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OSCILLATION OF TIME FRACTIONAL VECTOR DIFFUSION-WAVE EQUATION WITH FRACTIONAL DAMPING

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Abstract. In this paper, sufficient conditions for H-oscillation of solutions of a time fractional vector diffusion-wave equation with forced and fractional damping terms subject to the Neumann boundary condition are established by employing certain fractional differential inequality, where H is a unit vector in $\mathbb{R}^n$. The examples are given to illustrate the main results.

Keywords: fractional diffusion-wave equation, H-oscillation, vector differential equation.

Mathematics Subject Classification: 35B05, 35R11, 34K37.

1. INTRODUCTION

Interest on the study of fractional differential equation is on the rise because of its utility in the fields of science and engineering such as neural networks, population dynamics, electrical and mechanical engineering. In recent years, there has been a significant development in fractional order ordinary and partial differential equations, for example Kilbas *et al.*[6]. In particular, the oscillation theory of fractional differential equations attracted by many authors [1, 2, 5, 7, 8, 10, 15–17].

The H-oscillation for vector differential equation was introduced by Domshlak [3] in 1970. Few authors [9, 11–13] have discussed H-oscillation of vector partial differential equations. Prakash and Harikrishnan [14] have established criteria for H-oscillation of solutions of impulsive vector hyperbolic differential equations with delays. However, the concept of H-oscillation of vector partial differential equation studied for integer order only. In this paper, we establish sufficient conditions for H-oscillation of a class of time fractional vector diffusion-wave equation with forced and fractional damping terms subject to the Neumann boundary condition by using differential inequality method.

We establish the oscillation criteria for the time fractional vector diffusion-wave equation with forced and fractional damping terms of the form

$$D^{\alpha}_{+,t}U(x,t)+p(t)D^{\beta}_{+,t}U(x,t)+q(x,t)U(x,t) = a(t)\Delta U(x,t)+F(x,t), (x,t) \in G, \quad (E)$$

with the Neumann boundary condition

$$\frac{\partial U(x,t)}{\partial N} = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}_+, \qquad (B)$$

where $\alpha \in (1,2)$ and $\beta = \alpha - 1$ are constants, $G := \Omega \times \mathbb{R}_+$, $D^{\alpha}_{+,t}U$ is the Riemann–Liouville fractional derivative of order α of U with respect to t, Ω is a bounded domain in $\mathbb{R}$ with piecewise smooth boundary $\partial\Omega$, $\mathbb{R}_+ = [0,\infty)$, Δ is the Laplacian operator and N is the unit exterior normal vector to $\partial\Omega$.

By a solution of (E), we mean a function $U(x,t) : \Omega \times \mathbb{R}_+ \to \mathbb{R}^n$ such that $U(x,t)$, $D^{\alpha}_{+,t}U(x,t)$, $D^{\beta}_{+,t}U(x,t)$, $\partial_x U(x,t)$, and $\partial^2_{xx}U(x,t)$ are continuous on $\overline{\Omega} \times \mathbb{R}_+ \to \mathbb{R}^n$. Also the solution needs to satisfy (E) along with the boundary condition (B).

Throughout this paper, we assume that the following conditions hold:

(A_1) $a(t), p(t) \in C(\mathbb{R}_+;\mathbb{R}_+)$, $F \in (\overline{G}, \mathbb{R}^n)$;
(A_2) $q(x,t) \in (\overline{G}, \mathbb{R}_+)$, $q(t) = \min_{x\in\Omega} q(x,t)$.

This article is organized as follows. Section 2 gives the basic definitions and lemmas. In Section 3, we prove the main results. In Section 4, we present examples to illustrate the main results.

2. PRELIMINARIES

In this section, we present definitions of fractional derivatives, known lemmas and notations which are used in this paper.

Definition 2.1 ([6])**.** The Riemann–Liouville fractional partial derivative of order α where $0 < \alpha < 1$ with respect to t of a continuous function $u(x,t)$ in t is given by

$$(D^{\alpha}_{+,t}u)(x,t) := \frac{\partial}{\partial t}\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}u(x,s)ds,$$

where Γ is the gamma function.

Definition 2.2 ([6])**.** The left-sided Riemann–Liouville fractional integral of order $\alpha > 0$ of an integrable function $y : \mathbb{R}_+ \to \mathbb{R}$ on the half-axis $\mathbb{R}_+$ is given by

$$(I^{\alpha}_{+}y)(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds, \quad \text{for} \quad t > 0.$$

Definition 2.3 ([6]). The left-sided Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function $y : \mathbb{R}_+ \to \mathbb{R}$ of class C^α is given by

$$\begin{aligned}(D_+^\alpha y)(t) &:= \frac{d^{\lceil \alpha \rceil}}{dt^{\lceil \alpha \rceil}} \left(I_+^{\lceil \alpha \rceil - \alpha} y \right)(t) \\ &= \frac{1}{\Gamma(\lceil \alpha \rceil - \alpha)} \frac{d^{\lceil \alpha \rceil}}{dt^{\lceil \alpha \rceil}} \int\limits_0^t (t-s)^{\lceil \alpha \rceil - \alpha - 1} y(s) ds, \text{ for } t > 0,\end{aligned}$$

where $\lceil \alpha \rceil$ is the ceiling function of α.

Definition 2.4. The function $v(x,t)$, $(x,t) \in G$ is called eventually positive (negative), if there exists a number $\mu \geq 0$ such that $v(x,t) \geq 0 (v(x,t) \leq 0)$ for $(x,t) \in \Omega \times [\mu, \infty)$. The function $v(t)$ is called eventually positive (negative), if there exists a number $\mu \geq 0$ such that $v(t) \geq 0$ $(v(t) \leq 0)$ for $t \geq \mu$.

Definition 2.5. Let H be a fixed unit vector in $\mathbb{R}^n$. Then the vector solution $U(x,t)$ of (E), (B) is H-oscillatory in the domain G, if the inner product $\langle U(x,t), H \rangle$ has a zero in $\Omega \times [\mu, \infty)$ for any $\mu > 0$. Otherwise, the solution $U(x,t)$ is said to be H-nonoscillatory.

Definition 2.6. Let H be an arbitrary unit vector in $\mathbb{R}^n$. Then the vector solution $U(x,t)$ of (E), (B) is strongly H-oscillatory in the domain G, if the inner product $\langle U(x,t), H \rangle$ has a zero in $\Omega \times [\mu, \infty)$ for any $\mu > 0$.

Lemma 2.7 ([16]). *Let*

$$G(t) := \int\limits_0^t (t-\nu)^{-\alpha} y(\nu) d\nu,$$

for a continuous function $y(t)$, $\alpha \in (0,1)$ and $t > 0$. Then

$$G'(t) = \Gamma(1-\alpha)(D_+^\alpha y)(t).$$

For convenience, we use the following notations:

$$\begin{aligned}&u_H(x,t) = \langle U(x,t), H \rangle, \quad f_H(x,t) = \langle F(x,t), H \rangle, \\ &V(t) = \int\limits_\Omega u_H(x,t) dx, \quad F_H(t) = \int\limits_\Omega f_H(x,t) dx, \quad R(t) = \int\limits_0^t p(s) ds.\end{aligned}$$

3. OSCILLATION CRITERIA FOR (E), (B)

Case (i). *$f_H(x,t)$ is non zero, where H is a fixed unit vector.*

For this case, we assume the following condition hold:

(A_3) For $T \geq 0$, there exists $T \leq a < b \leq \tilde{a} < \tilde{b}$ such that

$$F_H(t) = \begin{cases} \leq 0, & t \in [a,b], \\ \geq 0, & t \in [\tilde{a}, \tilde{b}] \end{cases} \qquad \text{and} \quad \lim_{t \to \infty} \int\limits_{t_0}^t \frac{1}{e^{R(s)}} \int\limits_{t_0}^s e^{R(\nu)} F_H(s) d\nu ds = 0.$$

Lemma 3.1. *Let $U(x,t)$ be a solution of (E). Then $u_H(x,t)$ satisfies the following scalar time fractional partial differential inequalities:*

$$D^{\alpha}_{+,t}u_H(x,t) + p(t)D^{\beta}_{+,t}u_H(x,t) + q(t)u_H(x,t) \leq a(t)\Delta u_H(x,t) + f_H(x,t) \tag{3.1}$$

if $u_H(x,t)$ is eventually positive, and

$$D^{\alpha}_{+,t}u_H(x,t) + p(t)D^{\beta}_{+,t}u_H(x,t) + q(t)u_H(x,t) \geq a(t)\Delta u_H(x,t) + f_H(x,t) \tag{3.2}$$

if $u_H(x,t)$ is eventually negative.

Proof. Let $u_H(x,t)$ be eventually positive. Then from the inner product of (E) and H, we obtain

$$D^{\alpha}_{+,t}u_H(x,t) + p(t)D^{\beta}_{+,t}u_H(x,t) + q(x,t)u_H(x,t) = a(t)\Delta u_H(x,t) + f_H(x,t).$$

Using the condition (A_2), we have

$$D^{\alpha}_{+,t}u_H(x,t) + p(t)D^{\beta}_{+,t}u_H(x,t) + q(t)u_H(x,t) \leq a(t)\Delta u_H(x,t) + f_H(x,t).$$

Similarly, by letting $u_H(x,t)$ to be eventually negative, we easily obtain (3.2). □

From the inner product of boundary condition (B) with H, we have the following boundary condition:

$$\frac{\partial}{\partial N}u_H(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,\infty). \tag{B_1}$$

Theorem 3.2. *If the inequalities* (3.1) *and* (3.2) *have no eventually positive and eventually negative solutions, respectively, and that satisfies the boundary condition (B_1), then every solution $U(x,t)$ of the problem (E), (B) is H-oscillatory in G.*

Proof. Assume that the solution $U(x,t)$ of (E), (B) is a H-nonoscillatory, then by definition, the inner product $u_H(x,t)$ is eventually positive or eventually negative.

Suppose that $u_H(x,t)$ is eventually positive. Then by Lemma 3.1, $u_H(x,t)$ satisfies (3.1) and (B_1), which is a contradiction.

Similarly, we get a contradiction if $u_H(x,t)$ is eventually negative. The proof is complete. □

Theorem 3.3. *If the fractional differential inequalities*

$$D^{\alpha}_{+}V(t) + p(t)D^{\beta}_{+}V(t) + q(t)V(t) \leq F_H(t) \tag{3.3}$$

and

$$D^{\alpha}_{+}V(t) + p(t)D^{\beta}_{+}V(t) + q(t)V(t) \geq F_H(t) \tag{3.4}$$

have no eventually positive and no eventually negative solutions, respectively, then every solution $U(x,t)$ of (E), (B) is H-oscillatory in G.

Proof. Let $u_H(x,t)$ be an eventually positive solution of the inequality (3.1) satisfying the boundary condition (B_1) for $(x,t) \in \Omega \times [t_0, \infty)$, $t_0 \geq 0$. Then by Theorem 3.2 and assumptions, it is enough to prove that (3.1) has no eventually positive solution satisfying (B_1).

Integrating the inequality in (3.1) with respect to x over the domain Ω, we have

$$D_+^{\alpha}\left(\int\limits_{\Omega} u_H(x,t)dx\right) + p(t)D_+^{\beta}\left(\int\limits_{\Omega} u_H(x,t)dx\right) + q(t)\int\limits_{\Omega} u_H(x,t)dx$$
$$\leq a(t)\int\limits_{\Omega} \Delta u_H(x,t)dx + \int\limits_{\Omega} f_H(x,t)dx.$$

Using Green's formula and (B_1), we have

$$\int\limits_{\Omega} \Delta u_H(x,t)dx = \int\limits_{\partial\Omega} \frac{\partial u_H(x,t)}{\partial N}dS = 0$$

which implies that

$$D_+^{\alpha}V(t) + p(t)D_+^{\beta}V(t) + q(t)V(t) \leq F_H(t).$$

Therefore, $V(t)$ is a positive solution of (3.3), which is a contradiction.

Similarly, suppose that $u_H(x,t) < 0$ is a solution of (3.2) satisfying (B_1). Using the above procedure, we obtain a contradiction. □

Theorem 3.4. *Suppose that the conditions (A_1)–(A_3) hold and additionally*

$$\int\limits_{t_0}^{\infty} \frac{1}{e^{R(s)}}ds = \infty,$$

$$\liminf_{t\to\infty} \int\limits_{t_1}^{t} F_H(s)ds = -\infty \text{ for } t_1 \geq t_0. \tag{3.5}$$

and

$$\limsup_{t\to\infty} \int\limits_{t_1}^{t} F_H(s)ds = \infty \text{ for } t_1 \geq t_0. \tag{3.6}$$

Then each solution of (E), (B) oscillates in G.

Proof. To prove this theorem, it suffices to prove that either (3.3) has no eventually positive solutions or (3.4) has no eventually negative solutions.

Suppose that V is an eventually positive solution of (3.3). Then there exists $t_0 < a < b$ such that $V(t) > 0$ on $[t_0, \infty)$, $F_H(t) \le 0$ on $[a, b]$ and we have

$$\begin{aligned} D\Big[e^{R(t)} D_+^{\beta} V(t)\Big] &= e^{R(t)} D_+^{\alpha} V(t) + e^{R(t)} p(t) D_+^{\beta} V(t) \\ &= e^{R(t)} \{D_+^{\alpha} V(t) + p(t) D_+^{\beta} V(t)\} \\ &= -e^{R(t)} q(t) V(t) + e^{R(t)} F_H(t) < 0. \end{aligned}$$

Then $e^{R(t)} D_+^{\beta} V(t)$ is strictly decreasing on $[a, b]$ and thus $D_+^{\beta} V(t)$ is eventually of one sign. We claim $D_+^{\beta} V(t) > 0$ on $[t_1, b]$, where $a < t_1 < b$. Otherwise, assume there exists $t_1 < T < b$ such that $D_+^{\beta} V(t) < 0$ on $[T, b]$. Then for $t \in [T, b]$, by Lemma 2.7, we have

$$\frac{G'(t)}{\Gamma(1-\beta)} = D_+^{\beta} V(t) \le \frac{e^{R(T)} D_+^{\beta} V(T)}{e^{R(t)}}.$$

Integrating the above inequality from T to t, we have

$$G(t) \le G(T) + \Gamma(1-\beta) e^{R(T)} D_+^{\beta} V(T) \int_T^t \frac{1}{e^{R(s)}} ds.$$

Letting $t \to \infty$, we get $\lim_{t\to\infty} G(t) \le -\infty$ which is a contradiction since $G(t) > 0$ if $V(t) > 0$. Hence $D_+^{\beta} V(t) > 0$ on $[t_1, b]$. Therefore,

$$\begin{aligned} D\left[e^{R(t)} D_+^{\beta} V(t)\right] &\le e^{R(t)} F_H(t), \\ e^{R(t)} D_+^{\beta} V(t) &\le e^{R(t_1)} D_+^{\beta} V(t_1) + \int_{t_1}^{t} e^{R(t)} F_H(t). \end{aligned}$$

Thus, we get a contradiction to (3.5) since $e^{R(t)} D_+^{\beta} V(t)$ is eventually positive.

Assume that V is an eventually negative solution of (3.4). Then there exists $t_0 < \tilde{a} < \tilde{b}$ such that $V(t) < 0$ on $[t_0, \infty)$, $F_H(t) \ge 0$ on $[\tilde{a}, \tilde{b}]$ and we have

$$\begin{aligned} D\Big[e^{R(t)} D_+^{\beta} V(t)\Big] &= e^{R(t)} D_+^{\alpha} V(t) + e^{R(t)} p(t) D_+^{\beta} V(t) \\ &= e^{R(t)} \{D_+^{\alpha} V(t) + p(t) D_+^{\beta} V(t)\} \\ &= -e^{R(t)} q(t) V(t) + e^{R(t)} F_H(t) > 0. \end{aligned}$$

Then $e^{R(t)} D_+^{\beta} V(t)$ is strictly increasing on $[\tilde{a}, \tilde{b}]$ and thus $D_+^{\beta} V(t)$ is eventually of one sign. We claim $D_+^{\beta} V(t) < 0$ on $[t_1, \tilde{b}]$, where $\tilde{a} < t_1 < \tilde{b}$. Otherwise, assume there exists $t_1 < T < \tilde{b}$ such that $D_+^{\beta} V(t) > 0$ on $[T, \tilde{b}]$. Then for $t \in [T, \tilde{b}]$, by Lemma 2.7, we have

$$\frac{G'(t)}{\Gamma(1-\beta)} = D_+^{\beta} V(t) \ge \frac{e^{R(T)} D_+^{\beta} V(T)}{e^{R(t)}}.$$

Integrating the above inequality from T to t, we have

$$G(t) \geq G(T) + \Gamma(1-\beta)e^{R(T)}D_+^\beta V(T)\int_T^t \frac{1}{e^{R(s)}}ds.$$

Letting $t \to \infty$, we get $\lim_{t\to\infty} G(t) \geq \infty$ which is a contradiction since $G(t) < 0$ if $V(t) < 0$. Hence, $D_+^\beta V(t) > 0$ on $[t_1, \tilde{b}]$. Therefore,

$$D\left[e^{R(t)}D_+^\beta V(t)\right] \geq e^{R(t)}F_H(t),$$

$$e^{R(t)}D_+^\beta V(t) \geq e^{R(t_1)}D_+^\beta V(t_1) + \int_{t_1}^t e^{R(t)}F_H(t).$$

Thus, we get a contradiction to (3.6) since $e^{R(t)}D_+^\beta V(t)$ is eventually negative. □

Theorem 3.5. *Suppose that the conditions (A_1)–(A_3) hold and additionally*

$$\int_{t_0}^\infty \frac{1}{e^{R(s)}}ds < \infty, \quad \int_{t_0}^\infty \frac{1}{e^{R(s)}}\int_{t_0}^s e^{R(\nu)}q(\nu)d\nu ds = \infty,$$

$$\liminf_{t\to\infty}\int_{t_1}^t F_H(s)ds = -\infty \text{ for } t_1 \geq t_0. \tag{3.7}$$

and

$$\limsup_{t\to\infty}\int_{t_1}^t F_H(s)ds = \infty \text{ for } t_1 \geq t_0. \tag{3.8}$$

Then each solution of (E), (B) oscillates in G.

Proof. To prove this theorem, it suffices to prove that either (3.3) has no eventually positive solutions or (3.4) has no eventually negative solutions.

Suppose that V is an eventually positive solution of (3.3). Then there exists $t_0 < a < b$ such that $V(t) > 0$ on $[t_0, \infty)$, $F_H(t) \leq 0$ on $[a, b]$ and we have

$$\begin{aligned} D\left[e^{R(t)}D_+^\beta V(t)\right] &= e^{R(t)}D_+^\alpha V(t) + e^{R(t)}p(t)D_+^\beta V(t) \\ &= e^{R(t)}\{D_+^\alpha V(t) + p(t)D_+^\beta V(t)\} \\ &= -e^{R(t)}q(t)V(t) + e^{R(t)}F_H(t) < 0. \end{aligned}$$

Then $e^{R(t)}D_+^\beta V(t)$ is strictly decreasing on $[a, b]$ and thus $D_+^\beta V(t)$ is eventually of one sign. We claim $D_+^\beta V(t) > 0$ on $[t_1, b]$, where $a < t_1 < b$. Otherwise, assume there exists $t_1 < T < b$ such that $D_+^\beta V(t) < 0$ on $[T, b]$. Then there exists a constant $K > 0$ such that $V(t) \le K,\ T \le t \le b$. Consequently, we have

$$D\Big[e^{R(t)} D_+^\beta V(t)\Big] = -e^{R(t)} q(t)V(t) + e^{R(t)} F_H(t),$$

$$\begin{aligned} e^{R(t)} D_+^\beta V(t) &= e^{R(T)} D_+^\beta V(T) - \int_T^t e^{R(s)} q(s)V(s)ds + \int_T^t e^{R(s)} F_H(s)ds \\ &\le -K \int_T^t e^{R(s)} q(s)ds + \int_T^t e^{R(s)} F_H(s)ds, \end{aligned}$$

$$D_+^\beta V(t) \le \frac{-K}{e^{R(t)}} \int_T^t e^{R(s)} q(s)ds + \frac{1}{e^{R(t)}} \int_T^t e^{R(s)} F_H(s)ds.$$

By Lemma 2.7, we have

$$\frac{G'(t)}{\Gamma(1-\beta)} = D_+^\beta V(t) \le \frac{-K}{e^{R(t)}} \int_T^t e^{R(s)} q(s)ds + \frac{1}{e^{R(t)}} \int_T^t e^{R(s)} F_H(s)ds.$$

Integrating the above inequality from T to t, we have

$$\begin{aligned} G(t) \le G(T) + \Gamma(1-\beta)\Bigg[&- K \int_T^t \frac{1}{e^{R(s)}} \int_T^s e^{R(\nu)} q(\nu)d\nu ds \\ &+ \int_T^t \frac{1}{e^{R(s)}} \int_T^s e^{R(\nu)} F_H(\nu)d\nu ds \Bigg]. \end{aligned}$$

Letting $t \to \infty$, we get $\lim_{t\to\infty} G(t) \le -\infty$ which is a contradiction since $G(t) > 0$ if $V(t) > 0$. Hence $D_+^\beta V(t) > 0$ on $[t_1, b]$. Therefore,

$$D\left[e^{R(t)} D_+^\beta V(t)\right] \le e^{R(t)} F_H(t),$$

$$e^{R(t)} D_+^\beta V(t) \le e^{R(t_1)} D_+^\beta V(t_1) + \int_{t_1}^t e^{R(t)} F_H(t).$$

Thus, we get a contradiction to (3.7) since $e^{R(t)} D_+^\beta V(t)$ is eventually positive.

Assume that V is an eventually negative solution of (3.3). Then there exists $t_0 < \tilde{a} < \tilde{b}$ such that $V(t) < 0$ on $[t_0, \infty)$, $F_H(t) \ge 0$ on $[\tilde{a}, \tilde{b}]$ and we have

$$\begin{aligned} D\Big[e^{R(t)} D_+^\beta V(t)\Big] &= e^{R(t)} D_+^\alpha V(t) + e^{R(t)} p(t) D_+^\beta V(t) \\ &= e^{R(t)} \{D_+^\alpha V(t) + p(t) D_+^\beta V(t)\} \\ &= -e^{R(t)} q(t)V(t) + e^{R(t)} F_H(t) > 0. \end{aligned}$$

Then $e^{R(t)}D_+^\beta V(t)$ is strictly increasing on $[\tilde{a},\tilde{b}]$ and thus $D_+^\beta V(t)$ is eventually of one sign. We claim $D_+^\beta V(t)<0$ on $[t_1,\tilde{b}]$, where $\tilde{a}<t_1<\tilde{b}$. Otherwise, assume there exists $t_1<T<\tilde{b}$ such that $D_+^\beta V(t)>0$ on $[T,\tilde{b}]$. Then there exists a constant $K<0$ such that $V(t)\geq K,\ T\leq t\leq \tilde{b}$. Consequently, we have

$$\begin{aligned}
D\Big[e^{R(t)}D_+^\beta V(t)\Big] &= -e^{R(t)}q(t)V(t)+e^{R(t)}F_H(t),\\
e^{R(t)}D_+^\beta V(t) &= e^{R(T)}D_+^\beta V(T)-\int_T^t e^{R(s)}q(s)V(s)ds+\int_T^t e^{R(s)}F_H(s)ds\\
&\geq -K\int_T^t e^{R(s)}q(s)ds+\int_T^t e^{R(s)}F_H(s)ds,\\
D_+^\beta V(t) &\geq \frac{-K}{e^{R(t)}}\int_T^t e^{R(s)}q(s)ds+\frac{1}{e^{R(t)}}\int_T^t e^{R(s)}F_H(s)ds.
\end{aligned}$$

By Lemma 2.7, we have

$$\frac{G'(t)}{\Gamma(1-\beta)}=D_+^\beta V(t)\geq \frac{-K}{e^{R(t)}}\int_T^t e^{R(s)}q(s)ds+\frac{1}{e^{R(t)}}\int_T^t e^{R(s)}F_H(s)ds$$

Integrating the above inequality from T to t, we have

$$\begin{aligned}
G(t)\geq G(T)+\Gamma(1-\beta)\Bigg[&-K\int_T^t \frac{1}{e^{R(s)}}\int_T^s e^{R(\nu)}q(\nu)d\nu ds\\
&+\int_T^t \frac{1}{e^{R(s)}}\int_T^s e^{R(\nu)}F_H(\nu)d\nu ds\Bigg].
\end{aligned}$$

Letting $t\to\infty$, we get $\lim_{t\to\infty}G(t)\geq\infty$ which is a contradiction since $G(t)<0$ if $V(t)<0$. Hence $D_+^\beta V(t)<0$ on $[t_1,\tilde{b}]$. Therefore,

$$\begin{aligned}
D\left[e^{R(t)}D_+^\beta V(t)\right] &\geq e^{R(t)}F_H(t),\\
e^{R(t)}D_+^\beta V(t) &\geq e^{R(t_1)}D_+^\beta V(t_1)+\int_{t_1}^t e^{R(t)}F_H(t).
\end{aligned}$$

Thus, we get a contradiction to (3.8) since $e^{R(t)}D_+^\beta V(t)$ is eventually negative. $\square$

Case (ii). *$f_H(x,t)$ is zero, where H is a fixed unit vector.*

Lemma 3.6. *Let $U(x,t)$ be a solution of (E). Then $u_H(x,t)$ satisfies*

$$D^{\alpha}_{+,t}u_H(x,t) + p(t)D^{\beta}_{+,t}u_H(x,t) + q(t)u_H(x,t) \leq a(t)\Delta u_H(x,t)$$

if $u_H(x,t)$ is eventually positive, and

$$D^{\alpha}_{+,t}u_H(x,t) + p(t)D^{\beta}_{+,t}u_H(x,t) + q(t)u_H(x,t) \geq a(t)\Delta u_H(x,t)$$

if $u_H(x,t)$ is eventually negative.

Proof. The proof of this lemma is similar to that of Lemma 3.1. □

Theorem 3.7. *If the fractional differential inequality*

$$D^{\alpha}_{+}V(t) + p(t)D^{\beta}_{+}V(t) + q(t)V(t) \leq 0 \tag{3.9}$$

has no eventually positive solutions and the fractional differential inequality

$$D^{\alpha}_{+}V(t) + p(t)D^{\beta}_{+}V(t) + q(t)V(t) \geq 0 \tag{3.10}$$

has no eventually negative solutions, then every solution $U(x,t)$ of the problem (E), (B) is H-oscillatory in G.

Proof. The proof of this theorem is similar to that of Theorem 3.3. □

Theorem 3.8. *Suppose that the conditions (A_1)–(A_2) hold and additionally*

$$\int_{t_0}^{\infty} \frac{1}{e^{R(s)}}ds = \infty,$$

and

$$\limsup_{t\to\infty} \int_{T_1}^{t} P(\nu)d\nu \geq 1, \ \textit{for } t > T_1 > 0, \tag{3.11}$$

where $P(t) = \theta t q(t)$. Then each solution of (E), (B) oscillates in G.

Proof. To prove this theorem, it suffices to prove that either (3.9) has no eventually positive solutions or (3.10) has no eventually negative solutions. Suppose that V is an eventually positive solution of (3.9). Then there exists t_0 such that $V(t) > 0$ on $[t_0, \infty)$ and we have

$$\begin{aligned} D\Big[e^{R(t)}D^{\beta}_{+}V(t)\Big] &= e^{R(t)}D^{\alpha}_{+}V(t) + e^{R(t)}p(t)D^{\beta}_{+}V(t) \\ &= -e^{R(t)}q(t)V(t) < 0. \end{aligned}$$

Then $e^{R(t)}D^{\beta}_{+}V(t)$ is strictly decreasing on $[t_0, \infty)$, and thus $D^{\beta}_{+}V(t)$ is eventually of one sign. We claim $D^{\beta}_{+}V(t) > 0$ on $[t_1, \infty)$, where $t_1 > t_0$. Otherwise, assume there exists $T > t_1$ such that $D_+^\beta V(t) < 0$ on $[T, \infty)$. Then for $t \in [T, \infty)$, by Lemma 2.7, we have

$$\frac{G'(t)}{\Gamma(1-\beta)} = (D_+^\beta V(t)) \leq \frac{e^{R(T)} D_+^\beta V(T)}{e^{R(t)}}.$$

Integrating the above inequality from T to t, we have

$$G(t) \leq G(T) + \Gamma(1-\beta) e^{R(T)} D_+^\beta V(T) \int\limits_T^t \frac{1}{e^{R(s)}} ds.$$

Letting $t \to \infty$, we get $\lim_{t\to\infty} G(t) \leq -\infty$ which is a contradiction. Hence $D_+^\beta V(t) > 0$ for $t \geq T$ holds.

Therefore, there exists a ξ such that $V(t) - V(T) = D_+^\beta V(\xi)(t-T),\ \xi \in (T, t)$. Thus, we have

$$V(t) \geq (t-T) D_+^\beta V(\xi) \geq (t-T) D_+^\beta V(t), \text{ for } t > T. \tag{3.12}$$

For $\theta \in (0,1)$, we put $\mu = \frac{1}{1-\theta} > 1$. Then $\theta = 1 - \frac{1}{\mu}$ and

$$t - T \geq t - \frac{t}{\mu} = t\left(1 - \frac{1}{\mu}\right) = \theta t, \text{ for } t \geq \mu T = T_1. \tag{3.13}$$

Hence, from (3.12) and (3.13), we have

$$V(t) \geq \theta t D_+^\beta V(t), \text{ for } t \geq T_1. \tag{3.14}$$

It follows from (3.9) and (3.14) that

$$D_+^\alpha V(t) + p(t) D_+^\beta V(t) + \theta t q(t) D_+^\beta V(t) \leq 0.$$

Let $w(t) = e^{R(t)} D_+^\beta V(t)$. Then

$$D[w(t)] + \theta t q(t) w(t) \leq 0.$$

Integrating the above inequality from T_1 to t we obtain

$$w(t) - w(T_1) + \int\limits_{T_1}^t \theta s q(s) w(s) ds \leq 0,$$

$$\int\limits_{T_1}^t \theta s q(s) ds \leq 1 - \frac{w(t)}{w(T_1)} < 1,$$

$$\int\limits_{T_1}^t \theta s q(s) ds < 1.$$

Taking lim sup as $t \to \infty$ we get a contradiction to (3.11). $\square$

Theorem 3.9. *Suppose that the conditions (A_1)–(A_2) hold and additionally*

$$\int_{t_0}^{\infty} \frac{1}{e^{R(s)}} ds < \infty, \quad \int_{t_0}^{\infty} \frac{1}{e^{R(s)}} \int_{t_0}^{s} e^{R(\nu)} q(\nu) d\nu ds = \infty$$

and

$$\limsup_{t\to\infty} \int_{s}^{t} P(\nu) d\nu \geq 1, \ \textit{for } t > s > 0, \tag{3.15}$$

where $P(t) = \theta t q(t)$. Then each solution of (E), (B) oscillates in G.

Proof. To prove this theorem, it suffices to prove that (3.9) has no eventually positive solution. Suppose that V is an eventually positive solution of (3.9). Then there exists t_0 such that $V(t) > 0$ on $[t_0, \infty)$ and we have

$$\begin{aligned} D\left[e^{R(t)} D_+^{\beta} V(t)\right] &= e^{R(t)} D_+^{\alpha} V(t) + e^{R(t)} p(t) D_+^{\beta} V(t) \\ &= -e^{R(t)} q(t) V(t) < 0. \end{aligned}$$

Then $e^{R(t)} D_+^{\beta} V(t)$ is strictly decreasing on $[t_0, \infty)$, and thus $D_+^{\beta} V(t)$ is eventually of one sign. We claim $D_+^{\beta} V(t) > 0$ on $[t_1, \infty)$, where $t_1 > t_0$. Otherwise, assume there exists $T > t_1$ such that $D_+^{\beta} V(t) < 0$ on $[T, \infty)$. Then there exists a constant $K > 0$ such that $V(t) \leq K, \ t \geq T$. Consequently, we have

$$\begin{aligned} D\left[e^{R(t)} D_+^{\beta} V(t)\right] &= -e^{R(t)} q(t) V(t), \\ e^{R(t)} D_+^{\beta} V(t) &= e^{R(T)} D_+^{\beta} V(T) - \int_{T}^{t} e^{R(s)} q(s) V(s) ds, \\ D_+^{\beta} V(t) &\leq \frac{-K}{e^{R(t)}} \int_{T}^{t} e^{R(s)} q(s) ds. \end{aligned}$$

By Lemma 2.7, we have

$$\frac{G'(t)}{\Gamma(1-\beta)} = D_+^{\beta} V(t) \quad \leq \quad \frac{-K}{e^{R(t)}} \int_{T}^{t} e^{R(s)} q(s) ds$$

Integrating the above inequality from T to t, we have

$$G(t) \leq G(T) - K\Gamma(1-\beta) \int_{T}^{t} \frac{1}{e^{R(s)}} \int_{T}^{s} e^{R(\nu)} q(\nu) d\nu ds.$$

Letting $t \to \infty$, we get $\lim_{t\to\infty} G(t) \leq -\infty$ which is a contradiction. Hence $D_{+}^{\beta} V(t) > 0$ for $t \geq T$ holds. Further, we proceed the proof as that of Theorem 3.8, we get a contradiction to (3.15). $\square$

Remark 3.10. In the above results, if we take H to be an arbitrary unit vector in $\mathbb{R}^n$ instead of a fixed unit vector in $\mathbb{R}^n$, then we obtain the results for strongly H-oscillatory in G.

4. EXAMPLES

In this section, we give two examples to illustrate the results.

Example 4.1. Consider the time fractional vector diffusion-wave equation with forced and fractional damping terms

$$D_{+,t}^{\alpha} U(x,t) + t^2 D_{+,t}^{\beta} U(x,t) + U(x,t) = \Delta U(x,t) + F(x,t), \quad (x,t) \in \Omega \times \mathbb{R}_+, \tag{4.1}$$

where

$$F(x,t) = \begin{pmatrix} (\cos t - t^2 \sin t) \cos x \\ (\sin t + t^2 \cos t) \sin x \end{pmatrix}$$

with the boundary condition

$$U_x(0,t) = U_x(\pi,t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Here $\Omega = (0,\pi)$, $p(t) = t^2$, $a(t) = 1$, $q(x,t) = 1$.

Letting $H = \binom{0}{1}$, we have $f_H(x,t) = (\sin t + t^2 \cos t) \sin x$ and $F_H(t) = 2 \sin t + 2t^2 \cos t$. It is easy to see that all the conditions of Theorem 3.4 are satisfied. Hence, every solution of (4.1) is H-oscillatory in domain $(0,\pi) \times [0,\infty)$.

Example 4.2. Consider the time fractional vector diffusion-wave equation with forced and fractional damping terms

$$D_{+,t}^{\alpha} U(x,t) + t D_{+,t}^{\beta} U(x,t) + e^t U(x,t) = \Delta U(x,t) + F(x,t), \quad (x,t) \in \Omega \times \mathbb{R}_+, \tag{4.2}$$

where

$$F(x,t) = \begin{pmatrix} (1+t) \cos t \cos x \\ (t + e^t) \sin t \sin x \end{pmatrix}$$

with the boundary condition

$$U_x(0,t) = U_x(\pi,t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Here $\Omega = (0,\pi)$, $p(t) = 1$, $a(t) = 1$, $q(x,t) = e^t$.

Letting $H = \binom{1}{0}$, we have $f_H(x,t) = (1+t) \cos t \cos x$ and $F_H(t) = 0$. It is easy to see that all the conditions of Theorem 3.8 are satisfied. Hence every solution of (4.2) is H-oscillatory in domain $(0,\pi) \times [0,\infty)$.

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