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Research article

A real representation method for special least squares solutions of the quaternion matrix equation $(AXB, DXE) = (C, F)$

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Abstract: In this article, our interest is the quaternion matrix equation $(AXB, DXE) = (C, F)$, and we study its minimal norm centrohermitian least squares solution and skew centrohermitian least squares solution. By applying of the real representation matrices of quaternion matrices and relative properties, we convert the quaternion least squares problems with constrained variables into the corresponding real least squares problems with free variables, and then we obtain the solutions of corresponding problems. The final results can be expressed only by real matrices and vectors, and thus the corresponding algorithms only involves real operations and avoid complex quaternion operations. Therefore, they are portable and convenient. In the end, we give two examples to verify the effectiveness of the purposed algorithms.

Keywords: quaternion matrix equation; real representation; quaternion centrohermitian matrix; quaternion skew centrohermitian matrix; least squares solution
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1. Introduction

In this article, we apply the following notations. $\mathbb{R}$ is the real number field. $\mathbb{R}^{m\times n}$ is the set of all $m \times n$ real matrices, and the set $\mathbb{R}^{m\times 1}$ is also denoted as $\mathbb{R}^m$. $\mathbb{R}^{p\times q}_{CS}$, $\mathbb{R}^{p\times q}_{ACS}$ are the sets of all $p \times q$ real centrosymmetric and anti centrosymmetric matrices, respectively. $\mathbb{Q}$ is the quaternion skew-field, and $\mathbb{Q}^{m\times n}$ is the set of all $m \times n$ quaternion matrices. $\mathbb{Q}^{p\times q}_{CH}$, $\mathbb{Q}^{p\times q}_{SCH}$ are the sets of all $p \times q$ quaternion centrohermitian and skew centrohermitian matrices, respectively. C^T, $\overline{C}$, $C^\dagger$ and $R(C)$ denote the transpose, the conjugate, the Moore-Penrose inverse and the rank of the matrix C, respectively. I_k denotes the $k \times k$ identity matrix. $J_k = (e_k, e_{k-1}, \cdots, e_2, e_1)$, in which e_i is the i-th column of I_k. Let $A = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{R}^{m\times n}$, $vec(A) = (\alpha_1^T, \alpha_2^T, \cdots, \alpha_n^T)^T$ is the vec operator. $\|\cdot\|$ stands for the matrix Frobenius norm. Let $B = (b_{ij}) \in \mathbb{R}^{m\times n}, C \in \mathbb{R}^{p\times q}$, and then $B \otimes C = (b_{ij}C) \in \mathbb{R}^{mp\times nq}$ means the Kronecker product of B and C. The $rand(m, n)$ is a function, which can generate a $m \times n$ random matrix in MATLAB.

Matrix equations are the main content of numerical algebra. They are widely used in computational mathematics, control theory, quantum physics, quantum mechanics, color image processing and so on. With the rapid development of these fields, matrix equations are widely studied [1–13]. In this article, our interest is the following matrix equation

$$(AXB, DXE) = (C, F) \tag{1.1}$$

in which X is an unknown matrix, and others are given matrices with proper sizes. In recent years, this matrix equation has attracted much attention for its important role in pole assignment [14], measurement feedback [15], matrix programming problem [16] and so on. The main methods of solving (1.1) include the iterative method and the direct method. In this article, we are concerned with the latter.

Some significant results of the matrix equation (1.1) have been obtained by direct method. Navarra et al. [17] derived a representation of the general solution for the complex matrix equation (1.1). Liao et al. [18] gave an analytical expression of the minimal norm least squares solution for the real matrix equation (1.1) by the the generalized singular value decomposition. Yuan et al. [19, 20] considered the minimal norm Hermitian least squares solution, η-bi-Hermitian least squares solution and η-anti-bi-Hermitian least squares solution for the quaternion matrix equation (1.1) by the complex representation method. Wang et al. [21] studied the minimal norm Hermitian least squares solution of the complex matrix equation (1.1) by a product of matrices and vectors. Zhang et al. [22] studied the same problems as [21] by the real representation matrices of quaternion matrices. Şimşek et al. [23] purposed the precise solutions on the minimum residual and matrix nearness problems of the quaternion matrix equation (1.1) for centrohermitian and skew centrohermitian matrices.

In addition, centrohermitian and skew centrohermitian matrices have wide applications in linear system theory, information theory, numerical analysis and linear estimate theory [24, 25]. Therefore, for the quaternion matrix equation (1.1), it is valuable of the research of centrohermitian and skew centrohermitian solutions. It is generally known that the quaternion skew-filed $\mathbb{Q}$ is a non-commutative but associate algebra. Thus, many conclusions on real number field are invalid on the quaternion algebra. For example, the result $vec(AXB) = (B^T \otimes A)vec(X)$ does not hold on the quaternion algebra. Based on these, in the present paper, we will convert the quaternion least squares problems with constrained variables into the corresponding real least squares problems with free variables by using the real representations of quaternion matrices and relative properties, and give the solutions of corresponding problems.

We will consider the following two problems.

Problem I. Let $A \in \mathbb{Q}^{m\times p}$, $B \in \mathbb{Q}^{q\times s}$, $C \in \mathbb{Q}^{m\times s}$, $D \in \mathbb{Q}^{m\times p}$, $E \in \mathbb{Q}^{q\times t}$, $F \in \mathbb{Q}^{m\times t}$ and

$$\mathbf{Q}_{CH} = \{X | X \in \mathbb{Q}_{CH}^{p\times q},\ \min \|(AXB - C, DXE - F)\|\}.$$

Find $X_{CH} \in \mathbf{Q}_{CH}$ such that $\|X_{CH}\| = \min\limits_{X\in\mathbf{Q}_{CH}} \|X\|$.

Problem II. Let $A \in \mathbb{Q}^{m\times p}$, $B \in \mathbb{Q}^{q\times s}$, $C \in \mathbb{Q}^{m\times s}$, $D \in \mathbb{Q}^{m\times p}$, $E \in \mathbb{Q}^{q\times t}$, $F \in \mathbb{Q}^{m\times t}$ and

$$\mathbf{Q}_{SCH} = \{X | X \in \mathbb{Q}_{SCH}^{p\times q},\ \min \|(AXB - C, DXE - F)\|\}.$$

Find $X_{SCH} \in \mathbf{Q}_{SCH}$ such that $\|X_{SCH}\| = \min\limits_{X \in \mathbf{Q}_{SCH}} \|X\|$.

The solution X_{CH} is also named the minimal norm centrohermitian least squares solution of (1.1), and X_{SCH} is also named the minimal norm skew centrohermitian least squares solution of (1.1).

The rest of this article is arranged as follows. In Section 2, we provide some preliminary results. In Section 3, by using the real representations of quaternion matrices and relative properties, we convert the quaternion least squares problems with constrained variables into the corresponding real least squares problems with free variables, and then we obtain the solutions of Problems I and II. In Section 4, we first purpose two numerical algorithms for Problems I and II, respectively, and then we give two examples to verify our algorithms. Finally, in Section 5, we offer some brief comments.

2. Preliminaries

A quaternion $q \in \mathbb{Q}$ can be written as

$$q = q_1 + q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k},$$

where $q_1, q_2, q_3, q_4 \in \mathbb{R}$ and $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. The conjugate of q is defined as

$$\overline{q} \equiv q_1 - q_2\mathbf{i} - q_3\mathbf{j} - q_4\mathbf{k}.$$

Similarly, a quaternion matrix $Q \in \mathbb{Q}^{m\times n}$ can be written as

$$Q = Q_1 + Q_2\mathbf{i} + Q_3\mathbf{j} + Q_4\mathbf{k},$$

where $Q_1, Q_2, Q_3, Q_4 \in \mathbb{R}^{m\times n}$, and its conjugate matrix is $\overline{Q} \equiv Q_1 - Q_2\mathbf{i} - Q_3\mathbf{j} - Q_4\mathbf{k}$. The real representation matrix of Q can be defined as

$$Q^{\mathbf{R}} \equiv \begin{pmatrix} Q_1 & -Q_2 & -Q_3 & -Q_4 \\ Q_2 & Q_1 & -Q_4 & Q_3 \\ Q_3 & Q_4 & Q_1 & -Q_2 \\ Q_4 & -Q_3 & Q_2 & Q_1 \end{pmatrix} \in \mathbb{R}^{4m\times 4n}.$$

The Frobenius norm of Q is defined as

$$\|Q\| \equiv \sqrt{\|Q_1\|^2 + \|Q_2\|^2 + \|Q_3\|^2 + \|Q_4\|^2}.$$

Obviously, $Q^{\mathbf{R}}$ has specific structure. Now, the k-th row block of $Q^{\mathbf{R}}$ is denoted by $Q^{\mathbf{R}}_{r_k}$, and the k-th column block of $Q^{\mathbf{R}}$ is denoted by $Q^{\mathbf{R}}_{c_k}$. $Q^{\mathbf{R}}$, $Q^{\mathbf{R}}_{r_k}$, $Q^{\mathbf{R}}_{c_k}$ have the properties as below.

Lemma 2.1 ([26]). *Let $A, B \in \mathbb{Q}^{m\times n}, C \in \mathbb{Q}^{n\times t}, l \in \mathbb{R}$. Then*

(1) $A = B \Leftrightarrow A^{\mathbf{R}} = B^{\mathbf{R}} \Leftrightarrow A^{\mathbf{R}}_{r_k} = B^{\mathbf{R}}_{r_k} \Leftrightarrow A^{\mathbf{R}}_{c_k} = B^{\mathbf{R}}_{c_k}$,
(2) $(A + B)^{\mathbf{R}}_{r_k} = A^{\mathbf{R}}_{r_k} + B^{\mathbf{R}}_{r_k}, (A + B)^{\mathbf{R}}_{c_k} = A^{\mathbf{R}}_{c_k} + B^{\mathbf{R}}_{c_k}$,
(3) $(lA)^{\mathbf{R}}_{r_k} = lA^{\mathbf{R}}_{r_k}, \quad (lA)^{\mathbf{R}}_{c_k} = lA^{\mathbf{R}}_{c_k}$,
(4) $(AC)^{\mathbf{R}}_{r_k} = A^{\mathbf{R}}_{r_k}C^{\mathbf{R}}, \quad (AC)^{\mathbf{R}}_{c_k} = A^{\mathbf{R}}C^{\mathbf{R}}_{c_k}$,
(5) $\|A\| = \|A^{\mathbf{R}}_{r_k}\| = \|A^{\mathbf{R}}_{c_k}\|$,

where $k = 1, 2, 3, 4$.

For simplicity, we use the first column block of $Q^{\mathbf{R}}$ to solve Problems I and II, and which is denoted as $Q_c^{\mathbf{R}}$, i.e.,

$$Q_c^{\mathbf{R}} = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix}.$$

The following lemmas are useful for solving Problems I and II.

Lemma 2.2 ([27])**.** *Let* $X \in \mathbb{Q}^{p\times q}$*. Then* $vec(X^{\mathbf{R}}) = \mathcal{F} vec(X_c^{\mathbf{R}})$*, where*

$$\mathcal{F} = \begin{pmatrix} diag(I_{4p}, \cdots, I_{4p}) \\ diag(Q_p, \cdots, Q_p) \\ diag(R_p, \cdots, R_p) \\ diag(S_p, \cdots, S_p) \end{pmatrix} \in \mathbb{R}^{16pq\times 4pq},$$

and

$$Q_p = \begin{pmatrix} 0 & -I_p & 0 & 0 \\ I_p & 0 & 0 & 0 \\ 0 & 0 & 0 & I_p \\ 0 & 0 & -I_p & 0 \end{pmatrix}, R_p = \begin{pmatrix} 0 & 0 & -I_p & 0 \\ 0 & 0 & 0 & -I_p \\ I_p & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 \end{pmatrix}, S_p = \begin{pmatrix} 0 & 0 & 0 & -I_p \\ 0 & 0 & I_p & 0 \\ 0 & -I_p & 0 & 0 \\ I_p & 0 & 0 & 0 \end{pmatrix}.$$

Lemma 2.3. *Let* $X \in \mathbb{R}^{p\times q}$*. Then* $vec(X_c^{\mathbf{R}}) = \mathcal{M} \begin{pmatrix} vec(X_1) \\ vec(X_2) \\ vec(X_3) \\ vec(X_4) \end{pmatrix}$*,where*

$$\mathcal{M} = \begin{pmatrix} diag(\mathcal{I}_1, \mathcal{I}_1, \mathcal{I}_1, \mathcal{I}_1) \\ diag(\mathcal{I}_2, \mathcal{I}_2, \mathcal{I}_2, \mathcal{I}_2) \\ \vdots \\ diag(\mathcal{I}_q, \mathcal{I}_q, \mathcal{I}_q, \mathcal{I}_q) \end{pmatrix} \in \mathbb{R}^{4pq\times 4pq},$$

and $\mathcal{I}_1 = (I_p, 0, \cdots, 0), \mathcal{I}_2 = (0, I_p, \cdots, 0), \cdots, \mathcal{I}_q = (0, 0, \cdots, I_p) \in \mathbb{R}^{p\times pq}$.

For a real matrix $X = (x_{ij}) \in \mathbb{R}^{p\times q}$, if $x_{ij} = x_{(p-i+1)(q-j+1)}$, then X is called centrosymmetric matrix. If $x_{ij} = -x_{(p-i+1)(q-j+1)}$, then X is called anti centrosymmetric matrix. They can also be described equivalently as

$$X \in \mathbb{R}_{CS}^{p\times q} \Longleftrightarrow X = J_p X J_q, \quad X \in \mathbb{R}_{ACS}^{p\times q} \Longleftrightarrow X = -J_p X J_q.$$

Similarly, for a quaternion matrix $X = (x_{ij}) \in \mathbb{Q}^{p\times q}$, if $x_{ij} = \overline{x}_{(p-i+1)(q-j+1)}$, then X is called centrohermitian matrix. If $x_{ij} = -\overline{x}_{(p-i+1)(q-j+1)}$, then X is called skew centrohermitian matrix. And it is not difficult to get the following conclusions:

$$X \in \mathbb{Q}_{CH}^{p\times q} \Longleftrightarrow X = J_p \overline{X} J_q, \quad X \in \mathbb{Q}_{SCH}^{p\times q} \Longleftrightarrow X = -J_p \overline{X} J_q.$$

Furthermore, we can get the following Lemma 2.4.

Lemma 2.4. *Let* $X = X_1 + X_2\mathbf{i} + X_3\mathbf{j} + X_4\mathbf{k} \in \mathbb{Q}^{p\times q}$*. Then*

(1) $X \in \mathbb{Q}_{CH}^{p\times q} \Longleftrightarrow X_1 \in \mathbb{R}_{CS}^{p\times q}, X_2, X_3, X_4 \in \mathbb{R}_{ACS}^{p\times q}$,
(2) $X \in \mathbb{Q}_{SCH}^{p\times q} \Longleftrightarrow X_1 \in \mathbb{R}_{ACS}^{p\times q}, X_2, X_3, X_4 \in \mathbb{R}_{CS}^{p\times q}$.

Proof. By the definition of centrohermitian matrix, we have

$$\begin{aligned} X \in \mathbb{Q}_{CH}^{p\times q} &\Longleftrightarrow X = J_p\overline{X}J_q \\ &\Longleftrightarrow X_1 + X_2\mathbf{i} + X_3\mathbf{j} + X_4\mathbf{k} = J_p(X_1 - X_2\mathbf{i} - X_3\mathbf{j} - X_4\mathbf{k})J_q \\ &\Longleftrightarrow X_1 = J_pX_1J_q, X_2 = -J_pX_2J_q, X_3 = -J_pX_3J_q, X_4 = -J_pX_4J_q \\ &\Longleftrightarrow X_1 \in \mathbb{R}_{CS}^{p\times q}, X_2, X_3, X_4 \in \mathbb{R}_{ACS}^{p\times q}. \end{aligned}$$

Thus (1) holds. Similarly, we can show that (2) holds. □

Lemma 2.5. *Suppose* $X = (x_{ij}) \in \mathbb{R}^{p\times q}$*.*

(1) If $p = 2m + 1, q = 2n + 1$, let

$$\begin{aligned} \alpha_i &= (x_{1i}, x_{2i}, \cdots, x_{(m+1)i}), \quad i = 1, 2, \cdots, n + 1, \\ \alpha_j &= (x_{1j}, x_{2j}, \cdots, x_{mj}), \quad j = n + 2, \cdots, q, \end{aligned}$$

and $vec_{oo}^{CS}(X) = (\alpha_1, \alpha_2, \cdots, \alpha_q)^T$, then

$$X \in \mathbb{R}_{CS}^{p\times q} \Longleftrightarrow vec(X) = G_{oo}^{CS}\, vec_{oo}^{CS}(X),$$

where

$$G_{oo}^{CS} = \begin{pmatrix} I_{m+1} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & J_m \\ 0 & I_{m+1} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & J_m & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I_{m+1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & J_m & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & I_m & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & J_m & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & I_m & \cdots & 0 & 0 \\ 0 & 0 & \cdots & J_{m+1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & I_m \\ J_{m+1} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \in \mathbb{R}^{pq\times(2mn+m+n+1)}.$$

(2) If $p = 2m + 1, q = 2n$, let

$$\begin{aligned} \alpha_i &= (x_{1i}, x_{2i}, \cdots, x_{(m+1)i}), \quad i = 1, 2, \cdots, n, \\ \alpha_j &= (x_{1j}, x_{2j}, \cdots, x_{mj}), \quad i = n + 1, \cdots, q, \end{aligned}$$

and $vec_{oe}^{CS}(X) = (\alpha_1, \alpha_2, \cdots, \alpha_q)^T$, then

$$X \in \mathbb{R}_{CS}^{p\times q} \Longleftrightarrow vec(X) = G_{oe}^{CS}\, vec_{oe}^{CS}(X),$$

where

$$G_{oe}^{CS}=\begin{pmatrix} I_{m+1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & J_m \\ 0 & I_{m+1} & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & J_m & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I_{m+1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & J_m & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & I_m & \cdots & 0 & 0 \\ 0 & 0 & \cdots & J_{m+1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & I_m \\ J_{m+1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}\in \mathbb{R}^{pq\times(2mn+n)}.$$

(3) If $p=2m, q=2n+1$, let

$$\alpha_i=(x_{1i},x_{2i},\cdots,x_{mi}), i=1,2,\cdots,q,$$

and $vec_{eo}^{CS}(X)=(\alpha_1,\alpha_2,\cdots,\alpha_q)^T$, then

$$X\in \mathbb{R}_{CS}^{p\times q}\Longleftrightarrow vec(X)=G_{eo}^{CS}vec_{eo}^{CS}(X),$$

where

$$G_{eo}^{CS}=\begin{pmatrix} I_m & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & J_m \\ 0 & I_m & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & J_m & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I_m & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & J_m & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & I_m & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & J_m & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & I_m & \cdots & 0 & 0 \\ 0 & 0 & \cdots & J_m & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & I_m \\ J_m & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}\in \mathbb{R}^{pq\times(2mn+m)}.$$

(4) If $p=2m, q=2n$, let

$$\alpha_i=(x_{1i},x_{2i},\cdots,x_{mi}), i=1,2,\cdots,q,$$

and $vec_{ee}(X)=(\alpha_1,\alpha_2,\cdots,\alpha_q)^T$, then

$$X\in \mathbb{R}_{CS}^{p\times q}\Longleftrightarrow vec(X)=G_{ee}^{CS}vec_{ee}^{CS}(X),$$

where

$$G_{ee}^{CS}=\begin{pmatrix} I_m & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & J_m \\ 0 & I_m & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & J_m & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I_m & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & J_m & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & I_m & \cdots & 0 & 0 \\ 0 & 0 & \cdots & J_m & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & I_m \\ J_m & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}\in \mathbb{R}^{pq\times 2mn}.$$

Lemma 2.6. *Suppose $X = (x_{ij}) \in \mathbb{R}^{p\times q}$.*

(1) If $p = 2m+1, q = 2n+1$, let

$$\begin{aligned}\alpha_i &= (x_{1i}, x_{2i}, \cdots, x_{(m+1)i}), \quad i = 1, 2, \cdots, n,\\ \alpha_j &= (x_{1j}, x_{2j}, \cdots, x_{mj}), \quad j = n+1, \cdots, q,\end{aligned}$$

and $vec_{oo}^{ACS}(X) = (\alpha_1, \alpha_2, \cdots, \alpha_q)^T$, then

$$X \in \mathbb{R}_{ACS}^{p\times q} \Longleftrightarrow vec(X) = G_{oo}^{ACS} vec_{oo}^{ACS}(X),$$

where

$$G_{oo}^{ACS} = \begin{pmatrix} I_{m+1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -J_m\\ 0 & I_{m+1} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -J_m & 0\\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & I_{m+1} & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0 & -J_m & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & I_m & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & -J_m & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0 & I_m & \cdots & 0 & 0\\ 0 & 0 & \cdots & -J_{m+1} & 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & I_m\\ -J_{m+1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \in \mathbb{R}^{pq\times(2mn+m+n)}.$$

(2) If $p = 2m+1, q = 2n$, let

$$\begin{aligned}\alpha_i &= (x_{1i}, x_{2i}, \cdots, x_{(m+1)i}), \quad i = 1, 2, \cdots, n,\\ \alpha_j &= (x_{1j}, x_{2j}, \cdots, x_{mj}), \quad j = n+1, \cdots, q,\end{aligned}$$

and $vec_{oe}^{ACS}(X) = (\alpha_1, \alpha_2, \cdots, \alpha_q)^T$, then

$$X \in \mathbb{R}_{ACS}^{p\times q} \Longleftrightarrow vec(X) = G_{oe}^{ACS} vec_{oe}^{ACS}(X),$$

where

$$G_{oe}^{ACS} = \begin{pmatrix} I_{m+1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -J_m\\ 0 & I_{m+1} & \cdots & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0 & \cdots & -J_m & 0\\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & I_{m+1} & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & -J_m & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & I_m & \cdots & 0 & 0\\ 0 & 0 & \cdots & -J_{m+1} & 0 & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & I_m\\ -J_{m+1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \in \mathbb{R}^{pq\times(2mn+n)}.$$

(3) If $p = 2m, q = 2n+1$, let

$$\alpha_i = (x_{1i}, x_{2i}, \cdots, x_{mi}), i = 1, 2, \cdots, q,$$

and $vec_{eo}^{ACS}(X) = (\alpha_1, \alpha_2, \cdots, \alpha_q)^T$, then

$$X \in \mathbb{R}_{ACS}^{p\times q} \Longleftrightarrow vec(X) = G_{eo}^{ACS} vec_{eo}^{ACS}(X),$$

where

$$G_{eo}^{ACS}=\begin{pmatrix} I_m & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -J_m\\ 0 & I_m & \cdots & 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -J_m & 0\\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & I_m & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0 & -J_m & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & I_m & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & -J_m & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0 & I_m & \cdots & 0 & 0\\ 0 & 0 & \cdots & -J_m & 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & I_m\\ -J_m & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}\in\mathbb{R}^{pq\times(2mn+m)}.$$

(4) If $p=2m, q=2n$, let

$$\alpha_i=(x_{1i},x_{2i},\cdots,x_{mi}), i=1,2,\cdots,q,$$

and $vec_{ee}^{ACS}(X)=(\alpha_1,\alpha_2,\cdots,\alpha_q)^T$, then

$$X\in\mathbb{R}_{ACS}^{p\times q}\Longleftrightarrow vec(X)=G_{ee}^{ACS}vec_{ee}^{ACS}(X),$$

where

$$G_{ee}^{ACS}=\begin{pmatrix} I_m & 0 & \cdots & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -J_m\\ 0 & I_m & \cdots & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0 & \cdots & -J_m & 0\\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & I_m & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & -J_m & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & I_m & \cdots & 0 & 0\\ 0 & 0 & \cdots & -J_m & 0 & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & I_m\\ -J_m & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}\in\mathbb{R}^{pq\times 2mn}.$$

Lemma 2.7. *Let $X=X_1+X_2\mathbf{i}+X_3\mathbf{j}+X_4\mathbf{k}\in\mathbb{Q}^{p\times q}$. Then*

$$(1)\ X\in\mathbb{Q}_{CH}^{p\times q}\Longleftrightarrow\begin{pmatrix} vec(X_1)\\ vec(X_2)\\ vec(X_3)\\ vec(X_4)\end{pmatrix}=\mathcal{G}_{\mu\nu}^{CH}\begin{pmatrix} vec_{\mu\nu}^{CS}(X_1)\\ vec_{\mu\nu}^{ACS}(X_2)\\ vec_{\mu\nu}^{ACS}(X_3)\\ vec_{\mu\nu}^{ACS}(X_4)\end{pmatrix},$$

$$(2)\ X\in\mathbb{Q}_{SCH}^{p\times q}\Longleftrightarrow\begin{pmatrix} vec(X_1)\\ vec(X_2)\\ vec(X_3)\\ vec(X_4)\end{pmatrix}=\mathcal{G}_{\mu\nu}^{SCH}\begin{pmatrix} vec_{\mu\nu}^{ACS}(X_1)\\ vec_{\mu\nu}^{CS}(X_2)\\ vec_{\mu\nu}^{CS}(X_3)\\ vec_{\mu\nu}^{CS}(X_4)\end{pmatrix},$$

where

$$\mathcal{G}_{\mu\nu}^{CH}=\begin{pmatrix} G_{\mu\nu}^{CS} & 0 & 0 & 0\\ 0 & G_{\mu\nu}^{ACS} & 0 & 0\\ 0 & 0 & G_{\mu\nu}^{ACS} & 0\\ 0 & 0 & 0 & G_{\mu\nu}^{ACS}\end{pmatrix},\quad \mathcal{G}_{\mu\nu}^{SCH}=\begin{pmatrix} G_{\mu\nu}^{ACS} & 0 & 0 & 0\\ 0 & G_{\mu\nu}^{CS} & 0 & 0\\ 0 & 0 & G_{\mu\nu}^{CS} & 0\\ 0 & 0 & 0 & G_{\mu\nu}^{CS}\end{pmatrix}.$$

Here, p,q are odd numbers, and then $\mu=\nu=o$. p is odd number and q is even number, and then $\mu=o,\nu=e$. p is even number and q is odd number, and then $\mu=e,\nu=o$. p,q are even numbers, and then $\mu=\nu=e$.

Lemmas 2.3, 2.5 and 2.6 can be obtained by direct calculation. According to Lemmas 2.4–2.6, we can easily get Lemma 2.7, so we omit the details.

3. The solutions of Problems I and II

In this section, we first convert the quaternion least squares problems with constrained variables into the corresponding real least squares problems with free variables by applying the real representations of quaternion matrices and the relative properties, and then obtain the solutions of Problems I and II.

Theorem 3.1. *Suppose $A \in \mathbb{Q}^{m\times p}$, $B \in \mathbb{Q}^{q\times s}, C \in \mathbb{Q}^{m\times s}, D \in \mathbb{Q}^{m\times p}$, $E \in \mathbb{Q}^{q\times t}, F \in \mathbb{Q}^{m\times t}$ and $X = X_1 + X_2\mathbf{i} + X_3\mathbf{j} + X_4\mathbf{k} \in \mathbb{Q}_{CH}^{p\times q}$, and denote $\mathcal{K} = \begin{pmatrix} (B_c^{\mathbf{R}})^T \otimes A^{\mathbf{R}} \\ (E_c^{\mathbf{R}})^T \otimes D^{\mathbf{R}} \end{pmatrix}$. Then the set $\mathbf{Q}_{CH}$ in Problem I can be represented as*

$$\mathbf{Q}_{CH} = \{X | \begin{pmatrix} vec(X_1) \\ vec(X_2) \\ vec(X_3) \\ vec(X_4) \end{pmatrix} = \mathcal{G}_{\mu\nu}^{CH}(\mathcal{K}\mathcal{F}\mathcal{M}\mathcal{G}_{\mu\nu}^{CH})^{\dagger} \begin{pmatrix} vec(C_c^{\mathbf{R}}) \\ vec(F_c^{\mathbf{R}}) \end{pmatrix} + \mathcal{G}_{\mu\nu}^{CH}[I_r - (\mathcal{K}\mathcal{F}\mathcal{M}\mathcal{G}_{\mu\nu}^{CH})^{\dagger}(\mathcal{K}\mathcal{F}\mathcal{M}\mathcal{G}_{\mu\nu}^{CH})]y\}, \quad (3.1)$$

where $y \in \mathbb{R}^r$ is an arbitrary vector. The unique solution $X_{CH} = X_1 + X_2\mathbf{i} + X_3\mathbf{j} + X_4\mathbf{k} \in \mathbf{Q}_{CH}$ of Problem I satisfies

$$\begin{pmatrix} vec(X_1) \\ vec(X_2) \\ vec(X_3) \\ vec(X_4) \end{pmatrix} = \mathcal{G}_{\mu\nu}^{CH}(\mathcal{K}\mathcal{F}\mathcal{M}\mathcal{G}_{\mu\nu}^{CH})^{\dagger} \begin{pmatrix} vec(C_c^{\mathbf{R}}) \\ vec(F_c^{\mathbf{R}}) \end{pmatrix}. \quad (3.2)$$

Here, if $p = 2m + 1, q = 2n + 1$, then $\mu = \nu = o$ and $r = 8mn + 4m + 4n + 1$. If $p = 2m + 1$, $q = 2n$, then $\mu = o, \nu = e$ and $r = 8mn + 4n$. If $p = 2m, q = 2n + 1$, then $\mu = e, \nu = o$ and $r = 8mn + 4m$. If $p = 2m, q = 2n$, then $\mu = \nu = e$ and $r = 8mn$.

Proof. Because of

$$\|(AXB - C, DXE - F)\|^2 = \|AXB - C\|^2 + \|DXE - F\|^2,$$

we get

$$\min \|(AXB - C, DXE - F)\| \Longleftrightarrow \min \|AXB - C\|^2 + \|DXE - F\|^2.$$

Therefore, the set $\mathbf{Q}_{CH}$ of Problem I can also be represented as

$$\mathbf{Q}_{CH} = \{X | X \in \mathbb{Q}_{CH}^{p\times q}, \|AXB - C\|^2 + \|DXE - F\|^2 = \min\}.$$

According to Lemmas 2.1–2.3, we obtain

$$\begin{aligned} \|AXB - C\|^2 + \|DXE - F\|^2 &= \|A^{\mathbf{R}}X^{\mathbf{R}}B_c^{\mathbf{R}} - C_c^{\mathbf{R}}\|^2 + \|D^{\mathbf{R}}X^{\mathbf{R}}E_c^{\mathbf{R}} - F_c^{\mathbf{R}}\|^2 \\ &= \|((B_c^{\mathbf{R}})^T \otimes A^{\mathbf{R}})vec(X^{\mathbf{R}}) - vec(C_c^{\mathbf{R}})\|^2 \\ &\qquad + \|((E_c^{\mathbf{R}})^T \otimes D^{\mathbf{R}})vec(X^{\mathbf{R}}) - vec(F_c^{\mathbf{R}})\|^2 \\ &= \| \begin{pmatrix} (B_c^{\mathbf{R}})^T \otimes A^{\mathbf{R}} \\ (E_c^{\mathbf{R}})^T \otimes D^{\mathbf{R}} \end{pmatrix} vec(X^{\mathbf{R}}) - \begin{pmatrix} vec(C_c^{\mathbf{R}}) \\ vec(F_c^{\mathbf{R}}) \end{pmatrix} \|^2 \end{aligned}$$

$$= \|\mathcal{KF} vec(X_c^R) - \begin{pmatrix} vec(C_c^{\mathbf{R}}) \\ vec(F_c^{\mathbf{R}}) \end{pmatrix}\|^2$$

$$= \|\mathcal{KFM} \begin{pmatrix} vec(X_1) \\ vec(X_2) \\ vec(X_3) \\ vec(X_4) \end{pmatrix} - \begin{pmatrix} vec(C_c^{\mathbf{R}}) \\ vec(F_c^{\mathbf{R}}) \end{pmatrix}\|^2.$$

For $X = X_1 + X_2\mathbf{i} + X_3\mathbf{j} + X_4\mathbf{k} \in \mathbb{Q}_{CH}^{p\times q}$, according to Lemma 2.7, we get

$$\begin{pmatrix} vec(X_1) \\ vec(X_2) \\ vec(X_3) \\ vec(X_4) \end{pmatrix} = \mathcal{G}_{\mu\nu}^{CH} \begin{pmatrix} vec_{\mu\nu}^{CS}(X_1) \\ vec_{\mu\nu}^{ACS}(X_2) \\ vec_{\mu\nu}^{ACS}(X_3) \\ vec_{\mu\nu}^{ACS}(X_4) \end{pmatrix}.$$

Consequently

$$\|AXB - C\|^2 + \|DXE - F\|^2 = \|\mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH} \begin{pmatrix} vec_{\mu\nu}^{CS}(X_1) \\ vec_{\mu\nu}^{ACS}(X_2) \\ vec_{\mu\nu}^{ACS}(X_3) \\ vec_{\mu\nu}^{ACS}(X_4) \end{pmatrix} - \begin{pmatrix} vec(C_c^{\mathbf{R}}) \\ vec(F_c^{\mathbf{R}}) \end{pmatrix}\|^2.$$

For the real matrix equation

$$\mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH} \begin{pmatrix} vec_{\mu\nu}^{CS}(X_1) \\ vec_{\mu\nu}^{ACS}(X_2) \\ vec_{\mu\nu}^{ACS}(X_3) \\ vec_{\mu\nu}^{ACS}(X_4) \end{pmatrix} = \begin{pmatrix} vec(C_c^{\mathbf{R}}) \\ vec(F_c^{\mathbf{R}}) \end{pmatrix},$$

its least squares solutions can be represented as

$$\begin{pmatrix} vec_{\mu\nu}^{CS}(X_1) \\ vec_{\mu\nu}^{ACS}(X_2) \\ vec_{\mu\nu}^{ACS}(X_3) \\ vec_{\mu\nu}^{ACS}(X_4) \end{pmatrix} = (\mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH})^\dagger \begin{pmatrix} vec(C_c^{\mathbf{R}}) \\ vec(F_c^{\mathbf{R}}) \end{pmatrix} + [I_r - (\mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH})^\dagger(\mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH})]y,$$

where $y \in \mathbb{R}^r$ is an arbitrary vector. Then

$$\begin{pmatrix} vec(X_1) \\ vec(X_2) \\ vec(X_3) \\ vec(X_4) \end{pmatrix} = \mathcal{G}_{\mu\nu}^{CH}(\mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH})^\dagger \begin{pmatrix} vec(C_c^{\mathbf{R}}) \\ vec(F_c^{\mathbf{R}}) \end{pmatrix} + \mathcal{G}_{\mu\nu}^{CH}[I_r - (\mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH})^\dagger(\mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH})]y.$$

Here, if $p = 2m+1, q = 2n+1$, then $\mu = \nu = o$ and $r = 8mn + 4m + 4n + 1$. If $p = 2m+1$, $q = 2n$, then $\mu = o, \nu = e$ and $r = 8mn + 4n$. If $p = 2m, q = 2n+1$, then $\mu = e, \nu = o$ and $r = 8mn + 4m$. If $p = 2m, q = 2n$, then $\mu = \nu = e$ and $r = 8mn$.

Thus, we obtain the set $\mathbf{Q}_{CH}$ in (3.1), and the unique solution $X_{CH} = X_1 + X_2\mathbf{i} + X_3\mathbf{j} + X_4\mathbf{k} \in \mathbf{Q}_{CH}$ of Problem I satisfies (3.2). □

By Theorem 3.1, we can get the condition that there is a centrohermitian solution and the centrohermitian solution set of the quaternion matrix equation (1.1).

Corollary 3.2. *Suppose* $A \in \mathbb{Q}^{m\times p}$, $B \in \mathbb{Q}^{q\times s}$, $C \in \mathbb{Q}^{m\times s}$, $D \in \mathbb{Q}^{m\times p}$, $E \in \mathbb{Q}^{q\times t}$, $F \in \mathbb{Q}^{m\times t}$ *and* $X = X_1 + X_2\mathbf{i} + X_3\mathbf{j} + X_4\mathbf{k} \in \mathbb{Q}_{CH}^{p\times q}$, *and denote* $\mathcal{K} = \begin{pmatrix} (B_c^{\mathbf{R}})^T \otimes A^{\mathbf{R}} \\ (E_c^{\mathbf{R}})^T \otimes D^{\mathbf{R}} \end{pmatrix}$. *Then the quaternion matrix equation* (1.1) *has a centrohermitian solution* $X \in \mathbb{Q}_{CH}^{p\times q}$ *if and only if*

$$[I_{4m(s+t)} - \mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH}(\mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH})^{\dagger}]\begin{pmatrix} vec(C_c^{\mathbf{R}}) \\ vec(F_c^{\mathbf{R}}) \end{pmatrix} = 0. \tag{3.3}$$

When (1.1) *has a centrohermitian solution, the centrohermitian solution set of* (1.1) *is*

$$\mathbf{S}_{CH} = \{X | \begin{pmatrix} vec(X_1) \\ vec(X_2) \\ vec(X_3) \\ vec(X_4) \end{pmatrix} = \mathcal{G}_{\mu\nu}^{CH}(\mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH})^{\dagger}\begin{pmatrix} vec(C_c^{\mathbf{R}}) \\ vec(F_c^{\mathbf{R}}) \end{pmatrix} + \mathcal{G}_{\mu\nu}^{CH}[I_r - (\mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH})^{\dagger}(\mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH})]y\}, \tag{3.4}$$

where $y \in \mathbb{R}^r$ *is an arbitrary vector. In addition, if* (1.1) *has a centrohermitian solution, then* (1.1) *has a unique centrohermitian solution* $X \in \mathbf{S}_{CH}$ *if and only if*

$$R(\mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH}) = \begin{cases} 8mn + 4m + 4n + 1, & if\ p = 2m + 1,\ q = 2n + 1, \\ 8mn + 4n, & if\ p = 2m + 1,\ q = 2n, \\ 8mn + 4m, & if\ p = 2m,\ q = 2n + 1, \\ 8mn, & if\ p = 2m,\ q = 2n. \end{cases}$$

And the unique centrohermitian solution $X = X_1 + X_2\mathbf{i} + X_3\mathbf{j} + X_4\mathbf{k} \in \mathbf{S}_{CH}$ *satisfies*

$$\begin{pmatrix} vec(X_1) \\ vec(X_2) \\ vec(X_3) \\ vec(X_4) \end{pmatrix} = \mathcal{G}_{\mu\nu}^{CH}(\mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH})^{\dagger}\begin{pmatrix} vec(C_c^{\mathbf{R}}) \\ vec(F_c^{\mathbf{R}}) \end{pmatrix}. \tag{3.5}$$

Here, if $p = 2m + 1, q = 2n + 1$, *then* $\mu = \nu = o$ *and* $r = 8mn + 4m + 4n + 1$. *If* $p = 2m + 1$, $q = 2n$, *then* $\mu = o, \nu = e$ *and* $r = 8mn + 4n$. *If* $p = 2m, q = 2n + 1$, *then* $\mu = e, \nu = o$ *and* $r = 8mn + 4m$. *If* $p = 2m, q = 2n$, *then* $\mu = \nu = e$ *and* $r = 8mn$.

Proof. The quaternion matrix equation (1.1) has a centrohermitian solution X if and only if X satisfies $(AXB, DXE) = (C, F)$. Notice that

$$\begin{aligned} &(AXB, DXE) = (C, F) \\ \Longleftrightarrow &(AXB - C, DXE - F) = 0 \\ \Longleftrightarrow &\|AXB - C, DXE - F\| = 0 \\ \Longleftrightarrow &\|AXB - C\|^2 + \|DXE - F\|^2 = 0. \end{aligned}$$

In addition,

$$\|AXB - C\|^2 + \|DXE - F\|^2$$

$$
=\|\mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH}\begin{pmatrix} vec_{\mu\nu}^{CS}(X_1)\\ vec_{\mu\nu}^{ACS}(X_2)\\ vec_{\mu\nu}^{ACS}(X_3)\\ vec_{\mu\nu}^{ACS}(X_4)\end{pmatrix}-\begin{pmatrix} vec(C_c^{\mathbf{R}})\\ vec(F_c^{\mathbf{R}})\end{pmatrix}\|^2
$$

$$
=\|\mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH}(\mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH})^{\dagger}(\mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH})\begin{pmatrix} vec_{\mu\nu}^{CS}(X_1)\\ vec_{\mu\nu}^{ACS}(X_2)\\ vec_{\mu\nu}^{ACS}(X_3)\\ vec_{\mu\nu}^{ACS}(X_4)\end{pmatrix}-\begin{pmatrix} vec(C_c^{\mathbf{R}})\\ vec(F_c^{\mathbf{R}})\end{pmatrix}\|^2
$$

$$
=\|\mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH}(\mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH})^{\dagger}\begin{pmatrix} vec(C_c^{\mathbf{R}})\\ vec(F_c^{\mathbf{R}})\end{pmatrix}-\begin{pmatrix} vec(C_c^{\mathbf{R}})\\ vec(F_c^{\mathbf{R}})\end{pmatrix}\|^2
$$

$$
=\|[I_{4m(s+t)}-\mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH}(\mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH})^{\dagger}]\begin{pmatrix} vec(C_c^{\mathbf{R}})\\ vec(F_c^{\mathbf{R}})\end{pmatrix}\|^2.
$$

So, (1.1) has a centrohermitian solution if and only if

$$
[I_{4m(s+t)}-\mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH}(\mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH})^{\dagger}]\begin{pmatrix} vec(C_c^{\mathbf{R}})\\ vec(F_c^{\mathbf{R}})\end{pmatrix}=0.
$$

Thus (3.3) holds. When (1.1) has a centrohermitian solution, the centrohermitian solution of (1.1) is the general solution of the following real matrix equation

$$
\mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH}\begin{pmatrix} vec_{\mu\nu}^{CS}(X_1)\\ vec_{\mu\nu}^{ACS}(X_2)\\ vec_{\mu\nu}^{ACS}(X_3)\\ vec_{\mu\nu}^{ACS}(X_4)\end{pmatrix}=\begin{pmatrix} vec(C_c^{\mathbf{R}})\\ vec(F_c^{\mathbf{R}})\end{pmatrix}.
$$

So we obtain the formula in (3.4). In addition, (1.1) has a unique centrohermitian solution if and only if

$$
R(\mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH})=\begin{cases} 8mn+4m+4n+1, \quad if\ p=2m+1,\ q=2n+1,\\ 8mn+4n, \quad if\ p=2m+1,\ q=2n,\\ 8mn+4m, \quad if\ p=2m,\ q=2n+1,\\ 8mn, \quad if\ p=2m,\ q=2n.\end{cases}
$$

And the unique centrohermitian solution $X=X_1+X_2\mathbf{i}+X_3\mathbf{j}+X_4\mathbf{k}\in\mathbf{S}_{CH}$ satisfies (3.5). □

The method of solving Problem II is similar to that of Problem I, so we only give the conclusions and omit the details.

Theorem 3.3. *Suppose* $A\in\mathbb{Q}^{m\times p}$, $B\in\mathbb{Q}^{q\times s}$, $C\in\mathbb{Q}^{m\times s}$, $D\in\mathbb{Q}^{m\times p}$, $E\in\mathbb{Q}^{q\times t}$, $F\in\mathbb{Q}^{m\times t}$ *and* $X=X_1+X_2\mathbf{i}+X_3\mathbf{j}+X_4\mathbf{k}\in\mathbb{Q}_{SCH}^{p\times q}$, *and denote* $\mathcal{K}=\begin{pmatrix}(B_c^{\mathbf{R}})^T\otimes A^{\mathbf{R}}\\ (E_c^{\mathbf{R}})^T\otimes D^{\mathbf{R}}\end{pmatrix}$. *Then the set* $\mathbf{Q}_{SCH}$ *in Problem II can be represented as*

$$
\mathbf{Q}_{SCH}=\{X|\begin{pmatrix} vec(X_1)\\ vec(X_2)\\ vec(X_3)\\ vec(X_4)\end{pmatrix}=\mathcal{G}_{\mu\nu}^{SCH}(\mathcal{KFM}\mathcal{G}_{\mu\nu}^{SCH})^{\dagger}\begin{pmatrix} vec(C_c^{\mathbf{R}})\\ vec(F_c^{\mathbf{R}})\end{pmatrix}+\mathcal{G}_{\mu\nu}^{SCH}[I_r-(\mathcal{KFM}\mathcal{G}_{\mu\nu}^{SCH})^{\dagger}(\mathcal{KFM}\mathcal{G}_{\mu\nu}^{SCH})]y\},
\tag{3.6}
$$

where $y \in \mathbb{R}^r$ is an arbitrary vector. The unique solution $X_{SCH} = X_1 + X_2\mathbf{i} + X_3\mathbf{j} + X_4\mathbf{k} \in \mathbf{Q}_{SCH}$ of Problem II satisfies

$$\begin{pmatrix} vec(X_1) \\ vec(X_2) \\ vec(X_3) \\ vec(X_4) \end{pmatrix} = \mathcal{G}_{\mu\nu}^{SCH}(\mathcal{KFM}\mathcal{G}_{\mu\nu}^{SCH})^{\dagger} \begin{pmatrix} vec(C_c^{\mathbf{R}}) \\ vec(F_c^{\mathbf{R}}) \end{pmatrix}. \tag{3.7}$$

Here, if $p = 2m + 1, q = 2n + 1$, then $\mu = \nu = o$ and $r = 8mn + 4m + 4n + 3$. If $p = 2m + 1$, $q = 2n$, then $\mu = o, \nu = e$ and $r = 8mn + 4n$. If $p = 2m, q = 2n + 1$, then $\mu = e, \nu = o$ and $r = 8mn + 4m$. If $p = 2m, q = 2n$, then $\mu = \nu = e$ and $r = 8mn$.

Corollary 3.4. *Suppose $A \in \mathbb{Q}^{m\times p}$, $B \in \mathbb{Q}^{q\times s}, C \in \mathbb{Q}^{m\times s}, D \in \mathbb{Q}^{m\times p}$, $E \in \mathbb{Q}^{q\times t}, F \in \mathbb{Q}^{m\times t}$ and $X = X_1 + X_2\mathbf{i} + X_3\mathbf{j} + X_4\mathbf{k} \in \mathbb{Q}_{SCH}^{p\times q}$, and denote $\mathcal{K} = \begin{pmatrix} (B_c^{\mathbf{R}})^T \otimes A^{\mathbf{R}} \\ (E_c^{\mathbf{R}})^T \otimes D^{\mathbf{R}} \end{pmatrix}$. Then* (1.1) *has a skew centrohermitian solution $X \in \mathbb{Q}_{SCH}^{p\times q}$ if and only if*

$$[I_{4m(s+t)} - \mathcal{KFM}\mathcal{G}_{\mu\nu}^{SCH}(\mathcal{KFM}\mathcal{G}_{\mu\nu}^{SCH})^{\dagger}] \begin{pmatrix} vec(C_c^{\mathbf{R}}) \\ vec(F_c^{\mathbf{R}}) \end{pmatrix} = 0. \tag{3.8}$$

When (1.1) *has a skew centrohermitian solution, the skew centrohermitian solution set of* (1.1) *is*

$$\mathbf{S}_{SCH} = \{X| \begin{pmatrix} vec(X_1) \\ vec(X_2) \\ vec(X_3) \\ vec(X_4) \end{pmatrix} = \mathcal{G}_{\mu\nu}^{SCH}(\mathcal{KFM}\mathcal{G}_{\mu\nu}^{SCH})^{\dagger} \begin{pmatrix} vec(C_c^{\mathbf{R}}) \\ vec(F_c^{\mathbf{R}}) \end{pmatrix} + \mathcal{G}_{\mu\nu}^{SCH}[I_r - (\mathcal{KFM}\mathcal{G}_{\mu\nu}^{SCH})^{\dagger}(\mathcal{KFM}\mathcal{G}_{\mu\nu}^{SCH})]y\}, \tag{3.9}$$

where $y \in \mathbb{R}^r$ is arbitrary vector. In addition, if (1.1) *has a skew centrohermitian solution, then* (1.1) *has a unique skew centrohermitian solution $X \in \mathbf{S}_{SCH}$ if and only if*

$$R(\mathcal{KFM}\mathcal{G}_{\mu\nu}^{SCH}) = \begin{cases} 8mn + 4m + 4n + 3, & if\ p = 2m + 1,\ q = 2n + 1, \\ 8mn + 4n, & if\ p = 2m + 1,\ q = 2n, \\ 8mn + 4m, & if\ p = 2m,\ q = 2n + 1, \\ 8mn, & if\ p = 2m,\ q = 2n. \end{cases}$$

And the unique skew centrohermitian solution $X = X_1 + X_2\mathbf{i} + X_3\mathbf{j} + X_4\mathbf{k} \in \mathbf{S}_{SCH}$ satisfies

$$\begin{pmatrix} vec(X_1) \\ vec(X_2) \\ vec(X_3) \\ vec(X_4) \end{pmatrix} = \mathcal{G}_{\mu\nu}^{SCH}(\mathcal{KFM}\mathcal{G}_{\mu\nu}^{SCH})^{\dagger} \begin{pmatrix} vec(C_c^{\mathbf{R}}) \\ vec(F_c^{\mathbf{R}}) \end{pmatrix}. \tag{3.10}$$

Here, if $p = 2m + 1, q = 2n + 1$, then $\mu = \nu = o$ and $r = 8mn + 4m + 4n + 3$. If $p = 2m + 1$, $q = 2n$, then $\mu = o, \nu = e$ and $r = 8mn + 4n$. If $p = 2m, q = 2n + 1$, then $\mu = e, \nu = o$ and $r = 8mn + 4m$. If $p = 2m, q = 2n$, then $\mu = \nu = e$ and $r = 8mn$.

Remark 1. It is generally known that quaternion operations are more complex than real operations. In the present paper, we convert the quaternion least squares problems into the corresponding real least squares problems. The final results are expressed only by real matrices and real vectors, and thus they are portable and convenient.

Remark 2. In [23], the authors studied the matrix nearness problem of the quaternion matrix equation (1.1) by the complex representation matrices of quaternion matrices. In fact, if $X_0 = 0$ of Problem 2 in [23], then the solution of Problem 2 is also the solution of Problem I in this paper. We will compare the accuracy of the solution of Problem I computed by these two methods in Section 4.

4. Numerical algorithms and examples

In this section, we first purpose two numerical algorithms for Problems I and II according to the results in Sections 3, and then give two examples to verify our purposed algorithms.

Algorithm 4.1. *(For Problem I)*

(1) *Input* $A_1, A_2, A_3, A_4 \in \mathbb{R}^{m\times p}$, $B_1, B_2, B_3, B_4 \in \mathbb{R}^{q\times s}$, $C_1, C_2, C_3, C_4 \in \mathbb{R}^{m\times s}$, $D_1, D_2, D_3, D_4 \in \mathbb{R}^{m\times p}$, $E_1, E_2, E_3, E_4 \in \mathbb{R}^{q\times t}$, $F_1, F_2, F_3, F_4 \in \mathbb{R}^{m\times t}$, *and* $\mathcal{F}, \mathcal{M}$, $G_{\mu\nu}^{CS}$, $G_{\mu\nu}^{ACS}$.

(2) *Form* $A^{\mathbf{R}}$, $D^{\mathbf{R}}$, $B_c^{\mathbf{R}}$, $E_c^{\mathbf{R}}$, $C_c^{\mathbf{R}}$, $F_c^{\mathbf{R}}$. *Then we calculate* $\mathcal{G}_{\mu\nu}^{CH}$ *and* $\mathcal{K}$ *according to Lemma 2.7 and Theorem 3.1, respectively.*

(3) *Compute the unique solution* X_{CH} *of Problem I on the base of* (3.2).

Algorithm 4.2. *(For Problem II)*

(1) *Input* $A_1, A_2, A_3, A_4 \in \mathbb{R}^{m\times p}$, $B_1, B_2, B_3, B_4 \in \mathbb{R}^{q\times s}$, $C_1, C_2, C_3, C_4 \in \mathbb{R}^{m\times s}$, $D_1, D_2, D_3, D_4 \in \mathbb{R}^{m\times p}$, $E_1, E_2, E_3, E_4 \in \mathbb{R}^{q\times t}$, $F_1, F_2, F_3, F_4 \in \mathbb{R}^{m\times t}$, *and* $\mathcal{F}, \mathcal{M}$, $G_{\mu\nu}^{CS}$, $G_{\mu\nu}^{ACS}$.

(2) *Form* $A^{\mathbf{R}}$, $D^{\mathbf{R}}$, $B_c^{\mathbf{R}}$, $E_c^{\mathbf{R}}$, $C_c^{\mathbf{R}}$, $F_c^{\mathbf{R}}$. *Then we calculate* $\mathcal{G}_{\mu\nu}^{SCH}$ *and* $\mathcal{K}$ *according to Lemma 2.7 and Theorem 3.3, respectively.*

(3) *Compute the unique solution* X_{SCH} *of Problem II on the base of* (3.7).

Now, in order to verify the effectiveness of Algorithms 4.1–4.2, we give two examples. In the first example, we compute the errors between the actual solutions and the solutions obtained by purposed algorithms for Problems I and II. In the second example, we compare the error of the solution of Problem I computed by Algorithm 4.1 and the algorithm in [23].

Example 4.1. *Let the quaternion matrix equation* $(AXB, DXE) = (C, F)$, *where*

$$A = A_1 + A_2\mathbf{i} + A_3\mathbf{j} + A_4\mathbf{k},\ B = B_1 + B_2\mathbf{i} + B_3\mathbf{j} + B_4\mathbf{k},$$

$$D = D_1 + D_2\mathbf{i} + D_3\mathbf{j} + D_4\mathbf{k},\ E = E_1 + E_2\mathbf{i} + E_3\mathbf{j} + E_4\mathbf{k},$$

and

$$A_i = rand(m, p), B_i = rand(q, s), D_i = rand(m, p), E_i = rand(q, t), i = 1, 2, 3, 4.$$

(1) *Let* $X_{CH} = X_1 + X_2\mathbf{i} + X_3\mathbf{j} + X_4\mathbf{k} \in \mathbb{Q}^{p\times q}$, *in which* X_1 *is a real centrosymmetric matrix, and* X_2, X_3, X_4 *are real anti centrosymmetric matrices. Thus* $X_{CH} \in \mathbb{Q}_{CH}^{p\times q}$. *Let*

$$C = AXB, \quad F = DXE.$$

Therefore (1.1) *has a unique centrohermitian solution* X_{CH}, *and* X_{CH} *is also the unique solution of Problem I. According to Algorithm 4.1, we compute* X_{CH}^*. *Let* $m = k$, $s = t = 2k$. *It is noticed* *that the centrosymmetric matrix and anti centrosymmetric matrix have different forms with the change of matrix order. The order of* X_{CH} *can be divided into the following four cases:*

$$(a)\, p = 2k+1, q = 2k+1, \qquad (b)\, p = 2k+1, q = 2k,$$
$$(c)\, p = 2k, q = 2k+1, \qquad (d)\, p = 2k, q = 2k.$$

Let $k = 1 : 10$ *and the error* $\varepsilon = log10(\|X_{CH}^* - X_{CH}\|)$. *For the above four cases, the errors are shown in Figure 1.*

(2) *Let* $X_{SCH} = X_1 + X_2\mathbf{i} + X_3\mathbf{j} + X_4\mathbf{k} \in \mathbb{Q}^{p\times q}$, *in which* X_1 *is a real anti centrosymmetric matrix, and* X_2, X_3, X_4 *are real centrosymmetric matrices. Thus,* $X_{SCH} \in \mathbb{Q}_{SCH}^{p\times q}$. *Let*

$$C = AXB, \quad F = DXE.$$

Therefore (1.1) *has a unique skew centrohermitian solution* X_{SCH}, *and* X_{SCH} *is the unique solution of Problem II. According to Algorithm 4.2, we compute the solution* X_{SCH}^*. *Let* $m = k, s = t = 2k$, *and the values of* p, q *has the following four cases:*

$$(a)\, p = 2k+1, q = 2k+1, \qquad (b)\, p = 2k+1, q = 2k,$$
$$(c)\, p = 2k, q = 2k+1, \qquad (d)\, p = 2k, q = 2k.$$

Let $k = 1 : 10$ *and the error* $\varepsilon = log10(\|X_{SCH}^* - X_{SCH}\|)$. *The errors are shown in Figure 2.*

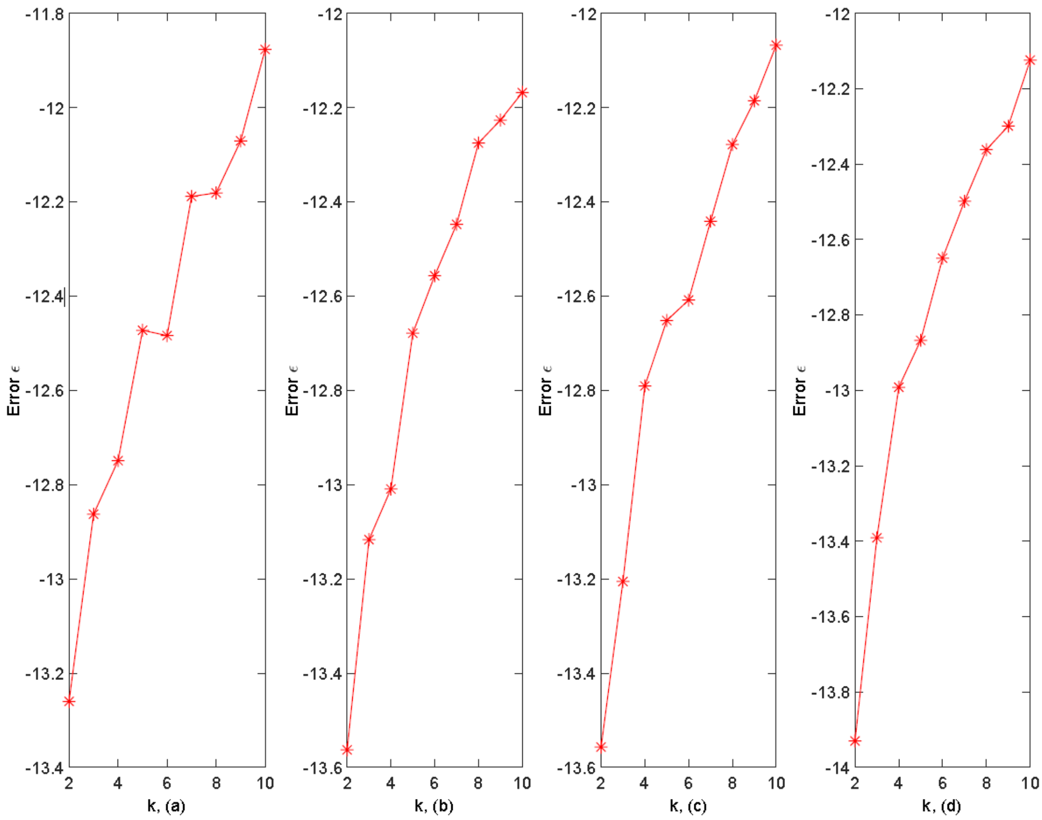

Figure 1. The error for Problem I.

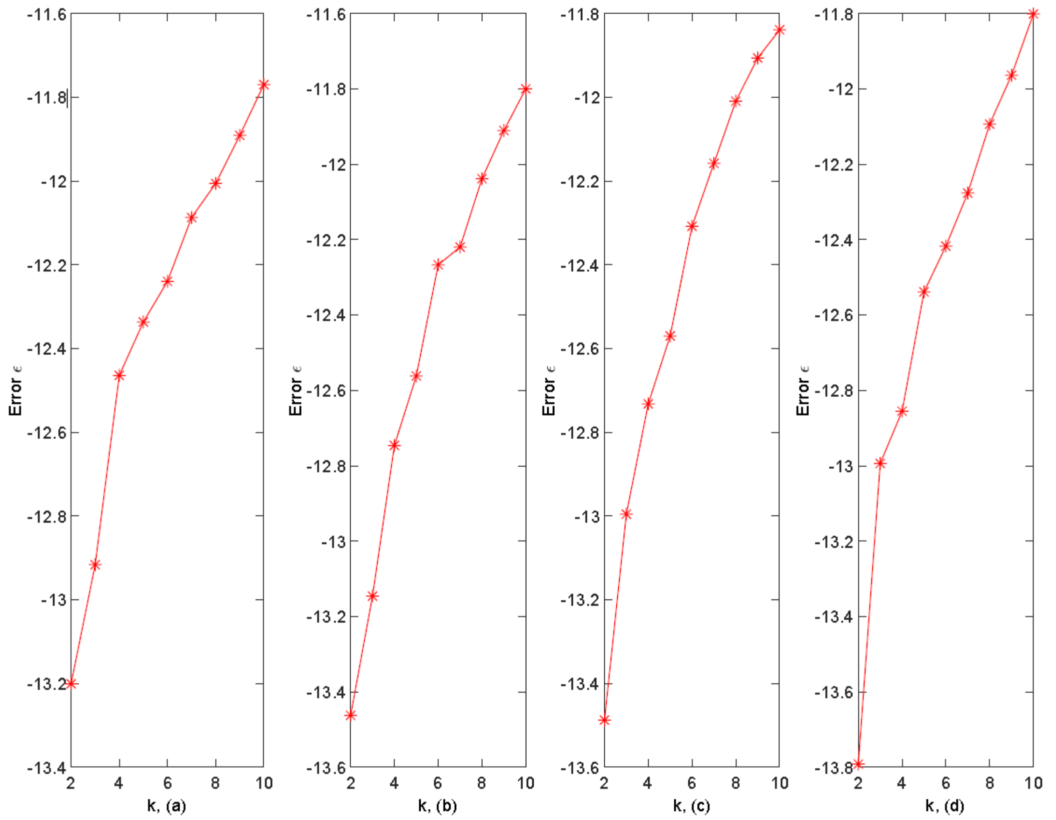

Figure 2. The error for Problem II.

From Figure 1, we know that the errors ε are no more than -11.8, which illustrate Algorithm 4.1 is effective. For the same number k, when $p = 2k + 1, q = 2k + 1$, the order of quaternion matrix is largest. So the corresponding norm is largest. When $p = 2k, q = 2k$, the order of quaternion matrix is smallest. Thus the corresponding norm is smallest. Figure 1 is consistent with theoretical analysis. Figure 2 also reflects Algorithm 4.2 is effective.

Example 4.2. *Consider the solution of Problem I, and the orders of the matrices are the same as (1) in Example 4.1. We compute the errors by Algorithm 4.1 and the algorithm of [23]. Let* $m = k, s = t = 2k, p = 2k, q = 2k + 1$ *and the error* $\varepsilon = log10(\|X^*_{CH} - X_{CH}\|)$. *The errors are shown in Figure 3.*

From Figure 3, we can know that the errors obtained by Algorithm 4.1 are smaller than those obtained by the algorithm in [23], thus our purposed algorithms are more effective.

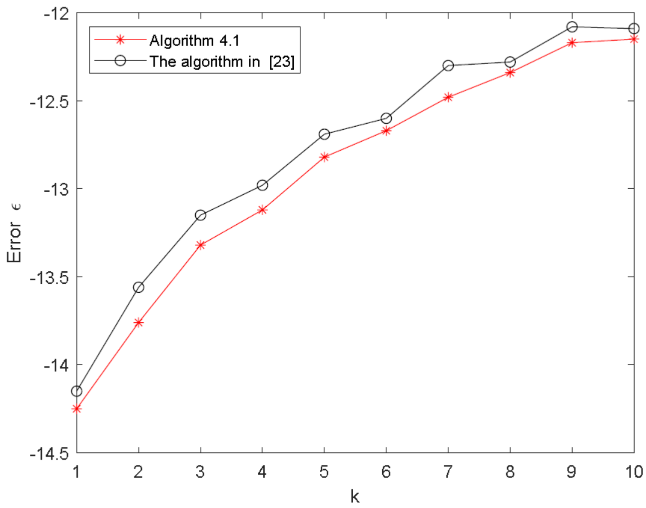

Figure 3. Error comparison of two methods for solving problem I.

5. Conclusions

In this article, we study the minimal norm centrohermitian least squares solution and skew centrohermitian least squares solution of quaternion matrix equation $(AXB, DXE) = (C, F)$ by using of the real representation matrices of quaternion matrices, and give the corresponding algorithms. This method is effective and convenient to analyze the problems of solution with special structures of quaternion matrix equation.

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Conflict of interest

The authors declare that there is no conflict of interest.

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