



Research article

Deterministic and stochastic model for the hepatitis C with different types of virus genome

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Abstract: In this paper, a deterministic and stochastic model for hepatitis C with different types of virus genomes is proposed and analyzed. Some sufficient conditions are obtained to ensure the stability of the deterministic equilibrium points. We perform a stochastic extension of the deterministic model to study the fluctuation between environmental factors. Firstly, the existence of a unique global positive solution for the stochastic model is investigated. Secondly, sufficient conditions for the extinction of the hepatitis C virus from the stochastic system are obtained. Theoretical and numerical results show that the smaller white noise can ensure the persistence of susceptible and infected populations while the larger white noise can lead to the extinction of disease. By introducing the basic reproduction number R_0 and the stochastic basic reproduction number R_0^s , the conditions that cause the disease to die out are indicated. The importance of environmental noise in the propagation of hepatitis C viruses is highlighted by these findings.

Keywords: hepatitis C; extinction; stochastic; stability; stationary distribution; Lyapunov function

Mathematics Subject Classification: 92D25, 92D30, 37C75, 37H30, 37L45

1. Introduction

Hepatitis C is a disease caused by a virus that infects the liver. The virus, called hepatitis C virus or HCV, is just one of the hepatitis viruses. Global statistics estimate that nearly 350 million people worldwide are infected with HCV, which significantly affects human health, especially the liver [16]. Mathematical models are currently a useful way to assess the transmissibility of many infectious diseases, predict future morbidity, and assess the efficacy of prevention and therapy [1, 11, 14, 17, 21, 34–36, 42, 46]. Many successful mathematical models investigating fundamental hepatitis C viral

dynamics have been produced in recent decades, including Feng et al. [12], Lestari et al. [23] and Li et al. [24] to cite only a few. Moneim and Mosa [31, 32], constructed a mathematical model to study the spread of HCV-subtype 4a amongst the Egyptian population. They divided the population into three groups, susceptible class, $S(t)$, the hepatitis C subtype 4a infective class, I_1 and the HCV from all other subtype classes, I_2 . They assumed that all types of the class I_2 could mutate to I_1 at a constant rate $\mu > 0$. A few years after the appearance of [31, 32], effective treatment for the hepatitis C virus appeared. For this, we will add to the mathematical model another class ($R(t)$), which represents those who have been cured of this virus. Following [8, 18, 19, 39, 47], the saturated incidence rate will replace the bilinear incidence rate. The hepatitis C virus becomes as follows.

$$\begin{aligned}\frac{dS}{dt} &= A - \frac{k_1 S I_1}{a + S} - \frac{k_2 S I_2}{a + S} - bS, \\ \frac{dI_1}{dt} &= \frac{k_1 S I_1}{a + S} + \mu I_1 I_2 - bI_1 - \gamma I_1, \\ \frac{dI_2}{dt} &= \frac{k_2 S I_2}{a + S} - \mu I_1 I_2 - bI_2 - \delta I_2, \\ \frac{dR}{dt} &= \gamma I_1 + \delta I_2 - bR,\end{aligned}\tag{1.1}$$

where A denotes the recruitment rate of susceptible individuals, a expresses the half-saturation constant for susceptible individuals with HCV. γ and δ are the recovery rates, k_1 is the transmission rate of virus C when susceptible individuals $S(t)$ contact with corresponding infected $I_1(t)$ individuals, and k_2 is the transmission rate when $S(t)$ contact with corresponding $I_2(t)$ individuals. Following [31, 32], we assume that the population is mixing in a homogenous manner, i.e., every person has the same chance of coming in contact with an infected person also we assume that the birth and death rates are equal and positive constant rate b .

The deterministic HCV system (1.1) ignores the possible importance of environmental noise. In reality, stochastic effects can be important during the transmission of hepatitis C viruses, because various cells and infective virus particles react differently in the same environment. Consequently, the deterministic models do not provide an adequate understanding of viral dynamics due to the stochastic behavior of viruses and the complexity of the immune system. The primary purpose of this paper is to propose and analyze a deterministic and stochastic model of HCV. The paper is arranged as follows: In Section 2, the dynamics of the deterministic system are verified. The stochastic extension of the deterministic model is performed in Section 3, and the existence of a unique global positive solution for the stochastic model is investigated, and the sufficient conditions for the extinction of the hepatitis C virus from the stochastic system are obtained. In Section 4, some numerical simulations are presented to verify the obtained theoretical results. Finally, Section 5 contains the conclusion.

2. Dynamics of deterministic system

In the following, two critical parameters R_{01} , and R_{02} , can be used to classify the dynamics of the hepatitis C virus model (1.1). The threshold parameter R_{01} defined by $R_{01} = \frac{Ak_1}{\rho(A+ab)}$, $\rho = b + \gamma$. While the threshold parameter R_{02} defined by $R_{02} = \frac{Ak_2}{\theta(A+ab)}$, $\theta = b + \delta$. Using the next generation method, one can obtain the basic reproduction number R_0 defined by $R_0 = \max \{R_{01}, R_{02}\}$.

The hepatitis C virus model (1.1) has the following four equilibrium points:

(1) The disease-free equilibrium point $E_0 = \left(\frac{A}{b}, 0, 0, 0\right)$, which always exists. At this point, all individuals are susceptible, and there is no infection present in the population.

(2) The equilibrium point $E_1 = (S_1, I_{11}, 0, R_1)$, where

$$S_1 = \frac{a\rho}{k_1 - \rho}, \quad I_{11} = \frac{(a + S_1)(A + ab)}{ak_1}(R_{01} - 1), \quad R_1 = \frac{\gamma(a + S_1)(A + ab)}{abk_1}(R_{01} - 1).$$

E_1 exists if $k_1 > \rho$ and $R_{01} > 1$.

(3) The equilibrium point $E_2 = (S_2, 0, I_{22}, R_2)$, where

$$S_2 = \frac{a\theta}{k_2 - \theta}, \quad I_{22} = \frac{(a + S_2)(A + ab)}{ak_2}(R_{02} - 1), \quad R_2 = \frac{\delta(a + S_2)(A + ab)}{abk_2}(R_{02} - 1).$$

E_2 exists if $k_2 > \theta$ and $R_{02} > 1$.

(4) The coexistence equilibrium point $E_3 = (S_3, I_{13}, I_{23}, R_3)$, where

$$S_3 = \frac{-\beta + \sqrt{\beta^2 + 4abA}}{2b}, \quad I_{13} = \frac{1}{\mu} \left(\frac{k_2 S_3}{a + S_3} - \theta \right), \quad I_{23} = \frac{1}{\mu} \left(\rho - \frac{k_1 S_3}{a + S_3} \right), \quad R_3 = \frac{1}{\mu} (\gamma I_{13} + \delta I_{23}),$$

$\beta = ab - A + \frac{\rho k_2}{\mu} - \frac{\theta k_1}{\mu}$. The coexistence equilibrium point E_3 exists if $\frac{k_2 S_3}{\theta(a + S_3)} > 1$ and $\frac{k_1 S_3}{\rho(a + S_3)} < 1$.

The boundedness of the solutions of model (1.1) is given as follows. Let $(S(t), I_1(t), I_2(t), R(t))$ be any solution of system (1.1) with non-negative initial conditions and assume that the total population size $N(t) = S(t) + I_1(t) + I_2(t) + R(t)$, then $\frac{dN}{dt} + bN = A$, consequentially $N(t) = \frac{A}{b} + N(0)e^{-bt}$, thus it follows that $0 \leq N(t) \leq \frac{A}{b}$, as $t \rightarrow \infty$.

The locally asymptotically stable equilibrium points of system (1.1) are now investigated. The Jacobian matrix is given as follows:

$$J = \begin{pmatrix} -\frac{k_1 a I_1}{(a+S)^2} - \frac{k_2 a I_2}{(a+S)^2} - b & -\frac{k_1 S}{a+S} & -\frac{k_2 S}{a+S} & 0 \\ \frac{k_1 a I_1}{(a+S)^2} & \frac{k_1 S}{a+S} + \mu I_2 - \rho & \mu I_1 & 0 \\ \frac{k_2 a I_2}{(a+S)^2} & -\mu I_2 & \frac{k_2 S}{a+S} + \mu I_1 - \theta & 0 \\ 0 & \gamma & \delta & -b \end{pmatrix}.$$

The eigenvalues of $J(E_0)$ are $-b$, $-b$, $\rho(R_{01} - 1)$, and $\theta(R_{02} - 1)$. Thus, E_0 is locally asymptotically stable if $R_0 < 1$. The eigenvalues of $J(E_1)$ are $\lambda_1 = -b$ and $\lambda_2 = \frac{k_2 S_1}{a + S_1} - \mu I_{11} - \theta$. The other roots are determined by

$$\lambda^2 + \left(\frac{k_1 a I_{11}}{(a + S_1)^2} + b \right) \lambda + \frac{k_1^2 a S_1 I_{11}}{(a + S_1)^3} = 0.$$

The eigenvalues λ_3 and λ_4 have negative real parts. Thus, if $\frac{k_2 S_1}{(\mu I_{11} + \theta)(a + S_1)} < 1$ then the equilibrium point E_1 is locally asymptotically stable. The eigenvalues of $J(E_2)$ are $\lambda_1 = -b$ and $\lambda_2 = \frac{k_1 S_2}{(a + S_2)} + \mu I_{22} - \rho$. The other roots are determined by

$$\lambda^2 + \left(\frac{k_2 a I_{22}}{(a + S_2)^2} + b \right) \lambda + \frac{k_2^2 a S_2 I_{22}}{(a + S_2)^3} = 0.$$

The eigenvalues λ_3 and λ_4 have negative real parts. Thus, if $\frac{k_1 S_2}{(a+S_2)} + \mu I_{22} < \rho$ then the equilibrium point E_2 is locally asymptotically stable. The stability of the coexistence equilibrium point $E_3 = (S_3, I_{13}, I_{23}, R_3)$ is investigated as follows

The first eigenvalues of $J(E_3)$ is $\lambda_1 = -b$. The other roots are determined by

$$\lambda^3 + c_1 \lambda^2 + c_2 \lambda + c_3 = 0, \quad (2.1)$$

where

$$\begin{aligned} c_1 &= \frac{a(ab + k_1 I_{13} + k_2 I_{23}) + 2abS_3 + bS_3^2}{(a + S_3)^2}, \\ c_2 &= \frac{I_{23} (ak_2^2 S_3 + \mu^2 I_{13} (a + S_3)^3) + ak_1^2 S_3 I_{13}}{(a + S_3)^3}, \\ c_3 &= \frac{\mu^2 I_{13} I_{23} (a(ab + k_1 I_{13} + k_2 I_{23}) + 2abS_3 + bS_3^2)}{(a + S_3)^2}. \\ c_1 c_2 - c_3 &= \frac{aS_3 (k_1^2 I_{13} + k_2^2 I_{23}) (a(ab + k_1 I_{13} + k_2 I_{23}) + 2abS_3 + bS_3^2)}{(a + S_3)^5}. \end{aligned}$$

It is clear that $c_1, c_2, c_3 > 0$ and $c_1 c_2 - c_3 > 0$. It follows from the Routh-Hurwitz criterion that all roots of (2.1) have negative real parts. Thus, the equilibrium point E_3 is locally asymptotically stable.

3. Dynamics of stochastic model

In this section, we will perform a stochastic extension of model (1.1). The hepatitis virus C deterministic model (1.1) will be extended to include the environmental noise as follows.

$$\begin{aligned} dS &= \left[A - \frac{k_1 S I_1}{a + S} - \frac{k_2 S I_2}{a + S} - bS \right] dt + \sigma_1 S dW_1, \\ dI_1 &= \left[\frac{k_1 S I_1}{a + S} + \mu I_1 I_2 - bI_1 - \gamma I_1 \right] dt + \sigma_2 I_1 dW_2, \\ dI_2 &= \left[\frac{k_2 S I_2}{a + S} - \mu I_1 I_2 - bI_2 - \delta I_2 \right] dt + \sigma_3 I_2 dW_3, \\ dR &= [\gamma I_1 + \delta I_2 - bR] dt + \sigma_4 R dW_4, \end{aligned} \quad (3.1)$$

where $W = \{W_1, W_2, W_3, W_4, t \geq 0\}$ represents the four-dimensional standard Brownian motions with $W_i(0) = 0$ and $\sigma_i^2 (i = 1, 2, 3, 4)$ denote the intensities of the white noise. The white noise is defined in a complete probability space $(\Omega, \mathcal{F}_{t \geq 0}, \mathbb{P})$ with a filtration $\mathcal{F}_{t \geq 0}$ satisfying the usual conditions. In many applications, the solution of the Itô stochastic differential equation must preserve the positivity of the solutions [9, 10, 37]. According to Theorem 2.2 and Corollary 1 in [9], the solutions of (3.1) emanating from nonnegative initial data (almost surely) remain nonnegative as long as they exist. The next theorem gives another approach according to [30] to prove the existence and uniqueness of a positive global solution of the system (3.1). This approach has recently been used in many papers, and one can highlight [4–7, 13, 17, 20, 22, 27, 28, 33, 38, 40–45, 48].

Theorem 1. For any given initial value $(S(0), I_1(0), I_2(0), R(0)) \in \mathbb{R}_+^4$, there exists a unique solution $(S(t), I_1(t), I_2(t), R(t))$ of system (3.1) for $t \geq 0$ and the global positive solution remains in \mathbb{R}_+^4 with probability one.

Proof. Firstly, one can consider the local solution $(S(t), I_1(t), I_2(t), R(t))$ of system (3.1) for $t \in [0, \tau_e)$, where τ_e is the explosion time [30]. By making the transformation of variables

$$x(t) = \ln S(t), \quad y(t) = \ln I_1(t), \quad z(t) = \ln I_2(t), \quad w = \ln R(t).$$

Using Itô formula, one can change system (3.1) as follows

$$\begin{aligned} dx(t) &= \left[\frac{A}{e^x} - \frac{k_1 e^y}{a + e^x} - \frac{k_2 e^z}{a + e^x} - b - \frac{\sigma_1^2}{2} \right] dt + \sigma_1 dW_1, \\ dy(t) &= \left[\frac{k_1 e^x}{a + e^x} + \mu e^z - \rho - \frac{\sigma_2^2}{2} \right] dt + \sigma_2 dW_2, \\ dz(t) &= \left[\frac{k_2 e^x}{a + e^x} - \mu e^y - \theta - \frac{\sigma_3^2}{2} \right] dt + \sigma_3 dW_3, \\ dw(t) &= \left[\gamma e^{y-w} - \delta e^{z-w} - b - \frac{\sigma_4^2}{2} \right] dt + \sigma_4 dW_4. \end{aligned} \tag{3.2}$$

For $X_0 = (x_0, y_0, z_0, w_0) \in \mathbb{R}_+^4$, the coefficients of system (3.2) satisfy the local Lipschitz conditions, consequently, there exists a unique local solution $(S(t), I_1(t), I_2(t), R(t)) = (e^{x(t)}, e^{y(t)}, e^{z(t)}, e^{w(t)})$ on $[0, \tau_e)$. To ensure that this solution is global, one needs to prove that $\tau_e = \infty$ a.s. Let $s_0 > 0$ be sufficiently large for every coordinate $(S(0), I_1(0), I_2(0), R(0))$ in the interval $[\frac{1}{s_0}, s_0]$. For each integer $s > s_0$, we define the stopping time

$$\tau_s = \inf \left\{ t \in [0, \tau_e) : \min \{S(t), I_1(t), I_2(t), R(t)\} \notin \left(\frac{1}{s}, s\right) \text{ or } \max \{S(t), I_1(t), I_2(t), R(t)\} \notin \left(\frac{1}{s}, s\right) \right\}. \tag{3.3}$$

From (3.3), one can note that τ_s is increasing as $s \rightarrow \infty$. Assume $\tau_\infty = \lim_{s \rightarrow \infty} \tau_s$, then $\tau_\infty \leq \tau_e$ almost sure. In the next, one needs to verify that $\tau_\infty = \infty$. If this is not true, then there exists a constant $T > 0$ and $\epsilon \in (0, 1)$ such that $\mathbb{P}(\tau_\infty \leq T) \geq \epsilon$. As a result, there exists an integer $s_1 \geq s_0$ such that $\mathbb{P}(\tau_s \leq T) \geq \epsilon$, $s \geq s_1$. Define the following C^2 positive definite function $V_1(S, I_1, I_2, R)$ as

$$V_1(S, I_1, I_2, R) = (S + 1 - \ln S) + (I_1 + 1 - \ln I_1) + (I_2 + 1 - \ln I_2) + (R + 1 - \ln R).$$

Using Itô's formula, one obtains

$$\begin{aligned} dV_1 &= \left[(S - 1) \left(\frac{A}{S} - \frac{k_1 I_1}{a + S} - \frac{k_2 I_2}{a + S} - b \right) + (I_1 - 1) \left(\frac{k_1 S}{a + S} + \mu I_2 - \rho \right) \right. \\ &\quad \left. + (I_2 - 1) \left(\frac{k_2 S}{a + S} - \mu I_1 - \theta \right) + \left(1 - \frac{1}{R} \right) (\gamma I_1 + \delta I_2 - bR) + \frac{1}{2} \sum_{i=1}^4 \sigma_i^2 \right] dt \\ &\quad + \sigma_1 (S - 1) dW_1 + \sigma_2 (I_1 - 1) dW_2 + \sigma_3 (I_2 - 1) dW_3 + \sigma_4 (R - 1) dW_4 \\ &\leq \left[A + 2b + \rho + \theta + \frac{1}{2} \sum_{i=1}^4 \sigma_i^2 + \left(\frac{k_1}{a} + \mu + \gamma \right) I_1 + \left(\frac{k_2}{a} + \delta \right) I_2 \right] dt \\ &\quad + \sigma_1 (S - 1) dW_1 + \sigma_2 (I_1 - 1) dW_2 + \sigma_3 (I_2 - 1) dW_3 + \sigma_4 (R - 1) dW_4. \end{aligned}$$

Using the inequality $x \leq 2(x + 1 - \ln x)$, for any $x > 0$, one obtains

$$\begin{aligned} dV_1 \leq & \left[A + 2b + \rho + \theta + \frac{1}{2} \sum_{i=1}^4 \sigma_i^2 + 2(S + 1 - \ln S) + 2 \left(\frac{k_1}{a} + \mu + \gamma \right) (I_1 + 1 - \ln I_1) \right. \\ & \left. + 2 \left(\frac{k_2}{a} + \delta \right) (I_2 + 1 - \ln I_2) + 2(R + 1 - \ln R) \right] dt + \sigma_1(S - 1)dW_1 + \sigma_2(I_1 - 1)dW_2 \\ & + \sigma_3(I_2 - 1)dW_3 + \sigma_4(R - 1)dW_4, \end{aligned}$$

which means that

$$dV_1 \leq K(1 + V_1)dt + \sigma_1(S - 1)dW_1 + \sigma_2(I_1 - 1)dW_2 + \sigma_3(I_2 - 1)dW_3 + \sigma_4(R - 1)dW_4, \quad (3.4)$$

where $K = \max \{K_1, K_2\}$, $K_1 = A + 2b + \rho + \theta + \frac{1}{2} \sum_{i=1}^4 \sigma_i^2$ and $K_2 = \max \left\{ 2, 2 \left(\frac{k_1}{a} + \mu + \gamma \right), 2 \left(\frac{k_2}{a} + \delta \right) \right\}$. For $t_1 \leq T$, integrating both sides of (3.4) from 0 to $t_1 \wedge \tau_s$ and then taking the expectation leads to

$$\begin{aligned} & EV_1(S(t_1 \wedge \tau_s), I_1(t_1 \wedge \tau_s), I_2(t_1 \wedge \tau_s), R(t_1 \wedge \tau_s)) \\ & \leq V_1(S(0), I_1(0), I_2(0), R(0)) + KE \int_0^{t_1 \wedge \tau_s} (1 + V_1) dt \\ & \leq V_1(S(0), I_1(0), I_2(0), R(0)) + KT + K \int_0^{t_1 \wedge \tau_s} EV_1(S(t_1 \wedge \tau_s), I_1(t_1 \wedge \tau_s), I_2(t_1 \wedge \tau_s), R(t_1 \wedge \tau_s)) dt. \end{aligned}$$

Following [25, 26, 29, 44], applying Grownwall's inequality, one gets

$$EV_1(S(t_1 \wedge \tau_s), I_1(t_1 \wedge \tau_s), I_2(t_1 \wedge \tau_s), R(t_1 \wedge \tau_s)) \leq [V_1(S(0), I_1(0), I_2(0), R(0)) + KT] e^{KT} = K_3.$$

The rest of the proof is similar to [25, 26, 29] and hence is omitted here. This complete the proof. \square

The above theorem shows that the stochastic HCV system (3.1) have positive global solution remain in \mathbb{R}_+^4 with probability one. The non-explosion characteristic in an epidemic model is essential but often insufficient. Hence, one needs to indicate that the solution will be ultimately bounded with a large probability. In the following, we will establish the stochastically ultimate boundedness property of the HCV system (3.1).

Lemma 2. Let $N(t) = S(t) + I_1(t) + I_2(t) + R(t)$, then for any given initial value $(S(0), I_1(0), I_2(0), R(0)) \in \mathbb{R}_+^4$, the following inequality holds:

$$\lim_{t \rightarrow \infty} N(t) < \infty \text{ a.s.}$$

Proof. It follows from HCV model (3.1) that

$$dN(t) = (A - bN(t))dt + \sigma_1 S(t)dW_1(t) + \sigma_2 I_1(t)dW_2(t) + \sigma_3 I_2(t)dW_3(t) + \sigma_4 R(t)dW_4(t) \quad (3.5)$$

Then, the solution of Eq (3.5) has the following form

$$N(t) = \frac{A}{b} + \left(N(0) - \frac{A}{b} \right) e^{-bt} + M(t),$$

where

$$M(t) = \int_0^t e^{-b(t-s)} \sigma_1 S(s) dW_1(s) + \int_0^t e^{-b(t-s)} \sigma_2 I_1(s) dW_2(s) + \int_0^t e^{-b(t-s)} \sigma_3 I_2(s) dW_3(s) \\ + \int_0^t e^{-b(t-s)} \sigma_4 R(s) dW_4(s),$$

which is a continuous local martingale with $M(0) = 0$, almost surely. Define $\Lambda(t) = \frac{A}{b}[1 - e^{-bt}]$ and $U(t) = N(0)[1 - e^{-bt}]$, then we have $N(t) = N(0) + \Lambda(t) - U(t) + M(t)$. Clearly, $\Lambda(t)$ and $U(t)$ are continuous adapted increasing processes on $t \geq 0$ with $\Lambda(0) = U(0) = 0$. Then by Theorem 3.9 in [30], we have $\lim_{t \rightarrow \infty} N(t) < \infty$ a.s. \square

Remark 3. It follows from the first equation of HCV system (3.1) that

$$S(t) - S(0) = At - b \int_0^t S(s) ds - \int_0^t \frac{k_1 S(s) I_1(s)}{a + S(s)} - \int_0^t \frac{k_2 S(s) I_2(s)}{a + S(s)} + \sigma_1 \int_0^t S(s) dW_1(s),$$

and it follows that

$$\langle S(t) \rangle \leq \lim_{t \rightarrow \infty} \left[\frac{A}{b} - \frac{S(t) - S(0)}{t} + \frac{\sigma_1}{t} \int_0^t S(s) dW_1(s) \right] \leq \frac{A}{b}.$$

Theorem 4. If $\frac{\sigma_1^2}{2} + \frac{1}{2} < b$, $k_1 + \frac{\sigma_2^2}{2} + \frac{1}{2} < \rho$, $k_2 + \frac{\sigma_3^2}{2} + \frac{1}{2} < \theta$, $\sigma_4^2 + \frac{1}{2} < b$, then the solutions of (3.1) are stochastically ultimately bounded.

Proof. For $(S(t), I_1(t), I_2(t), R(t)) \in \mathbb{R}_+^4$, define the following function

$$V_2(S(t), I_1(t), I_2(t), R(t)) = S(t)^2 + I_1(t)^2 + I_2(t)^2 + R(t)^2.$$

By Itô formula, one has

$$dV_2(S, I_1, I_2, R) = LV_2(S, I_1, I_2, R)dt + 2\sigma_1 S^2 dW_1 + 2\sigma_2 I_1^2 dW_2 + 2\sigma_3 I_2^2 dW_3 + 2\sigma_4 R^2 dW_4,$$

where

$$LV_2(S, I_1, I_2, R) = 2 \left(AS - \frac{k_1 S^2 I_1}{a + S} - \frac{k_2 S^2 I_2}{a + S} - bS^2 \right) + 2I_1^2 \left(\frac{k_1 S}{a + S} + \mu I_2 - \rho \right) \\ + 2I_2^2 \left(\frac{k_2 S}{a + S} - \mu I_1 - \theta \right) + 2(\gamma I_1 R + \delta I_2 R - bR^2) + \sigma_1^2 S^2 + \sigma_2^2 I_1^2 + \sigma_3^2 I_2^2 + \sigma_4^2 R^2 \\ \leq (\sigma_1^2 - 2b + 1)S^2 + (2k_1 + \sigma_2^2 - 2\rho + 1)I_1^2 + (2k_2 + \sigma_3^2 - 2\theta + 1)I_2^2 + (\sigma_4^2 - 2b + 1)R^2 \\ - [S(t)^2 + I_1(t)^2 + I_2(t)^2 + R(t)^2] + 2(AS + \mu I_1^2 I_2 + \gamma I_1 R + \delta I_2 R).$$

Assume $f(S(t), I_1(t), I_2(t), R(t)) = (\sigma_1^2 - 2b + 1)S^2 + (2k_1 + \sigma_2^2 - 2\rho + 1)I_1^2 + (2k_2 + \sigma_3^2 - 2\theta + 1)I_2^2 + (\sigma_4^2 - 2b + 1)R^2 + 2(AS + \mu I_1^2 I_2 + \gamma I_1 R + \delta I_2 R)$. According to Lemma 2 and Remark 3, one can find that the function $f(S(t), I_1(t), I_2(t), R(t))$ has an upper bound.

Let $N_1 = \sup f(S(t), I_1(t), I_2(t), R(t)) + 1$. As a result

$$dV_2(S, I_1, I_2, R) = (N_1 - V_2)dt + 2\sigma_1 S^2 dW_1 + 2\sigma_2 I_1^2 dW_2 + 2\sigma_3 I_2^2 dW_3 + 2\sigma_4 R^2 dW_4.$$

By Itô formula, one obtains

$$d(e^t V_2) \leq e^t N_1 dt + e^t \left[2\sigma_1 S^2 dW_1 + 2\sigma_2 I_1^2 dW_2 + 2\sigma_3 I_2^2 dW_3 + 2\sigma_4 R^2 dW_4 \right].$$

Integrating both sides of the above equation from 0 to t and then taking expectations, one gets

$$e^t V_2(S(t), I_1(t), I_2(t), R(t)) \leq V_2(S(0), I_1(0), I_2(0), R(0)) + N_1 e^t - N_1,$$

hence

$$\limsup_{t \rightarrow \infty} \mathbb{E}[|X(t)|^2] \leq N_1.$$

Using Chebyshev's inequality, one can obtain

$$\mathbb{P}[|X(t)| \geq H] \leq \frac{\mathbb{E}[|X(t)|^2]}{H^2},$$

where $H = \frac{\sqrt{N_1}}{\sqrt{\epsilon}}$, $\epsilon > 0$. Then

$$\limsup_{t \rightarrow \infty} \mathbb{P}[|X(t)| \geq H] \leq \frac{N_1}{H^2} = \epsilon.$$

This completes the proof. \square

In the following, we will establish the conditions for extinction of hepatitis C virus from the stochastic system (3.1). The threshold parameter R_{01}^s defined by $R_{01}^s = \frac{k_1}{\rho + \frac{\sigma_2^2}{2}}$, while the threshold parameter R_{02}^s defined by $R_{02}^s = \frac{k_2}{\theta + \frac{\sigma_3^2}{2}}$ and the threshold parameter R_{03}^s defined by $R_{03}^s = \frac{k_1 + k_2}{\rho + \theta}$. The stochastic basic reproduction number R_0^s defined by $R_0^s = \max\{R_{01}^s, R_{02}^s, R_{03}^s\}$.

Theorem 5. *The disease die out exponentially with probability one if $R_0^s < 1$.*

Proof. Let $V_3(I_1, I_2) = \ln(I_1 + I_2)$. Using Itô formula, one obtains

$$\begin{aligned} d(V_3(I_1, I_2)) &= \left[\frac{1}{(I_1 + I_2)} \left(\frac{k_1 S I_1}{a + S} + \frac{k_2 S I_2}{a + S} - \rho I_1 - \theta I_2 \right) - \frac{\sigma_2^2 I_1^2}{2(I_1 + I_2)^2} - \frac{\sigma_3^2 I_2^2}{2(I_1 + I_2)^2} \right] dt \\ &\quad + \frac{\sigma_2 I_1}{(I_1 + I_2)} dW_2 + \frac{\sigma_3 I_2}{(I_1 + I_2)} dW_3 \\ &\leq \frac{1}{(I_1 + I_2)^2} \left[\left(\frac{k_1 S}{(a + S)} - \rho - \frac{\sigma_2^2}{2} \right) I_1^2 + \left(\frac{k_2 S}{(a + S)} - \theta - \frac{\sigma_3^2}{2} \right) I_2^2 + \left(\frac{k_1 S}{(a + S)} - \rho \right) I_1 I_2 \right. \\ &\quad \left. + \left(\frac{k_2 S}{(a + S)} - \theta \right) I_1 I_2 \right] dt + \frac{\sigma_2 I_1}{(I_1 + I_2)} dW_2 + \frac{\sigma_3 I_2}{(I_1 + I_2)} dW_3 \\ &\leq \frac{1}{(I_1 + I_2)^2} \left[\left(k_1 - \rho - \frac{\sigma_2^2}{2} \right) I_1^2 + \left(k_2 - \theta - \frac{\sigma_3^2}{2} \right) I_2^2 + (k_1 - \rho) I_1 I_2 \right. \\ &\quad \left. + (k_2 - \theta) I_1 I_2 \right] dt + \frac{\sigma_2 I_1}{(I_1 + I_2)} dW_2 + \frac{\sigma_3 I_2}{(I_1 + I_2)} dW_3 \\ &\leq \left[\left(\rho + \frac{\sigma_2^2}{2} \right) (R_{01}^s - 1) + \left(\theta + \frac{\sigma_3^2}{2} \right) (R_{02}^s - 1) + (\rho + \theta) (R_{03}^s - 1) \right] dt \\ &\quad + \frac{\sigma_2 I_1}{(I_1 + I_2)} dW_2 + \frac{\sigma_3 I_2}{(I_1 + I_2)} dW_3 \\ &\leq \left[2(\rho + \theta) + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} \right] (R_0^s - 1) dt + \frac{\sigma_2 I_1}{(I_1 + I_2)} dW_2 + \frac{\sigma_3 I_2}{(I_1 + I_2)} dW_3, \end{aligned}$$

using integration from 0 to t , one obtains

$$\begin{aligned} \ln(I_1(t) + I_2(t)) &\leq \ln(I_1(0) + I_2(0)) + \left[2(\rho + \theta) + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2}\right](R_0^s - 1)t \\ &\quad + \int_0^t \frac{\sigma_2 I_1}{2(I_1 + I_2)} dW_2 + \int_0^t \frac{\sigma_3 I_2}{2(I_1 + I_2)} dW_3. \end{aligned}$$

Applying strong law of large numbers for local martingales one gets

$$\limsup_{t \rightarrow \infty} \frac{\ln(I_1(t) + I_2(t))}{t} \leq \left[2(\rho + \theta) + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2}\right](R_0^s - 1) < 0,$$

as a result, diseases $I_1(t)$ and $I_2(t)$ die out and tend to zero exponentially a.s., if $R_0^s < 1$. \square

Remark 6. *Theorem 3 indicates that the disease die out exponentially a.s., if $R_0^s < 1$ with the consequence that the recover class $R(t)$ also goes to extinction a.s.*

4. Numerical simulations

In this part, the numerical results will be compared with the theorems formulated in the previous sections. The interactions between populations classes will be simulated by the following parameters [32]: $N = 1000000$, $\mu = 0.02$, $a = 1$, $b = 0.02$, $k_1 = 0.00001$, $k_2 = 0.00002$, $\gamma = 0.001$, $\delta = 0.001$. To give some numerical finding to the HCV system (3.1), we use the Milstein method mentioned in [2, 3, 15]. The stochastic hepatitis virus C system (3.1) reduces to the following discrete system

$$\begin{aligned} S_{(j+1)} &= x_j + h \left(A - \frac{k_1 S_j I_{1j}}{a + S_j} - \frac{k_2 S_j I_{2j}}{a + S_j} - b S_j \right) + \sigma_1 S_j \sqrt{h} \epsilon_{1j} + \frac{\sigma_1^2}{2} S_j [\epsilon_{1j}^2 - 1] h, \\ I_{1(j+1)} &= I_{1j} + h \left(\frac{k_1 S_j I_{1j}}{a + S_j} + \mu I_{1j} I_{2j} - b I_{1j} - \gamma I_{1j} \right) + \sigma_2 I_{1j} \sqrt{h} \epsilon_{2j} + \frac{\sigma_2^2}{2} I_{1j} [\epsilon_{2j}^2 - 1] h, \\ I_{2(j+1)} &= I_{2j} + h \left(\frac{k_2 S_j I_{2j}}{a + S_j} - \mu I_{1j} I_{2j} - b I_{2j} - \delta I_{2j} \right) + \sigma_3 I_{2j} \sqrt{h} \epsilon_{3j} + \frac{\sigma_3^2}{2} I_{2j} [\epsilon_{3j}^2 - 1] h, \\ R_{(j+1)} &= R_j + h (\gamma I_{1j} + \delta I_{2j} - b R_j) + \sigma_4 R_j \sqrt{h} \epsilon_{4j} + \frac{\sigma_4^2}{2} R_j [\epsilon_{4j}^2 - 1] h, \end{aligned} \tag{4.1}$$

where h is a positive time increment and ϵ_{ij} , ($i = 1, 2, 3, 4$) are independent random Gaussian variables $N(0, 1)$. One can note that for the given parameters, the value of the stochastic basic reproduction number $R_0^s = 0.00095$. As a result, the conditions of Theorem 5 are verified and the disease die out exponentially with probability one if $R_0^s < 1$. Figure 1 represents the dynamical behavior of model (3.1) when the noise strength law ($\sigma_i = 0.01$). For $k_1 = k_2 = 0.0308$, the conditions of Theorem 5 are verified as $R_0^s = 1.130769 > 1$ as shown in Figure 2. It is shown that the trajectories of the stochastic system (3.1) oscillates around the coexistence equilibrium of the deterministic system (1.1).

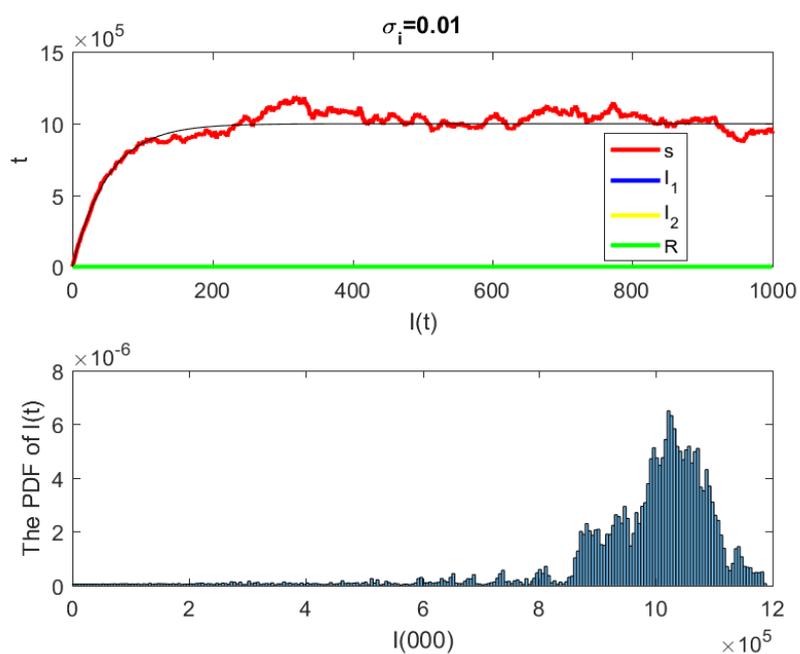


Figure 1. Fluctuation in susceptible population with with $R_0^s = 0.00095 < 1$.

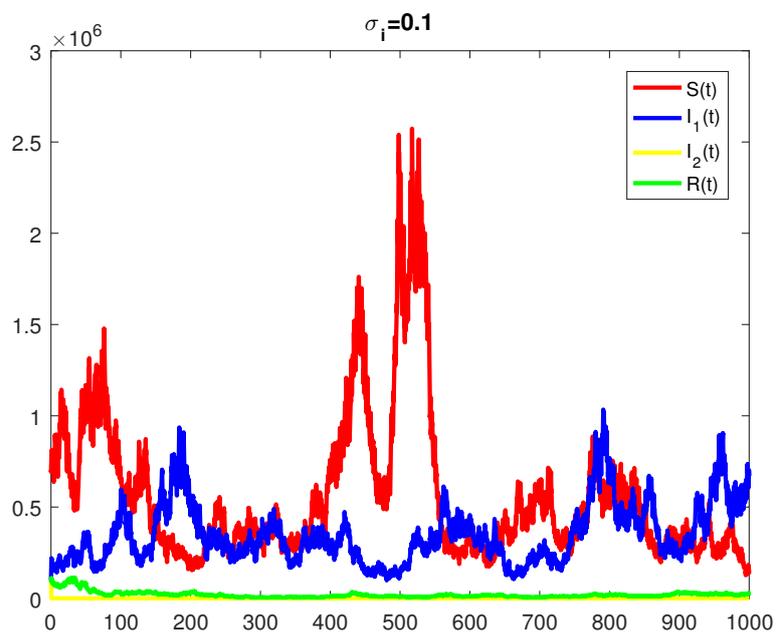


Figure 2. The evolution of infected and recovered individuals with $R_0^s = 1.3968 > 1$.

5. Conclusions

The novelty of this study is that we introduced a new compartment and saturated incidence rate into the classical hepatitis C virus genome model. A deterministic and stochastic model for hepatitis C with different types of virus genomes is proposed and analyzed. We perform a stochastic extension of the deterministic model to study the fluctuation between environmental factors. Firstly, the existence of a unique global positive solution for the stochastic model is investigated. Secondly, sufficient conditions for the extinction of hepatitis C virus from the stochastic system are obtained. Theoretical results are illustrated using numerical simulations. Theoretical and numerical results show that the smaller white noise can ensure the persistence of susceptible and infected populations while the larger white noise can lead to the extinction of populations. For the deterministic model, some sufficient conditions are obtained to ensure the stability of the equilibrium points. The results show that when the basic reproduction number $R_0 < 1$, the deterministic model is asymptotic stable around the free disease equilibrium point E_0 . By introducing the basic reproduction number R_0 and the stochastic basic reproduction number R_0^s , the conditions that cause the disease to die out are indicated. Biologically, it can be noted that by social distancing with hepatitis C patients, this can lead to a fluctuation in the value of k_1 and k_2 , and thus R_0^s becomes less than one, which leads to the disappearance of the epidemic.

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Conflict of interest

The authors declare that there is no conflict of interests.

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