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Research article

A new characterization of hyperbolic cylinder in anti-de Sitter space $\mathbb{H}_1^5(-1)$

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Abstract: By investigating complete Willmore maximal spacelike hypersurfaces with constant scalar curvature in anti-de Sitter space $\mathbb{H}_1^5(-1)$, we give a new characterization of hyperbolic cylinder $\mathbb{H}^2(-2)\times\mathbb{H}^2(-2)$ in $\mathbb{H}_1^5(-1)$.

Keywords: anti-de Sitter space; Willmore maximal spacelike hypersurfaces; constant scalar curvature
Mathematics Subject Classification: 53C42, 53C50

1. Introduction

Let $N_1^{n+1}(c)$ be an $(n+1)$-dimensional Lorentzian space form of constant sectional curvature c. According to $c > 0$, $c = 0$ or $c < 0$, it is denoted by $\mathbb{S}_1^{n+1}(c)$, $\mathbb{R}_1^{n+1}$ or $\mathbb{H}_1^{n+1}(c)$, respectively. A hypersurface M of $N_1^{n+1}(c)$ is said to be spacelike if the induced metric on M from that of $N_1^{n+1}(c)$ is positive definite. Moreover, M is called maximal if its mean curvature vanishes identically.

Calabi [1] first studied the Bernstein problem for complete maximal spacelike entire graphs in $\mathbb{R}_1^{n+1}$ and proved that it must be a hyperplane, when $n \le 4$. Later, Cheng and Yau [8] showed that this conclusion remains true for arbitrary n. For $c \ge 0$, Cheng and Yau [8] and Ishihara [10] proved that complete maximal spacelike submanifolds are totally geodesic. Furthermore, Ishihara [10] also proved the following:

Theorem 1.1. [10] *Let M be a complete maximal spacelike hypersurface in $\mathbb{H}_1^{n+1}(-1)$, and let S be the squared norm of the second fundamental form of M. Then,*

$$S \le n,$$

and $S = n$ if and only if $M = \mathbb{H}^m(-\frac{n}{m}) \times \mathbb{H}^{n-m}(-\frac{n}{n-m})(1 \le m \le n-1)$.

There are some interesting results related to the study of maximal spacelike hypersurfaces with constant scalar curvature or constant Gauss-Kronecker curvature in anti-de Sitter space $\mathbb{H}_1^{n+1}(-1)$ (see [2–7]).

Recently, Deng-Gu-Wei [9] proved that closed Willmore minimal hypersurfaces with constant scalar curvature in $\mathbb{S}^5(1)$ are isoparametric. Motivated by Deng-Gu-Wei's paper, in this paper we investigate complete Willmore maximal spacelike hypersurfaces with constant scalar curvature in anti-de Sitter space $\mathbb{H}^5_1(-1)$, and give a new characterization of hyperbolic cylinder $\mathbb{H}^2(-2) \times \mathbb{H}^2(-2)$ in $\mathbb{H}^5_1(-1)$.

Theorem 1.2. *Let M be a complete Willmore maximal spacelike hypersurface in anti-de Sitter space $\mathbb{H}^5_1(-1)$ with constant scalar curvature. If there exists a point with two distinct principal curvatures, then M is the hyperbolic cylinder $\mathbb{H}^2(-2) \times \mathbb{H}^2(-2)$.*

2. Preliminaries

In this section, we give some formulas and notations of maximal spacelike hypersurfaces in an $(n+1)$-dimensional Lorentzian space form $N^{n+1}_1(c)$ with constant sectional curvature c.

Let M be a connected spacelike hypersurface in $N^{n+1}_1(c)$. We choose a local frame of orthonormal vector fields $e_1, \cdots, e_{n+1}$ adapted to the indefinite Riemannian metric of $N^{n+1}_1(c)$ and the dual coframe $\{\omega_1, \cdots, \omega_{n+1}\}$ in such a way that, restricted to M, $e_1, \cdots, e_n$ are tangent to M and e_{n+1} is normal to M.

We will agree on the following index convention:

$$1 \le i, j, k, \cdots \le n; \qquad 1 \le A, B, C, \cdots \le n+1.$$

Then the connection forms $\{\omega_{AB}\}$ of $N^{n+1}_1(c)$ are characterized by the following structure equations:

$$\begin{aligned} d\omega_A &= -\sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \\ d\omega_{AB} &= -\sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \varepsilon_C \varepsilon_D K_{ABCD} \omega_C \wedge \omega_D, \end{aligned}$$

where $\varepsilon_i = 1$ for $1 \le i \le n$, $\varepsilon_{n+1} = -1$ and

$$K_{ABCD} = c\varepsilon_A \varepsilon_B (\delta_{AD}\delta_{BC} - \delta_{AC}\delta_{BD})$$

denote the components of the curvature tensor of $N^{n+1}_1(c)$.

Restricting to M, we have

$$\omega_{n+1} = 0. \tag{2.1}$$

It follows from Cartan's Lemma that

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \qquad h_{ij} = h_{ji}. \tag{2.2}$$

The second fundamental form h and the mean curvature H of M are defined by

$$h = -\sum_{i,j} h_{ij}\omega_i\omega_j, \qquad H = \frac{1}{n}\sum_i h_{ii}. \tag{2.3}$$

The squared norm of the second fundamental form of M is given by

$$S = \sum_{i,j} h_{ij}^2.$$

The structure equations of M are given by

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0,$$
$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l,$$

where R_{ijkl} are the components of the curvature tensor of M. Moreover, using the previous structure equations, we obtain the Gauss equation

$$R_{ijkl} = c(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) - (h_{il}h_{jk} - h_{ik}h_{jl}). \tag{2.4}$$

Let R_{jk} and R represent the components of the Ricci curvature and the scalar curvature of M, respectively. From (2.4), we get

$$R_{jk} = (n-1)c\delta_{jk} - nHh_{jk} + \sum_i h_{ik}h_{ji},$$
$$R = n(n-1)c - n^2H^2 + S.$$

The components of the covariant differential ∇h of h are defined by

$$\sum_k h_{ijk}\omega_k = dh_{ij} - \sum_k h_{ik}\omega_{kj} - \sum_k h_{jk}\omega_{ki}. \tag{2.5}$$

We have the Codazzi equation

$$h_{ijk} = h_{ikj}. \tag{2.6}$$

The squared norm of the covariant differential ∇h is given by

$$|\nabla h|^2 = \sum_{i,j,k} h_{ijk}^2.$$

We take the second covariant differential of h and define h_{ijkl} by

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} - \sum_m h_{mjk}\omega_{mi} - \sum_m h_{imk}\omega_{mj} - \sum_m h_{ijm}\omega_{mk}. \tag{2.7}$$

The Ricci identity is given by

$$\begin{aligned} h_{ijkl} - h_{ijlk} &= -\sum_m h_{im}R_{mjkl} - \sum_m h_{jm}R_{mikl} \\ &= (c - \lambda_k\lambda_l)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})(\lambda_i - \lambda_j). \end{aligned} \tag{2.8}$$

The Laplacian of h_{ij} is defined by

$$\Delta h_{ij} = \sum_k h_{ijkk}.$$

According to the equation (2.6) and the Ricci identity (2.8), we have

$$\Delta h_{ij} = \sum_k h_{kkij} - \sum_{k,m} h_{km}R_{mijk} - \sum_{k,m} h_{im}R_{mkjk}. \tag{2.9}$$

When M is a maximal spacelike hypersurface, we have

$$H = \frac{1}{n}\sum_i h_{ii} = 0. \tag{2.10}$$

It follows from (2.4) and (2.9) that

$$\Delta h_{ij} = (S + nc)h_{ij}. \tag{2.11}$$

According to the above discussion, we immediately obtain the following lemmas:

Lemma 2.1. *Let M be a maximal spacelike hypersurface in $N_1^{n+1}(c)$, then we have*

$$R = n(n-1)c + S. \tag{2.12}$$

Lemma 2.2. *Let M be a maximal spacelike hypersurface of $N_1^{n+1}(c)$ with constant scalar curvature, then we have*

$$|\nabla h|^2 = -S(S + nc). \tag{2.13}$$

Next, we consider the Willmore functional

$$W(\varphi) = \int_M (S - nH^2)^{\frac{n}{2}} dv$$

which vanishes if and only if $\varphi : M \to N_1^{n+1}(c)$ is umbilical. The critical submanifolds of the Willmore functional are called Willmore submanifolds. Recently, Sun and Chen [12] studied Willmore spacelike submanifolds in a Lorentzian space form, and got the Euler-Lagrange equation of Willmore spacelike submanifolds. For maximal Willmore spacelike hypersurfaces, they proved

Theorem 2.3. *Let M be a maximal spacelike hypersurface in $N_1^{n+1}(c)$. Then M is a maximal Willmore spacelike hypersurface if and only if*

$$S^{\frac{n-2}{2}} \sum_{i,j,k} h_{ik}h_{kj}h_{ij} - \sum_{i,j} (S^{\frac{n-2}{2}})_{i,j} h_{ij} = 0,$$

where $(S^{\frac{n-2}{2}})_{i,j}$ is the Hessian of $S^{\frac{n-2}{2}}$ with respect to the induced metric.

Combining Theorem 2.3 and Lemma 2.1 yields the following lemma:

Lemma 2.4. *Let M be a maximal Willmore spacelike hypersurface with constant scalar curvature R in $N_1^{n+1}(c)$. Then the function*

$$f_3 := \sum_{i,j,k} h_{ij}h_{jk}h_{ki} \equiv 0. \tag{2.14}$$

3. Two distinct principal curvatures at one point

In this paper, we only consider the case of $n = 4$. Let M be a Willmore maximal spacelike hypersurface of $\mathbb{H}_1^5(c)$ with constant scalar curvature R. Fixing an arbitrary point $p \in M$, we take orthonormal frames such that $h_{ij} = \lambda_i \delta_{ij}$ at p for all i, j. According to (2.10), (2.12) and (3.47), we have

$$\sum_i \lambda_i = 0, \qquad \sum_i \lambda_i^2 = R - n(n-1)c, \qquad \sum_i \lambda_i^3 = 0. \tag{3.1}$$

Without loss of generality, supposing $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ at the point p, then we have (see [11, 13])

$$\lambda_1 + \lambda_4 = 0, \qquad \lambda_2 + \lambda_3 = 0. \tag{3.2}$$

In this section, we first prove the following theorem.

Theorem 3.1. *Let M be a Willmore maximal spacelike hypersurface of $\mathbb{H}_1^5(c)$ with constant scalar curvature. If there exists a point with two distinct principal curvatures, then the second fundamental form of the hypersurface M is parallel.*

Assume that there are two distinct principal curvatures at p, then it follows from (3.2) that

$$\lambda_1 = \lambda_2 = \lambda > 0, \qquad \lambda_3 = \lambda_4 = -\lambda < 0. \tag{3.3}$$

Next, all discussions and calculations will be considered at the point p.

Lemma 3.2. *For any $k, l = 1, 2, 3, 4$, we have*

$$h_{11k} + h_{22k} = 0, \qquad h_{33k} + h_{44k} = 0. \tag{3.4}$$

$$h_{11kl} + h_{22kl} = -(h_{33kl} + h_{44kl}) = -\frac{1}{2\lambda} \sum_{i,j} h_{ijk} h_{ijl}. \tag{3.5}$$

Proof. Since $H = 0$, taking the first and second covariant derivative, we have

$$h_{11k} + h_{22k} + h_{33k} + h_{44k} = 0,$$
$$h_{11kl} + h_{22kl} + h_{33kl} + h_{44kl} = 0.$$

Similarly, by the fact that S is constant, we can get

$$\sum_{i,j} h_{ij} h_{ijk} = 0, \qquad \sum_{i,j} (h_{ijk} h_{ijl} + h_{ij} h_{ijkl}) = 0. \tag{3.6}$$

Using (3.3), we get

$$h_{11} = h_{22} = \lambda, \qquad h_{33} = h_{44} = -\lambda, \qquad h_{ij} = 0, \text{ for } i \neq j. \tag{3.7}$$

Substituting (3.7) into (3.6), we have

$$\lambda(h_{11k} + h_{22k} - h_{33k} - h_{44k}) = 0,$$
$$\sum_{i,j} h_{ijk} h_{ijl} + \lambda(h_{11kl} + h_{22kl} - h_{33kl} - h_{44kl}) = 0.$$

Therefore, these equalities in the lemma hold. □

In order to get the relations between the components h_{ijk}, we take the derivatives of the function f_3 defined on M. We have the following lemma:

Lemma 3.3. *The components of* ∇h *satisfy the following equations:*

$$h_{111}^2 + h_{222}^2 = h_{332}^2 + h_{234}^2 = h_{331}^2 + h_{134}^2, \tag{3.8}$$

$$h_{333}^2 + h_{444}^2 = h_{114}^2 + h_{124}^2 = h_{113}^2 + h_{123}^2, \tag{3.9}$$

$$h_{111}h_{123} + h_{222}h_{113} = -h_{332}h_{333} + h_{234}h_{444}, \tag{3.10}$$

$$h_{111}h_{113} - h_{222}h_{123} = h_{331}h_{333} - h_{134}h_{444}, \tag{3.11}$$

$$h_{111}h_{124} + h_{222}h_{114} = h_{234}h_{333} + h_{332}h_{444}, \tag{3.12}$$

$$-h_{111}h_{114} + h_{222}h_{124} = h_{134}h_{333} + h_{331}h_{444}, \tag{3.13}$$

$$h_{134}h_{234} + h_{331}h_{332} = 0, \tag{3.14}$$

$$h_{123}h_{124} + h_{113}h_{114} = 0. \tag{3.15}$$

Proof. It follows from Lemma 2.4 that $f_3 = 0$. Taking the second covariant derivative of f_3 and using (3.3) and (2.10), we get

$$0 = (f_3)_{mn} = 3\Big(\sum_i h_{iimn}\lambda^2 + 2\sum_{i,k} h_{ikm}h_{ikn}\lambda_i\Big). \tag{3.16}$$

According to the Eq (3.4) and substituting $(m, n) = (1, 1), (2, 2), (3, 3), (4, 4)$ into the Eq (3.16), we get

$$h_{111}^2 + h_{222}^2 = h_{331}^2 + h_{134}^2, \qquad h_{111}^2 + h_{222}^2 = h_{332}^2 + h_{234}^2,$$
$$h_{333}^2 + h_{444}^2 = h_{113}^2 + h_{123}^2, \qquad h_{333}^2 + h_{444}^2 = h_{114}^2 + h_{124}^2.$$

Therefore, the two equalities (3.8) and (3.9) hold.
Similarly, setting $(m, n) = (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)$ into the Eq (3.16), the other equalities in the lemma will be obtained. □

Lemma 3.4. *For any* $k = 1, 2, 3, 4$*, we have*

$$\sum_{i,j} h_{ijk}^2 = \frac{1}{4}|\nabla h|^2. \tag{3.17}$$

Proof. Combining (3.4), (3.8) and (3.9), we immediately obtain

$$\sum_{i,j} h_{ijk}^2 = 4(h_{111}^2 + h_{222}^2 + h_{333}^2 + h_{444}^2), \quad \forall\, k.$$

So, the equality (3.17) holds. □

For the components h_{ijkl} of the second covariant differential $\nabla^2 h$, we have

Lemma 3.5.

$$h_{1233} = h_{3312}, \qquad h_{1244} = h_{4412}, \tag{3.18}$$

$$h_{3411} = h_{1134}, \qquad h_{3422} = h_{2234}, \tag{3.19}$$

$$h_{1122} = h_{2211}, \qquad h_{3344} = h_{4433}, \tag{3.20}$$

$$h_{1111} = h_{2222}, \qquad h_{3333} = h_{4444}. \tag{3.21}$$

Proof. Using (3.3) and the Ricci identity (2.8), we immediately get (3.18)–(3.20). From (3.5) and Lemma 3.4, it follows that

$$h_{1122} + h_{2222} = h_{1111} + h_{2211}, \qquad h_{3344} + h_{4444} = h_{3333} + h_{4433}.$$

Thus, (3.21) holds. □

According to Lemma 3.4, we find that $(h_{111}^2 + h_{222}^2 + h_{333}^2 + h_{444}^2)$ is independent of the choice of frames. Moreover, for any fixed e_3 and e_4, $(h_{111}^2 + h_{222}^2)$ is invariant when rotating e_1 and e_2. Similarly, for any fixed e_1 and e_2, $(h_{333}^2 + h_{444}^2)$ is also invariant when rotating e_3 and e_4. Set

$$p_1 = h_{111}^2 + h_{222}^2, \qquad p_2 = h_{333}^2 + h_{444}^2.$$

Next, we will prove Theorem 3.1 by contradiction. We assume that at least one of p_1 and p_2 is nonzero. Therefore, we have three cases:

$$\begin{aligned} &\textbf{Case I}: && p_1 \neq 0, \quad p_2 \neq 0; \\ &\textbf{Case II}: && p_1 = 0, \quad p_2 \neq 0; \\ &\textbf{Case III}: && p_1 \neq 0, \quad p_2 = 0. \end{aligned}$$

Since **Case II** and **Case III** are similar, we only discuss **Case I** and **Case II**.

3.1. Case I does not occur

In **Case I**, for fixed e_3 and e_4, we rotate e_1 and e_2 such that $h_{111} = 0$. If $h_{134} \neq 0$, then we fix e_1 and e_2, and rotate e_3 and e_4 such that $h_{134} = 0$. By the above rotation transformation, we reselect the frame $\{e_1, e_2, e_3, e_4\}$ such that $h_{111} = h_{134} = 0$. Furthermore, this frame preserves $h_{ij} = \lambda_i \delta_{ij}$ at p. We discuss **Case I** under the frame if not specified. According to the assumption that $p_1 \neq 0$, $p_2 \neq 0$ and Lemma 3.3, we get

$$h_{134} = h_{111} = h_{233} = 0, \qquad h_{222}^2 = h_{133}^2 = h_{234}^2 \neq 0.$$

There are four subcases:

$$\begin{aligned} &\textbf{Subcase I}-(\textbf{i}): && h_{222} = h_{133} = h_{234} \neq 0; \\ &\textbf{Subcase I}-(\textbf{ii}): && -h_{222} = h_{133} = h_{234} \neq 0; \\ &\textbf{Subcase I}-(\textbf{iii}): && h_{222} = -h_{133} = h_{234} \neq 0; \\ &\textbf{Subcase I}-(\textbf{iv}): && h_{222} = h_{133} = -h_{234} \neq 0. \end{aligned}$$

The four subcases are similar, so we only discuss Subcase I-(i). Using Lemma 3.3, we obtain

$$\begin{aligned} &h_{233} = h_{134} = h_{111} = 0; \qquad && h_{222} = h_{133} = h_{234} \neq 0; \\ &h_{333} = -h_{123} = h_{114}; && h_{444} = h_{124} = h_{113}. \end{aligned} \tag{3.22}$$

Combining (3.5) and (3.22), it yields

Lemma 3.6. *In Subcase I-(i), we have*

$$h_{3312} + h_{4412} = 0; \;\; h_{1134} + h_{2234} = 0; \;\; h_{3334} + h_{4434} = 0. \tag{3.23}$$

Lemma 3.7. *In Subcase I-(i), we have*

$$h_{4433} - h_{3333} - 2h_{1233} = 0, \tag{3.24}$$

$$h_{2222} - h_{1122} - 2h_{3422} = 0, \tag{3.25}$$

$$h_{4412} - h_{3312} - 2h_{1122} = 0, \tag{3.26}$$

$$h_{3333} + 2h_{1233} + 3h_{4433} - 4h_{2234} = 0. \tag{3.27}$$

Proof. It follows from Lemma 2.4 and (2.10) that

$$\begin{aligned} 0 = (f_3)_{lmn} &= \sum_{i,j,k} (h_{ij}h_{jk}h_{ki})_{lmn} \\ &= 6\sum_{i,j,k} h_{ijl}h_{jkm}h_{kin} + 6\sum_{i,j} \lambda_i(h_{ijlm}h_{ijn} + h_{ijln}h_{ijm} + h_{ijmn}h_{ijl}). \end{aligned} \tag{3.28}$$

Setting $(l,m,n) = (4,4,4)$ in the Eq (3.28), we have

$$\sum_{i,j,k} h_{ij4}h_{jk4}h_{ki4} + 3\sum_{i,j} h_{ij44}h_{ij4}\lambda_i = 0.$$

Using (3.22), (3.3) and (3.4), we obtain

$$\begin{aligned} &\sum_{i,j,k} h_{ij4}h_{jk4}h_{ki4} = 0, \\ &\sum_{i,j} h_{ij44}h_{ij4}\lambda_i = \lambda(h_{1144}h_{114} + h_{2244}h_{224} - h_{3344}h_{334} - h_{4444}h_{444} \\ &\qquad\qquad + 2h_{1244}h_{124} - 2h_{3444}h_{344}), \end{aligned}$$

which implies

$$(h_{1144} - h_{2244} + 2h_{3444})h_{333} + (h_{3344} - h_{4444} + 2h_{1244})h_{444} = 0. \tag{3.29}$$

Putting $(l,m,n) = (3,3,3)$ into (3.28), we can get the following equation by a similar proof of the Eq (3.29):

$$(h_{1133} - h_{2233} + 2h_{3433})h_{444} + (h_{4433} - h_{3333} - 2h_{1233})h_{333} = 0. \tag{3.30}$$

Combining (3.29) and (3.30), we have the following homogeneous system of linear equations involving h_{333} and h_{444}:

$$\begin{cases} (h_{1144} - h_{2244} + 2h_{3444})h_{333} + (h_{3344} - h_{4444} + 2h_{1244})h_{444} = 0, \\ (h_{4433} - h_{3333} - 2h_{1233})h_{333} + (h_{1133} - h_{2233} + 2h_{3433})h_{444} = 0. \end{cases} \tag{3.31}$$

Using the Ricci identity (2.8) and (2.6), we get

$$h_{1133} - h_{3311} = h_{2233} - h_{3322} = h_{1144} - h_{4411} = h_{2244} - h_{4422} = 2\lambda(c + \lambda^2).$$

Thus, it follows from (3.5) and Lemma 3.4 that

$$h_{1144} - h_{2244} = -(h_{3311} - h_{3322}) = -(h_{1133} - h_{2233}). \tag{3.32}$$

Combining Lemmas 3.5 and 3.6, we obtain

$$h_{1144}-h_{2244}+2h_{3444}=-(h_{1133}-h_{2233}+2h_{3433}),$$
$$h_{4433}-h_{3333}-2h_{1233}=h_{3344}-h_{4444}+2h_{1244}.$$

Since (3.31) has a nonzero solution if and only if

$$h_{4433}-h_{3333}-2h_{1233}=h_{3344}-h_{4444}+2h_{1244}=0, \tag{3.33}$$

$$h_{1144}-h_{2244}+2h_{3444}=-(h_{1133}-h_{2233}+2h_{3433})=0. \tag{3.34}$$

Setting $(l,m,n)=(2,2,2)$ into the Eq (3.28), we have

$$\sum_{i,j,k}h_{ij2}h_{jk2}h_{ki2}+3\sum_{i,j}h_{ij22}h_{ij2}\lambda_i=0.$$

Using (3.22) and (3.4), we also can get the first term

$$\sum_{i,j,k}h_{ij2}h_{jk2}h_{ki2}=0.$$

Then we have

$$-h_{1122}+h_{2222}-2h_{3422}=0.$$

Putting $(l,m,n)=(1,1,2)$ into the Eq (3.28), we have

$$\sum_{i,j,k}h_{ij1}h_{jk1}h_{ki2}+\sum_{i,j}\lambda_i(h_{ij11}h_{ij2}+2h_{ij12}h_{ij1})=0.$$

Using (3.22) and Lemma 3.2, we have

$$\sum_{i,j,k}h_{ij1}h_{jk1}h_{ki2}=0,$$
$$\sum_{i,j}h_{ij11}h_{ij2}\lambda_i=(-h_{1111}+h_{2211}-2h_{3411})h_{222}\lambda,$$
$$\sum_{i,j}h_{ij12}h_{ij1}\lambda_i=(-h_{3312}+h_{4412}-2h_{1212})h_{222}\lambda.$$

Combining (3.25), (3.23) and Lemma 3.5, we can get the equality (3.26).

Similarly, taking $(l,m,n)=(4,4,3),(3,3,4)$ into the Eq (3.28), respectively, we can obtain

$$(h_{4444}-2h_{1244}+3h_{3443}-4h_{2243})h_{333}+(h_{1144}-h_{2244}+2h_{3343}+4h_{1243})h_{444}=0.$$

$$(h_{1133}-h_{2233}-2h_{3334}-4h_{1234})h_{333}+(h_{3333}+2h_{1233}+3h_{4433}-4h_{2234})h_{444}=0.$$

Combining (3.32), Lemmas 3.5 and 3.6, we obtain

$$h_{4444}-2h_{1244}+3h_{3443}-4h_{2243}=h_{3333}+2h_{1233}+3h_{4433}-4h_{2234},$$
$$h_{1144}-h_{2244}+2h_{3343}+4h_{1243}=-(h_{1133}-h_{2233}-2h_{3334}-4h_{1234}).$$

Then we can get the last equality in the lemma. □

Proposition 3.8. *Let M be a Willmore maximal spacelike hypersurface of $\mathbb{H}_1^5(c)$ with constant scalar curvature. If there exists a point with two distinct principal curvatures, then Subcase I-(i) does not occur.*

Proof. It follows from (3.24) and (3.5) that

$$h_{1233} = h_{4433} - \frac{1}{4\lambda}\sum_{i,j} h_{ij3}^2, \tag{3.35}$$

Substituting (3.35) into (3.27) and using Lemma 3.2, we have

$$h_{4433} = h_{2234}. \tag{3.36}$$

Using (3.25) and (3.5), we get

$$h_{3422} = h_{2222} + \frac{1}{4\lambda}\sum_{i,j} h_{ij2}^2. \tag{3.37}$$

On the other hand, it follows from (3.26) and (3.23) that

$$h_{1122} = h_{4412}. \tag{3.38}$$

Combining (3.35)–(3.38), Lemmas 3.4–3.6, we can obtain

$$h_{1233} = h_{2222}, \tag{3.39}$$

$$h_{1122} + h_{1233} = h_{4412} + h_{1233} = 0. \tag{3.40}$$

Hence, it follows from Lemma 3.4 that

$$h_{1122} + h_{1233} = h_{1122} + h_{2222} = -\frac{1}{8\lambda}\sum_{i,j,k} h_{ijk}^2 = 0,$$

which implies

$$h_{ijk} = 0, \quad \forall\ i, j, k.$$

This contradicts the hypothesis. Therefore, Subcase I-(i) does not occur. □

Remark 3.1. The above proposition holds in the other three subcases too. Hence, Case I does not occur.

3.2. Case II does not occur

In **Case II**, for fixed e_3 and e_4, we rotate e_1 and e_2 such that $h_{123} = 0$. If $h_{444} \neq 0$, then we fix e_1 and e_2, and rotate e_3 and e_4 such that $h_{444} = 0$. By the above rotation transformation, we reselect the frame $\{e_1, e_2, e_3, e_4\}$ such that $h_{123} = h_{444} = 0$. Furthermore, this frame preserves $h_{ij} = \lambda_i\delta_{ij}$ at p. We discuss **Case II** under the frame if not specified. According to the assumption that $p_1 = 0,\ p_2 \neq 0$ and Lemma 3.3, we get

$$h_{111} = h_{114} = h_{123} = h_{133} = h_{134} = h_{222} = h_{233} = h_{234} = h_{444} = 0, \tag{3.41}$$

$$h_{113}^2 = h_{124}^2 = h_{333}^2 \neq 0. \tag{3.42}$$

Lemma 3.9. *In Case II, we get*

$$h_{3333} = 3h_{4433} = h_{4444}. \tag{3.43}$$

$$h_{1133}h_{113} + h_{2233}h_{223} + 2h_{1234}h_{124} = 0. \tag{3.44}$$

Proof. Using (2.12) and (2.13), we find that $|\nabla h|^2$ is constant. Taking the covariant derivative of $|\nabla h|^2$, we get

$$\sum_{i,j,k} h_{ijkl}h_{ijk} = 0. \tag{3.45}$$

Setting $l = 3$, we get

$$(h_{3333} - 3h_{4433})h_{333} + 3(h_{1133} - h_{2233})h_{113} + 6h_{1234}h_{124} = 0. \tag{3.46}$$

Next, according to Lemma 2.4 and (2.10), we have

$$\Delta f_3 = 3(S + 4c)f_3 + 6\sum_{i,j,k} h_{ijk}h_{ijl}h_{kl} = 0. \tag{3.47}$$

Taking the covariant derivative of Δf_3, we get

$$\sum_{i,j,k} (h_{ijkm}h_{ijl}h_{kl} + h_{ijk}h_{ijlm}h_{kl} + h_{ijk}h_{ijl}h_{klm}) = 0. \tag{3.48}$$

Putting $m = 3$ into (3.48), we have

$$-(h_{3333} - 3h_{4433})h_{333} + (h_{1133} - h_{2233})h_{113} + 2h_{1234}h_{124} = 0. \tag{3.49}$$

Combining (3.46) and (3.49) and using Lemma 3.5, we immediately get (3.43) and (3.44). □

Lemma 3.10. *In Case II, we get*

$$h_{1234}h_{124} + h_{4433}h_{333} = 0, \tag{3.50}$$

$$h_{1234}h_{443} - h_{1122}h_{124} = 0, \tag{3.51}$$

$$(h_{1144} - h_{2244})h_{113} - (h_{3344} - h_{4444})h_{333} = 0, \tag{3.52}$$

$$(h_{1122} - h_{2222})h_{113} - h_{3322}h_{333} - h_{4422}h_{443} = 0. \tag{3.53}$$

Proof. The proof of the lemma is similar to Lemma 3.7. Putting $(l, m, n) = (3, 3, 3), (4, 1, 2), (4, 4, 3), (2, 2, 3)$ into (3.28), respectively, and using (3.42), (3.41) and Lemma 3.2, we can get (3.50)–(3.53). □

Proposition 3.11. *Let M be a Willmore maximal spacelike hypersurface of $\mathbb{H}^5_1(c)$ with constant scalar curvature. If there exists a point with two distinct principal curvatures, then Case II does not occur.*

Proof. It follows from (3.42) that there are two subcases:

$$\textbf{Subcase II} - \textbf{(i)}: \ h_{333} = h_{113} \neq 0;$$
$$\textbf{Subcase II} - \textbf{(ii)}: \ h_{333} = -h_{113} \neq 0.$$

The two subcases are similar, we only prove Subcase II-(i).

Using (3.5), (3.17), (3.43) and Lemma 2.2, we have

$$h_{3344} = h_{4433} = \frac{1}{32\lambda}|\nabla h|^2 = \frac{-S(S+4c)}{32\lambda} = -\frac{1}{2}\lambda(\lambda^2 + c). \tag{3.54}$$

From (3.52), it yields

$$h_{1144} - h_{2244} - h_{3344} + h_{4444} = 0. \tag{3.55}$$

From (3.5), we can obtain

$$h_{1144} + h_{2244} + h_{3344} + h_{4444} = 0, \tag{3.56}$$

$$h_{1122} + h_{2222} + h_{3322} + h_{4422} = 0. \tag{3.57}$$

Combining (3.56), (3.55) and (3.43), we have

$$h_{2244} = -h_{3344} = -h_{4433}. \tag{3.58}$$

Using (3.53), we have

$$h_{1122} - h_{2222} - h_{3322} + h_{4422} = 0. \tag{3.59}$$

Combining (3.59) and (3.57), we obtain

$$h_{1122} = -h_{4422}. \tag{3.60}$$

From (3.50), (3.51) and (3.42), we can deduce

$$h_{1122} = h_{4433}, \tag{3.61}$$

Which together with (3.60) and (3.58) give that

$$h_{4422} = h_{2244}.$$

On the other hand, it follows from the Ricci identity (2.8) and (3.54) that

$$h_{4422} = h_{2244} - 2\lambda(\lambda^2 + c) = h_{2244} + \frac{1}{8\lambda}|\nabla h|^2,$$

which implies

$$h_{ijk} = 0, \quad \forall\ i, j, k.$$

This contradicts the hypothesis. Therefore, Case II does not occur. □

Remark 3.2. From the above discussion, it follows that Case I, Case II and Case III do not occur. Hence, at the point p, we have

$$p_1 = h_{111}^2 + h_{222}^2 = 0, \qquad p_2 = h_{333}^2 + h_{444}^2 = 0.$$

By Lemmas 3.4 and 2.2, we deduce that the second fundamental form of the hypersurface M is parallel. Therefore, we prove Theorem 3.1.

Proof of Theorem 1.2. Combining Theorem 3.1, Lemmas 2.1 and 2.2, we get $S \equiv 4$ on M. Therefore, by Theorem 1.1, M must be the hyperbolic cylinder

$$\mathbb{H}^2(-2) \times \mathbb{H}^2(-2) = \left\{(x, y) \in \mathbb{R}_1^3 \times \mathbb{R}_1^3 \middle| \langle x, x\rangle = -\frac{1}{2}, \langle y, y\rangle = -\frac{1}{2}\right\},$$

where $\mathbb{H}^2(-2)$ denotes the hyperbolic surface with constant sectional curvature -2. Thus we complete the proof of Theorem 1.2. □

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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