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ON OSCILLATORY BEHAVIOUR OF THIRD-ORDER HALF-LINEAR DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. In this work, we study the oscillation and asymptotic behaviour of third-order nonlinear dynamic equations on time scales. The findings are obtained using an integral criterion as well as a comparison theorem with the oscillatory properties of a first-order dynamic equation. As a consequence, we give conditions which guarantee that all solutions to the aforementioned problem are only oscillatory, different from any other result in the literature. We propose novel oscillation criteria that improve, extend, and simplify existing ones in the literature. The results are associated with a numerical example. We point out that the results are new even for the case $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$.

Keywords: oscillation, asymptotic behaviour, dynamic equation on time scales, comparison method, Riccati technique.

Mathematics Subject Classification: 34C10, 34K11, 34N05, 39A10.

1. INTRODUCTION

This paper is concerned with oscillatory behaviour of all solutions of the half-linear third-order dynamic equations of the form

$$\left(r(t)\left(x^{\Delta\Delta}(t)\right)^{\alpha}\right)^{\Delta} + q(t)x^{\alpha}(\omega(t)) = 0, \tag{1.1}$$

where $\sup \mathbb{T} = \infty$, $t \in [t_0, \infty)_{\mathbb{T}} = [t_0, \infty) \cap \mathbb{T}$ with $t_0 \in \mathbb{T}$. A solution of (1.1) is a function $x(t)$ continuous on $[T_x, \infty)$, $T_x \geq t_0$, which satisfies (1.1) on $[T_x, \infty)_{\mathbb{T}}$. Solutions vanishing identically in some neighbourhood of infinity will be excluded from our consideration. Such a solution is said to be oscillatory if it is neither eventually positive nor eventually negative, and to be nonoscillatory otherwise.

Throughout the remaining part of the paper, we always assume that:

(A_1) α is a quotient of positive odd integers, $\alpha \geq 1$,
(A_2) $r(t)$ and $q(t) \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}_+)$ and $q(t) \not\equiv 0$,
(A_3) $\omega \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$ such that $\omega(t) \leq t$ and $\lim_{t\to\infty} \omega(t) = \infty$.

For $r(t) \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}_+)$, we let

$$\mathcal{R}(v, u) = \int_u^v \frac{\Delta s}{r^{1/\alpha}(s)} \quad \text{and} \quad \mathcal{R}(t, t_0) = \int_{t_0}^t \frac{\Delta s}{r^{1/\alpha}(s)} \to \infty \text{ as } t \to \infty. \tag{1.2}$$

There are numerous applications of nonlinear differential/difference equations in science and engineering to understand natural and physical phenomena. This can only be accomplished by establishing a theory before its experimental verification. A third-order delay differential/difference equation can also be used to model a wide range of applications, including control systems, boundary layer theory, radio technology, electrodynamics, neural networks, and population models (see, e.g., [3, 29, 38, 39, 42]). In addition to these, there are some natural mechanisms that occur both over a continuous and over a discrete time period, see, for example, [7]. So, it is more realistic to model a physical process that incorporates both continuous and discrete times. The idea of time-scale calculus, developed by Hilger [28] in 1990, allows the theory of differential equations and difference equations to be unified. Due to this, we are mainly focusing on studying dynamic equations on time scales in order to have a comprehensive analysis of the dynamical systems. We suggest the reader consult the works of Bohner and Peterson [1, 9, 10] for more information on the theory of dynamic equations on time scales and its applications, as well as basic concepts and notations.

The study of qualitative outcomes, notably oscillatory behaviours of dynamic equations on time scales, has been highly popular over the past few years, see, for example, [4–6, 8, 11–14, 18–26, 37, 44]. In particular, Erbe *et al.* [15] initiated the study of third-order dynamic equations on time scales of the form:

$$(c(t)(r(t)x^{\Delta}(t))^{\Delta})^{\Delta} + q(t)F(x(t)) = 0, \tag{1.3}$$

where $c, r, q \in C_{rd}(t_0, \infty)_{\mathbb{T}}$, $F \in C(\mathbb{R}, \mathbb{R})$ such that $F(u)/u \geq M > 0$, $xF(x) > 0$ for all $x \neq 0$. Upon using the Riccati transformation technique, they found some sufficient conditions such that every solution of (1.3) is oscillatory or converges to zero. In this context, one can figure out that many scholars have studied various generalizations of third-order dynamic equations and improved the oscillation conditions by using different methods such as the Riccati transformation technique, integral averaging method, comparison method, and inequality technique, see [16, 17, 34–36, 41, 43, 45] and references cited therein.

Inspired by the above discussion, the objective of this work is to find new improved conditions for the oscillation of (1.1) via the Riccati transformation technique as well as comparison with the oscillatory behaviour of first-order dynamic equations.

2. SOME LEMMAS

Lemma 2.1 ([22])**.** *Let* $q \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R}_+)$*,* $g \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$ *such that* $g(t) \le t$ *and* $g(t) \to \infty$ *as* $t \to \infty$*. The associated delay dynamic equation* (*i.e.,* $g(t) \le t$)

$$\mathcal{W}^{\Delta}(t) + q(t)\mathcal{W}(g(t))) = 0$$

has an eventually positive solution if the first-order delay inequality

$$\mathcal{W}^{\Delta}(t) + q(t)\mathcal{W}(g(t))) \le 0$$

does.

Next, we present the following preliminary lemmas to obtain the sign properties of possible nonoscillatory solutions of Eq. (1.1).

Lemma 2.2. *Let* (A_1)*–*(A_3) *and* (1.2) *hold. Then the equation* (1.1) *has no eventually positive solution satisfying* $x(t) > 0$ *and* $x^{\Delta\Delta}(t) < 0$ *eventually.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1) such that $x(t) > 0$, $x(\omega(t)) > 0$ for $t \ge t_1 > t_0$. Since $x^{\Delta\Delta}(t) < 0$ for $t \ge t_2$, then we can find a constant $c > 0$ and $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that $(r(t)(x^{\Delta\Delta}(t))^{\alpha} \le -c < 0$ for $t \ge t_2$, or

$$x^{\Delta\Delta}(t) \le \left(\frac{-c}{r(t)}\right)^{1/\alpha},$$

where $c = r(t_2)w^{\Delta\Delta}(t_2)$. Integrating the preceding inequality from t_2 to t, we obtain

$$x^{\Delta}(t) \le x^{\Delta}(t_2) - \int_{t_2}^{t} \frac{c}{r^{1/\alpha}(t)}.$$

An application of condition (1.2) gives $x^{\Delta}(t) \to -\infty$ as $t \to \infty$. Therefore, there exists $t_3 \in [t_2,\infty)_{\mathbb{T}}$ such that

$$x^{\Delta}(t) \le x^{\Delta}(t_3) < 0,$$

which on integration form t_3 to t yields $\lim_{t\to\infty} x(t) = -\infty$, a contradiction to the fact that $x(t) > 0$. □

Lemma 2.3. *Let* (A_1)*–*(A_3) *and* (1.2) *hold. If the delay equation*

$$\mathcal{W}^{\Delta}(t) + \frac{1}{\alpha}\left(\int_{t_1}^{\omega(t)} \mathcal{R}(s,t_1)\Delta s\right)^{\alpha} q(t)\mathcal{W}(\omega(t)) = 0 \tag{2.1}$$

is oscillatory, then (1.1) *has no eventually positive solution satisfying* $x(t) > 0$*,* $x^{\Delta}(t) > 0$ *and* $x^{\Delta\Delta}(t) > 0$ *eventually.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1.1) such that $x(t) > 0$, $x(\omega(t)) > 0$ for $t \geq t_1 > t_0$ satisfying $x^{\Delta}(t) > 0$ and $x^{\Delta\Delta}(t) > 0$ for $t \geq t_2$. From (1.1), we see that

$$r(t)(x^{\Delta\Delta}(t))^{\alpha} = (r^{1/\alpha}(t)x^{\Delta\Delta}(t))^{\alpha}.$$

Taking Δ-derivative of the above inequality, we get

$$\begin{aligned}[r(t)(x^{\Delta\Delta}(t))^{\alpha}]^{\Delta} &= [(r^{1/\alpha}(t)x^{\Delta\Delta}(t))^{\alpha}]^{\Delta} \\ &\geq \alpha(r^{1/\alpha}(t)x^{\Delta\Delta}(t))^{\alpha-1}(r^{1/\alpha}(t)x^{\Delta\Delta}(t))^{\Delta}.\end{aligned}$$

Using (2.2) in Eq. (1.1), we have

$$(r^{1/\alpha}(t)x^{\Delta\Delta}(t))^{\Delta} + \frac{1}{\alpha}(r^{1/\alpha}(t)x^{\Delta\Delta}(t))^{1-\alpha}q(t)x^{\alpha}(\omega(t)) \leq 0. \tag{2.2}$$

Indeed,

$$x^{\Delta}(t) - x^{\Delta}(t_1) = \int_{t_1}^{t} \frac{r^{1/\alpha}(s)x^{\Delta\Delta}(s)}{r^{1/\alpha}(s)}\Delta s \geq r^{1/\alpha}(t)x^{\Delta\Delta}(t)\int_{t_1}^{t}\frac{1}{r^{1/\alpha}(s)}\Delta s,$$

which implies that

$$x^{\Delta}(t) \geq \mathcal{R}(t,t_1)(r^{1/\alpha}(t)x^{\Delta\Delta}(t)).$$

Integrating this inequality from t_1 to t, one can easily get

$$x(t) \geq x(t) - x(t_1) \geq \int_{t_1}^{t} \mathcal{R}(s,t_1)(r^{1/\alpha}(s)x^{\Delta\Delta}(s))\Delta s$$

implies that

$$x(t) \geq (r^{1/\alpha}(t)x^{\Delta\Delta}(t))\int_{t_1}^{t}\mathcal{R}(s,t_1)\Delta s.$$

As a result,

$$x(\omega(t)) \geq (r^{1/\alpha}(\omega(t))x^{\Delta\Delta}(\omega(t)))\int_{t_1}^{\omega(t)}\mathcal{R}(s,t_1)\Delta s. \tag{2.3}$$

Since the function $r^{1/\alpha}(t)x^{\Delta\Delta}(t)$ is nonincreasing and $\alpha \geq 1$, then we have

$$(r^{1/\alpha}(t)x^{\Delta\Delta}(t))^{1-\alpha} \geq (r^{1/\alpha}(\omega(t))x^{\Delta\Delta}(\omega(t)))^{1-\alpha}. \tag{2.4}$$

Using (2.3) and (2.4) in (2.2), we get

$$[r^{1/\alpha}(t)x^{\Delta\Delta}(t)]^{\Delta} \le -\frac{1}{\alpha}(r^{1/\alpha}(t)x^{\Delta\Delta}(t))^{1-\alpha}q(t)x^{\alpha}(\omega(t))$$

$$\le -\frac{1}{\alpha}(r^{1/\alpha}(\omega(t))x^{\Delta\Delta}(\omega(t)))^{1-\alpha}q(t)\left(\int_{t_1}^{\omega(t)}\mathcal{R}(s,t_1)\Delta s\right)^{\alpha}$$

$$\times (r^{1/\alpha}(\omega(t))x^{\Delta\Delta}(\omega(t)))^{\alpha}$$

$$= -\frac{1}{\alpha}q(t)\left(\int_{t_1}^{\omega(t)}\mathcal{R}(s,t_1)\Delta s\right)^{\alpha} r^{1/\alpha}(\omega(t))x^{\Delta\Delta}(\omega(t)).$$

Setting $\mathcal{W}(t) = r^{1/\alpha}(t)x^{\Delta\Delta}(t)$, in the last inequality, we have

$$\mathcal{W}^{\Delta}(t) + \frac{1}{\alpha}\left(\int_{t_1}^{\omega(t)}\mathcal{R}(s,t_1)\Delta s\right)^{\alpha} q(t)\mathcal{W}(\omega(t)) \le 0.$$

It follows from Lemma 2.4 that the corresponding dynamic equation (2.1) also has a positive solution, which is a contradiction. □

Lemma 2.4. *Let* (A_1)–(A_3) *and* (1.2) *hold. Assume that there exists a nondecreasing function* $\eta(t) \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that*

$$\eta(t) > t, \quad \eta(\omega(t)) > \omega(t) \quad \text{and} \quad \eta(\eta(\omega(t))) < t. \tag{2.5}$$

If the delay equation

$$\mathcal{X}^{\Delta}(t) + \frac{1}{\alpha}q(t)\left(\int_{\omega(t)}^{\eta(\omega(t))}\mathcal{R}(\eta(s),s)\Delta s\right)^{\alpha}\mathcal{X}(\eta(\eta(\omega(t)))) = 0 \tag{2.6}$$

is oscillatory, then (1.1) *has no eventually positive solution satisfying* $x(t) > 0$, $x^{\Delta}(t) < 0$ *and* $x^{\Delta\Delta}(t) > 0$ *eventually.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1.1) such that $x(t) > 0$, $x(\omega(t)) > 0$ for $t \ge t_1 > t_0$ satisfying $x^{\Delta}(t) < 0$ and $x^{\Delta\Delta}(t) > 0$ for $t \ge t_2$. Clearly,

$$x^{\Delta}(\eta(t)) - x^{\Delta}(t) = \int_{t}^{\eta(t)}\frac{r^{1/\alpha}(s)x^{\Delta\Delta}(s)}{r^{1/\alpha}(s)}\Delta s$$

$$\ge r^{1/\alpha}(\eta(t))x^{\Delta\Delta}(\eta(t))\int_{t}^{\eta(t)}\frac{1}{r^{1/\alpha}(s)}\Delta s,$$

which implies that

$$-x^{\Delta}(t) \geq \mathcal{R}(\eta(t),t)(r^{1/\alpha}(\eta(t))x^{\Delta\Delta}(\eta(t))).$$

Integrating the last inequality from t to $\eta(t)$, we obtain

$$\begin{aligned} x(t) \geq x(t) - x(\eta(t)) &\geq \int_{t}^{\eta(t)} (r^{1/\alpha}(\eta(s))x^{\Delta\Delta}(\eta(s)))\mathcal{R}(\eta(s),s)\Delta s \\ &\geq (r^{1/\alpha}(\eta(\eta(t)))x^{\Delta\Delta}(\eta(\eta(t)))) \int_{t}^{\eta(t)} \mathcal{R}(\eta(s),s)\Delta s. \end{aligned}$$

As a result,

$$x(\omega(t)) \geq (r^{1/\alpha}(\eta(\eta(\omega(t))))x^{\Delta\Delta}(\eta(\eta(\omega(t))))) \int_{t}^{\eta(\omega(t))} \mathcal{R}(\eta(s),s)\Delta s. \tag{2.7}$$

Using (2.7) in (2.2), we get

$$\begin{aligned} &[r^{1/\alpha}(t)x^{\Delta\Delta}(t)]^{\Delta} \\ &+ \frac{1}{\alpha}(r^{1/\alpha}(\eta(\eta(\omega(t))))x^{\Delta\Delta}(\eta(\eta(\omega(t)))))^{1-\alpha} \left(\int_{\omega(t)}^{\eta(\omega(t))} \mathcal{R}(\eta(s),s)\Delta s \right)^{\alpha} \\ &\times q(t)(r^{1/\alpha}(\eta(\eta(\omega(t))))x^{\Delta\Delta}(\eta(\eta(\omega(t)))))^{\alpha} \leq 0, \end{aligned}$$

that is,

$$\begin{aligned} [r^{1/\alpha}(t)x^{\Delta\Delta}(t)]^{\Delta} + \frac{1}{\alpha} &\left(\int_{\omega(t)}^{\eta(\omega(t))} \mathcal{R}(\eta(s),s)\Delta s \right)^{\alpha} q(t) \\ &\times (r^{1/\alpha}(\eta(\eta(\omega(t))))x^{\Delta\Delta}(\eta(\eta(\omega(t))))) \leq 0. \end{aligned}$$

Setting $\mathcal{X}(t) = r^{1/\alpha}(t)x^{\Delta\Delta}(t)$, in the last inequality, we have

$$\mathcal{X}^{\Delta}(t) + \frac{1}{\alpha}q(t) \left(\int_{\omega(t)}^{\eta(\omega(t))} \mathcal{R}(\eta(s),s)\Delta s \right)^{\alpha} \mathcal{X}(\eta(\eta(\omega(t)))) \leq 0.$$

It follows from Lemma 2.4 that the corresponding dynamic equation (2.6) also has a positive solution, which is a contradiction. $\square$

Lemma 2.5. *Let* (A_1)–(A_3) *and* (1.2) *hold. If*

$$\limsup_{t\to\infty} \int_{\omega(t)}^{t} q(l) \left(\int_{\omega(s)}^{\omega(l)} \mathcal{R}(\omega(l), s)\Delta s \right)^{\alpha} \Delta l > 1, \tag{2.8}$$

then (1.1) *has no eventually positive solution satisfying* $x(t) > 0$, $x^{\Delta}(t) < 0$ *and* $x^{\Delta\Delta}(t) > 0$ *eventually.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1.1) such that $x(t) > 0$, $x(\omega(t)) > 0$ for $t \geq t_1 > t_0$ satisfying $x^{\Delta}(t) < 0$ and $x^{\Delta\Delta}(t) > 0$ for $t \geq t_2$. Clearly, for $v \geq u$,

$$\begin{aligned} x^{\Delta}(v) - x^{\Delta}(u) &= \int_{u}^{v} \frac{r^{1/\alpha}(s)x^{\Delta\Delta}(s)}{r^{1/\alpha}(s)}\Delta s \\ &\geq r^{1/\alpha}(v)x^{\Delta\Delta}(v) \int_{u}^{v} \frac{1}{r^{1/\alpha}(s)}\Delta s. \end{aligned}$$

As a result,

$$-x^{\Delta}(u) \geq \mathcal{R}(v,u)(r^{1/\alpha}(v)x^{\Delta\Delta}(v)),$$

which for $v > u \geq t_1$ is equivalent to

$$x(u) \geq x(u) - x(v) \geq (r^{1/\alpha}(v)x^{\Delta\Delta}(v)) \int_{u}^{v} \mathcal{R}(v,s)\Delta s.$$

For $t \geq s \geq t_2$ for some $t_2 > t_1$, setting $u = \omega(s)$ and $v = \omega(t)$ in the preceding inequality gives

$$x(\omega(s)) \geq (r^{1/\alpha}(\omega(t))x^{\Delta\Delta}(\omega(t))) \int_{\omega(s)}^{\omega(t)} \mathcal{R}(\omega(t), s)\Delta s. \tag{2.9}$$

Upon integrating (1.1) from $\omega(t)$ to t, we have

$$\begin{aligned} & r^{1/\alpha}(\omega(t))x^{\Delta\Delta}(\omega(t)) \\ & \geq r^{1/\alpha}(\omega(t))x^{\Delta\Delta}(\omega(t)) - r^{1/\alpha}(t)x^{\Delta\Delta}(t) \geq \int_{\omega(t)}^{t} q(s)x^{\alpha}(\omega(s))\Delta \\ & \geq (r^{1/\alpha}(\omega(t))x^{\Delta\Delta}(\omega(t))) \int_{\omega(t)}^{t} q(l) \left(\int_{\omega(s)}^{\omega(l)} \mathcal{R}(\omega(l), s)\Delta s \right)^{\alpha} \Delta l, \end{aligned}$$

which implies that

$$\int_{\omega(t)}^{t} q(l)\left(\int_{\omega(s)}^{\omega(l)} \mathcal{R}(\omega(l),s)\Delta s\right)^{\alpha} \Delta l \leq 1,$$

a contradiction to (2.9). □

Lemma 2.6. *Let* (A_1)–(A_3) *and* (1.2) *hold. Assume that equation* (1.1) *has eventually positive solution satisfying* $x(t) > 0$, $x^{\Delta}(t) < 0$ *and* $x^{\Delta\Delta}(t) > 0$ *eventually. If one of the following condition*

$$\int_{t_0}^{\infty} q(s) = \infty, \tag{2.10}$$

or,

$$\int_{t_0}^{\infty}\int_{u}^{\infty}\left(\frac{1}{r(v)}\int_{v}^{\infty} q(s)\right)^{\alpha} \Delta v\Delta u = \infty \tag{2.11}$$

hold, then $\lim_{t\to\infty} x(t) = 0$.

Proof. Indeed, $\lim_{t\to\infty} x(t) = \mathcal{L} \geq 0$ for $t \geq t_1$ for some $t_1 > t_0$. We assert that $\mathcal{L} = 0$. If not, then we can find $t_2 > t_1$ such that $x(t) \geq \mathcal{L} > 0$ and hence $x^{\alpha}(\omega(t)) \geq \mathcal{L}^{\alpha}$ for $t \geq t_2$. The rest of the proof follows from the proof of [35, Lemma 2.4]. This completes the proof. □

3. OSCILLATION RESULTS

We are ready to present our novel comparison theorem, which reduces the oscillation problem of third-order nonlinear dynamic equations (1.1) to a set of first-order linear delay dynamic equations. The assumption of the existence of a positive solution in equation (1.1) leads to a contradiction because the proof for the opposite case is similar.

Theorem 3.1. *Let* (A_1)–(A_3) *and* (1.2) *hold. Assume that there exist nondecreasing function* $\eta(t) \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that* (2.5) *holds. If the first-order delay dynamic equations*

$$\mathcal{Y}^{\Delta}(t) + \frac{1}{\alpha}q(t)\left(\int_{t_1}^{\omega(t)} \mathcal{R}(s,t_1)\Delta s\right)^{\alpha} \mathcal{Y}(\omega(t)) = 0$$

and

$$\mathcal{W}^{\Delta}(t) + \frac{1}{\alpha}q(t)\left(\int_{\omega(t)}^{\eta(\omega(t))} \mathcal{R}(\eta(s),s)\Delta s\right)^{\alpha} \mathcal{W}(\eta(\eta(\omega(t)))) = 0$$

are oscillatory for $t \geq t_1 > t_0$, *then every solution of equation* (1.1) *is oscillatory.*

Proof. On the contrary, assume that $x(t)$ is a nonoscillatory solution of (1.1) such that $x(t)$, and $x(\omega(t))$ positive eventually for $t \geq t_1 > t_0$. It follows from (1.1) that

$$(r(t)(x^{\Delta\Delta}(t))^{\alpha})^{\Delta} = -q(t)x^{\alpha}(\omega(t)) < 0. \tag{3.1}$$

Hence $r(t)(x^{\Delta\Delta}(t))^{\alpha}$ is nonincreasing and is of one sign. That is, there exists a $t_2 \geq t_1$ such that $x^{\Delta\Delta}(t) > 0$ or, $x^{\Delta\Delta}(t) < 0$ for $t \geq t_2$. We shall distinguish the following cases:

Case I. $x^{\Delta}(t) > 0$, $x^{\Delta\Delta}(t) > 0$. Following the line of proof of Lemma 2.3, we obtain the desired conclusion.

Case II. ($x^{\Delta}(t) < 0$, $x^{\Delta\Delta}(t) > 0$). Following the line of proof of Lemma 2.4, we obtain the desired conclusion.

This completes the proof of the theorem. □

Applying known oscillation criteria to first-order dynamic equations, one can obtain sufficient conditions for oscillation of (1.1). In particular, on the basis of [2, Theorem 1] and Theorem 3.1, the following corollary is immediate.

Corollary 3.2. *Let* (A_1)–(A_3) *and* (1.2) *hold. Assume that there exists nondecreasing function* $\eta(t) \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that* (2.5) *holds. If*

$$\liminf_{t\to\infty} \int\limits_{\omega(t)}^{t} q(u)\left(\int\limits_{t_1}^{\omega(u)} \mathcal{R}(s,t_1)\Delta s\right)^{\alpha} \Delta u = \infty \quad for\ t \geq t_1, \tag{3.2}$$

and

$$\liminf_{t\to\infty} \int\limits_{\eta(\eta(\omega(t)))}^{t} q(u)\left(\int\limits_{\omega(u)}^{\eta(\omega(u))} \mathcal{R}(\eta(s),s)\Delta s\right)^{\alpha} \Delta u > \frac{\alpha}{e}, \tag{3.3}$$

then every solution of equation (1.1) *is oscillatory.*

The following example is illustrative.

Example 3.3. For $\mathbb{T} = \mathbb{R}$, consider the third-order differential equation

$$\left(\frac{1}{t^3}\left(x''(t)\right)^3\right)' + \frac{C}{t^9}x^3\left(\frac{t}{8}\right) = 0,\ t \geq t_0, \tag{3.4}$$

where $\alpha = 3$, $r(t) = \frac{1}{t^3}$, $\omega(t) = \frac{t}{8}$, and $q(t) = \frac{C}{t^9}$ with a constant $C > 0$. We let $\eta(t) = 2t$, then $\eta(\omega(t)) = \frac{t}{4}$ and $\eta(\eta(\omega(t))) = \frac{t}{2}$. Now $\mathcal{R}(t,s) = \frac{t^2}{2} - \frac{s^2}{2}$ for $t > s \geq t_1$. A straightforward verification shows that all conditions of Corollary 3.2 are satisfied for certain appropriate value of the constant $C > 0$ and conclude that (3.4) is oscillatory.

Next, we have the following comparison result with third-order linear dynamic inequalities.

Theorem 3.4. *Let* (A_1)–(A_3) *and* (1.2) *hold. Assume that there exists nondecreasing function* $\eta(t) \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that* (2.5) *holds. If the inequality*

$$(r^{1/\alpha}(t)\mathcal{Y}^{\Delta\Delta}(t))^{\Delta} + \frac{1}{\alpha}\left(\int_{t_1}^{\omega(t)} \mathcal{R}(s,t_1)\Delta s\right)^{\alpha-1} q(t)\mathcal{Y}(\omega(t)) \le 0 \tag{3.5}$$

has no eventually positive nondecreasing solution for $t \ge t_1 > t_0$ *and the inequality*

$$(r^{1/\alpha}(t)\mathcal{W}^{\Delta\Delta}(t))^{\Delta} + \frac{1}{\alpha}\left(\int_{\omega(t)}^{\eta(\omega(t))} \mathcal{R}(\eta(s),s)\Delta s\right)^{\alpha-1} q(t)\mathcal{W}(\omega(t)) \le 0 \tag{3.6}$$

has no eventually positive nonincreasing solution, then equation (1.1) *is oscillatory.*

Proof. On the contrary, assume that $x(t)$ is a nonoscillatory solution of (1.1) such that $x(t)$, and $x(\omega(t))$ are positive eventually for $t \ge t_1 > t_0$. Proceeding as in the proof of Theorem 3.1, we obtain the two cases I and II, and the inequalities (2.3) and (2.7).

Case I. From (2.3), we can easily see that

$$r^{1/\alpha}(t)x^{\Delta\Delta}(t) \le r^{1/\alpha}(\omega(t))x^{\Delta\Delta}(\omega(t)) \le \left(\int_{t_1}^{\omega(t)} \mathcal{R}(s,t_1)\Delta s\right)^{-1} x(\omega(t)),$$

and so

$$(r^{1/\alpha}(t)x^{\Delta\Delta}(t))^{1-\alpha} \ge \left(\int_{t_1}^{\omega(t)} \mathcal{R}(s,t_1)\Delta s\right)^{\alpha-1} x^{1-\alpha}(\omega(t)).$$

Using the last inequality in (2.2), we obtain

$$(r^{1/\alpha}(t)x^{\Delta\Delta}(t))^{\Delta} + \frac{1}{\alpha}\left(\int_{t_1}^{\omega(t)} \mathcal{R}(s,t_1)\Delta s\right)^{\alpha-1} q(t)x^{1-\alpha}(\omega(t))x^{\alpha}(\omega(t)) \le 0,$$

or

$$(r^{1/\alpha}(t)x^{\Delta\Delta}(t))^{\Delta} + \frac{1}{\alpha}\left(\int_{t_1}^{\omega(t)} \mathcal{R}(s,t_1)\Delta s\right)^{\alpha-1} q(t)x(\omega(t)) \le 0.$$

By condition (3.5), we arrive at the desired contradiction.

Case II. From (2.7) we find that

$$x(\omega(t))\left(\int_{t}^{\eta(\omega(t))} \mathcal{R}(\eta(s),s)\Delta s\right)^{-1} \ge r^{1/\alpha}(\eta(\eta(\omega(t))))\mathcal{Y}^{\Delta\Delta}(\eta(\eta(\omega(t)))).$$

Using this inequality in (2.2), we get

$$(r^{1/\alpha}(t)x^{\Delta\Delta}(t))^{\Delta} + \frac{1}{\alpha}\left(\int_t^{\eta(\omega(t))} \mathcal{R}(\eta(s),s)\Delta s\right)^{\alpha-1} q(t)x^{1-\alpha}(\omega(t))x^{\alpha}(\omega(t)) \le 0,$$

or,

$$(r^{1/\alpha}(t)x^{\Delta\Delta}(t))^{\Delta} + \frac{1}{\alpha}\left(\int_t^{\eta(\omega(t))} \mathcal{R}(\eta(s),s)\Delta s\right)^{\alpha-1} q(t)x(\omega(t)) \le 0.$$

By (2.5), we arrived at the desired contradiction. This completes the proof. □

Next, we present the following interesting criterion for the oscillatory and asymptotic behaviour of equation (1.1). For this, we let

$$\mathcal{Q}(t) = \frac{1}{\alpha}\left(\int_{t_1}^{\omega(t)} \mathcal{R}(s,t_1)\Delta s\right)^{1-\alpha} q(t).$$

Theorem 3.5. *Let* (A_1)–(A_3) *and* (1.2) *hold. Assume that there exist nondecreasing function* $\eta(t), \Phi(t) \in C_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ *such that* (2.5) *holds and* $\omega^{\Delta} > 0$. *If* (2.11) *and*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\Phi(s)\mathcal{Q}(s) - \frac{(\Phi^{\Delta}(s))^2}{4\Phi(s)\omega^{\Delta}(s)\mathcal{R}(\omega(s),t_1)}\right]\Delta s = \infty \tag{3.7}$$

hold for $t \ge t_1$ *for some* $t_1 > t_0$, *then every solution* $x(t)$ *of equation* (1.1) *is oscillatory or converges to zero.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1) such that $x(t) > 0$ with $\lim_{t\to\infty} x(t) > 0$, and $x(\omega(t)) > 0$ for $t \ge t_1 > t_0$. Proceeding as in the proof of Theorem 3.1, we obtain the two cases I and II, and the inequalities (2.3) and (2.7).

Case I. Define the function $\mathcal{V}(t)$ by

$$\mathcal{V}(t) = \frac{\Phi(t)r^{1/\alpha}(t)x^{\Delta\Delta}(t)}{x(\omega(t))} \quad \text{for } t \ge t_1.$$

Then, $\mathcal{V}(t) > 0$ and applying (3.5), we get

$$\begin{aligned}\mathcal{V}^{\Delta}(t) &= (r^{1/\alpha}(t)x^{\Delta\Delta}(t))^{\Delta}\left[\frac{\Phi(t)}{x(\omega(t))}\right] \\ &\quad + r^{1/\alpha}(\sigma(t))x^{\Delta\Delta}(\sigma(t))\left[\frac{\Phi^{\Delta}(t)x(\omega(t)) - \Phi(t)x^{\Delta}(\omega(t))}{x(\omega(t))x(\omega(\sigma(t)))}\right] \\ &\le -\frac{\Phi(t)}{\alpha}\left(\int_{t_1}^{\omega(t)} \mathcal{R}(s,t_1)\Delta s\right)^{\alpha-1} q(t) + \Phi^{\Delta}(t)\frac{r^{1/\alpha}(\sigma(t))x^{\Delta\Delta}(\sigma(t))}{x(\omega(\sigma(t)))} \\ &\quad - \Phi(t)\frac{r^{1/\alpha}(\sigma(t))x^{\Delta\Delta}(\sigma(t))x^{\Delta}(\omega(t))}{x(\omega(t))x(\omega(\sigma(t)))},\end{aligned}$$

which implies that

$$\mathcal{V}^{\Delta}(t) \leq -\Phi(t)\mathcal{Q}(t) + \frac{\Phi^{\Delta}(t)}{\Phi(\sigma(t))}\mathcal{V}^{\Delta}(\sigma(t)) - \Phi(t)\frac{r^{1/\alpha}(\sigma(t))x^{\Delta\Delta}(\sigma(t))x^{\Delta}(\omega(t))}{x(\omega(t))x(\omega(\sigma(t)))}. \quad (3.8)$$

If $\sigma(t) > t$, then by [10, Theorem 1.14] we have

$$(x(\omega(t)))^{\Delta} = \frac{x(\omega(\sigma(t))) - x(\omega(t))}{\sigma(t) - t} = \frac{x(\omega(\sigma(t))) - x(\omega(t))}{\omega(\sigma(t)) - \omega(t)}\omega^{\Delta}(t) \geq x^{\Delta}(\xi)\omega^{\Delta}(t), \quad (3.9)$$

where $\xi \in [\omega(t), \omega(\sigma(t)))$. If $\sigma(t) = t$, then we have $\omega(\sigma(t)) = \sigma(\omega(t)) = \omega(t)$ and

$$(x(\omega(t)))^{\Delta} = x'(\omega(t))\omega'(t). \quad (3.10)$$

Using (3.9) and (3.10) in (3.8), we get

$$\begin{aligned}\mathcal{V}^{\Delta}(t) &\leq -\Phi(t)\mathcal{Q}(t) + \frac{\Phi^{\Delta}(t)}{\Phi(\sigma(t))}\mathcal{V}^{\Delta}(\sigma(t)) - \Phi(t)\omega^{\Delta}(t)\frac{r^{1/\alpha}(\sigma(t))x^{\Delta\Delta}(\sigma(t))x^{\Delta}(\xi)}{x(\omega(t))x(\omega(\sigma(t)))}\\ &= -\Phi(t)\mathcal{Q}(t) + \frac{\Phi^{\Delta}(t)}{\Phi(\sigma(t))}\mathcal{V}^{\Delta}(\sigma(t)) - \Phi(t)\omega^{\Delta}(t)\frac{r^{1/\alpha}(\sigma(t))x^{\Delta\Delta}(\sigma(t))x^{\Delta}(\xi)}{x^2(\omega(\sigma(t)))}\\ &\quad\times \frac{x(\omega(\sigma(t)))}{x(\omega(t))}.\end{aligned}$$

Using the fact that $\omega(t)$ and $x(t)$ is nondecreasing, we have $x(\omega(\sigma(t))) \geq x(\omega(t))$. Therefore,

$$\begin{aligned}\mathcal{V}^{\Delta}(t) &\leq -\Phi(t)\mathcal{Q}(t) + \frac{\Phi^{\Delta}(t)}{\Phi(\sigma(t))}\mathcal{V}^{\Delta}(\sigma(t)) - \Phi(t)\omega^{\Delta}(t)\frac{r^{1/\alpha}(\sigma(t))x^{\Delta\Delta}(\sigma(t))x^{\Delta}(\xi)}{x^2(\omega(\sigma(t)))}\\ &= -\Phi(t)\mathcal{Q}(t) + \frac{\Phi^{\Delta}(t)}{\Phi(\sigma(t))}\mathcal{V}^{\Delta}(\sigma(t)) - \frac{\Phi(t)\omega^{\Delta}(t)}{\Phi^2(\sigma(t))}\mathcal{V}^2(\sigma(t))\frac{x^{\Delta}(\xi)}{r^{1/\alpha}(\sigma(t))x^{\Delta\Delta}(\sigma(t))}.\end{aligned}$$

Again for $\xi \in [\omega(t), \omega(\sigma(t)))$, we have

$$x^{\Delta}(\xi) \geq x^{\Delta}(\omega(t)).$$

Using this in the last inequality, we get

$$\mathcal{V}^{\Delta}(t) \leq -\Phi(t)\mathcal{Q}(t) + \frac{\Phi^{\Delta}(t)}{\Phi(\sigma(t))}\mathcal{V}^{\Delta}(\sigma(t)) - \frac{\Phi(t)\omega^{\Delta}(t)}{\Phi^2(\sigma(t))}\mathcal{V}^2(\sigma(t))\frac{x^{\Delta}(\omega(t))}{r^{1/\alpha}(\sigma(t))x^{\Delta\Delta}(\sigma(t))}.$$

Since $x^{\Delta}(\omega(t)) \geq \mathcal{R}(\omega(t), t_1)(r^{1/\alpha}(\omega(t))x^{\Delta\Delta}(\omega(t)))$, then using the fact that $\sigma(t) \geq t \geq \omega(t)$ and $r^{1/\alpha}(t)x^{\Delta\Delta}(t)$ is nonincrasing for $t \geq t_1$, we get

$$x^{\Delta}(\omega(t)) \geq \mathcal{R}(\omega(t), t_1)(r^{1/\alpha}(\sigma(t))x^{\Delta\Delta}(\sigma(t))),$$

or

$$\frac{x^{\Delta}(\omega(t))}{x^{\Delta\Delta}(\sigma(t))} \geq \mathcal{R}(\omega(t), t_1)r^{1/\alpha}(\sigma(t)).$$

Therefore,

$$\mathcal{V}^{\Delta}(t) \leq -\Phi(t)\mathcal{Q}(t) + \frac{\Phi^{\Delta}(t)}{\Phi(\sigma(t))}\mathcal{V}^{\Delta}(\sigma(t)) - \frac{\Phi(t)\omega^{\Delta}(t)\mathcal{R}(\omega(t),t_1)}{\Phi^2(\sigma(t))}\mathcal{V}^2(\sigma(t)).$$

Hence,

$$\mathcal{V}^{\Delta}(t) \leq -\Phi(t)\mathcal{Q}(t) + \frac{\Phi^{\Delta}(t)}{\Phi(\sigma(t))}\mathcal{V}^{\Delta}(\sigma(t)) - \frac{\mathcal{P}(t)}{\Phi^2(\sigma(t))}\mathcal{V}^2(\sigma(t)), \tag{3.11}$$

where $\mathcal{P}(t) = \Phi(t)\omega^{\Delta}(t)\mathcal{R}(\omega(t),t_1)$. On simplification (3.11) yields

$$\mathcal{V}^{\Delta}(t) \leq -\Phi(t)\mathcal{Q}(t) + \frac{(\Phi^{\Delta}(t))^2}{4\mathcal{P}(t)} - \left[\frac{(\mathcal{P}(t))^{1/2}}{\Phi(\sigma(t))}\mathcal{V}(\sigma(t)) - \frac{\Phi^{\Delta}(t)}{2}(\mathcal{P}(t))^{-1/2}\right]^2$$

implies that

$$\mathcal{V}^{\Delta}(t) \leq -\Phi(t)\mathcal{Q}(t) + \frac{(\Phi^{\Delta}(t))^2}{4\mathcal{P}(t)}.$$

Integrating this inequality from t_1 to t, we arrived at the desired contradiction. Indeed,

$$\int_{t_1}^{t}\left[\Phi(s)\mathcal{Q}(s) - \frac{(\Phi^{\Delta}(s))^2}{4\mathcal{P}(s)}\right]\Delta s \leq \mathcal{V}(t_1) - \mathcal{V}(t) \leq \mathcal{V}(t_1) < \infty.$$

Case II. This case follows from the proof of Lemma 2.6.

This completes the proof. □

Following Theorem 3.5 and Lemma 2.5, finally, we present the following interesting criterion for the oscillatory behaviour of Eq. (1.1).

Theorem 3.6. *Let* (A_1)–(A_3) *and* (1.2) *hold. Assume that there exist nondecreasing function* $\eta(t), \Phi(t) \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that* (2.5) *holds and* $\omega^{\Delta} > 0$. *If* (2.8) *and* (3.7) *hold, then every solution* $x(t)$ *of equation* (1.1) *is oscillatory.*

Remark 3.7.

(1) Theorem 2.5 of [30] by Han *et al.* guarantees that every solution $x(t)$ of (1.1) with $r(t) = 1$ is either oscillatory or satisfies $\lim_{t\to\infty} x(t)$ exists. However, Theorem 3.6 we obtained ensures that all solutions of (1.1) are oscillatory only, which means that our results improve or generalise those in [30].
(2) A similar observation can be made for [35, Theorem 2.1] by Li *et al.*
(3) Finally, the results reported in [45, Theorem 2.1] by Yu and Wang is not applicable to Eq. (3.4).

4. CONCLUSION

This research successfully establishes new results concerning the oscillation of Eq. (1.1). To prove the main theorem, we developed a novel technique based on a simple inequality and certain comparative results. As an application of the main results, Corollary 3.2 as well as an example have been provided to demonstrate the validity and relevance of our findings. It is hoped that the four new lemmas proved here will have future applications in the field of oscillation theory. Because there are numerous results in the literature on the oscillation of first-order dynamic equations, several conditions for the oscillation of equation (1.1) might be formulated based on the findings of this article. Finally, we propose the following possible next directions for this research.

1. When $n \geq 3$ is an odd natural integer, it will be interesting to investigate Eq. (1.1).
2. It will be interesting to investigate Eq. (1.1) in the context of condition

$$\mathcal{R}^*(t, t_0) = \int_{t_0}^{t} \frac{\Delta s}{r^{1/\alpha}(s)} < \infty \quad \text{as} \quad t \to \infty.$$

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