AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(7): 12620–12634.
DOI:10.3934/math.2022698
Received: 08 March 2022
Revised: 10 April 2022
Accepted: 17 April 2022
Published: 28 April 2022

*Research article*

# Approximation of the initial value for damped nonlinear hyperbolic equations with random Gaussian white noise on the measurements

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**Abstract:** The main goal of this work is to study a regularization method to reconstruct the solution of the backward non-linear hyperbolic equation $u_{tt}+\alpha\Delta^2 u_t+\beta\Delta^2 u = \mathcal{F}(x,t,u)$ come with the input data are blurred by random Gaussian white noise. We first prove that the considered problem is ill-posed (in the sense of Hadamard), i.e., the solution does not depend continuously on the data. Then we propose the Fourier truncation method for stabilizing the ill-posed problem. Base on some priori assumptions for the true solution we derive the error and a convergence rate between a mild solution and its regularized solutions. Also, a numerical example is provided to confirm the efficiency of theoretical results.

**Keywords:** wave equations; hyperbolic equations; Gaussian white noise; random noise; regularized solution; ill-posed
**Mathematics Subject Classification:** 35L15, 35R60

## 1. Introduction

Given an open bounded domain $\Omega \subset \mathbb{R}^d$ which has a smooth boundary $\Gamma$, and a positive real number $T$. We consider the non-linear hyperbolic partial different equation with the strong damping $\alpha\Delta^2 u_t$, as follows

$$u_{tt} + \alpha\Delta^2 u_t + \beta\Delta^2 u = \mathcal{F}(x,t,u), \quad (x,t) \in \Omega \times (0,T), \tag{1.1}$$

associated with the final value functions

$$u(x, T) = \rho(x), \quad u_t(x, T) = \xi(x), \quad x \in \Omega, \tag{1.2}$$

and the Dirichlet boundary condition

$$u(x, t) = 0, \quad (x, t) \in \Gamma \times (0, T), \tag{1.3}$$

where $\alpha, \beta$ are positive constants, and the source $\mathcal{F}(x, t, u)$ is a given function of the variable $u$.

As we all know, the amplitude of a wave is related to the amount of energy it carries. A high amplitude wave carries a large amount of energy and vice versa. A wave propagates through a certain environment, its energy will decrease as time goes on, so wave amplitude also decreases (called damped wave). The damped wave equations are widely used in science and engineering, especially in physics. They can describe how waves propagate. It applies to all kinds of waves, from water waves [8] to sound and vibrations [13, 21], and even light and radio waves [10].

Let us briefly describe some previous results related to the Problem (1.1). In recent years, much attention has been paid to the study on the properties and asymptotic behavior of the solution on Problem (1.1) subject to the initial conditions $u(x, 0) = \rho(x)$, $u_t(x, 0) = \xi(x)$ (pioneering works [1, 2, 5, 9, 15]). However, to the best of our knowledge, *there are not any result on backward problem* (1.1)–(1.3).

In practice we usually do not have these final value functions, instead they are suggested from the experience of the researcher. A more reliable way is to use their observed values. However, we all know that observations always come with random errors, these errors are derived from the ability of the measuring device (measurement error). It is therefore natural that observations are observed usually in the presence of some noise. In this paper, we will consider the case where these perturbation are an additive stochastic white noise

$$\rho^{\epsilon}(x) = \rho(x) + \epsilon W(x), \quad \xi^{\epsilon}(x) = \xi(x) + \epsilon W(x), \tag{1.4}$$

where $\epsilon$ is the amplitude of the noise and $W(x)$ is a Gaussian white noise process. Suppose further that even the observations (1.4) cannot be observed exactly, but they can only be observed in discretized form

$$\langle \rho^{\epsilon}, \varphi_p \rangle = \langle \rho, \varphi_p \rangle + \epsilon \langle W, \varphi_p \rangle, \quad \langle \xi^{\epsilon}, \varphi_p \rangle = \langle \xi, \varphi_p \rangle + \epsilon \langle W, \varphi_p \rangle, \quad p = 1, \ldots, N, \tag{1.5}$$

where $\{\varphi_p\}$ is a orthonormal basic of Hilbert space $\mathcal{H}$; $\langle , \rangle$ denotes the inner product in $\mathcal{H}$; $W_p := \langle W, \varphi_p \rangle$ are standard normal distribution; and $\langle \rho^{\epsilon}, \varphi_p \rangle$ are independent random variables for orthonormal functions $\varphi_p$. For more detail on the white noise model see, [3, 11, 12].

It is well-known that Problem (1.1)–(1.4) is ill-posed in the sense of Hadamard (if the solution exists, then it does not depend continuously on the final values), and regularization methods for it are required. The aim of this paper is to recover the unknown final value functions $\rho$, $\xi$ from indirect and noisy discrete observations (1.5) and then we use them to establish a regularized solution by the Fourier truncation method. To the best of our knowledge, the present paper may be the fist study for ill-posed problem for hyperbolic equations with Gaussian white noise. We have learned more ideas from these articles [14, 17, 18, 20], but the detailed technique is different.

The organizational structure of this paper is as follows. Section 2 introduces some preliminary materials. Section 3 uses the Fourier series to obtain the mild solution and analyse the ill-posedness of problem. Section 4 presents an example of an ill-posed problem with random noise. In Section 5, we draw into main results: first we propose a new regularized solution, and then we give the convergent estimates between a mild solution and a regularized solution under some priori assumptions on the exact solution. To end this section, we discuss a regularization parameter choice rule. Finally, Section 6 reports numerical implementations to support our theoretical results and to show the validity of the proposed reconstruction method.

## 2. Preliminaries

Throughout this paper, let us denote the Hilbert space $\mathcal{H} := L^2(\Omega)$, and $\langle \cdot, \cdot \rangle$ is the inner product of $\mathcal{H}$. Since $\Omega$ is the bounded open set, there exists a Hilbert orthonormal basic $\{\varphi_p\}_{p=1}^{\infty}$ in $\mathcal{H}$ ($\varphi_p \in \mathcal{H}_0^1(\Omega) \cap C^\infty(\Omega)$) and a sequence $\{\lambda_p\}_{p=1}^{\infty}$ of real, $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lim\limits_{p\to\infty} \lambda_p = +\infty$, such that $-\Delta\varphi_p(x) = \lambda_p \varphi_p(x)$ for $x \in \Omega$ and $\varphi_p(x) = 0$ for $x \in \partial\Omega$. We say that $\lambda_p$ are the eigenvalues of of $-\Delta$ and $\varphi_p$ are the associated eigenfunctions. The Sobolev class of function is defined as follows

$$\mathcal{H}^\mu = \left\{ f \in \mathcal{H} : \sum_{p=1}^{\infty} \lambda_p^\mu \left\langle f, \varphi_p \right\rangle^2 < \infty \right\}.$$

It is a Hilbert space endowed with the norm $\|f\|_{\mathcal{H}^\mu}^2 = \sum_{p=1}^{\infty} \lambda_p^\mu \left\langle f, \varphi_p \right\rangle^2$. For $\tau, \nu > 0$, following [4, 6], we introduce the special Gevrey classes of functions

$$\mathcal{G}_{\sigma,\nu} = \left\{ f \in \mathcal{H} : \sum_{p=1}^{\infty} e^{\sigma\lambda_p} \lambda_p^\nu \langle f, \mathrm{e}_p \rangle^2 < +\infty \right\}.$$

We remark that $G_{\sigma,\nu}$ is also the Hilbert space endowed with the norm $\|f\|_{G_{\sigma,\nu}}^2 = \sum_{p=1}^{\infty} e^{\sigma\lambda_p} \lambda_p^\nu \langle f, \mathrm{e}_i \rangle^2$.

**Definition 2.1** (Bochner space [22])**.** *Given a probability measure space $(\widetilde{\Omega}, \mathcal{M}, \mu)$, a Hilbert space $\mathcal{H}$. The Bochner space $L^2(\widetilde{\Omega}, \mathcal{H}) \equiv L^2((\widetilde{\Omega}, \mathcal{M}, \mu); \mathcal{H})$ is defined to be the functions $u : \widetilde{\Omega} \mapsto \mathcal{H}$ such that the corresponding norm is finite*

$$\|u\|_{L^2(\widetilde{\Omega},\mathcal{H})} := \left( \int_{\widetilde{\Omega}} \|u(\omega)\|_{\mathcal{H}}^2 \mathrm{d}\mu(\omega) \right)^{1/2} = \left( \mathbb{E}\|u\|_{\mathcal{H}}^2 \right)^{1/2} < +\infty. \tag{2.1}$$

**Definition 2.2** (Reconstruction of the final value functions)**.** *Given $\rho, \xi \in \mathcal{H}^\mu$ ($\mu > 0$), which have sequences of $n$ (is known as **sample size**) discrete observations $\langle \rho^\epsilon, \varphi_p \rangle$ and $\langle \xi^\epsilon, \varphi_p \rangle$, $p = 1, \ldots, n$. Non-parametric estimation of $\rho$ and $\xi$ are suggested as*

$$\widetilde{\rho}_n(x) = \sum_{p=1}^{n} \langle \rho^\epsilon, \varphi_p \rangle \varphi_p(x), \quad \widetilde{\xi}_n^\epsilon(x) = \sum_{p=1}^{n} \langle \xi^\epsilon, \varphi_p \rangle \varphi_p(x). \tag{2.2}$$

**Lemma 2.1.** *Given $\rho, \xi \in \mathcal{H}^\mu$ ($\mu > 0$), then the estimation errors are*

$$\mathbb{E}\|\widetilde{\rho}_n - \rho\|_{\mathcal{H}}^2 \leq \epsilon^2 n + \frac{1}{\lambda_n^\mu} \|\rho\|_{\mathcal{H}^\mu}^2, \quad \mathbb{E}\|\widetilde{\xi}^N - \xi\|_{\mathcal{H}}^2 \leq \epsilon^2 n + \frac{1}{\lambda_n^\mu} \|\xi\|_{\mathcal{H}^\mu}^2. \tag{2.3}$$

*Here $n(\epsilon) := n$ depends on $\epsilon$ and satisfies that* $\lim_{\epsilon\to 0^+} n(\epsilon) = +\infty$.

*Proof.* Our proof starts with the observation that

$$\mathbf{E}\|\widetilde{\rho}^n - \rho\|_{\mathcal{H}} = \mathbb{E}\left(\sum_{p=1}^{n}\langle\rho^\epsilon - \rho, \varphi_p\rangle^2\right) + \sum_{p=n+1}^{\infty}\langle\rho, \varphi_p\rangle^2 = \epsilon\mathbb{E}\left(\sum_{p=1}^{n} W_p^2\right) + \sum_{p=n+1}^{\infty}\lambda_p^{-\mu}\lambda_p^{\mu}\langle\rho, \varphi_p\rangle^2$$
$$\leq \epsilon\mathbb{E}\left(\sum_{p=1}^{n} W_p^2\right) + \frac{1}{\lambda_n^\mu}\sum_{p=n+1}^{\infty}\lambda_p^{\mu}\langle\rho, \varphi_p\rangle^2.$$

The assumption $W_p = \langle W, \varphi_p\rangle \overset{iid}{\sim} N(0,1)$ implies that $\mathbb{E}W_p^2 = 1$. We then have the desired the first result. The same conclusion can be drawn for the remaining case. □

## 3. Mild solution

Taking the inner product on both side of (1.1) and (1.2) with $\varphi_p$, and set $u_p(t) = \langle u(\cdot, t), \varphi_p\rangle$, $\rho_p(t) = \langle\rho, \varphi_p\rangle$, $\xi_p(t) = \langle\xi, \varphi_p\rangle$, and $\mathcal{F}_p(u) = \langle\mathcal{F}(\cdot, t, u(\cdot, t)), \varphi_p\rangle$, then

$$\begin{cases} u''{}_p(t) + \alpha\lambda_p^2 u_p'(t) + \beta\lambda_p^2 u_p = \mathcal{F}_p(u)(t), \\ u_p(T) = \rho_p, \quad u_p{}'(T) = \xi_p. \end{cases} \tag{3.1}$$

*In this work we assume that* $\Delta_p := \alpha^2\lambda_p^4 - 4\beta\lambda_p^2 > 0$ then a quadratic equation $k^2 - \alpha\lambda_p^2 k + \beta\lambda_p^2 = 0$ has two different solutions $k_p^- = \frac{\alpha\lambda_p^2 - \sqrt{\Delta_p}}{2}, k_p^+ = \frac{\alpha\lambda_p^2 + \sqrt{\Delta_p}}{2}$. Multiplying both sides the first equation of System (3.1) by $\phi_p(\tau) = \frac{e^{(\tau - t)k_p^+} - e^{(\tau - t)k_p^-}}{\sqrt{\Delta_p}}$, and integrating both sides from $t$ to $T$,

$$\int_t^T \phi_p(\tau)u_p''(\tau)\mathrm{d}\tau + \alpha\lambda_p^2\int_t^T \phi_p(\tau)u_p'(\tau)\mathrm{d}\tau + \beta\lambda_p^2\int_t^T \phi_p(\tau)u_p\mathrm{d}\tau = \int_t^T \phi_p(\tau)\mathcal{F}_p(u)(\tau)\mathrm{d}\tau. \tag{3.2}$$

The left hand side of (3.2) now becomes

$$\Big[\phi_p(\tau)u_p'(\tau) - \phi_p'(\tau)u_p(\tau) + \alpha\lambda_p^2\phi_p(\tau)u_p(\tau)\Big]_t^T + \int_t^T \Big[\phi_p''(\tau) - \alpha\lambda_p^2\phi_p'(\tau) + \beta\lambda_p^2\phi_p(\tau)\Big]u_p(\tau)\mathrm{d}\tau.$$

Since $k_p^-$, $k_p^+$ satisfy the equation $k^2 - \alpha\lambda_p^2 k + \beta\lambda_p^2 = 0$, then $\phi''(\tau) - \alpha\lambda_p^2\phi_p'(\tau) + \beta\lambda_p^2 = 0$. Hence, (3.2) becomes

$$\Big[\phi_p(\tau)u_p'(\tau) - \phi_p'(\tau)u_p(\tau) + \alpha\lambda_p^2\phi_p(\tau)u_p(\tau)\Big]_t^T = \int_t^T \phi_p(\tau)\mathcal{F}_p(u)(\tau)\mathrm{d}\tau. \tag{3.3}$$

It is worth noticing that $\phi_p(t) = 0$, $\phi_p'(t) = 1$ and $-\phi_p'(T) + \alpha\lambda_p^2\phi_p(T) = \frac{k_p^- e^{(T-t)k_p^+} - k_p^+ e^{(T-t)k_p^-}}{\sqrt{\Delta_p}}$. Therefore, (3.3) now becomes

$$u_p(t) = \frac{k_p^+ e^{(T-t)k_p^-} - k_p^- e^{(T-t)k_p^+}}{\sqrt{\Delta_p}}\rho_p - \frac{e^{(T-t)k_p^+} - e^{(T-t)k_p^-}}{\sqrt{\Delta_p}}\xi_p + \int_t^T \frac{k_p^+ e^{(\tau-t)k_p^-} - k_p^- e^{(\tau-t)k_p^+}}{\sqrt{\Delta_p}}\mathcal{F}_p(u)(\tau)\mathrm{d}\tau.$$

**Lemma 3.1.** *Let $\rho, \xi \in \mathcal{H}$. Suppose that the given problem* (1.1)–(1.3) *has a solution $u \in C([0,T], \mathcal{H})$, then the mild solution is represented in terms of the Fourier series as follows*

$$u(x,t) = \mathbf{R}(T-t)\rho(x) - \mathbf{S}(T-t)\xi(x) + \int_t^T \mathbf{S}(\tau - t)\mathcal{F}(x,\tau,u)\mathrm{d}\tau, \tag{3.4}$$

*where the operators $\mathbf{R}(t)f$ and $\mathbf{S}(t)f$ are*

$$\mathbf{R}(t)f = \sum_{p=1}^{\infty}\left(\frac{k_p^+ \mathrm{e}^{tk_p^-} - k_p^- \mathrm{e}^{tk_p^+}}{\sqrt{\Delta_p}}\langle f, \varphi_p\rangle\right)\varphi_p(x); \quad \mathbf{S}(t)f = \sum_{p=1}^{\infty}\left(\frac{\mathrm{e}^{tk_p^+} - \mathrm{e}^{tk_p^-}}{\sqrt{\Delta_p}}\langle f, \varphi_p\rangle\right)\varphi_p(x). \tag{3.5}$$

## 4. The ill-posedness of the problem

In this section, we present an example of Problem (1.1)–(1.3) with random noise (1.4) which is ill-posed in the sense of Hadamard (does not depend continuously on the final data). We consider the particular case as follows

$$\begin{cases} \tilde{u}^n_{tt} + \alpha\Delta^2\tilde{u}^n_t + \beta\Delta^2\tilde{u}^n = \mathcal{F}(\tilde{u}^n), & (x,t) \in \Omega \times (0,T), \\ \tilde{u}^n(x,T) = 0, & x \in \Omega, \\ \tilde{u}^n_t(x,T) = \widetilde{\xi}^{\epsilon}_n(x), & x \in \Omega, \\ \tilde{u}^n(x,t) = 0, & (x,t) \in \Gamma \times (0,T), \end{cases} \tag{4.1}$$

where $\mathcal{F}(\tilde{u}^n)(x,t) = \sum_{p=1}^{\infty} \frac{\mathrm{e}^{-\alpha\lambda_p^2 T}}{2T^2}\langle \tilde{u}^n(\cdot,t), \varphi_p\rangle\varphi_p(x)$. For simple computation, we assume that $\Omega = (0,\pi)$. It immediately follows that $\lambda_p = p^2$. We assume further that the function $\xi(x) = 0$ (unknown) has observations $\langle \xi^\epsilon, \varphi_p\rangle = \epsilon\langle W, \varphi_p\rangle$, $p = 1, \ldots, n$. Then the statistical estimate of $\xi(x)$ is in the form.

$$\widetilde{\xi}^{\epsilon}_n(x) = \sum_{p=1}^{n} \epsilon\langle W, \varphi_p\rangle\varphi_p(x). \tag{4.2}$$

Using Lemma 3.1, System (4.1) has the mild solution

$$\tilde{u}^n(x,t) = -\mathbf{S}(T-t)\widetilde{\xi}^{\epsilon}_n + \int_t^T \mathbf{S}(\tau - t)\mathcal{F}(\tilde{u}^n)(\tau)\mathrm{d}\tau. \tag{4.3}$$

We first show that this nonlinear integral equation has unique solution $\tilde{u}^n \in L^\infty([0,T]; L^2(\tilde{\Omega}, \mathcal{H}))$. Indeed, let us denote

$$\mathbf{\Phi}(u)(x,t) = -\mathbf{S}(T-t)\widetilde{\xi}^{\epsilon}_n + \int_t^T \mathbf{S}(\tau - t)\mathcal{F}(u)(\tau)\mathrm{d}\tau. \tag{4.4}$$

Let $u_1, u_2 \in L^\infty([0,T]; L^2(\tilde{\Omega}, \mathcal{H}))$. Using the Hölder inequality and Parseval's identity, we obtain

$$\begin{aligned} \mathbb{E}\|\mathbf{\Phi}(u_1)(\cdot,t) - \mathbf{\Phi}(u_2)(\cdot,t)\|^2_{\mathcal{H}} &= \mathbb{E}\left\|\int_t^T \mathbf{S}(\tau - t)(\mathcal{F}(u_1)(\cdot,\tau) - \mathcal{F}(u_2)(\cdot,\tau))\mathrm{d}\tau\right\|^2_{\mathcal{H}} \\ &\leq T\mathbb{E}\int_t^T \sum_{p=1}^{\infty}\Bigg(\underbrace{\frac{\mathrm{e}^{(\tau-t)k_p^+} - \mathrm{e}^{(\tau-t)k_p^-}}{\sqrt{\Delta_p}}}_{\mathbf{\Pi}_p(\tau)}\underbrace{\left\langle \mathcal{F}(v_1)(\cdot,\tau) - \mathcal{F}(v_2)(\cdot,\tau), \varphi_p\right\rangle}_{\mathbf{\Pi}^{\mathcal{F}}_p(\tau)}\Bigg)^2\mathrm{d}\tau. \end{aligned}$$

Since $\left|\mathrm{e}^{-(\tau-t)k_p^-} - \mathrm{e}^{-(\tau-t)k_p^+}\right| \leq (\tau - t)|k_p^+ - k_p^-| \leq T\sqrt{\Delta_p}$ and $(\tau - t)(k_p^+ + k_p^-) \leq T\alpha\lambda_p^2$, then

$$|\mathbf{\Pi}_p(\tau)| = \mathrm{e}^{(\tau-t)(k_p^+ + k_p^-)} \frac{\left|\mathrm{e}^{-(\tau-t)k_p^-} - \mathrm{e}^{-(\tau-t)k_p^+}\right|}{\sqrt{\Delta_p}} \leq T\mathrm{e}^{\alpha\lambda_p^2 T}. \tag{4.5}$$

From defining the function $\mathcal{F}$ as above, it follows that $\mathbf{\Pi}_p^{\mathcal{F}}(\tau) = \frac{\mathrm{e}^{-\alpha\lambda_p^2 T}}{2T^2}\langle u_1(\cdot,\tau) - u_2(\cdot,\tau), \varphi_p\rangle$. Thus

$$\begin{aligned}\mathbb{E}\|\mathbf{\Phi}(u_1)(\cdot,t) - \mathbf{\Phi}(u_2)(\cdot,t)\|_{\mathcal{H}}^2 &\leq \frac{1}{4T}\mathbb{E}\int_t^T \sum_{p=1}^{\infty}\langle u_1(\cdot,\tau) - u_2(\cdot,\tau), \varphi_p\rangle^2 \mathrm{d}\tau \\ &\leq \frac{1}{4}\|u_1 - u_2\|_{L^\infty([0,T];L^2(\tilde{\Omega},\mathcal{H}))}^2.\end{aligned}$$

Hence, we have that $\|\mathbf{\Phi}(u_1) - \mathbf{\Phi}(u_2)\|_{L^\infty([0,T];L^2(\tilde{\Omega},\mathcal{H}))}^2 \leq \frac{1}{4}\|u_1 - u_2\|_{L^\infty([0,T];L^2(\tilde{\Omega},\mathcal{H}))}^2$. This means that $\mathbf{\Phi}$ is a contraction. The Banach fixed point theorem leads to a conclude that $\mathbf{\Phi}(u) = u$ has a unique solution $u \in L^\infty([0,T]; L^2(\tilde{\Omega},\mathcal{H}))$.

We then point out that System (4.1) does not depend continuously on the final data. We start by

$$\mathbb{E}\|\tilde{u}^n(\cdot,t)\|_{\mathcal{H}}^2 \geq \mathbb{E}\left\|\mathbf{S}(T-t)\widetilde{\xi_n^\epsilon}\right\|_{\mathcal{H}}^2 - \frac{1}{2}\mathbb{E}\left\|\int_t^T \mathbf{S}(\tau - t)\mathcal{F}(\tilde{u}^n)(\tau)\mathrm{d}\tau\right\|_{\mathcal{H}}^2. \tag{4.6}$$

It is easy to verify that

$$\mathbb{E}\left\|\int_t^T \mathbf{S}(\tau - t)\mathcal{F}(\tilde{u}^n)(\tau)\mathrm{d}\tau\right\|_{\mathcal{H}}^2 \leq \frac{1}{4}\mathbb{E}\|u(\cdot,t)\|_{\mathcal{H}}^2.$$

This leads to

$$\mathbb{E}\|\tilde{u}^n(\cdot,t)\|_{\mathcal{H}}^2 \geq \frac{8}{9}\mathbb{E}\left\|\mathbf{S}(T-t)\widetilde{\xi_n^\epsilon}\right\|_{\mathcal{H}}^2. \tag{4.7}$$

It is worth recalling that $\mathbb{E}\left\langle\widetilde{\xi_n^\epsilon}, \varphi_p\right\rangle^2 = \epsilon^2$, so

$$\mathbb{E}\|\mathbf{S}(T-t)\widetilde{\xi_n^\epsilon}\|_{\mathcal{H}}^2 = \sum_{p=1}^{n}\left[\frac{\mathrm{e}^{(T-t)k_p^+} - \mathrm{e}^{(T-t)k_p^-}}{k_p^+ - k_p^-}\right]^2 \mathbb{E}\left\langle\widetilde{\xi_n^\epsilon}, \varphi_p\right\rangle^2 \geq \left[\frac{\mathrm{e}^{(T-t)k_n^+} - \mathrm{e}^{(T-t)k_n^-}}{k_n^+ - k_n^-}\right]^2 \epsilon^2. \tag{4.8}$$

We note that $k_n^+ - k_n^- = \sqrt{\Delta_p} = \sqrt{\alpha^2\lambda_n^4 - \beta\lambda_n^2} > \sqrt{\alpha^2\lambda_1^4 - \beta\lambda_1^2}$, then we have

$$\left[\frac{\mathrm{e}^{(T-t)k_n^+} - \mathrm{e}^{(T-t)k_n^-}}{k_n^+ - k_n^-}\right]^2 = \frac{\mathrm{e}^{2(T-t)k_n^+}\left[1 - \mathrm{e}^{-(T-t)(k_n^+ - k_n^-)}\right]^2}{(k_n^+ - k_n^-)^2} \geq \frac{\mathrm{e}^{2(T-t)k_n^+}\left[1 - \mathrm{e}^{-(T-t)\sqrt{\alpha^2\lambda_1^4 - 4\beta\lambda_1^2}}\right]^2}{\alpha^2\lambda_n^4 - 4\beta\lambda_n^2}.$$

The function $h(t) = \mathrm{e}^{(T-t)k_n^+}\left[1 - \mathrm{e}^{-(T-t)\sqrt{\alpha^2\lambda_1^4 - 4\beta\lambda_1^2}}\right]$ is a decreasing function with respect to variable $t \in [0,T]$, so $\sup_{0\leq t\leq T} h(t) = h(0)$. This leads to

$$\sup_{0\leq t\leq T}\frac{h^2(t)}{\alpha^2\lambda_n^4 - 4\beta\lambda_n^2} = \frac{\mathrm{e}^{2Tk_n^+}\left[1 - \mathrm{e}^{-T\sqrt{\alpha^2\lambda_1^4 - 4\beta\lambda_1^2}}\right]^2}{\alpha^2\lambda_n^4 - 4\beta\lambda_n^2} \geq \frac{\mathrm{e}^{2T\lambda_n}\left[1 - \mathrm{e}^{-T\sqrt{\alpha^2\lambda_1^4 - 4\beta\lambda_1^2}}\right]^2}{\alpha^2\lambda_n^4 - 4\beta\lambda_n^2}. \tag{4.9}$$

Combining (4.7)–(4.9) yields

$$\mathbb{E}\|\tilde{u}^n(\cdot,t)\|^2_{\mathcal{H}} \geq \frac{8}{9}\frac{e^{2T\lambda_n}\Big[1-e^{-T\sqrt{\alpha^2\lambda_1^4-4\beta\lambda_1^2}}\Big]^2}{\alpha^2\lambda_n^4-4\beta\lambda_n^2}\epsilon^2 \geq \frac{8}{9}\frac{e^{2Tn^2}\Big[1-e^{-T\sqrt{\alpha^2-4\beta}}\Big]^2}{\alpha^2 n^8-4\beta n^4}\epsilon^2.$$

Let us choose $n(\epsilon) := n = \sqrt{\frac{1}{2T}\ln(\frac{1}{\epsilon^3})}$. When $\epsilon \to 0^+$, we have $\mathbb{E}\|\widetilde{\xi}^{\epsilon}_n\|^2_{\mathcal{H}} = \epsilon^2 n(\epsilon) \to 0$. However,

$$\mathbb{E}\|\tilde{u}^n\|^2_{C([0,T];L^2(\Omega))} = \frac{8}{9}\frac{\frac{1}{\epsilon}\Big[1-e^{-T\sqrt{\alpha^2-4\beta}}\Big]^2}{\alpha^2\left[\frac{1}{2T}\ln(\frac{1}{\epsilon^3})\right]^4-4\beta\left[\frac{1}{2T}\ln(\frac{1}{\epsilon^3})\right]^2} \to +\infty.$$

Thus, we can conclude that Problem (1.1)–(1.3) with random noise (1.4) which is ill-posed in the sense of Hadamard.

## 5. Main results

To come up with a regularized solution, we first denote a truncation operator $\Bbbk_N f = \sum_{p=1}^{N}\langle f,\varphi_p\rangle\varphi_p(x)$ for all $f \in \mathcal{H}$. Now, let us consider a problem as follows

$$\begin{cases} \widetilde{U}^N_{tt} + \alpha\Delta^2\widetilde{U}^N_t + \beta\Delta^2\widetilde{U}^N = \Bbbk_N\mathcal{F}(x,t,\widetilde{U}^N), & (x,t)\in\Omega\times(0,T),\\ \widetilde{U}^N(x,T) = \Bbbk_N\widetilde{\rho}_n(x), & x\in\Omega,\\ \widetilde{U}^N_t(x,T) = \Bbbk_N\widetilde{\xi}^{\epsilon}_n(x), & x\in\Omega,\\ \widetilde{U}^N(x,t) = 0, & (x,t)\in\Gamma\times(0,T), \end{cases} \tag{5.1}$$

where $\widetilde{\rho}_n(x)$, $\widetilde{\xi}^{\epsilon}_n(x)$ as in Definition 2.2 and $N$, $n$ are called *the regularized parameter* and *the sample size* respectively. Applying Lemma 3.1, Problem (5.1) has the mild solution

$$\widetilde{U}^N(x,t) = \mathbf{R}^N(T-t)\widetilde{\rho}^{\epsilon}_n(x) - \mathbf{S}^N(T-t)\widetilde{\xi}^{\epsilon}_n(x) + \int_t^T \mathbf{S}^N(\tau-t)\mathcal{F}(x,\tau,\widetilde{U}^N)d\tau, \tag{5.2}$$

where

$$\mathbf{R}^N(t)f = \sum_{p=1}^{N}\frac{k_p^+e^{tk_p^-}-k_p^-e^{tk_p^+}}{\sqrt{\Delta_p}}\langle f,\varphi_p\rangle\varphi_p(x);\quad \mathbf{S}^N(t)f = \sum_{p=1}^{N}\frac{e^{tk_p^+}-e^{tk_p^-}}{\sqrt{\Delta_p}}\langle f,\varphi_p\rangle\varphi_p(x). \tag{5.3}$$

The non-linear integral equation is called *the regularized solution* of Problem (1.1)–(1.3) with the perturbation random model (1.4). And $N$ serves as *the regularization parameter*.

**Lemma 5.1** ( [16,19]). *Given $f \in \mathcal{H}$ and $t \in [0,T]$. We have the following estimates:*

$$\|\mathbf{R}^N(t)f\|^2_{\mathcal{H}} \leq \mathbf{C_R}e^{2\alpha t k_N^2}\|f\|^2_{\mathcal{H}};\quad \|\mathbf{S}^N(t)f\|^2_{\mathcal{H}} \leq \mathbf{C_S}e^{2\alpha t k_N^2}\|f\|^2_{\mathcal{H}}. \tag{5.4}$$

*where $\mathbf{C_R}$, $\mathbf{C_S}$ are constants dependent on $\alpha$, $T$.*

**Theorem 5.1.** *Given the functions $\rho, \xi \in \mathcal{H}$. Assume that $\mathcal{F} \in C(\Omega \times [0,T] \times \mathbb{R})$ satisfies the globally Lipschitz property with respect to the third variable i.e., there exists a constant $\mathbf{L} > 0$ independent of $x, t, u_1, u_2$ such that*

$$\left\|\mathcal{F}(\cdot, t, u_1(\cdot, t)) - \mathcal{F}(\cdot, t, u_2(\cdot, t))\right\|_{\mathcal{H}} \leq \mathbf{L}\left\|u_1(\cdot, t) - u_2(\cdot, t)\right\|_{\mathcal{H}}.$$

*Then the nonlinear integral equation* (5.2) *has a unique solution $\widetilde{U}^N \in L^\infty([0,T], L^2(\widetilde{\Omega}; \mathcal{H}))$.*

*Proof.* Define the operator $\mathbf{P} : L^\infty([0,T], L^2(\widetilde{\Omega}; \mathcal{H})) \mapsto L^\infty([0,T], L^2(\widetilde{\Omega}; \mathcal{H}))$ as following

$$\mathbf{P}(v)(x,t) = \mathbf{R}^N(T-t)\widetilde{\rho}_n^\epsilon(x) - \mathbf{S}^N(T-t)\widetilde{\xi}_n^\epsilon(x) + \int_t^T \mathbf{S}^N(\tau - t)\mathcal{F}(x, \tau, v)\mathrm{d}\tau.$$

For integer $m \geq 1$, we shall begin with showing that for any $v_1, v_2 \in L^\infty([0,T], L^2(\widetilde{\Omega}; \mathcal{H}))$

$$\mathbb{E}\|\mathbf{P}^m(v_1)(\cdot,t) - \mathbf{P}^m(v_2)(\cdot,t)\|_{\mathcal{H}}^2 \leq \left[\mathbf{L}^2\mathbf{C_S}T\mathrm{e}^{2\alpha T\lambda_N^2}\right]^m \frac{(T-t)^m}{m!}\|v_1 - v_2\|^2_{L^\infty([0,T],L^2(\widetilde{\Omega};\mathcal{H}))}. \tag{5.5}$$

We now proceed by induction on $m$. For the base case ($m = 1$),

$$\begin{aligned}\mathbb{E}\|\mathbf{P}(v_1)(\cdot,t) - \mathbf{P}(v_2)(\cdot,t)\|_{\mathcal{H}}^2 &= \mathbb{E}\left\|\int_t^T \mathbf{S}^N(\tau - t)(\mathcal{F}(x,\tau,v_1) - \mathcal{F}(x,\tau,v_2))\mathrm{d}\tau\right\|_{\mathcal{H}}^2 \\ &\leq T\mathbb{E}\int_t^T \mathbf{C_S}\mathrm{e}^{2\alpha t\lambda_N^2}\|\mathcal{F}(\cdot,\tau,v_1) - \mathcal{F}(\cdot,\tau,v_2)\|_{\mathcal{H}}^2\mathrm{d}\tau \\ &\leq T(T-t)\mathbf{L}^2\mathbf{C_S}\mathrm{e}^{2\alpha T\lambda_N^2}\|v_1 - v_2\|^2_{L^\infty([0,T],L^2(\widetilde{\Omega};\mathcal{H}))},\end{aligned}$$

where we apply Lemma 5.1 and the Lipschitz condition of $\mathcal{F}$. Thus it is correct for $m = 1$. For the inductive hypothesis, it is true for $m = m_0$. We show that (5.5) is true for $m + 1$.

$$\begin{aligned}\mathbb{E}\|\mathbf{P}^{m+1}(v_1)(\cdot,t) - \mathbf{P}^{m+1}(v_2)(\cdot,t)\|_{\mathcal{H}}^2 &= \mathbb{E}\,\|\mathbf{P}(\mathbf{P}^m(v_1))(\cdot,t) - \mathbf{P}(\mathbf{P}^m(v_2))(\cdot,t)\|_{\mathcal{H}}^2 \\ &= \mathbb{E}\left\|\int_t^T \mathbf{S}^N(\tau - t)(\mathcal{F}(x,\tau,\mathbf{P}^m(v_1)) - \mathcal{F}(x,\tau,\mathbf{P}^m(v_2)))\mathrm{d}\tau\right\|_{\mathcal{H}}^2 \\ &\leq T\mathbb{E}\int_t^T \mathbf{C_S}\mathrm{e}^{2\alpha t\lambda_N^2}\|\mathcal{F}(\cdot,\tau,\mathbf{P}^m(v_1)) - \mathcal{F}(\cdot,\tau,\mathbf{P}^m(v_2))\|_{\mathcal{H}}^2\mathrm{d}\tau \\ &\leq \left[\mathbf{L}^2\mathbf{C_S}T\mathrm{e}^{2\alpha T\lambda_N^2}\right]^{m+1}\|v_1 - v_2\|^2_{L^\infty([0,T],L^2(\widetilde{\Omega};\mathcal{H}))}\int_t^T \frac{(T-\tau)^m}{m!}\mathrm{d}\tau.\end{aligned} \tag{5.6}$$

From the inductive hypothesis, we have

$$\mathbb{E}\|\mathbf{P}^{m+1}(v_1)(\cdot,t) - \mathbf{P}^{m+1}(v_2)(\cdot,t)\|_{\mathcal{H}}^2 \leq \left[\mathbf{L}^2\mathbf{C_S}T\mathrm{e}^{2\alpha T\lambda_N^2}\right]^{m+1}\|v_1 - v_2\|^2_{L^\infty([0,T],L^2(\widetilde{\Omega};\mathcal{H}))}\int_t^T \frac{(T-\tau)^m}{m!}\mathrm{d}\tau.$$

Hence, by the principle of mathematical induction, Formula (5.5) holds. We realize that,

$$\lim_{m\to\infty}\frac{\left[\mathbf{L}^2\mathbf{C_S}T\mathrm{e}^{2\alpha T\lambda_N^2}\right]^m}{m!} = 0,$$

and therefore, there will exist a positive number $m = m_0$, such that $\mathbf{P}^{m_0}$ is a contraction. It means that $\mathbf{P}^{m_0}(\widetilde{U}^N) = \widetilde{U}^N$ has a unique solution $\widetilde{U}^N \in \mathbf{L}^\infty([0,T]; L^2(\widetilde{\Omega}, \mathcal{H}))$. This leads to $\mathbf{P}(\mathbf{P}^{m_0}(\widetilde{U}^N)) = \mathbf{P}(\widetilde{U}^N)$. Since $\mathbf{P}(\mathbf{P}^{m_0}(\widetilde{U}^N)) = \mathbf{P}^{m_0}(\mathbf{P}(\widetilde{U}^N))$, it follows that $\mathbf{P}^{m_0}(\mathbf{P}(\widetilde{U}^N)) = \mathbf{P}(\widetilde{U}^N)$. Hence $\mathbf{P}(\widetilde{U}^N)$ is a fixed point of $\mathbf{P}^{m_0}$. By the uniqueness of the fixed point of $\mathbf{P}^{m_0}$, we conclude that $\mathbf{P}(\widetilde{U}^N) = \widetilde{U}^N$ has a unique solution $\widetilde{U}^N \in \mathbf{L}^\infty([0,T]; L^2(\widetilde{\Omega}, \mathcal{H}))$. □

**Theorem 5.2.** *Let $\rho, \xi \in \mathcal{H}^\mu$, ($\mu > 0$). Assume that System* (1.1)–(1.3) *has the exact solution $u \in C([0,T]; \mathcal{G}_{\sigma,2})$, where $\sigma > 2\alpha T$. Given $\varepsilon > 0$, the following estimate holds*

$$\mathbb{E}\|\widetilde{U}^N(\cdot,t) - u(\cdot,t)\|^2_{\mathcal{H}} \leq 2\mathrm{e}^{-2\alpha t\lambda_N^2}\Big(2\lambda_N^{-2}\|u\|_{L^\infty([0,T];\mathcal{G}_{\sigma,2})}\Big)\mathrm{e}^{2\mathbf{C_S}\mathbf{L}^2T(T-t)}$$
$$+ 2\mathrm{e}^{2\alpha(T-t)\lambda_N^2}\Big[3\mathbf{C_R}\Big(\epsilon^2 n + \frac{1}{\lambda_n^\mu}\|\rho\|^2_{\mathcal{H}^\mu}\Big) + 3\mathbf{C_S}\Big(\epsilon^2 n + \frac{1}{\lambda_n^\mu}\|\xi\|^2_{\mathcal{H}^\mu}\Big)\Big]\mathrm{e}^{3\mathbf{C_S}\mathbf{L}^2T(T-t)}, \tag{5.7}$$

*where the regularization parameter $N(\epsilon) := N$ and the sample size $n(\epsilon) := n$ are choosen such that*

$$\lim_{\epsilon\to 0^+} N(\epsilon) = +\infty, \quad \lim_{\epsilon\to 0^+} \epsilon^2 n(\epsilon)\mathrm{e}^{2\alpha T\lambda^2_{N(\epsilon)}} = \lim_{\epsilon\to 0^+} \frac{\mathrm{e}^{2\alpha T\lambda^2_{N(\epsilon)}}}{\lambda^\mu_{n(\epsilon)}} = 0. \tag{5.8}$$

**Remark 5.1.** *The order of convergence of* (5.7) *is*

$$\mathrm{e}^{-2\alpha t\lambda^2_{N(\epsilon)}} \max\left\{\epsilon^2 n(\epsilon)\mathrm{e}^{2\alpha T\lambda^2_{N(\epsilon)}}; \frac{\mathrm{e}^{2\alpha T\lambda^2_{N(\epsilon)}}}{\lambda^\mu_{n(\epsilon)}}; \frac{1}{\lambda^2_{N(\epsilon)}}\right\}. \tag{5.9}$$

*There are many ways to choose the parameters $n(\epsilon), N(\epsilon)$, that satisfies* (5.8). *Since $\lambda_{n(\epsilon)} \sim (n(\epsilon))^{2/d}$ [7], one of the ways we can do by choosing the regularization parameter $N(\epsilon)$ such that $\lambda_{N(\epsilon)}$ satisfies $\mathrm{e}^{2\alpha T\lambda^2_{N(\epsilon)}} = (n(\epsilon))^a$, where $0 < a < 2\mu/d$. Then we obtain $\lambda^2_{N(\epsilon)} = \frac{a}{2\alpha T}\ln(n(\epsilon))$. The sample size $n(\epsilon)$ is chosen as $n(\epsilon) = (1/\epsilon)^{b/(a+1)}$, $(0 < b < 2)$. In this case, the error will be of order*

$$\epsilon^{\frac{tb}{T(a+1)}} \max\left\{\epsilon^{2-b}; \epsilon^{\frac{b}{a+1}(\frac{2\mu}{d}-a)}; \frac{ab}{2\alpha(a+1)}\ln\frac{1}{\epsilon}\right\}.$$

*Proof of Theorem 5.2.* Let us define the integral equation

$$u^N = \mathbf{R}^N(T-t)\rho(x) - \mathbf{S}^N(T-t)\xi(x) + \int_t^T \mathbf{S}^N(\tau - t)\mathcal{F}(x,\tau,u^N)\mathrm{d}\tau.$$

Then, we have

$$\mathbb{E}\|\widetilde{U}^N(\cdot,t) - u(\cdot,t)\|^2_{\mathcal{H}} \leq 2\mathbb{E}\|\widetilde{U}^N(\cdot,t) - u^N(\cdot,t)\|^2_{\mathcal{H}} + 2\mathbb{E}\|u^N(\cdot,t) - u(\cdot,t)\|^2_{\mathcal{H}}. \tag{5.10}$$

For easy tracking, we divide the above estimate into two main steps:
**Step 1.** We have

$$\mathbb{E}\|\widetilde{U}^N(\cdot,t) - u^N(\cdot,t)\|^2_{\mathcal{H}} \leq 3\mathbb{E}\|\mathbf{R}^N(T-t)(\widetilde{\rho}^\epsilon_n - \rho)\|^2_{\mathcal{H}} + 3\mathbb{E}\|\mathbf{S}^N(T-t)(\widetilde{\xi}^\epsilon_n - \xi)\|^2_{\mathcal{H}}$$
$$+ 3\mathbb{E}\Big\|\int_t^T \mathbf{S}^N(\tau-t)(\mathcal{F}(x,\tau,\widetilde{U}^N) - \mathcal{F}(x,\tau,u^N))\mathrm{d}\tau\Big\|^2_{\mathcal{H}}.$$

By Höder's inequality and the results in Lemma 5.1, we have

$$\begin{aligned}\mathbb{E}\|\widetilde{U}^N(\cdot,t)-u^N(\cdot,t)\|_{\mathcal{H}}^2 \leq & 3\mathbf{C_R}e^{2\alpha(T-t)\lambda_N^2}\mathbb{E}\|\widetilde{\rho}_n^\epsilon-\rho\|_{\mathcal{H}}^2+3\mathbf{C_S}e^{2\alpha(T-t)\lambda_N^2}\mathbb{E}\|\widetilde{\xi}_n^\epsilon-\xi\|_{\mathcal{H}}^2\\ &+3T\mathbb{E}\int_t^T \mathbf{C_S}e^{2\alpha(\tau-t)\lambda_N^2}\left\|\mathcal{F}(\cdot,\tau,\widetilde{U}^N)-\mathcal{F}(\cdot,\tau,u^N))\right\|_{\mathcal{H}}^2 d\tau.\end{aligned}$$

Use the results of Lemma 2.1 and the Lipschitz property of $\mathcal{F}$, we have

$$\begin{aligned}\mathbb{E}\|\widetilde{U}^N(\cdot,t)-u^N(\cdot,t)\|_{\mathcal{H}}^2 \leq & 3\mathbf{C_R}e^{2\alpha(T-t)\lambda_N^2}\left(\epsilon^2 n+\frac{1}{\lambda_n^\mu}\|\rho\|_{\mathcal{H}^\mu}^2\right)+3\mathbf{C_S}e^{2\alpha(T-t)\lambda_N^2}\left(\epsilon^2 n+\frac{1}{\lambda_n^\mu}\|\xi\|_{\mathcal{H}^\mu}^2\right)\\ &+3\mathbf{C_S}\mathbf{L}^2T\mathbb{E}\int_t^T e^{2\alpha(\tau-t)\lambda_N^2}\left\|\widetilde{U}^N(\cdot,\tau)-u^N(\cdot,\tau)\right\|_{\mathcal{H}}^2 d\tau. \end{aligned}\tag{5.11}$$

Multiplying both sides (5.11) to $e^{2\alpha t\lambda_N^2}$, we derive that

$$\begin{aligned}e^{2\alpha t\lambda_N^2}\mathbb{E}\|\widetilde{U}^N(\cdot,t)-u^N(\cdot,t)\|_{\mathcal{H}}^2 \leq & 3\mathbf{C_R}e^{2\alpha T\lambda_N^2}\left(\epsilon^2 n+\frac{1}{\lambda_n^\mu}\|\rho\|_{\mathcal{H}^\mu}^2\right)+3\mathbf{C_S}e^{2\alpha T\lambda_N^2}\left(\epsilon^2 n+\frac{1}{\lambda_n^\mu}\|\xi\|_{\mathcal{H}^\mu}^2\right)\\ &+3\mathbf{C_S}\mathbf{L}^2T\mathbb{E}\int_t^T e^{2\alpha\tau\lambda_N^2}\left\|\widetilde{U}^N(\cdot,\tau)-u^N(\cdot,\tau)\right\|_{\mathcal{H}}^2 d\tau.\end{aligned}$$

Gronwall's inequality leads to

$$\begin{aligned}&e^{2\alpha t\lambda_N^2}\mathbb{E}\|\widetilde{U}^N(\cdot,t)-u^N(\cdot,t)\|_{\mathcal{H}}^2\\ &\leq e^{2\alpha T\lambda_N^2}\left[3\mathbf{C_R}\left(\epsilon^2 n+\frac{1}{\lambda_n^\mu}\|\rho\|_{\mathcal{H}^\mu}^2\right)+3\mathbf{C_S}\left(\epsilon^2 n+\frac{1}{\lambda_n^\mu}\|\xi\|_{\mathcal{H}^\mu}^2\right)\right]e^{3\mathbf{C_S}\mathbf{L}^2T(T-t)}.\end{aligned}\tag{5.12}$$

**Step 2.** To evaluate the remining term, we define the truncation version of the solution $u$ as following

$$\chi_u^N(x,t)=\mathbf{R}^N(T-t)\rho(x)-\mathbf{S}^N(T-t)\xi(x)+\int_t^T \mathbf{S}^N(\tau-t)\mathcal{F}(x,\tau,u)d\tau.$$

Then, we have

$$\|u^N(\cdot,t)-u(\cdot,t)\|_{\mathcal{H}}^2\leq 2\|u^N(\cdot,t)-\chi_u^N(\cdot,t)\|_{\mathcal{H}}^2+2\|\chi_u^N(\cdot,t)-u(\cdot,t)\|_{\mathcal{H}}^2.\tag{5.13}$$

**Sub-step 1.1.** By Höder's inequality, Lemma 5.1 and the Lipschitz property of $\mathcal{F}$, we have

$$\begin{aligned}\|u^N(\cdot,t)-\chi_u^N(\cdot,t)\|_{\mathcal{H}}^2 &= \left\|\int_t^T \mathbf{S}^N(\tau-t)(\mathcal{F}(x,\tau,u^N)-\mathcal{F}(x,\tau,u))d\tau\right\|_{\mathcal{H}}^2\\ &\leq T\int_t^T \mathbf{C_S}e^{2\alpha(\tau-t)\lambda_N^2}\left\|\mathcal{F}(\cdot,\tau,u^N)-\mathcal{F}(\cdot,\tau,u)\right\|_{\mathcal{H}}^2 d\tau\\ &\leq \mathbf{C_S}\mathbf{L}^2T\mathbb{E}\int_t^T e^{2\alpha(\tau-t)\lambda_N^2}\left\|u^N(\cdot,\tau)-u(\cdot,\tau)\right\|_{\mathcal{H}}^2 d\tau.\end{aligned}\tag{5.14}$$

Since $u\in C([0,T];\mathcal{G}_{\sigma,2})$, then

$$\|\chi_u^N(\cdot,t)-u(\cdot,t)\|_{\mathcal{H}}^2=\sum_{p=N+1}^{\infty}\langle u(\cdot,t),\varphi_p\rangle^2\leq e^{-2\alpha t\lambda_N}\lambda_N^{-2}\sum_{p=N+1}^{\infty}e^{2\alpha t\lambda_p}\lambda_p^2\langle u(\cdot,t),\varphi_p\rangle^2$$

$$\leq \mathrm{e}^{-2\alpha t\lambda_N}\lambda_N^{-2}\|u(\cdot,t)\|_{L^\infty([0,T];\mathcal{G}_{\sigma,2})}. \tag{5.15}$$

Substituting (5.14) and (5.15) into (5.13), we have

$$\|u^N(\cdot,t)-u(\cdot,t)\|_{\mathcal{H}}^2 \leq 2\mathbf{C_S}\mathbf{L}^2 T\mathbb{E}\int_t^T \mathrm{e}^{2\alpha(\tau-t)\lambda_N^2}\Big\|u^N(\cdot,\tau)-u(\cdot,\tau)\Big\|_{\mathcal{H}}^2 \mathrm{d}\tau + 2\mathrm{e}^{-2\alpha t\lambda_N}\lambda_N^{-2}\|u\|_{L^\infty([0,T];\mathcal{G}_{\sigma,2})}.$$

Multiplying both sides above formula to $\mathrm{e}^{2\alpha t\lambda_N}$, we have

$$\mathrm{e}^{2\alpha t\lambda_N^2}\|u^N(\cdot,t)-u(\cdot,t)\|_{\mathcal{H}}^2 \leq 2\mathbf{C_S}\mathbf{L}^2 T\mathbb{E}\int_t^T \mathrm{e}^{2\alpha\tau\lambda_N^2}\Big\|u^N(\cdot,\tau)-u(\cdot,\tau)\Big\|_{\mathcal{H}}^2 \mathrm{d}\tau + 2\lambda_N^{-2}\|u\|_{L^\infty([0,T];\mathcal{G}_{\sigma,2})}.$$

Using Gronwall's inequality, we obtain

$$\mathrm{e}^{2\alpha t\lambda_N^2}\|u^N(\cdot,t)-u(\cdot,t)\|_{\mathcal{H}}^2 \leq \Big(2\lambda_N^{-2}\|u\|_{L^\infty([0,T];\mathcal{G}_{\sigma,2})}\Big)\mathrm{e}^{2\mathbf{C_S}\mathbf{L}^2T(T-t)}. \tag{5.16}$$

The proof is completed by combining (5.10), (5.12) and (5.16). □

## 6. Numerical experiments

We propose the general scheme of our numerical calculation. For simplicity, we fix $T = 1$ and $\Omega = (0,\pi)$. The eigenelements of the Dirichlet problem for the Laplacian in $\Omega$ have the following form:

$$\varphi_p = \sqrt{\frac{2}{\pi}}\sin(px), \quad \lambda_p = p^2, \quad \text{for } p = 1, 2, \ldots$$

### 6.1. General numerical scheme

To find a numerical solution to Eq (5.2), we first need to define a set of $Nx \times Nt$ grid points in the domain $\Omega \times [0,T]$. Let $\Delta x = \pi/Nx$ is the time step, $\Delta t = 1/Nt$ is the spatial step, the coordinates of the mesh points are $x_j = j\Delta x$, $j = 0,\ldots,Nx$, and $t_i = i\Delta t$, $i = 0,\ldots,Nt$, and the values of the regularized solution $\widetilde{U}^N(x,t)$ at these grid points are $\widetilde{U}^N(x_j,t_i) \approx \widetilde{U}_{ij}$, where we denote $\widetilde{U}_{ij}$ by the numerical estimate of the regularized solution $\widetilde{U}^N(x,t)$ of at the point $(x_j,t_i)$.

**Initialization step.** The numerical process starts when time $t = T$. Since $\widetilde{U}^N(x,T) = \mathbf{R}^N(0)\widetilde{\rho}_n^\epsilon$, then

$$\widetilde{U}_{Ntj} \approx \widetilde{U}^N(x,T) = \sum_{p=1}^N \langle\tilde{\rho}_n^\epsilon, \varphi_p\rangle\varphi_p(x_j) = \sum_{p=1}^N \langle\rho^\epsilon, \varphi_p\rangle\varphi_p(x_j), \quad j = 1,\ldots,Nx. \tag{6.1}$$

**Iteration steps.** For $t_i < T$, we want to determine

$$\widetilde{U}^N(x,t_i) = \mathbf{R}^N(T-t_i)\widetilde{\rho}_n^\epsilon - \mathbf{S}^N(T-t_i)\widetilde{\xi}_n^\epsilon + \underbrace{\int_{t_i}^T \mathbf{S}^N(\tau-t_i)\mathcal{F}(\widetilde{U}^N)(\tau)\mathrm{d}\tau}_{\mathbf{I}(t_i)}, \tag{6.2}$$

where $\mathbf{I}(t_i)$ is performed in backward time as following

$$\mathbf{I}(t_i) = \sum_{p=1}^N \Big[\int_{t_i}^T \frac{\mathrm{e}^{(\tau-t_i)k_p^+} - \mathrm{e}^{(\tau-t_i)k_p^-}}{\sqrt{\Delta_p}}\mathcal{F}_p(\widetilde{U}^N)(\tau)\mathrm{d}\tau\Big]\varphi_p(x)$$

$$= \sum_{p=1}^{N} \Big[ \sum_{k=i}^{Nt-1} \int_{t_k}^{t_{k+1}} \frac{e^{(\tau-t_i)k_p^+} - e^{(\tau-t_i)k_p^-}}{\sqrt{\Delta_p}} \mathcal{F}_p(\widetilde{U}^N)(t_{k+1}) d\tau \Big] \varphi_p(x).$$

It is worth pointing out that, the Simpson's rule leads to the approximation

$$\mathcal{F}_p(\widetilde{U}^N)(t_i) = \left\langle \mathcal{F}(\widetilde{U}^N)(\cdot, t_i), \varphi_p \right\rangle \approx \frac{\Delta}{3} \sum_{h=1}^{Nx} C_h \big[ \mathcal{F}(\widetilde{U}^N(x_h, t_i)) \varphi_p(x_h) \big],$$

where

$$C_h = \begin{cases} 1, & \text{if } h = 0 \text{ or } h = Nx, \\ 2, & \text{if } h \neq 0, h \neq Nx \text{ and } h \text{ is odd}, \\ 4, & \text{if } h \neq 0, h \neq Nx \text{ and } h \text{ is even}. \end{cases}$$

**Error estimation.** We use the absolute error estimation between the regularized solution and the exact solution as follows

$$\mathbf{Err}(t_i) = \left( \frac{1}{Nx+1} \sum_{j=0}^{Nx} \left| u(x_j, t_i) - \widetilde{U}^N(x_j, t_i) \right|^2 \right)^{1/2}. \tag{6.3}$$

### *6.2. Test case*

In this example, we fixed $\alpha = 0.3, \beta = 0.01$ and present the inputs

$$\rho(x) = e^{-2} \sin x + e^{-1} \sin 2x; \quad \xi(x) = -2e^{-2} \sin x - e^{-1} \sin 2x,$$

and source data $\mathcal{F}(x, t, u) = f(x, t) + \frac{1}{1+u^2}$, where

$$f(x,t) = \frac{(4 - 2\alpha + \beta) \left( e^{-2t} \sin x + e^{-4t} \sin x \sin^2 2x + e^{-6t} \sin^3 x + 2e^{-5t} \sin^2 x \sin 2x \right)}{1 + e^{-2t} \sin^2 2x + e^{-4t} \sin^2 x + 2e^{-3t} \sin x \sin 2x}$$
$$+ \frac{(1 - 16\alpha + 16\beta) \left( e^{-t} \sin 2x + e^{-3t} \sin^2 2x + e^{-5t} \sin x \sin 2x + 2e^{-4t} \sin x \sin^2 2x \right)}{1 + e^{-2t} \sin^2 2x + e^{-4t} \sin^2 x + 2e^{-3t} \sin x \sin 2x}.$$

It is easy to check that the exact solution of Problem (1.1)–(1.3) is given by $u(x,t) = e^{-t} \sin 2x + e^{-2t} \sin x$.

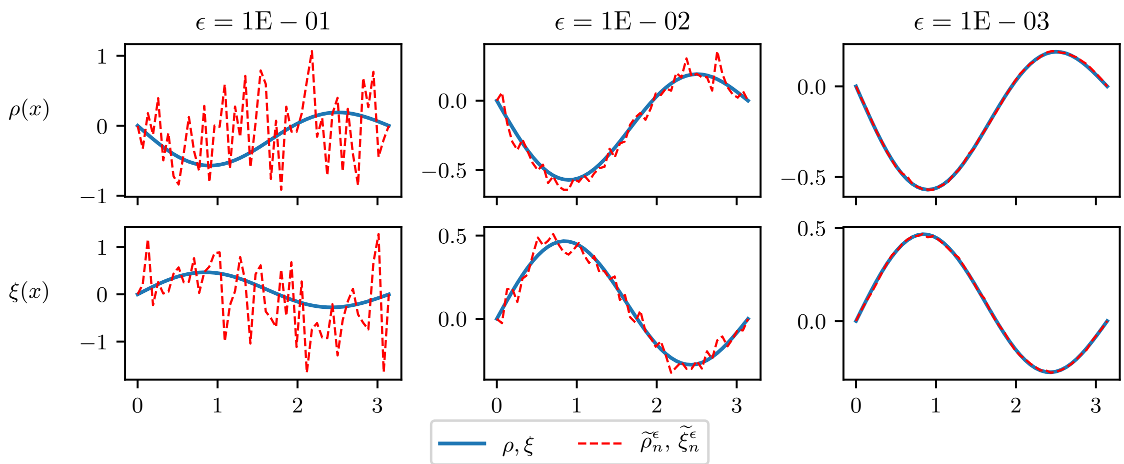


**Figure 1.** Comparison between: $\rho$ and $\widetilde{\rho}_n^\epsilon$; $\xi$ and $\widetilde{\xi}_n^\epsilon$ ($n = 100$).

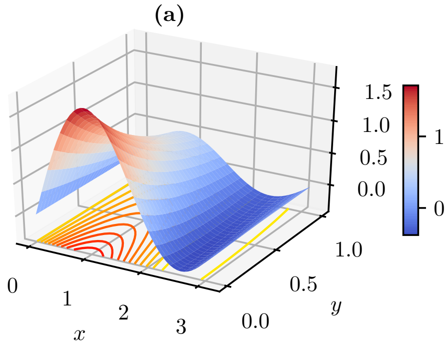


**Figure 2.** For $\epsilon = 1E - 03$: **(a)** - the exact solution; **(b)** - the regularized solution.

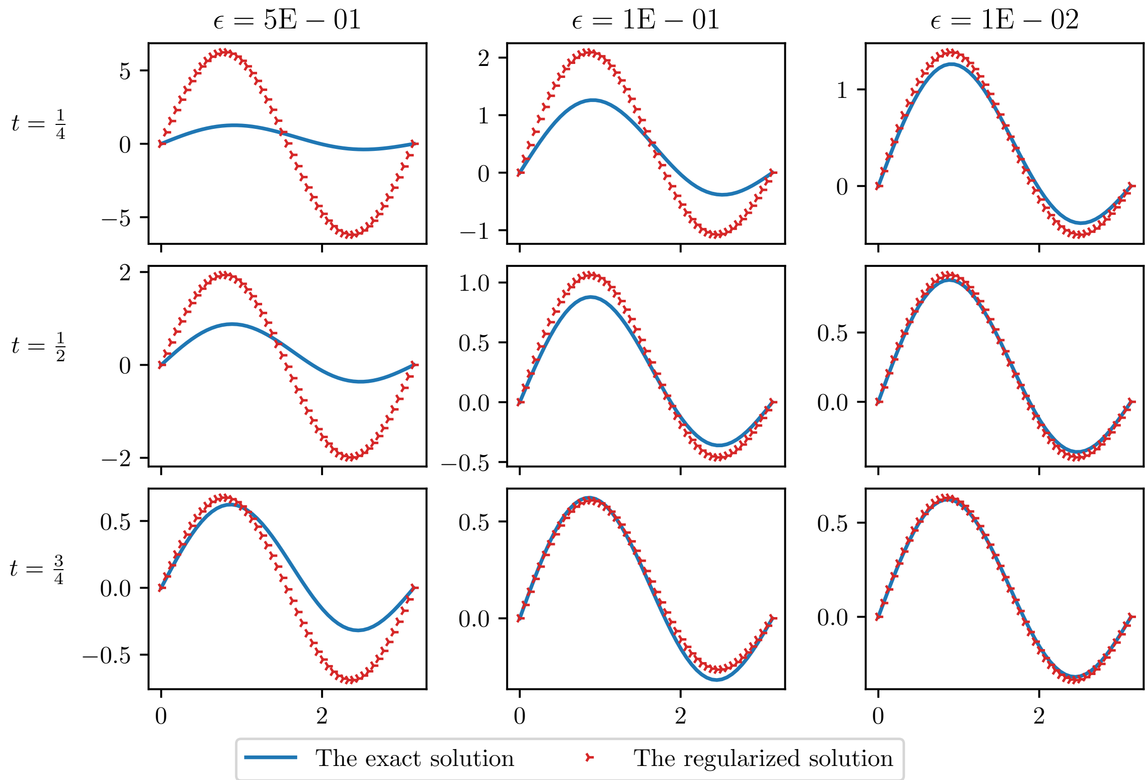


**Figure 3.** Comparison between regularized solution and exact solution.

Figure 1 compares $\rho(x)$, $\xi(x)$ with their estimates $\widetilde{\rho}_n^\epsilon(x)$, $\widetilde{\xi}_n^\epsilon(x)$, respectively. When $\epsilon$ tends to 0, the estimates are consistent with that of the exact ones. Figure 2 presents a 3D graph of the exact solution $u$ and the regularized solution for the case $\epsilon = 1E - 03$. Figure 3 displays the numerical convergence for different values of $\epsilon$ and t.

Table 1 shows the values of $\mathbf{Err}(t)$ from (6.3) calculated numerically. As a conclusion, our proposed regularization method works properly and the numerical solution method is also feasible in practice.

**Table 1.** Errors between regularized solution and exact solution for $t = \frac{1}{4}; \frac{1}{2}; \frac{3}{4}$.

| $\epsilon$ | $\mathbf{Err}(\frac{1}{4})$ | $\mathbf{Err}(\frac{1}{2})$ | $\mathbf{Err}(\frac{3}{4})$ |
|---|---|---|---|
| 5E − 01 | 1.071213E + 00 | 2.464307E − 01 | 1.124781E − 01 |
| 1E − 01 | 3.161161E − 02 | 5.976654E − 03 | 2.063242E − 03 |
| 1E − 02 | 1.143085E − 04 | 1.761839E − 05 | 4.912298E − 05 |
| 1E − 03 | 5.851911E − 06 | 2.820096E − 07 | 2.488333E − 10 |

## Acknowledgements

This research is supported by Industrial University of Ho Chi Minh City (IUH) under grant number 130/HD-DHCN. Nguyen Anh Tuan thanks the Van Lang University for the support.

## Conflict of interest

The authors declare no conflict of interest.

## References

1. M. Aassila, A. Guesmia, Energy decay for a damped nonlinear hyperbolic equation, *Appl. Math. Lett.*, **12** (1999), 49–52. https://doi.org/10.1016/S0893-9659(98)00171-2
2. G. Chen, F. Da, Blow-up of solution of Cauchy problem for three-dimensional damped nonlinear hyperbolic equation, *Nonlinear Anal. Theor.*, **71** (2009), 358–372. https://doi.org/10.1016/j.na.2008.10.132
3. P.-L. Chow, I. A. Ibragimov, R. Z. Khasminskii, Statistical approach to some ill-posed problems for linear partial differential equations, *Probab. Theory Relat. Fields*, **113** (1999), 421–441. https://doi.org/10.1007/s004400050212
4. C. Cao, M. A. Rammaha, E. S. Titi, Gevrey regularity for nonlinear analytic parabolic equations on the sphere, *J. Dyn. Differ. Equ.*, **12** (2000), 411–433. https://doi.org/10.1023/A:1009072526324
5. G. Chen, Y. Wang, Z. Zhao, Blow-up of solution of an initial boundary value problem for a damped nonlinear hyperbolic equation, *Appl. Math. Lett.*, **17** (2004), 491–497. https://doi.org/10.1016/S0893-9659(04)90116-4
6. C. Foias, R. Temam, Gevrey class regularity for the solutions of the Navier-Stokes equations, *J. Funct. Anal.*, **87** (1989), 359–369. https://doi.org/10.1016/0022-1236(89)90015-3
7. L. Hörmander, *Linear partial differential operators*, Berlin, Heidelberg: Springer, 1963. https://doi.org/10.1007/978-3-642-46175-0
8. A. Imani, D. Ganji, H. B. Rokni, H. Latifizadeh, E. Hesameddini, M. H. Rafiee, Approximate traveling wave solution for shallow water wave equation, *Appl. Math. Model.*, **36** (2012), 1550–1557. https://doi.org/10.1016/j.apm.2011.09.030
9. W. Liu, K. Chen, Existence and general decay for nondissipative hyperbolic differential inclusions with acoustic/memory boundary conditions, *Math. Nachr.*, **289** (2016), 300–320. https://doi.org/10.1002/mana.201400343

10. J. C. Maxwell, VIII. A dynamical theory of the electromagnetic field, *Phil. Trans. R. Soc.*, **155** (1865), 459–512. https://doi.org/10.1098/rstl.1865.0008

11. P. Mathé, S. V. Pereverzev, Optimal discretization of inverse problems in hilbert scales. Regularization and self-regularization of projection methods, *SIAM J. Numer. Anal.*, **38** (2001), 1999–2021. https://doi.org/10.1137/S003614299936175X

12. D. Picard, G. Kerkyacharian, Estimation in inverse problems and second-generation wavelets, In: *Proceedings of the international congress of mathematicians*, European Mathematical Society Publishing House, Madrid, August 22–30, 2006, 713–739. https://doi.org/10.4171/022-3/37

13. A. D. Pierce, *Acoustics: an introduction to its physical principles and applications*, Cham: Springer, 2019. https://doi.org/10.1007/978-3-030-11214-1

14. N. D. Phuong, N. H. Tuan, D. Baleanu, T. B. Ngoc, On Cauchy problem for nonlinear fractional differential equation with random discrete data, *Appl. Math. Comput.*, **362** (2019), 124458. https://doi.org/10.1016/j.amc.2019.05.029

15. C. Song, Z. Yang, Existence and nonexistence of global solutions to the cauchy problem for a nonlinear beam equation, *Math. Method. Appl. Sci.*, **33** (2010), 563–575. https://doi.org/10.1002/mma.1175

16. N. H. Tuan, V. V. Au, N. H. Can, Regularization of initial inverse problem for strongly damped wave equation, *Appl. Anal.*, **97** (2018), 69–88. https://doi.org/10.1080/00036811.2017.1359560

17. N. H. Tuan, D. Baleanu, T. N. Thach, D. O'Regan, N. H. Can, Final value problem for nonlinear time fractional reaction-diffusion equation with discrete data, *J. Comput. Appl. Math.*, **376** (2020), 112883. https://doi.org/10.1016/j.cam.2020.112883

18. N. A. Triet, T. T. Binh, N. D. Phuong, D. Baleanu, N. H. Can, Recovering the initial value for a system of nonlocal diffusion equations with random noise on the measurements, *Math. Method. Appl. Sci.*, **44** (2021), 5188–5209. https://doi.org/10.1002/mma.7102

19. N. H. Tuan, D. V. Nguyen, V. V. Au, D. Lesnic, Recovering the initial distribution for strongly damped wave equation, *Appl. Math. Lett.*, **73** (2017), 69–77. https://doi.org/10.1016/j.aml.2017.04.014

20. N. H. Tuan, E. Nane, D. O'Regan, N. D. Phuong, Approximation of mild solutions of a semilinear fractional differential equation with random noise, *Proc. Amer. Math. Soc.*, **148** (2020), 3339–3357. https://doi.org/10.1090/proc/15029

21. G. F. Wheeler, W. P. Crummett, The vibrating string controversy, *Am. J. Phys.*, **55** (1987), 33–37. https://doi.org/10.1119/1.15311

22. G. A. Zou, B. Wang, Stochastic Burgers' equation with fractional derivative driven by multiplicative noise, *Comput. Math. Appl.*, **74** (2017), 3195–3208. https://doi.org/10.1016/j.camwa.2017.08.023

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