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Research article

Effect of fractional temporal variation on the vibration of waves on elastic substrates with spatial non-homogeneity

Ahmed SM Alzaidi[1], Ali M Mubaraki[1] and Rahmatullah Ibrahim Nuruddeen[2,*]

[1] Department of Mathematics and Statistics, Collage of Science, Taif University, P. O. Box 11099, Taif 21944, Saudi Arabia

[2] Department of Mathematics, Faculty of Science, Federal University Dutse, P. O. Box 7156 Dutse, Jigawa State, Nigeria

* **Correspondence:** Email: rahmatullah.n@fud.edu.ng; Tel: +2349084181812.

Abstract: The current manuscript examines the effect of the fractional temporal variation on the vibration of waves on non-homogeneous elastic substrates by applying the Laplace integral transform and the asymptotic approach. Four different non-homogeneities, including linear and exponential forms, are considered and scrutinized. In the end, it is reported that the fractional temporal variation significantly affects the respective vibrational fields greatly as the vibrations increase with a decrease in the fractional-order μ. Besides, the two approaches employed for the cylindrical substrates are also shown to be in good agreement for very small non-homogeneity parameter α. More so, the present study is set to play a vital role in the fields of material science, and non-homogenization processes to state a few.

Keywords: Laplace transform; non-homogeneous substrates; elastic waves; fractional order
Mathematics Subject Classification: 73D05, 26A33

1. Introduction

The vibration of waves in diverse elastic media has been a topic of much concern since long ago [1–4]. Various researchers working in wave-related areas are keen to identify new concepts and discoveries that could be taken to the next developmental stage. This very reason is what makes this area to be rich enough to soak up areas like thermo-dynamic, magneto-dynamics, hygro-dynamic, plasticity, and many engineering fields to mention a few, see [5–10] and the references therewith. More so, as the vibration of elastic waves is encountered in different shapes and media, it is easy to find a variety of literature on the vibration of waves in different elastic media like rods, plates, shells, panels, substrates, laminates, composites and multilayered structures, to state a few [11–18].

Additionally, a lot of solution approaches have in the past years been introduced to augment the further conceptual understanding of vibration phenomena in various media. To mention a few of the available approaches, one could find the asymptotic analysis approach [19], normal mode analysis approach [20], integral transform approach [21,22], auxiliary function technique [23], eigenfunctions expansion method [24,25], and many numerical approaches like the finite element methods among others [26,27]. Furthermore, with the reiteration of contemporary fractional derivative theories [28–30], a range of mathematicians and scientists have directed their interest to the examination of dissimilar processes amidst the presence of fractional-order operators. In view of this, so many physical problems were found to be best described with fractional differential operators. Here, we give the instances of the application of fractional operators in modeling diffusion processes and electrical circuits in [31–33], respectively. We also state the fractional calculus relevance in Black-Scholes model [34] and coupled wave-diffusion model in [35], while fractional versions of wave model in irregular domains and diffusion equation were numerically examined in [36,37], sequentially. Equally, fractional calculus has found its way to modeling chaotic systems [38], financial market models [39], and dispersive nonlinear models of fluid dynamics [40], just to mention a few.

However, the current study examines the effect of the fractional temporal variation on the vibration of elastic waves on non-homogeneous rectangular and cylindrical finite substrates, via mainly the application of the Laplace integral transform. The study shall also examine four types of material non-homogeneities comprising exponential and linear cases. Moreover, the inversion of the analytical Laplace transform is set to be carried out with the help of a well-known numerical Laplace inversion scheme in the literature; for various applications of the Laplace transform, and the generalized integral transform-based methods, one could read the above cited references and the references therewith. Additionally, as the vibration of waves in the cylindrical substrate is set to yield Bessel-typed functions, an asymptotic approach will further be employed to yet affirm the exactness of the Laplace solution for small non-homogeneity parameter α. Moreover, needless to mention, the present study is set to play a vital role in the field of material science, design and construction of non-homogeneous structures, and also in the non-homogenization processes, among others.

Besides, the current study is arranged in the following format: Section 2 gives specific preliminary definitions. Section 3 gives the formulation of the governing problem. Sections 4 and 5 present the study of vibration of waves in non-homogeneous rectangular and cylindrical substrates, respectively, while Sections 6 and 7 are reserved for the discussion of results and conclusion, sequentially.

2. Preliminary definitions

The present section outlines the methodology to be adopted while tackling the formulated problem and further mentions specific important definitions related to Caputo's fractional-order derivative.

Definition 2.1. *(Laplace transform) [21] The Laplace transform of the function $f(t)$ is formally defined as*

$$L\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt, \qquad Re(s) > 0. \tag{2.1}$$

Theorem 2.1. *(Existence theorem for the Laplace transform) [21] "If a function $f(t)$ is continuous or piecewise continuous in every finite interval $(0, T)$, and of exponential order e^{bt}, then the Laplace* *transform of $f(t)$ exists for all s provided $Re(s) > b$".*

Theorem 2.2. *(Convergence theorem of the Laplace transform) [21] "If $f(t) = O(e^{bt})$ as $t \to \infty$, then the Laplace integral*

$$\int_0^\infty f(t)e^{-st}dt,$$

is uniformly convergent with respect to s provided $s \geq b_1$, where $b_1 > b$ " (the proofs of Theorems 2.1 and 2.2 have been reported in [21]).

Definition 2.2. *(Inverse Laplace transform) The inverse Laplace transform of the function $F(s)$ is defined by the following complex integral formula:*

$$L^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i}\int_{-i\infty+c}^{i\infty+c} F(s)e^{st}ds. \tag{2.2}$$

Moreover, to invert the Laplace transform using the formula given above, we will use the numerical Laplace inversion scheme by Abate and Valkó [41] whenever the analytical process fails. In fact, this inversion process was a modification to the original Talbot [42] procedure via the application of the trapezoidal rule via multi-precision computing. Therefore, with the step-size π/M and $\psi_k = k\pi/M$, the above Bromwich integral approximates as follows [41]:

$$f(t, M) = \frac{w}{M}\left(\frac{1}{2}F(w)e^{wt} + \sum_{j=1}^{M-1} \Re\left[e^{ts(\psi_k)}F(s(\psi_k))(1 + i\sigma(\psi_k))\right]\right), \tag{2.3}$$

where the parameter w is experimentally fixed as $w = 2M/(5t)$, and $s(\psi)$ is the chosen path over $-\pi < \psi < +\pi$. For the convergence of this algorithm, one could refer to [41] and the references therein.

Definition 2.3. *Given the integrable function $f(t)$, the L transform of the n^{th} ($n \in \mathbb{N}$) derivative of $f(t)$ is given by*

$$L\{f^{(n)}(t)\} = s^n L\{f(t)\} - \sum_{k=0}^{n-1} s^{n-k-1}f^{(k)}(0). \tag{2.4}$$

Definition 2.4. *(Caputo's fractional derivative) [28–30] The Caputo's fractional derivative of the fractional order $\mu > 0$ for a function $f(t)$ is defined by*

$$D_t^\mu(f(t)) = \frac{1}{\Gamma(n-\mu)}\int_0^\infty \frac{f^{(n)}(s)}{(t-s)^{-n+\mu+1}}ds, \quad n-1 < \mu \leqslant n, \tag{2.5}$$

where $f(t) = 0$ for $t < 0$, while $\Gamma(.)$ is the gamma function given for $n(> -1) \in \mathbb{R}$ as $\Gamma(n+1) = n!$.

Definition 2.5. *(Laplace transform for the Caputo's fractional derivative) The Laplace transform for the Caputo's fractional derivative given in Eq (2.1) via Eq (2.5) reads as follows:*

$$L\{D_t^\mu(f(t))\} = s^\mu L\{f(t)\} - \sum_{k=1}^{n} s^{\mu-k-1}f^{(k)}(0), \qquad n-1 < \mu \leq n. \tag{2.6}$$

Definition 2.6. *(Mittag-Leffler function) [30] The Mittag-Leffler function for one parameter μ is defined as*

$$E_{\mu}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(1+\mu n)}, \ \ \mu > 0, \ t \in \mathbb{C}. \tag{2.7}$$

Definition 2.7. *(Bessel equation) The Bessel equation is an ordinary differential equation given by*

$$x^2 \frac{d^2u}{dx^2} + x\frac{du}{dx} + (x^2 - \nu^2)u = 0, \tag{2.8}$$

which admits the following solution:

$$u(x) = A_1 J_{\nu}(x) + A_2 Y_{\nu}(x), \tag{2.9}$$

where A_1 and A_2 are constants, $\nu(\geq 0)$ is the order of the Bessel equation, while $J_{\nu}(x)$ and $Y_{\nu}(x)$ are known as the Bessel functions of first and second kind, correspondingly.

Definition 2.8. *(Generalized form of Bessel equation) [43] The form of the Bessel equation is obtained from the Bessel equation above using the following change of variables:*

$$x = ay^p, \qquad v(y) = y^q u(x),$$

where a, r and s are constants. Thus, the application of the chain rule gives the generalized form of the Bessel equation as follows:

$$y^2 \frac{d^2v}{dy^2} + (1-2q)y\frac{dv}{dy} + [(q^2 - p^2\nu^2) + a^2p^2y^{2p}]v = 0, \tag{2.10}$$

which admits the following solution:

$$v(y) = y^q[A_1 J_{\nu}(ay^p) + A_2 Y_{\nu}(ay^p)], \tag{2.11}$$

where $J_{\nu}(.)$ and $Y_{\nu}(.)$ are the Bessel functions for the first and second kinds, sequentially.

3. Problem formulation

Consider a non-homogeneous elastic substrate of finite length $0 \leq x \leq l$, and having an arbitrary shape. As the substrate is non-homogeneous, the material properties are assumed to be space-dependent, that is, Young's modulus $E(x)$ and the density $\rho(x)$, while the cross-sectional area a remains constant. Moreover, if the fractional temporal derivative is further presumed via Caputo's definition with regards to the acceleration: $\frac{\partial^2 u}{\partial t^2}$, the vibrational field follows the following equation of motion [44]:

$$\frac{\partial}{\partial x}\left(aE(x)\frac{\partial u}{\partial x}\right) = a\rho(x)\frac{\partial^{\mu} u}{\partial t^{\mu}}, \qquad 1 < \mu \leq 2. \tag{3.1}$$

Furthermore, vibration is assumed to be initially at rest using the following initial conditions:

$$u(x,0) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = 0, \quad 0 \leq x \leq l. \tag{3.2}$$

Also, for the boundary conditions, one end of the substrate is tied to a time varying condition $f(t)$, and the other end is fixed as prescribed below:

$$u(0,t) = f(t), \quad u(l,t) = 0, \quad t \geq 0. \tag{3.3}$$

Additionally, as the present study aims to study the effect of the fractional temporal variation on the vibration of elastic waves on the non-homogeneous substrate, the study shall also examine four types of material non-homogeneities comprising the cases as follows:

(1) Type-1:

$$E(x) = E_0 e^{-\alpha x}, \qquad \rho(x) = \rho_0. \tag{3.4}$$

(2) Type-2:

$$E(x) = E_0 e^{-\alpha x}, \qquad \rho(x) = \rho_0 e^{-\alpha x}. \tag{3.5}$$

(3) Type-3:

$$E(x) = E_0(1 + \alpha x), \qquad \rho(x) = \rho_0. \tag{3.6}$$

(4) Type-4:

$$E(x) = E_0(1 + \alpha x), \qquad \rho(x) = \rho_0(1 + \alpha x). \tag{3.7}$$

In the above equations, α is a material non-homogeneity parameter (real constant), E_0 is a constant Young's modulus, while ρ_0 is a constant density. Besides, material non-homogeneity plays a vital role in the dynamic of wave propagation in various elastic media. Various forms of material non-homogeneity could be found in advanced books/papers on solid mechanics and engineering, as they are greatly utilized in the design and fabrication of composite structures. Moreover, the absence of material non-homogeneity reduces Eq (3.1) to the classical one-dimensional equation of motion with fractional temporal variation as follows:

$$E_0 \frac{\partial^2 u}{\partial x^2} = \rho_0 \frac{\partial^\mu u}{\partial t^\mu}, \qquad 1 < \mu \leq 2. \tag{3.8}$$

4. Vibration of waves in non-homogeneous rectangular substrate

The present section examines the effect of the fractional temporal variation on the vibration of elastic waves on the non-homogeneous rectangular substrate. The shape of the substrate here is found to be rectangular by types-1 and 2 non-homogeneities.

4.1. Vibration of waves with type-1 non-homogeneity

The governing equation of motion earlier given in Eq (3.1) through the use of the non-homogeneity type-1 now transforms to the following:

$$\frac{\partial^2 u}{\partial x^2} - \alpha \frac{\partial u}{\partial x} = \frac{1}{c^2} e^{\alpha x} \frac{\partial^\mu u}{\partial t^\mu}, \qquad 1 < \mu \leq 2, \tag{4.1}$$

where $c = \sqrt{E_0/\rho_0}$ is the transverse shear speed, and α is the non-homogeneity parameter.

Next, applying the Laplace transform to the above equation in t, we get

$$\frac{d^2 U}{dx^2} - \alpha \frac{dU}{dx} - \frac{s^\mu}{c^2} e^{\alpha x} U = 0, \qquad 1 < \mu \leq 2, \tag{4.2}$$

of which the auxiliary equation takes the following form:

$$m^2 - \alpha m - \frac{s^\mu}{c^2} e^{\alpha x} = 0, \tag{4.3}$$

which admits the following roots:

$$m_1 = \frac{\alpha - \sqrt{\alpha^2 + 4\frac{s^\mu}{c^2} e^{\alpha x}}}{2}, \qquad m_2 = \frac{\alpha + \sqrt{\alpha^2 + 4\frac{s^\mu}{c^2} e^{\alpha x}}}{2}. \tag{4.4}$$

Therefore, Eq (4.2) admits the following solution:

$$U(x, s) = A_1 e^{m_1 x} + A_2 e^{m_2 x}, \tag{4.5}$$

where A_1 and A_2 are constants to be determined. Thus, upon using the transformed version of the boundary conditions earlier given in Eq (3.3), with $L\{f(t)\} = F(s)$, the solution of Eq (4.2) is found to be

$$U(x, s) = \frac{F(s)}{e^{lm_1} - e^{lm_2}} \left(e^{lm_1 + m_2 x} - e^{lm_2 + m_1 x} \right), \tag{4.6}$$

such that after taking the inverse Laplace transform yields the following closed-form solution for the problem:

$$u(x, t) = \frac{1}{2\pi i} \int_{-i\infty + c}^{i\infty + c} \frac{F(s)}{e^{lm_1} - e^{lm_2}} \left(e^{lm_1 + m_2 x} - e^{lm_2 + m_1 x} \right) e^{st} ds. \tag{4.7}$$

The closed-form solution obtained above will in the subsequent section be evaluated numerically via the application of the numerical Laplace inversion scheme by Abate and Valkó [41].

4.2. Vibration of waves with type-2 non-homogeneity

The governing equation of motion given in Eq (3.1) through the use of the non-homogeneity type-2 now transforms to the following:

$$\frac{\partial^2 u}{\partial x^2} - \alpha \frac{\partial u}{\partial x} = \frac{1}{c^2} \frac{\partial^\mu u}{\partial t^\mu}, \qquad 1 < \mu \leq 2, \tag{4.8}$$

where $c = \sqrt{E_0/\rho_0}$ is the transverse shear speed, and α is the non-homogeneity parameter.

Then, applying the Laplace transform to the above equation in t, we get

$$\frac{d^2 U}{dx^2} - \alpha \frac{dU}{dx} - \frac{s^\mu}{c^2} U = 0, \qquad 1 < \mu \leq 2, \tag{4.9}$$

of which the auxiliary equation takes the following form:

$$m^2 - \alpha m - \frac{s^\mu}{c^2} = 0, \tag{4.10}$$

which admits the following roots:

$$m_3 = \frac{\alpha - \sqrt{\alpha^2 + 4\frac{s^\mu}{c^2}}}{2}, \qquad m_4 = \frac{\alpha + \sqrt{\alpha^2 + 4\frac{s^\mu}{c^2}}}{2}. \tag{4.11}$$

Therefore, Eq (4.9) admits the following solution:

$$U(x,s) = B_1 e^{m_3 x} + B_2 e^{m_4 x}, \tag{4.12}$$

where B_1 and B_2 are constants to be determined. Thus, upon using the transformed version of the boundary conditions earlier given in Eq (13), the solution of Eq (4.12) is found to be

$$U(x,s) = \frac{F(s)}{e^{lm_3} - e^{lm_4}} \left(e^{lm_3 + m_4 x} - e^{lm_4 + m_3 x} \right), \tag{4.13}$$

such that upon taking the inverse Laplace transform yields the following closed-form solution for the problem:

$$u(x,t) = \frac{1}{2\pi i} \int_{-i\infty + c}^{i\infty + c} \frac{F(s)}{e^{lm_3} - e^{lm_4}} \left(e^{lm_3 + m_4 x} - e^{lm_4 + m_3 x} \right) e^{st} ds. \tag{4.14}$$

The closed-form solution obtained above will in the subsequent section be evaluated numerically via the application of the numerical Laplace inversion scheme by Abate and Valkó [41].

5. Vibration of waves in non-homogeneous cylindrical substrate

The present section examines the effect of the fractional temporal variation on the vibration of elastic waves on the non-homogeneous cylindrical substrate. The shape of the substrate here is found to be cylindrical by types-3 and 4 non-homogeneities.

5.1. Vibration of waves with type-3 non-homogeneity

The governing equation of motion given in Eq (3.1) through the use of the non-homogeneity type-3 now transforms to the following:

$$(1 + \alpha x) \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\partial u}{\partial x} = \frac{1}{c^2} \frac{\partial^\mu u}{\partial t^\mu}, \qquad 1 < \mu \leq 2, \tag{5.1}$$

where $c = \sqrt{E_0/\rho_0}$ is the transverse shear speed, and α is the non-homogeneity parameter.

Then, applying the Laplace transform to the above equation in t, we get

$$(1 + \alpha x) \frac{d^2 U}{dx^2} + \alpha \frac{dU}{dx} - \frac{s^\mu}{c^2} U = 0, \qquad 1 < \mu \leq 2. \tag{5.2}$$

Now, let $y = 1 + \alpha x$, then, the above equation transforms to the following equation:

$$y \frac{d^2 U}{dy^2} + \frac{dU}{dy} - \frac{s^\mu}{\alpha^2 c^2} U = 0, \qquad 1 < \mu \leq 2. \tag{5.3}$$

Multiplying the above equation by y becomes

$$y^2 \frac{d^2 U}{dy^2} + y \frac{dU}{dy} - y \frac{s^\mu}{\alpha^2 c^2} U = 0, \qquad 1 < \mu \leq 2. \tag{5.4}$$

In fact, the above equation transforms into a modified Bessel from Eq (2.10), and admits the following solution:

$$U(x,s) = C_1 I_0 \left(\frac{2 s^k}{c\alpha} \sqrt{y} \right) + C_2 K_0 \left(\frac{2 s^k}{c\alpha} \sqrt{y} \right), \tag{5.5}$$

where $k = \frac{\mu}{2}$, C_1 and C_2 are constants to be determined, and $I_0(.)$ and $K_0(.)$ are modified Bessel functions for the first and second kinds, correspondingly. Thus, upon using the transformed version of the boundary conditions earlier given in Eq (3.3), the solution of Eq (5.4) is found to be

$$U(y,s) = R_1(s)\left(I_0\left(\frac{2s^k\sqrt{l\alpha+1}}{c\alpha}\right)K_0\left(\frac{2s^k}{c\alpha}\sqrt{y}\right) - K_0\left(\frac{2s^k\sqrt{l\alpha+1}}{c\alpha}\right)I_0\left(\frac{2s^k}{c\alpha}\sqrt{y}\right)\right), \tag{5.6}$$

where

$$R_1(s) = \frac{F(s)}{K_0\left(\frac{2s^k}{c\alpha}\right)I_0\left(\frac{2s^k\sqrt{l\alpha+1}}{c\alpha}\right) - I_0\left(\frac{2s^k}{c\alpha}\right)K_0\left(\frac{2s^k\sqrt{l\alpha+1}}{c\alpha}\right)}. \tag{5.7}$$

Therefore, on taking the inverse Laplace transform, the following closed-form solution is obtained:

$$u(x,t) = \frac{1}{2\pi i}\int_{-i\infty+c}^{i\infty+c} U(y,s)e^{st}ds. \tag{5.8}$$

Moreover, as the modified Bessel function of the first $I_0(.)$ is unbounded as $s \to \infty$, the above solution thus vanished. However, to approximately overcoming this defect, we resort to using the asymptotic approximations of both $I_0(.)$ and $K_0(.)$ by considering the non-homogeneity parameter α to be very small, that is, $\alpha \ll 1$ [42]. Hence, we asymptotically express these functions as follows:

$$I_0(r) \sim \frac{e^r}{\sqrt{2\pi r}}, \qquad K_0(r) \sim e^{-r}\sqrt{\frac{\pi}{2r}}, \tag{5.9}$$

and further rewrite Eq (5.6) asymptotically as follows:

$$\tilde{U}(y,s) \sim R_2(s)\left(e^{\frac{4\sqrt{y}s^k}{\alpha c}} - e^{\frac{4\sqrt{\alpha l+1}s^k}{\alpha c}}\right)y^{-1/4}e^{-\frac{2(\sqrt{y}-1)s^k}{\alpha c}}, \tag{5.10}$$

where

$$R_2(s) = \frac{F(s)}{e^{\frac{4s^k}{\alpha c}} - e^{\frac{4\sqrt{\alpha l+1}s^k}{\alpha c}}}. \tag{5.11}$$

Therefore, on taking the inverse Laplace transform of Eq (5.10), the following asymptotic closed-form solution is obtained:

$$\tilde{u}(y,t) \sim \frac{1}{2\pi i}\int_{-i\infty+c}^{i\infty+c} \tilde{U}(y,s)e^{st}ds. \tag{5.12}$$

Hence, the exact closed-form solution given in Eq (5.6) and that of the asymptotic one obtained in Eq (5.12) will in the subsequent section be evaluated numerically via the application of the numerical Laplace inversion scheme by Abate and Valkó [41]. More so, we shall compare the two solutions to be able to make sense of the two approaches.

5.2. *Vibration of waves with type-4 non-homogeneity*

The governing equation of motion earlier given in Eq (3.1) through the use of the non-homogeneity type-4 now transforms to the following:

$$(1+\alpha x)\frac{\partial^2 u}{\partial x^2} + \alpha\frac{\partial u}{\partial x} = \frac{1}{c^2}(1+\alpha x)\frac{\partial^\mu u}{\partial t^\mu}, \qquad 1 < \mu \le 2, \tag{5.13}$$

where $c = \sqrt{E_0/\rho_0}$ is the transverse shear speed, and α is the non-homogeneity parameter.

Then, applying the Laplace transform to the above equation in t, we get

$$(1+\alpha x)\frac{d^2U}{dx^2} + \alpha\frac{dU}{dx} - (1+\alpha x)\frac{s^\mu}{c^2}U = 0, \qquad 1 < \mu \leq 2. \tag{5.14}$$

Now, let us let $y = 1 + \alpha x$, then, the above equation transforms to the following equation:

$$y\frac{d^2U}{dy^2} + \frac{dU}{dy} - y\frac{s^\mu}{\alpha^2c^2}U = 0, \qquad 1 < \mu \leq 2, \tag{5.15}$$

such that after multiplying by y becomes

$$y^2\frac{d^2U}{dy^2} + y\frac{dU}{dy} - y^2\frac{s^\mu}{\alpha^2c^2}U = 0, \qquad 1 < \mu \leq 2. \tag{5.16}$$

In fact, the above equation is a modified Bessel from Eq (2.10), and admits the following solution:

$$U(x,s) = D_1 I_0\left(\frac{s^k}{c\alpha}y\right) + D_2 K_0\left(\frac{s^k}{c\alpha}y\right), \tag{5.17}$$

where $k = \frac{\mu}{2}$, D_1 and D_2 are constants to be determined, and $I_0(.)$ and $K_0(.)$ are modified Bessel functions for the first and second kinds, correspondingly. Thus, upon using the transformed version of the boundary conditions earlier given in Eq (3.3), the solution of Eq (5.15) is found to be

$$U(y,s) = R_3(s)\left(I_0\left(\frac{s^k(l\alpha+1)}{c\alpha}\right)K_0\left(\frac{s^k}{c\alpha}y\right) - K_0\left(\frac{s^k(l\alpha+1)}{c\alpha}\right)I_0\left(\frac{s^k}{c\alpha}y\right)\right), \tag{5.18}$$

where

$$R_3(s) = \frac{F(s)}{K_0\left(\frac{s^k}{c\alpha}\right)I_0\left(\frac{s^k(l\alpha+1)}{c\alpha}\right) - I_0\left(\frac{s^k}{c\alpha}\right)K_0\left(\frac{s^k(l\alpha+1)}{c\alpha}\right)}. \tag{5.19}$$

Therefore, on taking the inverse Laplace transform of Eq (5.18), the following closed-form solution is thus obtained:

$$u(x,t) = \frac{1}{2\pi i}\int_{-i\infty+c}^{i\infty+c} U(y,s)e^{st}ds. \tag{5.20}$$

Moreover, as preceded, we asymptotically express the above results to as follows:

$$\tilde{U}(y,s) \sim R_4(s)\left(e^{\frac{2ys^k}{\alpha c}} - e^{\frac{2(\alpha l+1)s^k}{\alpha c}}\right)y^{-1/2}e^{-\frac{(y+1)s^k}{\alpha c}}, \tag{5.21}$$

where

$$R_4(s) = \frac{F(s)}{1 - e^{\frac{2ls^k}{c}}}. \tag{5.22}$$

Therefore, on taking the inverse Laplace transform of Eq (5.21), the following closed-form solution is obtained

$$\tilde{u}(y,t) \sim \frac{1}{2\pi i}\int_{-i\infty+c}^{i\infty+c} \tilde{U}(y,s)e^{st}ds. \tag{5.23}$$

Thus, the exact closed-form solution given in Eq (5.20) and that of the asymptotic one obtained in Eq (5.23) will in the subsequent section be evaluated numerically via the application of the numerical Laplace inversion scheme by Abate and Valkó [41]. More so, we shall compare the two solutions to be able to make sense of the two approaches.

6. Discussion of results

The present study examines the effect of the fractional temporal variation on the vibration of elastic waves on the non-homogeneous substrate. Four types of material non-homogeneities comprising exponential and linear are independently studied, by applying the Laplace integral transform. The governing model typically models a scenario of a vibrating finite elastic substrate with one end fixed, and the other end tied to a time varying fixed condition $f(x)$. Additionally, the substrate is considered to be non-homogeneous. Physically, such scenarios arise as a result of material impurities, prescribed in-homogeneities or the effect of external influence like corrosion, which gradually alters the material constituents of a given structure, see [45] for consideration of an exponentially varying non-homogeneous beam. Remarkably, the homogeneous substrate is obtained when the material properties are assumed constants, that is, Young's modulus $E(x) = E_0$ and the density $\rho(x) = \rho_0$. In such a situation, the vibrational field of a finite homogeneous (isotropic) rectangular substrate is presided over by the following expression:

$$u(x,t) = \frac{1}{2\pi i}\int_{-i\infty+c}^{i\infty+c} \frac{F(s)}{1-e^{\frac{2ls^k}{c}}}\left(e^{\frac{xs^k}{c}} - e^{\frac{s^k}{c}(2l-x)}\right)e^{st}ds, \quad k = \frac{\mu}{2}, \quad 0 \le x \le l, \tag{6.1}$$

and further reduces to the expression for a semi-infinite rectangular substrate as follows:

$$u(x,t) = \frac{1}{2\pi i}\int_{-i\infty+c}^{i\infty+c} F(s)e^{st-\frac{xs^k}{c}}ds, \qquad 0 \le x < \infty, \tag{6.2}$$

which obviously reduces at $\mu = 2$ to the following closed-form expression:

$$u(x,t) = \frac{1}{2\pi i}\int_{-i\infty+c}^{i\infty+c} F(s)e^{s\left(t-\frac{x}{c}\right)}ds, \qquad 0 \le x < \infty.$$

Moreover, as the types-3 and 4 homogeneities posed the modified Bessel function of the first kind $I_0(.)$ which is unbounded as $s \to \infty$, this development necessitates yet asymptotic approximation method to overcome this defect by considering the non-homogeneity parameter α to be very small, that is, $\alpha \ll 1$ [44]. More so, to avoid having any encounter with ∞ while inverting the Laplace transform, we have chosen the function $f(t)$ in Eq (3.3) in the form

$$f(t) = \begin{cases} \sin(2t), & 0 \le t \le 1, \\ 0, & t > 1, \end{cases} \tag{6.3}$$

such that (after applying the Laplace transform on $f(t)$)

$$F(s) = \frac{e^{-s}\left(2e^s - s\sin(2) - 2\cos(2)\right)}{s^2+4}. \tag{6.4}$$

Furthermore, we make consideration to the physical data of aluminum material for the substrate as tabulated in Table 1.

Table 1. Physical data of aluminum material [12].

Young's modulus E_0 ($\times 10^{10}\text{Nm}^{-2}$)	Density ρ_0 ($\times 10^3\text{kgm}^{-3}$)	Transverse shear speed $c = \sqrt{E_0/\rho_o}$ ($\times 10^3$m/s)
2.700	2.643	3.128

So, in Figures 1–4, we portray the nature of wave vibration on non-homogeneous elastic substrates with types-1–4 non-homogeneities with the fractional temporal variation, while Figures 5 and 6 compare the obtained approximate exact closed-form solution and the approximate asymptotic solution in the cases of types-3 and 4 non-homogeneities.

More explicitly, Figure 1 shows the effect of the fractional temporal variation on the vibration of waves on a rectangular elastic substrate in the presence of type-1 non-homogeneity. Firstly, the prescribed boundary conditions are well satisfied at the two endpoints of the rectangular substrate, that is, the vibrational field decreases down the thickness of the substrate x and progresses steadily to rest at 0. Secondly, it is noted that the vibrational field increases with a decrease in the fractional temporal order μ. This is very obvious as the vibrational field happens to be minimum when $\mu = 2$. Proceeding to Figure 2, the same interpretation of Figure 1 applies to Figure 2 only that the non-homogeneity is type-2. What's more, one could easily observe the effect of the exponential term $e^{\alpha x}$ from Figure 1 against its absence in Figure 2.

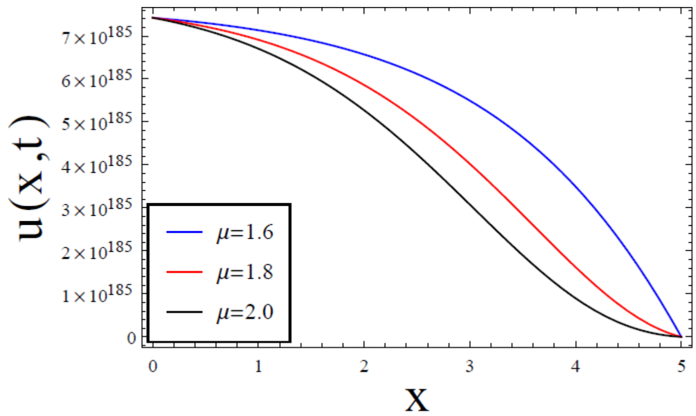

Figure 1. Effect of fractional temporal variation on the vibration of waves on a rectangular substrate with type-1 non-homogeneity when $t = 0.075$, $\alpha = 0.65$.

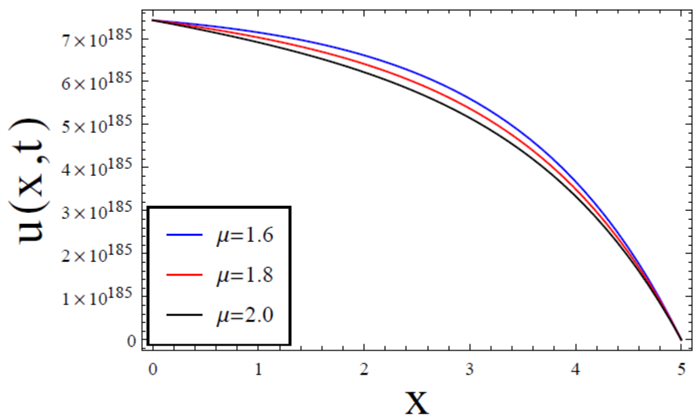

Figure 2. Effect of fractional temporal variation on the vibration of waves on a rectangular substrate with type-2 non-homogeneity when $t = 0.075$, $\alpha = 0.65$.

Figures 3 and 4 show the effect of the fractional temporal variation on the vibration of waves on a cylindrical elastic substrate in the presence of types-1 and 2 non-homogeneity, respectively. Firstly, the prescribed boundary conditions are well satisfied at the two end points of the cylindrical substrate, that is, the vibrational field decreases down the thickness of the substrate x and progresses steadily to rest at 0. Additionally, the vibrational fields in both figures increase with a decrease in the fractional temporal order μ. Moreover, the two figures could equally be seen as a rescaling of each other considering the modified Bessel functions posed by the two cases, regardless of the transformation made with regards to type-3 non-homogeneity.

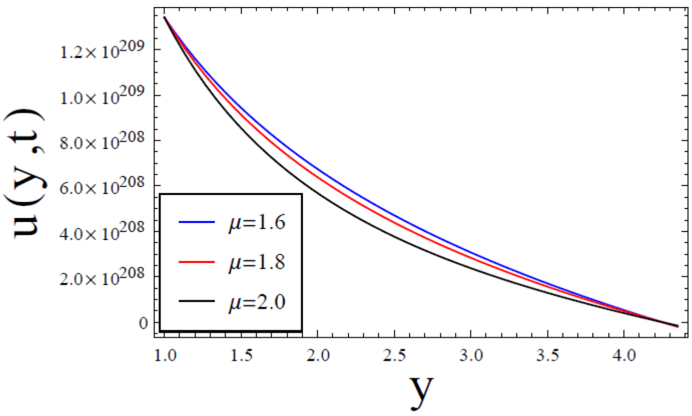

Figure 3. Effect of fractional temporal variation on the vibration of waves on a cylindrical substrate with type-3 non-homogeneity when $t = 0.067$, $\alpha = 0.65$.

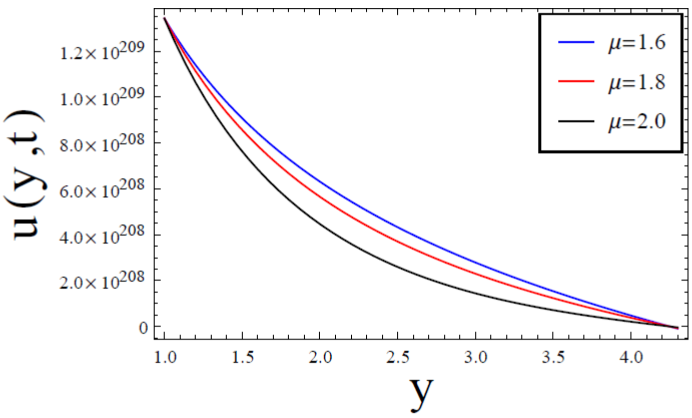

Figure 4. Effect of fractional temporal variation on the vibration of waves on a cylindrical substrate with type-4 non-homogeneity when $t = 0.067$, $\alpha = 0.65$.

Finally, Figures 5 and 6 compare the approximate exact closed-form solution and that of the asymptotic one obtained in the cases of types-3 and 4 non-homogeneities, respectively, by plotting the two vibrational fields. In these plots, we have considered the non-homogeneity parameter α to be very small, that is, $\alpha \ll 1$. This is, of course, due to the existence of the Laplace inversion of the modified Bessel function of the second kind, $K_0(.)$. However, we have bypassed this defect by defining a finite domain for $f(t)$. In both figures, one could observe an agreement between the two results, even though the agreement slacks a bit as it progresses down the thickness of the cylindrical substrate. Moreover, the numerical comparisons of the respective absolute error differences between the approximate exact and approximate asymptotic solutions are reported in Table 2 for the types-3 and 4 non-homogeneities, respectively. From this table, one could notice an increase in the error as the propagation is progressed. This is obvious in favor of the asymptotic approximation of the modified Bessel functions $I_0(.)$ and $K_0(.)$. Thus, one could see the advantage of the employed numerical Laplace inversion scheme over the asymptotic approach, as the asymptotic approach truncated a large part of the result. More so, upon expanding the asymptotic expression in Binomial series, yet another approximate closed-form solution is revealed amidst the presence of Heaviside function [44].

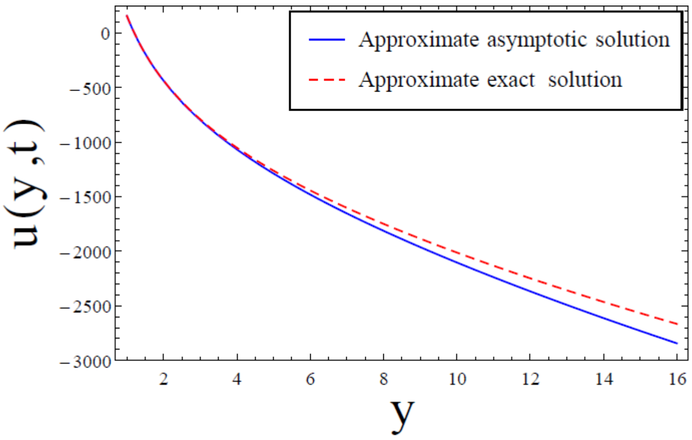

Figure 5. Comparison of the approximate asymptotic and approximate exact solutions of the cylindrical substrate with type-3 non-homogeneity when $t = 0.85$, $\alpha = 0.04$, $\mu = 1.9$.

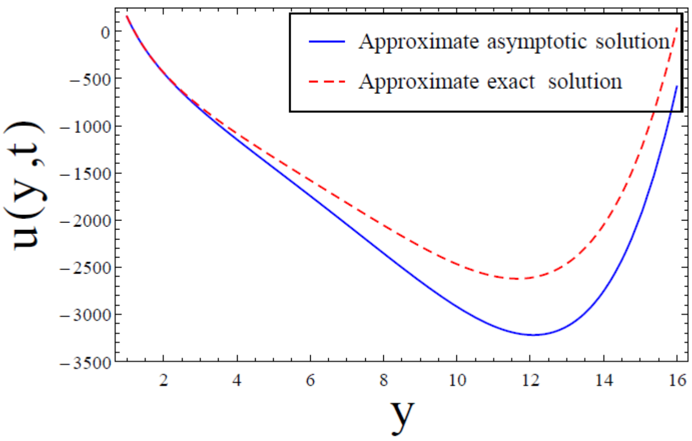

Figure 6. Comparison of the approximate asymptotic and approximate exact solutions of the cylindrical substrate with type-4 non-homogeneity when $t = 0.85$, $\alpha = 0.04$, $\mu = 1.9$.

Table 2. Absolute error difference between the approximate exact and approximate asymptotic solutions for the non-homogeneous cylindrical substrate when $t = 0.85$, $\alpha = 0.04$, $\mu = 1.9$.

x	Type-3 non-homogeneity $\lvert u(y,t) - \tilde{u}(y,t)\rvert$	Type-4 non-homogeneity $\lvert u(y,t) - \tilde{u}(y,t)\rvert$
1.00	3.97904×10^{-13}	7.38964×10^{-13}
1.35	2.03081×10^{-2}	8.11841×10^{-2}
1.70	2.73346×10^{-2}	1.09407×10^{0}
2.05	1.21177×10^{0}	4.86138×10^{0}
2.40	2.77234×10^{0}	1.11580×10^{1}
2.75	4.86571×10^{0}	1.96628×10^{1}
3.10	7.40119×10^{0}	3.00552×10^{1}
3.45	1.03012×10^{1}	4.20699×10^{1}
3.80	1.35025×10^{1}	5.55016×10^{1}
4.15	1.69543×10^{1}	7.01959×10^{1}

7. Conclusions

To conclude the current manuscript, a vibration problem for non-homogeneous rectangular and cylindrical elastic substrates has been formulated and examined by applying the Laplace integral transform. Four different non-homogeneities, including linear and exponential forms, were considered and securitized, taking into account the effect of the fractional temporal variation. A numerical Laplace inversion scheme was further utilized to reverse the transformed solution back to its original domain, in addition to the deployment of an asymptotic approach, specifically to tackle the vibration of waves in cylindrical substrates. Above and beyond, the effect of the fractional temporal variation has been noted to increase the vibrational fields greatly with a decrease in the fractional temporal order μ. Also, with regards to the cylindrical substrates with were governed by types-3 and 4 non-homogeneities, the approximate exact closed-form solution and that of the asymptotic one are found to be in good agreement for very small non-homogeneity parameter α. Moreover, the present study is set to play a vital role in the field of material science, design and construction of non-homogeneous structures, and also in the non-homogenization processes, among others.

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Conflict of interest

The authors declare that they have no competing interests.

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