

## STOCHASTIC WIENER FILTER IN THE WHITE NOISE SPACE

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**Abstract.** In this paper we introduce a new approach to the study of filtering theory by allowing the system's parameters to have a random character. We use Hida's white noise space theory to give an alternative characterization and a proper generalization to the Wiener filter over a suitable space of stochastic distributions introduced by Kondratiev. The main idea throughout this paper is to use the nuclearity of this space in order to view the random variables as bounded multiplication operators (with respect to the Wick product) between Hilbert spaces of stochastic distributions. This allows us to use operator theory tools and properties of Wiener algebras over Banach spaces to proceed and characterize the Wiener filter equations under the underlying randomness assumptions.

**Keywords:** Wiener filter, white noise space, Wick product, stochastic distribution.

**Mathematics Subject Classification:** 93E11, 60H40, 46F25, 13J99.

### 1. INTRODUCTION

The fundamental question in filtering theory is how to recover a signal from incomplete or distorted information. The Wiener filter, under stationarity assumptions, is the best linear minimum mean square error filter. In order to lay the foundation for the Wiener filter (see [25]), one needs to know the spectral structure of the noise and of the signal in advance. In the non-causal case, the Wiener filter is given in terms of the two sided  $z$ -transform

$$H(z) = \frac{S_{uy}(z)}{S_y(z)},$$

where  $S_y(z)$  and  $S_{uy}(z)$  are the  $z$ -transforms of the autocorrelation function of the output process and the crosscorrelation function of the output and the input signals,

respectively. The causal Wiener filter is based on factorization theorems over Wiener algebras and is expressed by

$$H(z) = [S_{uy}(z)S_y^-(z)^{-1}]_+ S_y^+(z)^{-1},$$

where the operator  $[S]_+$  denotes the causal part of an element  $S$ . The theory is well-developed and complete as long as the spectral structure is known and non-random. However, when uncertainty is allowed in the spectral structure, the relation is not clear. Some solutions to this question may be found in [14, 15, 22, 24]. The main approaches to handle such uncertainty is to examine the average spectral behavior or, alternatively, to determine thresholds such that the estimated signal remains bounded in some sense.

These two approaches have some disadvantages. The first, considering the mean behavior of the spectral structure, misses some of the stochastic data and, furthermore, some of the results can cause divergence in data recovery. The former, detecting thresholds to ensure bounded result tends to be too restricted and conceals some of the stochastic behavior of the spectral structure.

At this stage, it is worth mentioning a different approach where positive definite functions  $R(n)$  on the integers and with values continuous operators from a Banach space into its anti-dual play an important role in the theory of stochastic processes and related topics ([5, 17, 19, 20, 23] and [3]). An important feature of Banach spaces (not always used in the above references) is that they possess the factorization property: a positive operator from a Banach space into its anti-dual can be factorized via a Hilbert space; see [9] and e.g. [3] for a proof.

In [1] a study of linear stochastic systems within the framework of the white noise space has been initiated. The usual input-output relation was replaced by

$$y_n = \sum_{m \in \mathbb{Z}} h_{n-m} \circ u_m, \quad n \in \mathbb{Z}, \quad (1.1)$$

where  $y_n, u_n$  and  $h_n$  are random variables in the space of stochastic distribution  $\mathcal{S}_{-1}$  introduced by Kondratiev (see [13, 16]), and  $\circ$  denotes the Wick product. This approach was continued in [2] with the study of the state-space equations within the space of stochastic distributions.

In this paper we study the Wiener filter when the system parameters are elements in the space of stochastic distributions  $\mathcal{S}_{-1}$  rather than deterministic numbers. In this way, the randomness is inherited naturally in the system. We replace the point-wise product by the Wick product. Using the nuclearity, we show that the autocorrelation function becomes a sequence of bounded operators in a given Hilbert space  $H$ , while its spectral density (namely, the  $z$ -domain) belongs to  $\mathcal{W}_{\mathcal{B}(H)}$ , the Wiener algebra over the Banach space of bounded operators over Hilbert space. Some of the fundamental factorization theorems in  $\mathcal{W}_{\mathcal{B}(H)}$  are utilized in order to characterize the Wiener filter in the space of stochastic distributions. As a consequence, the randomness assumption reflects to uncertainty under the spectral mapping.

In classical linear system theory,  $z$ -transforms of the underlying sequences are usually assumed to converge, and play an important role in the arguments. For instance, the transfer function is the  $z$ -transform of the impulse response  $h_0, h_1, \dots$ . In the present

setting it is important to note that if  $(x_n)$  is a second-order stationary process in the probability space  $\mathbf{L}_2(\Omega, \mathcal{B}, P)$ , the power series  $\sum_{n=0}^{\infty} z^n x_n$  converges in  $\mathbf{L}_2(\Omega, \mathcal{B}, P)$ . In the present approach, we replace these two convergences by convergence in  $\mathcal{S}_{-1}$  (see Subsection 4.2).

Finally, we mention that the suggested approach is a proper generalization of the conventional Wiener filter. As soon as one assumes that the system is described by a deterministic impulse response sequence (that is, the  $h_n$  in (1.1) are complex numbers and not stochastic distributions) and input and output are now discrete signals with values in the  $L^2$  white noise space, the Wick product becomes the scalar multiplication and we consider the Wiener algebra with scalar coefficients. See Remark 5.3 below for more details.

We now turn to the outline of the paper. To develop filtering theory over Hida's white noise space, we first give in Section 2 a short, but necessary, survey of the foundation of the white noise space theory and of the relevant tools. The Wiener algebra over a Banach space and an associated factorization theorem are presented in Section 3. Section 4 is dedicated to the construction of a framework of stochastic processes over the space of stochastic distributions. The non-causal Wiener filter within the white noise space is studied in Section 5. Finally, in Section 6, we develop the causal Wiener filter in the white noise space setting.

**Remark 1.1.** We restrict this paper to the case of discrete time models. Similar results may be obtained for continuous models.

## 2. THE WHITE NOISE SPACE AND THE KONDRATIEV SPACE

### 2.1. THE WHITE NOISE SPACE

To begin with, consider the Schwartz space  $\mathcal{S}$  of *real-valued* smooth functions which, together with their derivatives, decrease rapidly to zero at infinity. For  $s \in \mathcal{S}$ , let  $\|s\|$  denote its  $\mathbf{L}_2(\mathbb{R})$  norm. The function

$$K(s_1 - s_2) = e^{-\frac{\|s_1 - s_2\|^2}{2}}$$

is positive (in the sense of reproducing kernels) for  $s_1, s_2$  belonging to  $\mathcal{S}$ . Since  $\mathcal{S}$  is nuclear, we may use an extension of Bochner's Theorem for nuclear spaces, due to Minlos (see [21], [11, Théorème 3, p. 311]): there exists a probability measure  $P$  on  $\mathcal{S}'$  such that

$$K(s) = \int_{\mathcal{S}'} e^{-i\langle s', s \rangle} dP(s'),$$

where  $\langle s', s \rangle$  denotes the duality between  $\mathcal{S}$  and  $\mathcal{S}'$ . The triple  $\mathcal{W} \stackrel{\text{def}}{=} (\mathcal{S}', \mathcal{F}, dP)$ , where  $\mathcal{F}$  is the Borel  $\sigma$ -algebra, is called the white noise space. For a given  $s \in \mathcal{S}$ , the formula  $Q_s(w) = \langle w, s \rangle$ , defined on  $\mathcal{S}'$ , is a centered Gaussian variable with covariance  $\|s\|^2$ .

A certain orthogonal basis of the real space  $\mathbf{L}_2(\mathcal{S}', \mathcal{F}, dP)$  plays a special role and is constructed in terms of Hermite functions. Its elements are denoted by  $H_\alpha$  and given by

$$H_\alpha = \prod_j h_{\alpha_j}(Q_{\xi_j})$$

where  $h_k$  is the  $k$ -th Hermite polynomial and where  $\xi_j$  is the  $j$ -th normalized Hermite function. Moreover, the index  $\alpha$  runs through the set  $\ell$  of sequences  $(\alpha_1, \alpha_2, \dots)$ , whose entries are in

$$\mathbb{N}_0 = \{0, 1, \dots\},$$

and  $\alpha_k \neq 0$  for all but a finite number of indices  $k$  and (see [13]). With the multi-index notation

$$\alpha! = \alpha_1! \alpha_2! \dots,$$

we have

$$\|H_\alpha\|_{\mathcal{W}}^2 = \alpha!. \quad (2.1)$$

The Wick product in  $\mathcal{W}$  is defined by

$$H_\alpha \circ H_\beta = H_{\alpha+\beta}, \quad \alpha, \beta \in \ell.$$

## 2.2. THE KONDRATIEV SPACE

The space  $\mathbf{L}_2(\mathcal{W})$  is not stable under the Wick product. This motivates the definition of the Kondratiev space, denoted by  $\mathcal{S}_{-1}$ , which contains  $\mathbf{L}_2(\mathcal{W})$ . We first introduce a family of Hilbert spaces  $\mathcal{H}_k$  containing  $\mathbf{L}_2(\mathcal{W})$ . Let  $k \in \mathbb{N}$  and let  $\mathcal{H}_k$  be the collection of formal series

$$f(\omega) = \sum_{\alpha \in \ell} c_\alpha H_\alpha(\omega), \quad c_\alpha \in \mathbb{C}, \quad (2.2)$$

such that

$$\|f\|_k \stackrel{\text{def}}{=} \left( \sum_{\alpha \in \ell} |c_\alpha|^2 (2\mathbb{N})^{-k\alpha} \right)^{1/2} < \infty, \quad (2.3)$$

where

$$(2\mathbb{N})^\alpha = (2 \times 1)^{\alpha_1} (2 \times 2)^{\alpha_2} (2 \times 3)^{\alpha_3} \dots$$

The product makes sense since only a finite number of  $\alpha_i \neq 0$ . We note that

$$\mathcal{W} \subseteq \mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \dots \subseteq \mathcal{H}_k \subseteq \dots$$

The Kondratiev space  $\mathcal{S}_{-1}$  is the inductive limit of the spaces  $\mathcal{H}_k$ ,

$$\mathcal{S}_{-1} \stackrel{\text{def}}{=} \bigcup_{k \in \mathbb{N}} \mathcal{H}_k,$$

and is stable under the Wick product, see below. The Kondratiev space of stochastic test functions,  $\mathcal{S}_1$ , is the space of all functions of the form

$$f(\omega) = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha(\omega), \quad c_\alpha \in \mathbb{C},$$

such that  $\sum_{\alpha \in \mathcal{J}} (2\mathbb{N})^{k\alpha} (\alpha!)^2 |c_\alpha|^2 < \infty$  for all  $k \in \mathbb{N}_0$ . It is a countably normed space, contained in  $\mathbf{L}_2(\mathcal{W})$ . The spaces  $\mathcal{S}_{-1}$  and  $\mathcal{S}_1$  are nuclear spaces, dual of each other, and together with  $\mathcal{W}$ , form a Gelfand triple,

$$\mathcal{S}_1 \subseteq \mathcal{W} \subseteq \mathcal{S}_{-1}.$$

The Våge's inequality, presented below, plays a major role and shows that  $\mathcal{S}_{-1}$  is stable under the Wick product. It allows us to consider the multiplication operator (with the appropriate assumptions) as bounded operators.

**Theorem 2.1** ([13, Våge's inequality, p. 118]). *Fix  $k, \ell \in \mathbb{N}$  with  $k > \ell + 1$ . Let  $h \in \mathcal{H}_\ell$  and let  $f \in \mathcal{H}_k$ . Then,*

$$\|h \circ f\|_k \leq A(k - \ell) \|h\|_\ell \|f\|_k.$$

where

$$A(k - \ell) \stackrel{\text{def}}{=} \left( \sum_{\alpha} 2^{\mathbb{N}(\ell - k)\alpha} \right)^{\frac{1}{2}} < \infty.$$

For a proof that  $A(k - \ell)$  is finite, see [13, Prop. 2.3.3, p. 31].

Finally, in order to present a well defined generalization to the nonrandom parameter case, the observation below proves that the Wick product is reduced to a pointwise multiplication when one of the multipliers is nonrandom.

**Lemma 2.2** ([13]). *Let  $F, G \in \mathcal{S}_{-1}$  with  $G = g_0 \in \mathbb{C}$ . Then*

$$F \circ G = F \cdot g_0.$$

We conclude with the following remark.

**Remark 2.3.** The Kondratiev space is not a metric space, yet we note that convergent sequences in  $\mathcal{S}_{-1}$  have the following important property (see [10, p. 58, Théorème 4], in a more general setting). A sequence  $(x_n)_{n \in \mathbb{Z}}$  converges to some  $x$  in  $\mathcal{S}_{-1}$  if and only if there exist  $N, k > 0$  such that for all  $n > N$ ,  $x_n$  and  $x$  are belong to  $\mathcal{H}_k$  and

$$\|x_n - x\|_k \rightarrow 0.$$

This property is used in Section 4 to provide motivation to one of our hypothesis, namely Condition 4.2.

### 3. WIENER ALGEBRAS OVER BANACH SPACES

It is well known that the causal Wiener filter is related to factorizations over Wiener algebras. While in the classical case, the Wiener algebra consists of the elements of the form

$$w = \sum_{n=-\infty}^{\infty} w_n z^n, \quad w_n \in \mathbb{C}^{p \times p},$$

such that  $\sum_{n=-\infty}^{\infty} \|w_n\| < \infty$ , in our case we replace  $\mathbb{C}^{p \times p}$  by  $\mathcal{B}(H)$  and consider the Wiener algebra associated with bounded operators over a Hilbert space. We denote this space by  $\mathcal{W}(\mathcal{B})$ .

Gohberg and Leiterer studied in [6, 7] the Wiener algebras  $\mathcal{W}(\mathcal{B})$  when  $\mathcal{B}$  is the Banach algebra of linear bounded operators in a Hilbert space. In their papers, they prove spectral factorization theorems in  $\mathcal{W}(\mathcal{B})$  (see also the unpublished manuscript [4] for a related result). In the present paper we use a different version of the factorization theorem [8, Lemma II.1.1]. Let  $H$  be a complex Hilbert space and let  $\mathcal{B}(H)$  be the Banach algebra of bounded linear operators on  $H$ . We denote by  $\mathcal{W}_{\mathcal{B}(H)}$  the Wiener algebra on the unit circle  $\mathbb{T}$  with Fourier coefficients in  $\mathcal{B}(H)$ .

**Theorem 3.1** ([4] and [8, Lemma II.1.1], Spectral decomposition – Operator-valued version). *Let  $H$  be a separable Hilbert space and let  $\mathcal{B} = \mathcal{B}(H)$  be the associated Banach algebra of bounded operators. Let  $W \in \mathcal{W}(\mathcal{B})$  and assume that  $W(e^{it})$  is positive definite for all  $t \in \mathbb{R}$ . Then there exists  $W_+ \in \mathcal{W}_+(\mathcal{B})$  such that  $W_+^{-1} \in \mathcal{W}_+(\mathcal{B})$  and  $W = W_+^* W_+$ .*

#### 4. $\mathcal{S}_{-1}$ VALUED STOCHASTIC PROCESSES

We define wide-sense stationary stochastic processes in the white noise space setting. The main idea is to look at random variables as multiplication operators between Hilbert spaces of stochastic distributions.

##### 4.1. THE CLASSICAL CASE

We first look at the equivalent definitions in the classical case of second-order discrete-time stochastic processes. We express the autocorrelation function in terms of multiplication operators. Consider a probability space  $\mathbf{L}_2(\Omega, \mathcal{A}, P)$ . We associate to each element  $y \in \mathbf{L}_2(\Omega, \mathcal{A}, P)$ , the multiplication operator

$$M_y : \mathbb{C} \longrightarrow \mathbf{L}_2(\Omega, \mathcal{A}, P).$$

defined by

$$M_y(1) = y.$$

In the next lemma, we denote by  $E(y)$  the expectation of the random variable  $y \in \mathbf{L}_1(\Omega, \mathcal{A}, P)$ .

**Proposition 4.1.** *Let  $(y_n)_{n \in \mathbb{Z}}$  be a second-order discrete-time stochastic process. Then*

$$E(y_{n_1} y_{n_2}^*) = (M_{y_{n_2}})^* M_{y_{n_1}}, \quad n_1, n_2 \in \mathbb{Z}. \quad (4.1)$$

*Proof.* Indeed, by definition of the adjoint operator,

$$\begin{aligned} \langle M_{y_{n_2}}^* M_{y_{n_1}} 1, 1 \rangle_{\mathbb{C}} &= \langle M_{y_{n_1}} 1, M_{y_{n_2}} 1 \rangle_{\mathbf{L}_2(\Omega, \mathcal{A}, P)} \\ &= \langle y_{n_1}, y_{n_2} \rangle_{\mathbf{L}_2(\Omega, \mathcal{A}, P)} = E(y_{n_1} y_{n_2}^*). \end{aligned} \quad \square$$

To justify the assumption given in Definition 4.2 below, we first consider the classical setting. Let  $(x_n)_{n \in \mathbb{Z}}$  be a second-order stationary stochastic process in a probability space  $\mathbf{L}_2(\Omega, \mathcal{B}, \mathcal{P})$ , with correlation function  $r(n - m)$ . Then the series

$$X(z) = \sum_{n=0}^{\infty} z^n x_n, \quad |z| < 1,$$

converges in  $\mathbf{L}_2(\Omega, \mathcal{B}, \mathcal{P})$  and we have

$$E[X(z)\overline{X(w)}] = \frac{\phi(z) + \phi(w)^*}{2(1 - z\bar{w})},$$

where

$$\phi(z) = r(0) + 2 \sum_{n=1}^{\infty} r(n)z^n.$$

#### 4.2. STATIONARY PROCESSES OVER THE WHITE NOISE SPACE

In our setting, we consider a sequence  $(x_n)_{n=0}^{\infty}$  in  $\mathcal{S}_{-1}$  and assume that the sum  $X(z) = \sum_{n=0}^{\infty} z^n x_n$  converges in  $\mathcal{S}_{-1}$ . Then, by Remark 2.3, there exists  $\ell > 0$  such that the sum  $X(z)$  converges in  $\mathcal{H}_{\ell}$ .

Hence, the following definition makes sense.

**Definition 4.2.** Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence of elements in  $\mathcal{H}_{\ell}$  for some fixed  $\ell \in \mathbb{N}$ . The stochastic sequence  $(x_n)_{n \in \mathbb{N}}$  is called wide-sense stationary if

$$M_{x_n}^* M_{x_m} = R_x(m - n),$$

depends only on  $m - n$ .

Here we also assumed that the  $z$ -transform of the impulse response coefficients  $(h_n)_{n \in \mathbb{Z}}$  converges in  $\mathcal{S}_{-1}$ , and therefore it is also convergent in  $\mathcal{H}_{\ell}$  for some  $\ell \in \mathbb{N}$ .

**Definition 4.3.** Let  $(x_n)_{n \in \mathbb{N}}$  and  $(y_n)_{n \in \mathbb{N}}$  be two sequences of elements of  $\mathcal{H}_{\ell}$  for some  $\ell \in \mathbb{N}$ . The stochastic sequences are called jointly wide-sense stationary if the stochastic sequence  $z_n = [x_n \ y_n]^T$  is a stationary process in the sense of Definition 4.2. In such a case, the cross-correlation operator

$$M_{y_n}^* M_{x_m} = R_{xy}(m - n)$$

depends only on  $m - n$ .

**Theorem 4.4.** *Let  $(x_n)_{n \in \mathbb{Z}}$ , a sequence of elements in  $\mathcal{H}_k$ , be a wide-sense stationary process in the white noise space such that*

$$\sum_{m \in \mathbb{Z}} \|M_{x_n}^* M_{x_{n+m}}\|_{\mathcal{H}_k} < \infty, \quad n \in \mathbb{Z},$$

*and let  $(h_n)_{n \in \mathbb{Z}}$  be a sequence of elements in  $\mathcal{H}_\ell$  such that*

$$\sum_{n \in \mathbb{Z}} \|M_{h_n}\|_{\mathcal{H}_\ell} < \infty. \quad (4.2)$$

*Then, the output  $(y_n)_{n \in \mathbb{Z}}$  sequence of the LTI stochastic system characterized by the impulse response  $(h_n)_{n \in \mathbb{Z}}$ , i.e.*

$$y_n = \sum_{m \in \mathbb{Z}} h_{n-m} \circ x_m, \quad n \in \mathbb{Z}, \quad (4.3)$$

*belongs to  $\mathcal{H}_k$  and is stationary.*

To prove Theorem 4.4, we first recall F. Mertens' Theorem on convolutions of sequences.

**Theorem 4.5** ([18]). *Assume  $(a_n)_{n \in \mathbb{N}}$  and  $(b_n)_{n \in \mathbb{N}}$  are two sequences of complex numbers such that*

$$\sum_{n=0}^{\infty} |a_n| < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} |b_n| < \infty,$$

*and let  $(c_n)_{n \in \mathbb{N}}$  be defined by  $c_n = \sum_{m=0}^n a_{n-m} b_m$ . Then*

$$\sum_{n=0}^{\infty} |c_n| < \infty$$

*and*

$$\sum_{n=0}^{\infty} c_n = \left( \sum_{n=0}^{\infty} a_n \right) \left( \sum_{n=0}^{\infty} b_n \right).$$

*Proof of Theorem 4.4.* We first note that

$$M_{y_m}^* M_{y_{n+m}} = M_{\sum_{t \in \mathbb{Z}} x_{m-t} \circ h_t}^* M_{\sum_{s \in \mathbb{Z}} x_{n+m-s} \circ h_s} = \sum_{t,s=1}^{\infty} M_{h_t}^* M_{x_{m-t}}^* M_{x_{n+m-s}} M_{h_s},$$

therefore

$$M_{y_m}^* M_{y_{n+m}} = \sum_{t,k=1}^{\infty} M_{h_t}^* R_x(k) M_{h_{n+t-k}}$$



depends only on  $n$  and, by definition, the stationarity of the output signal follows. Furthermore, we have

$$\begin{aligned} & \sum_{t,s=-\infty}^{\infty} \|M_{h_t}^* M_{x_{m-t}}^* M_{x_{n+m-s}} M_{h_s}\|_{\mathcal{H}_k} \\ & \leq \sum_{t,s=-\infty}^{\infty} \|M_{h_t}^*\|_{\mathcal{H}_k} \|M_{x_{m-t}}^* M_{x_{n+m-s}}\|_{\mathcal{H}_k} \|M_{h_s}\|_{\mathcal{H}_k}. \end{aligned} \quad (4.4)$$

Then, using Theorem 4.5 twice, the expression

$$\sum_{t,s=1}^{\infty} M_{h_t}^* M_{x_{m-t}}^* M_{x_{n+m-s}} M_{h_s} \quad (4.5)$$

is absolutely convergent, i.e.

$$\sum_{t,s=1}^{\infty} \|M_{h_t}^* M_{x_{m-t}}^* M_{x_{n+m-s}} M_{h_s}\|_{\mathcal{H}_k} < \infty. \quad \square$$

We note that, when the system parameters are assumed deterministic (in view of Lemma 2.2), the definitions and the results are reduced to the classical case.

### 4.3. THE SPECTRUM

In the study of stochastic processes, a specific case is of interest, namely when the joint probability distribution does not change while shifting in time (moving the indices by a constant). In particular, the autocorrelation operator depends only on the difference of the indices. This property underlines the notion of wide sense stationary processes. In order to proceed, the counterparts of known definitions should be given within the white noise space framework. As in the classical case, the spectrum is given in term of the Fourier series.

**Definition 4.6.** Let  $\ell \in \mathbb{N}$  and let  $(x_n)_{n \in \mathbb{Z}}$  be a stationary process with elements in  $\mathcal{H}_\ell$ . Then the spectrum of a stationary process is the operator-valued function defined by

$$S_x(e^{i\omega}) = \sum_{m=-\infty}^{\infty} M_{x_n}^* M_{x_{n+m}} e^{i\omega m} = \sum_{m=-\infty}^{\infty} R_x(m) e^{i\omega m},$$

where we assume

$$\sum_{m \in \mathbb{Z}} \|M_{x_n}^* M_{x_{n+m}}\|_{\mathcal{H}_k} < \infty.$$

The above assumption is required in order to ensure the expression  $\sum_{m=-\infty}^{\infty} R_x(m) e^{i\omega m}$  belongs to the operator-valued Wiener algebra.

The following result describes the relationship between the input spectrum and the output spectrum through a linear time invariant (LTI) system in the white noise space setting (4.3).

**Theorem 4.7.** *Let  $(x_n)_{n \in \mathbb{Z}}$  be a wide-sense stationary processes in  $\mathcal{S}_{-1}$ , such that*

$$\sum_{m \in \mathbb{Z}} \|M_{x_n}^* M_{x_{n+m}}\|_{\mathcal{H}_k} < \infty, \quad n \in \mathbb{Z},$$

*and let  $(h_n)_{n \in \mathbb{Z}}$  be a sequence of elements in  $\mathcal{H}_\ell$  such that Condition (4.2) holds. Then the spectrum of  $(y_n)_{n \in \mathbb{Z}}$  is well-defined and is given by*

$$S_y(e^{i\omega}) = \mathcal{Z}(M_h^*)(e^{-i\omega}) S_x(e^{i\omega}) \mathcal{Z}(M_h)(e^{i\omega}),$$

*where  $\mathcal{Z}$  denotes the element in the Wiener algebra associated to a sequence of operators, or equivalently,*

$$\mathcal{Z}(M_{h_n})(e^{i\omega}) \stackrel{\text{def}}{=} \sum_{n \in \mathbb{Z}} M_{h_n} e^{i\omega n}.$$

*Proof.* Using Theorem 4.4,  $S_y(e^{i\omega})$  is well defined and belongs to the associated Wiener algebra (over  $\mathcal{H}_k$ ). A direct calculation, using (4.4), completes the proof:

$$\begin{aligned} S_y(e^{i\omega}) &= \sum_{n \in \mathbb{Z}} M_{y_n}^* M_{y_{n+m}} e^{i\omega n} \\ &= \sum_{n \in \mathbb{Z}} \left( \sum_{t,s \in \mathbb{Z}} M_{h_t}^* M_{x_{n-t}}^* M_{x_{n+m-s}} M_{h_s} \right) e^{i\omega n} \\ &= \sum_{n \in \mathbb{Z}} \left( \sum_{t,s \in \mathbb{Z}} M_{h_t}^* R_x(n+t-s) M_{h_s} \right) e^{i\omega n} \\ &= \sum_{t,s \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} M_{h_t}^* R_x(n+t-s) M_{h_s} e^{i\omega(n+t-s)} \\ &= \sum_{t,s \in \mathbb{Z}} \sum_{u \in \mathbb{Z}} M_{h_t}^* R_x(u) M_{h_s} e^{i\omega(u-t+s)} \\ &= \left( \sum_{t \in \mathbb{Z}} M_{h_t}^* e^{-i\omega t} \right) \left( \sum_{u \in \mathbb{Z}} R_x(u) e^{i\omega u} \right) \left( \sum_{s \in \mathbb{Z}} M_{h_s} e^{i\omega s} \right) \\ &= \mathcal{Z}(M_h^*)(e^{-i\omega}) S_x(e^{i\omega}) \mathcal{Z}(M_h)(e^{i\omega}). \quad \square \end{aligned}$$

## 5. OPERATOR-VALUED STOCHASTIC WIENER FILTER

In this section, we follow the classical approach of the Wiener filter construction, see [12, Chapter 7], and adapt it to the white noise space framework.

There are two equivalent fundamental starting points which both lead to the classical Wiener filter. The first is by derivation of the mean square error and the second is based on the orthogonality between the error and the filter input. We develop the white noise space approach to the Wiener filter based on the orthogonality property. The main result is presented below in Theorem 5.2 and the stochastic filter is in (5.6).

The orthogonality in the setting of the white noise space is defined as follows:

**Definition 5.1.** Let  $\ell, k \in \mathbb{N}$  be such that  $k > \ell + 1$ . The elements  $x, y \in \mathcal{H}_\ell$  are orthogonal in the white noise space sense if

$$\langle M_x \eta, M_y \xi \rangle_{\mathcal{H}_k} = 0, \quad \eta, \xi \in \mathcal{H}_k,$$

that is,

$$\langle x \circ \eta, y \circ \xi \rangle_{\mathcal{H}_k} = 0, \quad \eta, \xi \in \mathcal{H}_k.$$

This definition makes it possible to give a geometric interpretation of multiplication operators via the geometry of  $\mathcal{H}_k$ .

We consider two Hilbert space valued, discrete time, stationary and jointly stationary random sequences  $(u_i)_{i \in \mathbb{Z}}$  and  $(y_i)_{i \in \mathbb{Z}}$  belong  $\mathcal{H}_k$ . Then the estimated process, by assumption, is an output of an LTI system, for every  $j$ , and hence is of the form

$$\hat{u}_j \stackrel{\text{def}}{=} \sum_{m=-\infty}^{\infty} y_m \circ K_{j,m}, \quad j \in \mathbb{Z}. \quad (5.1)$$

The main goal is to describe  $(K_{j,m}) \in \mathcal{H}_\ell$  such that error is minimal. The orthogonality of the error and the output is

$$(\hat{u}_j - u_j) \perp y_l, \quad l, j \in \mathbb{Z},$$

or, equivalently (by 5.1),

$$\left( \sum_{m=-\infty}^{\infty} y_m \circ K_{j,m} - u_j \right) \perp y_l, \quad l \in \mathbb{Z}.$$

By definition we have

$$\left\langle (M_{\hat{u}_j} - M_{u_j})\eta, M_{y_l}\xi \right\rangle_{\mathcal{H}_k} = 0, \quad \eta, \xi \in \mathcal{H}_k, l \in \mathbb{Z},$$

which yields the following

$$\left\langle M_{\hat{u}_j}\eta, M_{y_l}\xi \right\rangle_{\mathcal{H}_k} = \left\langle M_{u_j}\eta, M_{y_l}\xi \right\rangle_{\mathcal{H}_k} \quad \text{for all } \eta, \xi \in \mathcal{H}_k \text{ and } l \in \mathbb{Z}. \quad (5.2)$$

Then, by using (5.2) and (5.1), we have

$$\begin{aligned} M_{y_l}^* M_{u_j} &= M_{y_l}^* M_{\hat{u}_j} \\ &= M_{y_l}^* M \sum_{m=-\infty}^{\infty} y_m \circ K_{j,m} \\ &= \sum_{m=-\infty}^{\infty} M_{y_l}^* M_{y_m} M_{K_{j,m}} \\ &= \sum_{m=-\infty}^{\infty} R_y(m-l) M_{K_{j,m}}. \end{aligned}$$

This implies a autocorrelation operator-valued equality, also known as the Wiener–Hopf equations, which are given by

$$R_{uy}(j-l) = \sum_{m=-\infty}^{\infty} R_y(m-l)M_{K_{j,m}}, \quad l \in \mathbb{Z}. \quad (5.3)$$

We have an infinite set of linear equations indexed by  $l$  and we show, by stationarity, that this set reduces to an equation solvable by the Fourier series representation, namely, its spectrum. We first simplify the indices of the filter  $K$ . By the following change of variables  $m-l = m'$  and  $j-l = j'$ , we have

$$R_{uy}(j') = \sum_{m'=-\infty}^{\infty} R_y(m')M_{K_{j'+l,m'+l}}, \quad l \in \mathbb{Z}. \quad (5.4)$$

The left-hand side of (5.4) is independent of  $l$ . This implies that there exists a sequence  $(\widehat{K}_{j-l})_{j-l \in \mathbb{N}}$  such that

$$M_{K_{j'+l,m'+l}} = M_{K_{j',m'}} = M_{\widehat{K}_{j'-m'}}.$$

Hence (5.3) is reduced to

$$R_{uy}(j) = \sum_{m=-\infty}^{\infty} R_y(m)M_{\widehat{K}_{j-l}},$$

and, as a consequence, (5.1) may be rewritten as

$$\widehat{u}_j = \sum_{m=-\infty}^{\infty} y_m \circ \widehat{K}_{j-m}.$$

Note that this expression implicitly contains double convolution, one is obvious and the second is due to the Wick product. By applying the Fourier series to the autocorrelation function, and noticing that convolution becomes point-wise multiplication in the Wiener algebra, we have

$$S_{uy}(e^{i\omega}) = S_y(e^{i\omega})\widehat{K}(e^{i\omega}).$$

We wish to invert  $S_y(e^{i\omega})$ . To that end, we assume that

$$\sum_{n=-\infty}^{\infty} \|R_x(n)\|_{\mathcal{H}_k} < \infty \quad \text{and} \quad \sum_{n=-\infty}^{\infty} \|R_y(n)\|_{\mathcal{H}_k} < \infty, \quad (5.5)$$

where  $\|\cdot\|_{\mathcal{H}_k}$  denotes the operator norm in  $\mathcal{H}_k$ . Under these assumptions,  $S_y$  belongs to the associated Wiener algebra. If, furthermore, we assume that  $S_y(e^{i\omega})$  is positive definite for every  $\omega \in \mathbb{R}$ , by the operator-valued factorization theorem, Theorem 3.1,  $S_y^{-1}$  exists in the associated Wiener algebra. In this case, we conclude that the Wiener algebra element associated with the operators of the linear filter is given by

$$\widehat{K}(e^{i\omega}) = S_y^{-1}(e^{i\omega})S_{uy}(e^{i\omega}).$$

We summarize the above in the following theorem.

**Theorem 5.2** (Linear optimal discrete-time filter - The WNS version). *Fix  $k, \ell \geq 0$  such that  $k > \ell + 1$  and assume that conditions (5.5) are in force. Let  $(u_n)_{n \in \mathbb{Z}}$  and  $(y_n)_{n \in \mathbb{Z}}$  be two stationary processes over  $\mathcal{H}_\ell$  such that they are jointly stationary. Then the linear filter (5.1) is given by*

$$\hat{K}(e^{i\omega}) = S_y^{-1}(e^{i\omega}) S_{uy}(e^{i\omega}) \quad (5.6)$$

and belongs to  $\mathcal{W}_{\mathcal{B}(\mathcal{H}_k)}$ .

**Remark 5.3** (Reduction to the classical case). Let us assume that the filter  $(K_{j,m})_{j,m \in \mathbb{N}}$  in (5.1) is a sequence of complex number or, equivalently, represents the impulse response of a non-random system. Hence, see (2.2), the Wick product reduces to the pointwise multiplication operator. Furthermore, assuming that  $(y_m)_{m \in \mathbb{N}}$  and  $(u_m)_{m \in \mathbb{N}}$  are sequences which belong to  $\mathbf{L}_2(\mathcal{S}', \mathcal{F}, dP)$ , the operators  $M_{x_m}^* M_{x_{m+n}}$  (see the equality in (4.1)) reduce now to scalar operators, i.e.  $\lambda_n I$ , where  $\lambda_n \in \mathbb{C}$ .

Hence,  $R_y(n)$  and  $R_{uy}(n)$  are now scalar-valued functions instead of operator-valued functions. Consequently,  $S_y$  and  $S_{uy}$  are members of the Wiener algebra with complex coefficients. As a consequence, we conclude that the suggested stochastic model is in fact a generalization of the classical Wiener filter. The result for the causal Wiener filter in the next section follows similarly.

## 6. STOCHASTIC CAUSAL WIENER FILTER

A problem arises when one restricts interest to causal filters. In such cases, following the notations and the analysis in [12], denoting the estimated signal by  $\hat{u}_{j|j}$ , we have

$$\hat{u}_j = \sum_{m=-\infty}^j y_m \circ K_{j,m}.$$

The orthogonality principle becomes

$$(\hat{u}_j - u_j) \perp y_l \quad \text{for all } l.$$

The double indices of  $K_{j,m}$  may be reduced to a single index  $K_m$ . Using the same method used in the previous section we can rewrite the Wiener–Hopf equations (5.3) as

$$R_{uy}(j) = \sum_{m=-\infty}^j R_y(m) M_{K_{j-m}} = \sum_{m=0}^{\infty} R_y(j-m) M_{K_m} \quad \text{for all } j \geq 0,$$

or, equivalently, as

$$R_{uy}(j) = \sum_{m=-\infty}^{\infty} R_y(m) M_{K_{j-m}} = \sum_{m=-\infty}^{\infty} R_y(j-m) M_{K_m} \quad \text{for all } j \geq 0, \quad (6.1)$$

where

$$K_m = 0 \quad \text{for all } m < 0.$$

Equation (6.1) is difficult to solve because the equality is valid only for  $j \geq 0$ . As a result, the Fourier series is not defined.

In order to overcome the restriction in (6.1) that  $j$  is positive, we define the sequence  $(g_j)_{j \in \mathbb{Z}}$  by

$$g_j \stackrel{\text{def}}{=} R_{uy}(j) - \sum_{m=0}^{\infty} R_y(j-m)M_{K_m}, \quad -\infty < j < \infty, \quad (6.2)$$

which, by (6.1),  $g_m = 0$  for all  $m \geq 0$ . Since  $g_m$  is defined for all  $-\infty < m < \infty$ , we now may consider the Fourier series of equation (6.2) to obtain

$$G(e^{i\omega}) = S_{uy}(e^{i\omega}) - S_y(e^{i\omega})K(e^{i\omega}). \quad (6.3)$$

If  $S_y(e^{i\omega}) > 0$ , then by the operator-valued spectral factorization, Theorem 3.1, we have the following factorization:

$$S_y(e^{i\omega}) = W_S(e^{i\omega})^* W_S(e^{i\omega}). \quad (6.4)$$

Substituting (6.4) in (6.3) leads to

$$W_S(e^{i\omega})^{-*} G(e^{i\omega}) = W_S(e^{i\omega})^{-*} S_{uy}(e^{i\omega}) - W_S(e^{i\omega}) K(e^{i\omega}). \quad (6.5)$$

We note that, since  $g_j$  is strictly anticausal,  $G(e^{i\omega}) \in \mathcal{W}_-(\mathcal{B})$  and, similarly, since  $K_m = 0$  for all  $m < 0$ , we have  $K(e^{i\omega}) \in \mathcal{W}_+(\mathcal{B})$ . Furthermore, by (3.1),  $W_S(e^{i\omega}) \in \mathcal{W}_+(\mathcal{B})$  and  $W_S(e^{i\omega})^{-*} \in \mathcal{W}_-(\mathcal{B})$ . We therefore conclude that we have

$$W_S(e^{i\omega}) K(e^{i\omega}) \in \mathcal{W}_+(\mathcal{B}) \quad (6.6)$$

and

$$W_S(e^{i\omega})^{-*} G(e^{i\omega}) \in \mathcal{W}_-(\mathcal{B}). \quad (6.7)$$

We denote by  $\mathcal{C}(W)$  the causal part of an element  $W$  in the corresponding Wiener algebra. Taking the causal part of Equation 6.5, one has, using (6.6) and (6.7), the following

$$\begin{aligned} \mathcal{C}(W_S(e^{i\omega})^{-*} G(e^{i\omega})) \\ = \mathcal{C}(W_S(e^{i\omega})^{-*} S_{uz}(e^{i\omega})) - \mathcal{C}(W_S(e^{i\omega}) K(e^{i\omega})). \end{aligned} \quad (6.8)$$

By the spectral factorization, we have  $W_S(e^{i\omega}) \in \mathcal{W}_+(\mathcal{B})$  and  $W_S(e^{i\omega})^{-1} \in \mathcal{W}_+(\mathcal{B})$ . It follows that the left hand side of (6.8) is strictly anticausal while the term on the right hand side is causal, and so

$$\mathcal{C}(W_S(e^{i\omega})^{-*} G(e^{i\omega})) = 0 \quad (6.9)$$

and

$$\mathcal{C}(W_S(e^{i\omega}) K(e^{i\omega})) = W_S(e^{i\omega}) K(e^{i\omega}). \quad (6.10)$$

Substituting (6.9) and (6.10) into (6.8) and multiplying by  $W_S(e^{i\omega})^{-1}$  on the left, leads to

$$K(e^{i\omega}) = W_S(e^{i\omega})^{-1} \mathcal{C}(W_S(e^{i\omega})^{-*} S_{uz}(e^{i\omega})), \quad (6.11)$$

where the components in (6.11) are operator-valued functions.

**Example 6.1** (The case of additive noise). Let us consider the question of filtering  $(x_n)_{n \in \mathbb{N}}$  from  $(y_n)_{n \in \mathbb{N}}$ , where  $(y_n)_{n \in \mathbb{N}}$  is given by

$$y_n = x_n + v_n.$$

Here  $(v_n)_{n \in \mathbb{N}}$ , the system noise, has spectrum  $S_v(e^{i\omega}) = \mathcal{V}_0$  where  $\mathcal{V}$  is a constant positive operator. Equivalently, we have

$$M_{v_n}^* M_{v_m} \stackrel{\text{def}}{=} R_v(m - n) = \mathcal{V}_0 \delta(m - n).$$

We also assume that  $\langle v_n \circ \xi, x_m \circ \eta \rangle_{\mathcal{H}_k} = 0$  for all  $\xi, \eta \in \mathcal{H}_k$ , i.e.  $R_{xv}(n) = 0$  and so  $S_{xv}(e^{i\omega}) = 0$ . As consequences, we may conclude the following

$$S_{xv}(e^{i\omega}) = 0, \quad S_y(e^{i\omega}) = S_x(e^{i\omega}) + \mathcal{V}_0,$$

and

$$S_{xy}(e^{i\omega}) = S_x(e^{i\omega}).$$

Hence, in the additive noise case, the non-causal Wiener filter (5.6) becomes

$$K(e^{i\omega}) = S_y(e^{i\omega})^{-1} S_{uy}(e^{i\omega}) = (S_x(e^{i\omega}) + \mathcal{V}_0)^{-1} S_{uy}(e^{i\omega}).$$

The causal Wiener filter, developed above in (6.11), is now given by

$$\begin{aligned} K(e^{i\omega}) &= W_S(e^{i\omega})^{-1} \mathcal{C}(W_S(e^{i\omega})^{-*} S_{xy}(e^{i\omega})) \\ &= W_S(e^{i\omega})^{-1} \mathcal{C}(W_S(e^{i\omega})^{-*} (S_y(e^{i\omega}) - \mathcal{V}_0)) \\ &= W_S(e^{i\omega})^{-1} \mathcal{C}(W_S(e^{i\omega}) - W_S(e^{i\omega})^{-*} \mathcal{V}_0). \end{aligned}$$

Finally, since

$$\mathcal{C}(W_S(e^{i\omega})) = W_S(e^{i\omega}) \quad \text{and} \quad \mathcal{C}(W_S(e^{i\omega})^{-*} \mathcal{V}_0) = \mathcal{V}_0,$$

we may conclude that

$$K(e^{i\omega}) = I_{\mathcal{H}_k} - W_S(e^{i\omega})^{-1} \mathcal{V}_0.$$

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