak{w}}\right)$ is convex on $\left[\frac{1}{\tau},\frac{1}{\varsigma}\right]$. Thus

$$\frac{\breve{h}\left(\frac{1}{\tau}\right)-\breve{h}\left(\frac{\varsigma+\tau}{2\varsigma\tau}\right)}{2}\leq\frac{\varsigma-\tau}{4\varsigma\tau}\breve{h}'\left(\frac{1}{\tau}\right)$$

and

$$\frac{\breve{h}\left(\frac{1}{\varsigma}\right)-\breve{h}\left(\frac{\varsigma+\tau}{2\varsigma\tau}\right)}{2}\leq\frac{\tau-\varsigma}{4\varsigma\tau}\breve{h}'\left(\frac{1}{\varsigma}\right).$$

Adding the above inequalities

$$\frac{\breve{h}\left(\frac{1}{\varsigma}\right)+\breve{h}\left(\frac{1}{\tau}\right)}{2}-\breve{h}\left(\frac{\varsigma+\tau}{2\varsigma\tau}\right)\leq\frac{(\tau-\varsigma)\left(\breve{h}'\left(\frac{1}{\varsigma}\right)-\breve{h}'\left(\frac{1}{\tau}\right)\right)}{4\varsigma\tau}. \tag{2.39}$$

The inequality (2.39) becomes

$$\frac{\zeta(\varsigma)+\zeta(\tau)}{2}-\zeta\left(\frac{2\varsigma\tau}{\varsigma+\tau}\right)\leq\frac{(\tau-\varsigma)\left(\tau^2\zeta'(\tau)-\varsigma^2\zeta'(\varsigma)\right)}{4\varsigma\tau}. \tag{2.40}$$

Multiplying (2.40) both sides by $\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}$ and integrating over $[\varsigma,\tau]$, we get

$$\begin{aligned}&\frac{\zeta(\varsigma)+\zeta(\tau)}{2}\int_{\varsigma}^{\tau}\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}-\zeta\left(\frac{2\varsigma\tau}{\varsigma+\tau}\right)\int_{\varsigma}^{\tau}\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}\\&\qquad\leq\frac{(\tau-\varsigma)\left(\tau^2\zeta'(\tau)-\varsigma^2\zeta'(\varsigma)\right)}{4\varsigma\tau}\int_{\varsigma}^{\tau}\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}.\end{aligned} \tag{2.41}$$

From (2.17) and (2.41) we get (2.21). □

Corollary 1. *Suppose that the assumption of Theorem 17 are satisfied and* $\vartheta(\mathfrak{w})=\frac{\varsigma\tau}{\tau-\varsigma}$, $\mathfrak{w}\in[\varsigma,\tau]$, *then*

(i) The inequalities

$$\begin{aligned}\zeta\left(\frac{2\varsigma\tau}{\varsigma+\tau}\right)\leq 2\int_{\frac{4\varsigma\tau}{\varsigma+3\tau}}^{\frac{4\varsigma\tau}{3\varsigma+\tau}}\frac{\zeta(\mathfrak{w})}{\mathfrak{w}^2}\leq\int_0^1\mathbb{S}(\iota)\,d\iota\\\leq\frac{1}{2}\left[\zeta\left(\frac{2\varsigma\tau}{\varsigma+\tau}\right)+\frac{\varsigma\tau}{\tau-\varsigma}\int_{\varsigma}^{\tau}\frac{\zeta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}\right]\end{aligned} \tag{2.42}$$

holds.

(ii) The inequalities

$$\begin{aligned}0\leq\frac{\varsigma\tau}{\tau-\varsigma}\int_{\varsigma}^{\tau}\frac{\zeta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}-\mathbb{S}(\iota)\\\leq(1-\iota)\left[\frac{\zeta(\varsigma)+\zeta(\tau)}{2}-\frac{\varsigma\tau}{\tau-\varsigma}\int_{\varsigma}^{\tau}\frac{\zeta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}\right]\end{aligned} \tag{2.43}$$

hold for all $\iota\in[0,1]$.

(iii) The inequalities

$$0\leq\frac{\zeta(\varsigma)+\zeta(\tau)}{2}-\mathbb{S}(\iota)\leq\frac{(\tau-\varsigma)\left(\tau^2\zeta'(\tau)-\varsigma^2\zeta'(\varsigma)\right)}{4\varsigma\tau} \tag{2.44}$$

are valid for all $\iota\in[0,1]$.

In the following theorems, we discuss inequalities for the functions $\mathbb{S}$, $\mathbb{S}_{\vartheta}$, $\breve{\mathcal{G}}_1$, $\mathbb{T}$ and $\mathbb{T}_{\vartheta}$ as considered above:

Theorem 18. *Let ζ, ϑ, $\breve{\mathcal{G}}_1$,$\mathbb{S}_\vartheta$ be defined as above. Then*

(i) The inequality

$$\mathbb{S}_\vartheta(\iota) \le \breve{\mathcal{G}}_1(\iota)\int_\varsigma^\tau \frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w} \tag{2.45}$$

holds for all $\iota \in [0,1]$.

(ii) The inequalities

$$\begin{aligned}
2\int_{\frac{4\varsigma\tau}{\varsigma+3\tau}}^{\frac{4\varsigma\tau}{3\varsigma+\tau}} \zeta(\mathfrak{w})\,\vartheta\left(\frac{2\varsigma\tau\mathfrak{w}}{4\varsigma\tau-(\varsigma+\tau)\mathfrak{w}}\right)\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w} \\
&\le \frac{1}{2}\left[\zeta\left(\frac{4\varsigma\tau}{3\varsigma+\tau}\right)+\zeta\left(\frac{4\varsigma\tau}{\varsigma+3\tau}\right)\right]\int_\varsigma^\tau \frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w} \\
&\le \frac{\tau-\varsigma}{\varsigma\tau}\int_0^1 \lambda_1(\iota)\,\vartheta\left(\frac{\varsigma\tau}{(1-\iota)\varsigma+\iota\tau}\right)d\iota \\
&\le \frac{1}{2}\left[\zeta\left(\frac{2\varsigma\tau}{\varsigma+\tau}\right)+\frac{\zeta(\varsigma)+\zeta(\tau)}{2}\right]\int_\varsigma^\tau \frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}
\end{aligned} \tag{2.46}$$

hold.

(iii) If ζ is differentiable on $[\varsigma,\tau]$ and ϑ is bounded on $[\varsigma,\tau]$, then, for all $\iota \in [0,1]$, then

$$0 \le \mathbb{S}_\vartheta(\iota) - \zeta\left(\frac{2\varsigma\tau}{\varsigma+\tau}\right)\int_\varsigma^\tau \frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w} \le \left(\frac{\tau-\varsigma}{\varsigma\tau}\right)\left[\breve{\mathcal{G}}_1(\iota)-\mathbb{S}(\iota)\right]\|\vartheta\|_\infty, \tag{2.47}$$

where $\|\vartheta\|_\infty = \sup_{\mathfrak{w}\in[\varsigma,\tau]} \vartheta(\mathfrak{w})$.

Proof. (i) Using integration by substitution and the assumptions on ϑ, we have that the following identity holds on $[0,1]$:

$$\breve{\mathcal{G}}_1(\iota)\int_\varsigma^\tau \frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w} = \int_\varsigma^{\frac{2\varsigma\tau}{\varsigma+\tau}}\left[\zeta\left(\frac{2\varsigma\tau}{2\varsigma\iota+(1-\iota)(\varsigma+\tau)}\right) + \zeta\left(\frac{2\varsigma\tau}{2\tau\iota+(1-\iota)(\varsigma+\tau)}\right)\right]\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}. \tag{2.48}$$

By Lemma 2, the following inequality holds for all $\mathfrak{w} \in \left[\varsigma, \frac{2\varsigma\tau}{\varsigma+\tau}\right]$ with

$$\mathfrak{w}_1 = \frac{2\varsigma\tau\mathfrak{w}}{2\varsigma\tau\iota+(1-\iota)(\varsigma+\tau)\mathfrak{w}},\ \mathfrak{w}_2 = \frac{2\varsigma\tau\mathfrak{w}}{2\iota(\varsigma\mathfrak{w}+\tau\mathfrak{w}-\varsigma\tau)+(1-\iota)(\varsigma+\tau)\mathfrak{w}},$$
$$\breve{\varkappa}_1 = \frac{2\varsigma\tau}{2\tau\iota+(1-\iota)(\varsigma+\tau)}\text{and } \breve{\varkappa}_2 = \frac{2\varsigma\tau}{2\varsigma\iota+(1-\iota)(\varsigma+\tau)}:$$

$$\zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{2\varsigma\tau\iota+(1-\iota)(\varsigma+\tau)\mathfrak{w}}\right)+\zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{2\iota(\varsigma\mathfrak{w}+\tau\mathfrak{w}-\varsigma\tau)+(1-\iota)(\varsigma+\tau)\mathfrak{w}}\right)$$

$$\leq \zeta\left(\frac{2\varsigma\tau}{2\tau\iota+(1-\iota)(\varsigma+\tau)}\right)+\zeta\left(\frac{2\varsigma\tau}{2\varsigma\iota+(1-\iota)(\varsigma+\tau)}\right). \quad (2.49)$$

Multiplying both sides of (2.49) with $\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}$, integrating over $\left[\varsigma,\frac{2\varsigma\tau}{\varsigma+\tau}\right]$ and using (2.34) and (2.49), we obtain (2.45).

(ii) We can observe that

$$\frac{1}{2}\left[\zeta\left(\frac{4\varsigma\tau}{3\varsigma+\tau}\right)+\zeta\left(\frac{4\varsigma\tau}{\varsigma+3\tau}\right)\right]\int_{\varsigma}^{\tau}\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}$$
$$=\left[\zeta\left(\frac{4\varsigma\tau}{3\varsigma+\tau}\right)+\zeta\left(\frac{4\varsigma\tau}{\varsigma+3\tau}\right)\right]\int_{\varsigma}^{\frac{2\varsigma\tau}{\varsigma+\tau}}\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}. \quad (2.50)$$

By using harmonic symmetric assumption on ϑ, we get

$$2\int_{\frac{4\varsigma\tau}{\varsigma+3\tau}}^{\frac{4\varsigma\tau}{3\varsigma+\tau}}\zeta(\mathfrak{w})\,\vartheta\left(\frac{2\varsigma\tau\mathfrak{w}}{4\varsigma\tau-(\varsigma+\tau)\mathfrak{w}}\right)\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}$$
$$=\int_{\varsigma}^{\frac{2\varsigma\tau}{\varsigma+\tau}}\left[\zeta\left(\frac{4\varsigma\tau\mathfrak{w}}{2\varsigma\tau+(\varsigma+\tau)\mathfrak{w}}\right)+\zeta\left(\frac{4\varsigma\tau\mathfrak{w}}{3(\varsigma+\tau)\mathfrak{w}-2\varsigma\tau}\right)\right]\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}. \quad (2.51)$$

We can also see that the following identity holds:

$$\frac{\tau-\varsigma}{\varsigma\tau}\int_0^1\breve{\mathcal{G}}_1(\iota)\,\vartheta\left(\frac{\varsigma\tau}{(1-\iota)\varsigma+\iota\tau}\right)d\iota=\frac{\tau-\varsigma}{\varsigma\tau}$$
$$\times\left[\int_{\frac{1}{2}}^{1}\zeta\left(\frac{1}{2\varsigma\iota+(1-\iota)(\varsigma+\tau)}\right)\vartheta\left(\frac{\varsigma\tau}{(1-\iota)\varsigma+\iota\tau}\right)d\iota\right.$$
$$+\int_0^{\frac{1}{2}}\zeta\left(\frac{1}{2\varsigma\iota+(1-\iota)(\varsigma+\tau)}\right)\vartheta\left(\frac{\varsigma\tau}{(1-\iota)\tau+\iota\varsigma}\right)d\iota$$
$$+\int_0^{\frac{1}{2}}\zeta\left(\frac{1}{2\tau\iota+(1-\iota)(\varsigma+\tau)}\right)\vartheta\left(\frac{\varsigma\tau}{(1-\iota)\tau+\iota\varsigma}\right)d\iota$$
$$\left.+\int_{\frac{1}{2}}^{1}\zeta\left(\frac{1}{2\tau\iota+(1-\iota)(\varsigma+\tau)}\right)\vartheta\left(\frac{\varsigma\tau}{(1-\iota)\varsigma+\iota\tau}\right)d\iota\right]=\frac{1}{2}\int_{\varsigma}^{\frac{2\varsigma\tau}{\varsigma+\tau}}\left[\zeta\left(\frac{2\mathfrak{w}\varsigma}{\varsigma+\mathfrak{w}}\right)\right.$$
$$\left.+\zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{2\varsigma\mathfrak{w}+\tau\mathfrak{w}-\varsigma\tau}\right)+\zeta\left(\frac{2\tau\mathfrak{w}}{\tau+\mathfrak{w}}\right)+\zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{\varsigma\mathfrak{w}+2\tau\mathfrak{w}-\varsigma\tau}\right)\right]\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}. \quad (2.52)$$

Finally we also have

$$\frac{1}{2}\left[\zeta\left(\frac{2\varsigma\tau}{\varsigma+\tau}\right)+\frac{\zeta(\varsigma)+\zeta(\tau)}{2}\right]\int_{\varsigma}^{\tau}\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}$$
$$=\left[\zeta\left(\frac{2\varsigma\tau}{\varsigma+\tau}\right)+\frac{\zeta(\varsigma)+\zeta(\tau)}{2}\right]\int_{\varsigma}^{\frac{2\varsigma\tau}{\varsigma+\tau}}\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}. \quad (2.53)$$

By Lemma 2, the following inequalities hold for all $\mathfrak{w} \in \left[\varsigma, \frac{2\varsigma\tau}{\varsigma+\tau}\right]$:
The inequality

$$\zeta\left(\frac{4\varsigma\tau\mathfrak{w}}{2\varsigma\tau+(\varsigma+\tau)\mathfrak{w}}\right)+\zeta\left(\frac{4\varsigma\tau\mathfrak{w}}{3(\varsigma+\tau)\mathfrak{w}-2\varsigma\tau}\right)\leq\zeta\left(\frac{4\varsigma\tau}{3\varsigma+\tau}\right)+\zeta\left(\frac{4\varsigma\tau}{\varsigma+3\tau}\right) \tag{2.54}$$

holds with the choices of $\mathfrak{w}_1 = \frac{4\varsigma\tau\mathfrak{w}}{2\varsigma\tau+(\varsigma+\tau)\mathfrak{w}}$, $\mathfrak{w}_2 = \frac{4\varsigma\tau\mathfrak{w}}{3(\varsigma+\tau)\mathfrak{w}-2\varsigma\tau}$, $\breve{\varkappa}_1 = \frac{4\varsigma\tau}{3\varsigma+\tau}$ and $\breve{\varkappa}_2 = \frac{4\varsigma\tau}{\varsigma+3\tau}$.
The inequality

$$\zeta\left(\frac{4\varsigma\tau}{3\varsigma+\tau}\right)\leq\frac{1}{2}\left[\zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{2\varsigma\mathfrak{w}+\tau\mathfrak{w}-\varsigma\tau}\right)+\zeta\left(\frac{2\varsigma\mathfrak{w}}{\varsigma+\mathfrak{w}}\right)\right] \tag{2.55}$$

holds with the choices of $\mathfrak{w}_1 = \mathfrak{w}_2 = \frac{4\varsigma\tau}{3\varsigma+\tau}$, $\breve{\varkappa}_1 = \frac{2\varsigma\mathfrak{w}}{\varsigma+\mathfrak{w}}$, $\breve{\varkappa}_2 = \frac{2\varsigma\tau\mathfrak{w}}{2\varsigma\mathfrak{w}+\tau\mathfrak{w}-\varsigma\tau}$.
The inequality

$$\zeta\left(\frac{4\varsigma\tau}{\varsigma+3\tau}\right)\leq\frac{1}{2}\left[\zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{\varsigma\mathfrak{w}+2\tau\mathfrak{w}-\varsigma\tau}\right)+\zeta\left(\frac{2\tau\mathfrak{w}}{\tau+\mathfrak{w}}\right)\right] \tag{2.56}$$

holds with the choices of $\mathfrak{w}_1 = \mathfrak{w}_2 = \frac{4\varsigma\tau}{\varsigma+3\tau}$, $\breve{\varkappa}_1 = \frac{2\tau\mathfrak{w}}{\tau+\mathfrak{w}}$, $\breve{\varkappa}_2 = \frac{2\varsigma\tau\mathfrak{w}}{\varsigma\mathfrak{w}+2\tau\mathfrak{w}-\varsigma\tau}$.
The inequality

$$\zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{2\varsigma\mathfrak{w}+\tau\mathfrak{w}-\varsigma\tau}\right)+\zeta\left(\frac{2\varsigma\mathfrak{w}}{\varsigma+\mathfrak{w}}\right)\leq\zeta(\varsigma)+\zeta\left(\frac{2\tau\varsigma}{\varsigma+\tau}\right) \tag{2.57}$$

holds with the choices of $\mathfrak{w}_1 = \frac{2\varsigma\mathfrak{w}}{\varsigma+\mathfrak{w}}$, $\mathfrak{w}_2 = \frac{2\varsigma\tau\mathfrak{w}}{2\varsigma\mathfrak{w}+\tau\mathfrak{w}-\varsigma\tau}$, $\breve{\varkappa}_1 = \varsigma$, $\breve{\varkappa}_2 = \frac{2\tau\varsigma}{\varsigma+\tau}$.
The inequality

$$\zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{\varsigma\mathfrak{w}+2\tau\mathfrak{w}-\varsigma\tau}\right)+\zeta\left(\frac{2\tau\mathfrak{w}}{\tau+\mathfrak{w}}\right)\leq\zeta\left(\frac{2\tau\varsigma}{\varsigma+\tau}\right)+\zeta(\tau) \tag{2.58}$$

holds with the choices of $\mathfrak{w}_1 = \frac{2\tau\mathfrak{w}}{\tau+\mathfrak{w}}$, $\mathfrak{w}_2 = \frac{2\varsigma\tau\mathfrak{w}}{\varsigma\mathfrak{w}+2\tau\mathfrak{w}-\varsigma\tau}$, $\breve{\varkappa}_1 = \frac{2\tau\varsigma}{\varsigma+\tau}$, $\breve{\varkappa}_2 = \tau$.
Multiplying (2.54)–(2.58) by $\vartheta(\mathfrak{w})$, integrating them over $\left[\varsigma, \frac{2\varsigma\tau}{\varsigma+\tau}\right]$ and using (2.50)–(2.53), we get (2.46).
(iii) By integration by parts, we get

$$\begin{aligned}
&\iota\int_{\frac{1}{\tau}}^{\frac{\varsigma+\tau}{2\varsigma\tau}}\left[\left(\mathfrak{w}-\frac{\varsigma+\tau}{2\varsigma\tau}\right)\breve{h}'\left(\iota\mathfrak{w}+(1-\iota)\left(\frac{\varsigma+\tau}{2\varsigma\tau}\right)\right)\right.\\
&\qquad\left.+\left(\frac{\varsigma+\tau}{2\varsigma\tau}-\mathfrak{w}\right)\breve{h}'\left(\iota\left(\frac{1}{\varsigma}+\frac{1}{\tau}-\mathfrak{w}\right)+(1-\iota)\left(\frac{\varsigma+\tau}{2\varsigma\tau}\right)\right)\right]d\mathfrak{w}\\
&\qquad=\iota\int_{\frac{1}{\tau}}^{\frac{1}{\varsigma}}\left(\mathfrak{w}-\frac{\varsigma+\tau}{2\varsigma\tau}\right)\breve{h}'\left(\iota\mathfrak{w}+(1-\iota)\left(\frac{\varsigma+\tau}{2\varsigma\tau}\right)\right)d\mathfrak{w}\\
&\qquad=\frac{\tau-\varsigma}{2\varsigma\tau}\left[\zeta\left(\frac{2\varsigma\tau}{2\tau\iota+(1-\iota)(\varsigma+\tau)}\right)+\zeta\left(\frac{2\varsigma\tau}{2\varsigma\iota+(1-\iota)(\varsigma+\tau)}\right)\right]\\
&\qquad\quad-\int_{\varsigma}^{\tau}\frac{1}{\mathfrak{w}^2}\zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{2\varsigma\tau\iota+(1-\iota)(\varsigma+\tau)}\right)d\mathfrak{w}=\left(\frac{\tau-\varsigma}{\varsigma\tau}\right)\left[\breve{\mathcal{G}}_1(\iota)-\mathbb{S}(\iota)\right].
\end{aligned} \tag{2.59}$$

Using the convexity of $\breve{h}$ and the hypothesis of ϑ, the inequality holds for all $\iota \in [0, 1]$ and $\mathfrak{w} \in \left[\frac{1}{\tau}, \frac{\varsigma+\tau}{2\varsigma\tau}\right]$:

$$\left[\breve{h}\left(\iota\mathfrak{w}+(1-\iota)\left(\frac{\varsigma+\tau}{2\varsigma\tau}\right)\right)-\breve{h}\left(\frac{\varsigma+\tau}{2\varsigma\tau}\right)\right]\vartheta\left(\frac{1}{\mathfrak{w}}\right)$$
$$+\left[\breve{h}\left(\iota\left(\frac{1}{\varsigma}+\frac{1}{\tau}-\mathfrak{w}\right)+(1-\iota)\left(\frac{\varsigma+\tau}{2\varsigma\tau}\right)\right)-\breve{h}\left(\frac{\varsigma+\tau}{2\varsigma\tau}\right)\right]\vartheta\left(\frac{1}{\mathfrak{w}}\right)$$
$$\leq \iota\left(\mathfrak{w}-\frac{\varsigma+\tau}{2\varsigma\tau}\right)\breve{h}'\left(\iota\mathfrak{w}+(1-\iota)\left(\frac{\varsigma+\tau}{2\varsigma\tau}\right)\right)\vartheta\left(\frac{1}{\mathfrak{w}}\right)$$
$$+\iota\left(\frac{\varsigma+\tau}{2\varsigma\tau}-\mathfrak{w}\right)\breve{h}'\left(\iota\left(\frac{1}{\varsigma}+\frac{1}{\tau}-\mathfrak{w}\right)+(1-\iota)\left(\frac{\varsigma+\tau}{2\varsigma\tau}\right)\right)\vartheta\left(\frac{1}{\mathfrak{w}}\right)$$
$$=\iota\left(\frac{\varsigma+\tau}{2\varsigma\tau}-\mathfrak{w}\right)\left[\breve{h}'\left(\iota\left(\frac{1}{\varsigma}+\frac{1}{\tau}-\mathfrak{w}\right)+(1-\iota)\left(\frac{\varsigma+\tau}{2\varsigma\tau}\right)\right)\right.$$
$$\left.-\breve{h}'\left(\iota\mathfrak{w}+(1-\iota)\left(\frac{\varsigma+\tau}{2\varsigma\tau}\right)\right)\right]\vartheta\left(\frac{1}{\mathfrak{w}}\right)$$
$$\leq \iota\left(\frac{\varsigma+\tau}{2\varsigma\tau}-\mathfrak{w}\right)\left[\breve{h}'\left(\iota\left(\frac{1}{\varsigma}+\frac{1}{\tau}-\mathfrak{w}\right)+(1-\iota)\left(\frac{\varsigma+\tau}{2\varsigma\tau}\right)\right)\right.$$
$$\left.-\breve{h}'\left(\iota\mathfrak{w}+(1-\iota)\left(\frac{\varsigma+\tau}{2\varsigma\tau}\right)\right)\right]\|\vartheta\|_{\infty}. \quad (2.60)$$

Integrating (2.60), using (2.59) and (2.17), we get (2.47). □

Corollary 2. *According to the assumptions of Theorem 18 with* $\vartheta(\mathfrak{w})=\frac{\varsigma\tau}{\tau-\varsigma}$, $\mathfrak{w}\in[\varsigma,\tau]$, *then*

(i) The inequality

$$\mathbb{S}(\iota)\leq \breve{\mathcal{G}}_1(\iota)$$

holds for all $\iota\in[0,1]$.

(ii) The inequalities

$$\frac{2\varsigma\tau}{\tau-\varsigma}\int_{\frac{4\varsigma\tau}{\varsigma+3\tau}}^{\frac{4\varsigma\tau}{3\varsigma+\tau}}\zeta(\mathfrak{w})\vartheta\left(\frac{2\varsigma\tau\mathfrak{w}}{4\varsigma\tau-(\varsigma+\tau)\mathfrak{w}}\right)\frac{d\mathfrak{w}}{\mathfrak{w}^2}$$
$$\leq\frac{1}{2}\left[\zeta\left(\frac{4\varsigma\tau}{3\varsigma+\tau}\right)+\zeta\left(\frac{4\varsigma\tau}{\varsigma+3\tau}\right)\right]\leq\frac{\tau-\varsigma}{\varsigma\tau}\int_0^1\breve{\mathcal{G}}_1(\iota)$$
$$\times\vartheta\left(\frac{\varsigma\tau}{(1-\iota)\varsigma+\iota\tau}\right)d\iota\leq\frac{1}{2}\left[\zeta\left(\frac{2\varsigma\tau}{\varsigma+\tau}\right)+\frac{\zeta(\varsigma)+\zeta(\tau)}{2}\right] \quad (2.61)$$

hold.

(iii) The inequality

$$0\leq\mathbb{S}(\iota)-\zeta\left(\frac{2\varsigma\tau}{\varsigma+\tau}\right)d\mathfrak{w}\leq\left(\frac{\tau-\varsigma}{\varsigma\tau}\right)\left[\breve{\mathcal{G}}_1(\iota)-\mathbb{S}(\iota)\right] \quad (2.62)$$

holds for all $\iota\in[0,1]$.

Theorem 19. *Let* ζ, ϑ, $\breve{\mathcal{G}}_1$, $\mathbb{S}_\vartheta$, $\mathbb{T}_\vartheta$ *be defined as above. Then*

(i) $\mathbb{T}_\vartheta$ *is harmonic convex on* $(0,1]$.

(ii) The inequalities

$$\breve{\mathcal{G}}_1(\iota)\int_{\varsigma}^{\tau}\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}\le\mathbb{T}_{\vartheta}(\iota)$$
$$\le(1-\iota)\int_{\varsigma}^{\tau}\frac{\zeta(\mathfrak{w})\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}+\iota\cdot\frac{\zeta(\varsigma)+\zeta(\tau)}{2}\int_{\varsigma}^{\tau}\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}$$
$$\le\frac{\zeta(\varsigma)+\zeta(\tau)}{2}\int_{\varsigma}^{\tau}\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w},\quad(2.63)$$

$$\mathbb{S}_{\vartheta}(1-\iota)\le\mathbb{T}_{\vartheta}(\iota)\quad(2.64)$$

and

$$\frac{\mathbb{S}_{\vartheta}(\iota)+\mathbb{S}_{\vartheta}(1-\iota)}{2}\le\mathbb{T}_{\vartheta}(\iota)\quad(2.65)$$

hold for all $\iota\in[0,1]$.

(iii) The following bound is true:

$$\sup_{\iota\in[0,1]}\mathbb{T}_{\vartheta}(\iota)=\frac{\zeta(\varsigma)+\zeta(\tau)}{2}\int_{\varsigma}^{\tau}\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w}.\quad(2.66)$$

Proof. (i) Since ζ is harmonic convex and ϑ is nonnegative, we see that $\mathbb{T}_{\vartheta}$ is harmonic convex on $(0,1]$.

(ii) We observe that the following identity holds on $[0,1]$:

$$\mathbb{T}_{\vartheta}(\iota)=\frac{1}{2}\int_{\varsigma}^{\frac{2\varsigma\tau}{\varsigma+\tau}}\left[\zeta\left(\frac{\tau\mathfrak{w}}{\iota\mathfrak{w}+(1-\iota)\tau}\right)+\zeta\left(\frac{\varsigma\tau\mathfrak{w}}{\varsigma\mathfrak{w}\iota+(1-\iota)(\varsigma\mathfrak{w}+\tau\mathfrak{w}-\varsigma\tau)}\right)\right.$$
$$\left.+\zeta\left(\frac{\varsigma\mathfrak{w}}{\iota\mathfrak{w}+(1-\iota)\varsigma}\right)+\zeta\left(\frac{\varsigma\tau\mathfrak{w}}{\tau\mathfrak{w}\iota+(1-\iota)(\varsigma\mathfrak{w}+\tau\mathfrak{w}-\varsigma\tau)}\right)\right]\vartheta(\mathfrak{w})\,d\mathfrak{w}.\quad(2.67)$$

By Lemma 2, the following inequalities hold for all $\mathfrak{w}\in\left[\varsigma,\frac{2\varsigma\tau}{\varsigma+\tau}\right]$:

$$2\zeta\left(\frac{2\varsigma\tau}{2\varsigma\iota+(1-\iota)(\varsigma+\tau)}\right)$$
$$\le\zeta\left(\frac{\tau\mathfrak{w}}{\iota\mathfrak{w}+(1-\iota)\tau}\right)+\zeta\left(\frac{\varsigma\tau\mathfrak{w}}{\varsigma\mathfrak{w}\iota+(1-\iota)(\varsigma\mathfrak{w}+\tau\mathfrak{w}-\varsigma\tau)}\right)\quad(2.68)$$

with

$$\mathfrak{w}_1=\mathfrak{w}_2=\frac{2\varsigma\tau}{2\varsigma\iota+(1-\iota)(\varsigma+\tau)},\ \breve{\varkappa}_1=\frac{\tau\mathfrak{w}}{\iota\mathfrak{w}+(1-\iota)\tau}$$
$$\text{and }\breve{\varkappa}_2=\frac{\varsigma\tau\mathfrak{w}}{\varsigma\mathfrak{w}\iota+(1-\iota)(\varsigma\mathfrak{w}+\tau\mathfrak{w}-\varsigma\tau)}.$$

$$2\zeta\left(\frac{2\varsigma\tau}{2\tau\iota+(1-\iota)(\varsigma+\tau)}\right)$$
$$\le\zeta\left(\frac{\varsigma\mathfrak{w}}{\iota\mathfrak{w}+(1-\iota)\varsigma}\right)+\zeta\left(\frac{\varsigma\tau\mathfrak{w}}{\tau\mathfrak{w}\iota+(1-\iota)(\varsigma\mathfrak{w}+\tau\mathfrak{w}-\varsigma\tau)}\right)\quad(2.69)$$

with

$$\mathfrak{w}_1 = \mathfrak{w}_2 = \frac{2\varsigma\tau}{2\tau\iota + (1-\iota)(\varsigma+\tau)}, \breve{\varkappa}_1 = \frac{\varsigma\mathfrak{w}}{\iota\mathfrak{w} + (1-\iota)\varsigma}$$
$$\text{and } \breve{\varkappa}_2 = \frac{\varsigma\tau\mathfrak{w}}{\tau\mathfrak{w}\iota + (1-\iota)(\varsigma\mathfrak{w} + \tau\mathfrak{w} - \varsigma\tau)}.$$

Multiplying the inequalities (2.68) and (2.69) by $\vartheta(\mathfrak{w})$, integrating them over $\mathfrak{w}$ on $\left[\varsigma, \frac{2\varsigma\tau}{\varsigma+\tau}\right]$ and using identities (2.48) and (2.67), we derive the first inequality of (2.63). Using the harmonic convexity of ζ and the inequality (2.17), the last part of (2.63) holds. Using again the harmonic convexity of ζ, we get

$$\begin{aligned}
\mathbb{S}_{\vartheta}(1-\iota) = \int_{\varsigma}^{\tau} \zeta\left(\frac{2\varsigma\tau\mathfrak{w}}{2\varsigma\tau(1-\iota) + \iota(\varsigma+\tau)\mathfrak{w}}\right)\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w} \\
= \int_{\varsigma}^{\tau} \zeta\left(\frac{1}{\frac{1}{2}\left(\frac{\iota\mathfrak{w}+(1-\iota)\varsigma}{\varsigma\mathfrak{w}}\right) + \frac{1}{2}\left(\frac{\iota\mathfrak{w}+(1-\iota)\tau}{\tau\mathfrak{w}}\right)}\right)\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w} \\
\leq \frac{1}{2}\int_{\varsigma}^{\tau}\left[\zeta\left(\frac{\varsigma\mathfrak{w}}{\iota\mathfrak{w} + (1-\iota)\varsigma}\right) + \zeta\left(\frac{\varsigma\mathfrak{w}}{\iota\mathfrak{w} + (1-\iota)\varsigma}\right)\right]\frac{\vartheta(\mathfrak{w})}{\mathfrak{w}^2}d\mathfrak{w} = \mathbb{T}_{\vartheta}(\iota).
\end{aligned} \tag{2.70}$$

From (2.45), (2.63) and 2.70), we get (2.65).
(iii) (2.66) holds due to the inequality (2.63). □

3. Conclusions

The subject of mathematical inequalities using convex functions has been seen to be an emerging topic during the past more than three decades. The researchers are trying to find new generalizations of convex functions and as a result new results are being adding to the theory of inequalities. In the current research we have used harmonic convex functions to generalize a number of results that hold for convex functions. In order to get the novel results in this study, we defined some new mappings over the interval [0, 1]. We have discussed some interesting properties of these mappings and obtained new refinements of the Hermite-hadamard and Fejér type inequalities already proven for harmonic convex functions. We believe that the results of this paper could be a source of inspiration for mathematicians working in this field and young researchers thinking to start their career in this fascinating field of mathematics.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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