Journal of Applied Sciences Research

2020 January -February; 16(1): pages 9-19
DOI: 10.22587/jasr.2020.16.1.2

Research Article AENSI Publications

Cubic Transmuted Fréchet Distribution

Rania M. Shalabi

The Higher Institute of Managerial Science, Culture and Science City, 6th of October, Giza, Egypt

Correspondence Author: Rania M. Shalabi, The Higher Institute of Managerial Science, Culture and Science City, 6th of October, Giza, Egypt

Received date: 11 January 2020, Accepted date: 24 February 2020, Online date: 28 February 2020

Abstract

In this paper, a Cubic Transmuted Fréchet distribution (CTFD) to expand the work of cubic transmuted distribution families. CTFD increases the ability of the transmuted distributions to be flexible and facilitate the modelling of more complex data. Some statistical properties of CTFD are discussed. To estimate the parameters for CTFD model, the maximum likelihood estimation technique, Bootstrap (BtSp), the least-squares (LS) and the weighted least-squares (WLS) methods are considered. Finally, in order to appear the interest of the Cubic Transmuted Fréchet Distribution, two real datasets are used to examine the ability to apply it and to observe the performance of estimation techniques on CTFD, Fréchet and transmuted Fréchet distributions. The observed results showed that CTFD gives better fit than Fréchet and transmuted Fréchet distributions for the applied data sets.

Keywords: Cubic transmuted; Fréchet Distribution; Rényi Entropy; Shannon Entropy

INTRODUCTION

The Fréchet Distribution is named after French mathematician Maurice René Fréchet [1], who developed it in the 1920s as a maximum value distribution. Also known as inverse or the reciprocal of Weibull distribution. In 1927, he had identified before one possible limit distribution for the largest order statistic. The Fréchet distribution is applied to extreme events such as annually maximum one-day rainfalls and river discharges. Also, for analysis of several extreme events ranging from accelerated life testing to earthquakes, floods, sea currents, horse racing, human lifespans and wind speeds and a variety of engineering applications.

Based on Fréchet random variables, Nadarajah and Kotz [2] discussed the sociological models. Zaharim et al. [3] applied it to analyze the wind speed data. Mubarak [4] studied the Fréchet progressive type-II censored data with binomial removals. The Fréchet Distribution, also called the extreme value distribution (EVD) Type II, is used to model maximum values in a data set. It is one of four EVDs in common use. The other three are the Gumbel Distribution, the Weibull Distribution, and the Generalized Extreme Value Distribution. The generalized extreme value (GEV) distribution is a family of continuous probability distributions developed within extreme value theory to combine the Gumbel, Fréchet, and Weibull families also known as type I, II and III extreme value distributions. The lifetimes of the test items are assumed to follow a Fréchet distribution. The probability density function (PDF) and the cumulative distribution function (CDF) for Fréchet distribution are defined as follow;

$$g(x,\alpha,\theta)=\alpha\theta^{\alpha}x^{-\alpha-1}e^{-\left(\frac{x}{\theta}\right)^{-\alpha}}\quad x>0\ ,\alpha,\theta>0 \tag{1}$$

$$G(x,\theta)=exp\left\{-\left(\frac{x}{\theta}\right)^{-\alpha}\right\},\ \ x>0\ ,\alpha,\theta>0 \tag{2}$$

Some extensions of the Fréchet distribution have previously been proposed. The Beta Fréchet Distribution, Barreto-Souza et al. [5], Weibull Fréchet distribution, Afify et al. [6] and the Logistic Fréchet distribution, Tahir et al. [7] are remarkable examples. The Fréchet distribution has been studied and extended by several authors. A generalization of Fréchet distribution by Mahmoud and Mandouh [8] uses transmutation approach, developed by Shaw and Buckley [9], giving transmuted Fréchet distribution. And as application of Geetha and Poongothai [10] studied the growth hormone during acute sleep deprivation to by using the

Citation: Rania M. Shalabi, 2020. Cubic Transmuted Fréchet Distribution. Journal of Applied Sciences Research., 16(1): 9-19. DOI: 10.22587/jasr.2020.16.1.2

transmuted Fréchet distribution. The quadratic transmuted Fréchet distribution is obtained by using (1) in quadratic transmuted family of Mahmoud and Mandouh [8] given as

$$F(x) = (1 + \lambda)G(x) - \lambda[G(x)]^2, \qquad (3)$$

where $\lambda \in [-1, 1]$ is transmutation parameter. The quadratic transmuted family of distributions given in (3) extends any baseline distribution *G(x)* giving larger applicability. The quadratic transmuted distributions are, generally, not capable to handle the bi-modality of the data. Recently, Rahman et al. [11]. have proposed an extension of the quadratic transmuted distributions by adding one more parameter and have named the resulting family the cubic transmuted family of distributions. The cdf of cubic transmuted family of distributions is given as

$$F(x) = (1 + \lambda_1)G(x) + (\lambda_2 - \lambda_1)[G(x)]^2 - \lambda_2[G(x)]^3 \qquad (4)$$

where $\lambda_i \in [-1, 1], i = 1,2$ are the transmutation parameters and obey the condition $-2 \leq \lambda_1 + \lambda_2 \leq 1$.

In this paper, the cubic transmutation family of distributions of Rahman et al. [11] used to generalize the Fréchet distribution and have named the resulting distribution as cubic transmuted Fréchet distribution.

This paper is organized as follows, the cubic transmuted Fréchet distribution is defined in Section 1. In Section 2 statistical properties have been discussed; like moments, generating functions, quantiles, random number generation, reliability function and Shannon entropy; for the cubic transmuted Fréchet distribution along with the distribution of order statistics in Section 3. Section 4 provides parameter estimation of cubic transmuted Fréchet distribution. In Section 5, the simulation study and two real-life data sets have been applied. Finally, in Sect. 6, some conclusions are declared.

1. CUBIC TRANSMUTED FRÉCHET DISTRIBUTION

The cubic transmuted Fréchet distribution is useful to analyze complex data arising in the distribution of wealth. The cubic transmuted Fréchet distribution is defined as follows;

The CDF of a cubic transmuted Fréchet distribution is obtained by using (1) in (3)

$$F(x) = (1 + \lambda_1)e^{-(\frac{x}{\theta})^{-\alpha}} + (\lambda_2 - \lambda_1)e^{-2(\frac{x}{\theta})^{-\alpha}} - \lambda_2 e^{-3(\frac{x}{\theta})^{-\alpha}} \quad x > 0\ , \alpha, \theta > 0 \qquad (5)$$

$\lambda_i \in [-1, 1], i = 1,2$ and $-2 \leq \lambda_1 + \lambda_2 \leq 1$.

It is seen that the distribution function given in (5) reduces to the CDF of transmuted Fréchet distribution, given in (4), for $\lambda_2 = 0$. Also the cubic transmuted Fréchet distribution given in (5) reduces to the Fréchet distribution, given in (1), for $\lambda_1 = \lambda_2 = 0$ as it should be.

Lemma 1: Suppose a continuous random variable X follows a Fréchet distribution with parameters $\alpha, \theta \in R^+$, then the density function of the cubic transmuted Fréchet distribution is given by

$$f(x) = \alpha\theta^\alpha x^{-\alpha-1}e^{-(\frac{x}{\theta})^{-\alpha}}\left[(1 + \lambda_1) + 2(\lambda_2 - \lambda_1)e^{-2(\frac{x}{\theta})^{-\alpha}} - 3\lambda_2 e^{-3(\frac{x}{\theta})^{-\alpha}}\right], x > 0\ , \alpha, \theta > 0 \qquad (6)$$

Proof: The proof is simple and density function can be easily obtained by differentiating (5) with respect to *x*.

Figures 1 and 2, show the pdf and CDF of the cubic transmuted Fréchet distribution for different values of parameters.

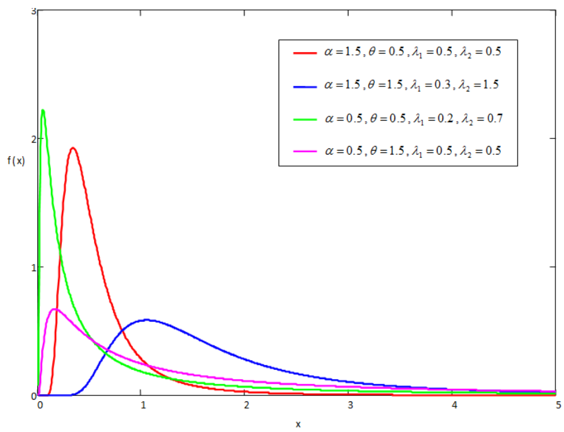


Fig. 1 The *pdf* of the Cubic Transmuted Fréchet distribution for different values of the parameters

Citation: Rania M. Shalabi, 2020. Cubic Transmuted Fréchet Distribution. Journal of Applied Sciences Research., 16(1): 9-19. DOI: 10.22587/jasr.2020.16.1.2

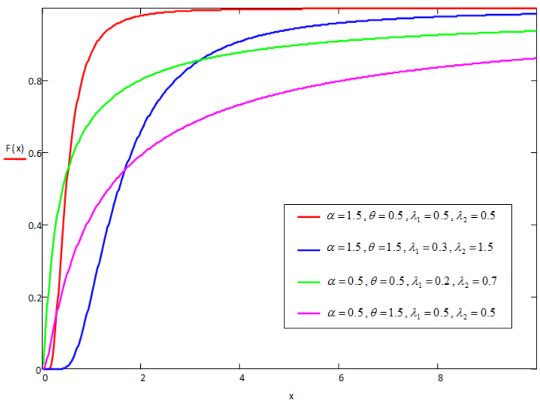


Fig. 2 The *cdf* of the Cubic Transmuted Fréchet distribution for different values of the parameters

2. STATISTICAL PROPERTIES

2.1 Quantile, Median and Mode

The quantile of the $CTFD(\theta,\alpha)$ is obtained by solving the following equation, with respect to x_q

$$p\left(X \le x_q\right) = q \ \ , 0 < q < 1 \tag{7}$$

The quantile of the cubic transmuted Fréchet distribution can be obtained as a nonnegative solution of the following nonlinear equation;

$$(1+\lambda_1)e^{-\left(\frac{x}{\theta}\right)^{-\alpha}} + (\lambda_2-\lambda_1)e^{-2\left(\frac{x}{\theta}\right)^{-\alpha}} - \lambda_2 e^{-3\left(\frac{x}{\theta}\right)^{-\alpha}} = q. \tag{8}$$

The median of the cubic transmuted Fréchet distribution can be obtained from equation (8). Also, the mode of the CTFD distribution can be obtained by deriving its PDF given in (6) with respect to x and equal it to zero. Thus the mode of the $CTFD(\theta,\alpha)$can be obtained as a nonnegative solution of the following nonlinear equation.

$$(\acute{f}_1 \times f_2 \times f_3) + (f_1 \times \grave{f}_2 \times f_3 + f_1 \times f_2 \times \grave{f}_3) = 0 \tag{9}$$

Where

$f_1 = \alpha\theta^{\alpha}x^{-\alpha-1}$, $f_2 = e^{-\left(\frac{x}{\theta}\right)^{-\alpha}}$, $f_3 = \left[(1+\lambda_1) + 2(\lambda_2-\lambda_1)e^{-2\left(\frac{x}{\theta}\right)^{-\alpha}} - 3\lambda_2 e^{-3\left(\frac{x}{\theta}\right)^{-\alpha}}\right]$,

$\acute{f}_1 = \frac{d}{dx}f_1 = 1\alpha(\alpha+1)\theta^{\alpha}x^{-\alpha-2}$,

$\acute{f}_2 = \frac{d}{dx}f_2 = \alpha\theta^{\alpha}x^{-\alpha-1}e^{-\left(\frac{x}{\theta}\right)^{-\alpha}}$ And

$\acute{f}_3 = \frac{d}{dx}f_3 = \alpha\theta^{\alpha}x^{-\alpha-1}\left[4(\lambda_2-\lambda_1)e^{-2\left(\frac{x}{\theta}\right)^{-\alpha}} - 9\lambda_2 e^{-3\left(\frac{x}{\theta}\right)^{-\alpha}}\right]$

It is not feasible to get an explicit solution of the equation (9) in the common case. In this case, numerical methods should be used.

2.2 The Moments

The r^th moments theorem of the $CTFD(\theta,\alpha)$ is ;

Theorem 2.1.: The r^th moments of a random variable $X \sim CTFD\ (\theta,\alpha)$, is given by

$$E(x^r) = \theta^r\left[1+\lambda_1+2^{\frac{r}{\alpha}}(\lambda_2-\lambda_1)+3^{\frac{r}{\alpha}}\lambda_2\right]\Gamma\left(1-\frac{r}{\alpha}\right) \tag{10}$$

Proof. The r^th moment of the positive random variable X with probability density function $f(x)$ is given by

$$E(x^r) = \int_0^{\infty} x^r f(x)\,dx. \tag{11}$$

Substituting from (6) into (11),

$$E(x^r) = \int_0^{\infty} \alpha\theta^{\alpha}x^{r-\alpha-1}e^{-\left(\frac{x}{\theta}\right)^{-\alpha}}\left[(1+\lambda_1)+2(\lambda_2-\lambda_1)e^{-2\left(\frac{x}{\theta}\right)^{-\alpha}} - 3\lambda_2 e^{-3\left(\frac{x}{\theta}\right)^{-\alpha}}\right]dx \tag{12}$$

Using the transformation $y = \left(\frac{x}{\theta}\right)^{-\alpha}$ $and\ r < \alpha$

Then $dy = \frac{-\alpha}{\theta}\left(\frac{x}{\theta}\right)^{-\alpha-1} dx$ and $0 < y < \infty$

With substitution by this transformation in (12) using Gamma integration;

$$E(x^r) = \theta^r \left[1 + \lambda_1 + 2^{\frac{r}{\alpha}}(\lambda_2 - \lambda_1) + 3^{\frac{r}{\alpha}}\lambda_2\right] \Gamma\left(1 - \frac{r}{\alpha}\right)$$

The mean and variance of the distribution are stated as

$$\mu = E(x) = \theta\left[1 + \lambda_1 + 2^{\frac{1}{\alpha}}(\lambda_2 - \lambda_1) + 3^{\frac{1}{\alpha}}\lambda_2\right] \Gamma\left(1 - \frac{1}{\alpha}\right) \ ,\alpha > 1$$

$$\sigma^2 = \left\{\theta^2\left[1 + \lambda_1 + 2^{\frac{2}{\alpha}}(\lambda_2 - \lambda_1) + 3^{\frac{2}{\alpha}}\lambda_2\right] \Gamma\left(1 - \frac{2}{\alpha}\right) - \left[\theta\left[1 + \lambda_1 + 2^{\frac{1}{\alpha}}(\lambda_2 - \lambda_1) + 3^{\frac{1}{\alpha}}\lambda_2\right] \Gamma\left(1 - \frac{1}{\alpha}\right)\right]^2\right\} \ ,\alpha > 2$$

2.3 Moment Generating Function

Theorem 2.2.: The moment generating function $M_x(t)$ of a random variable $X \sim CTFD\ (\theta, \alpha)$, is given by

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\theta^r \left[1 + \lambda_1 + 2^{\frac{r}{\alpha}}(\lambda_2 - \lambda_1) + 3^{\frac{r}{\alpha}}\lambda_2\right] \Gamma\left(1 - \frac{r}{\alpha}\right) \tag{13}$$

Proof.: The moment generating function of the positive random variable X with probability density function $f(x)$ is given by

$$M_x(t) = \int_0^{\infty} e^{tx} f(x)\, dx.$$

Using series expansion of e^{tx},

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\int_0^{\infty} x^r f(x)\, dx = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(x^r) \tag{14}$$

Substituting from (13) into (14),

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\theta^r \left[1 + \lambda_1 + 2^{\frac{r}{\alpha}}(\lambda_2 - \lambda_1) + 3^{\frac{r}{\alpha}}\lambda_2\right] \Gamma\left(1 - \frac{r}{\alpha}\right)$$

2.4 Characteristic Function

The characteristic function theorem of the cubic transmuted Fréchet distribution is stated as follow;

Theorem 2.3: Suppose that the random variable X have the $CTFD\ (\theta, \alpha)$, then characteristic function, $\Phi_x(t)$, is

$$\Phi_x(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!}\theta^r \left[1 + \lambda_1 + 2^{\frac{r}{\alpha}}(\lambda_2 - \lambda_1) + 3^{\frac{r}{\alpha}}\lambda_2\right] \Gamma\left(1 - \frac{r}{\alpha}\right) \tag{15}$$

Where $i = \sqrt{-1}$ and $t \in R$.

2.5 Reliability Analysis

The reliability function is defined as $R(t) = 1 - F(t)$ and for the cubic transmuted Fréchet distribution, it is given that,

$$R(t) = 1 - (1 + \lambda_1)e^{-(\frac{t}{\theta})^{-\alpha}} - (\lambda_2 - \lambda_1)e^{-2(\frac{t}{\theta})^{-\alpha}} + \lambda_2 e^{-3(\frac{t}{\theta})^{-\alpha}} \quad t > 0 \ ,\alpha, \theta > 0 \tag{16}$$

Citation: Rania M. Shalabi, 2020. Cubic Transmuted Fréchet Distribution. Journal of Applied Sciences Research., 16(1): 9-19. DOI: 10.22587/jasr.2020.16.1.2

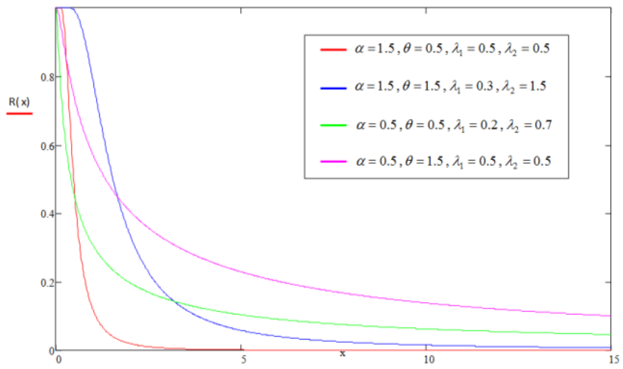


Fig. 3 Reliability function of the Cubic Transmuted Fréchet distribution for different values of the parameters

The hazard function is stated by

$$h(t) = \frac{\alpha\theta^{\alpha}t^{-\alpha-1}e^{-(\frac{t}{\theta})^{-\alpha}}\left[(1+\lambda_1)+2(\lambda_2-\lambda_1)e^{-2(\frac{t}{\theta})^{-\alpha}}-3\lambda_2 e^{-3(\frac{t}{\theta})^{-\alpha}}\right]}{1-(1+\lambda_1)e^{-(\frac{t}{\theta})^{-\alpha}}-(\lambda_2-\lambda_1)e^{-2(\frac{t}{\theta})^{-\alpha}}+\lambda_2 e^{-3(\frac{t}{\theta})^{-\alpha}}} \quad t>0,\alpha,\theta>0 \qquad (17)$$

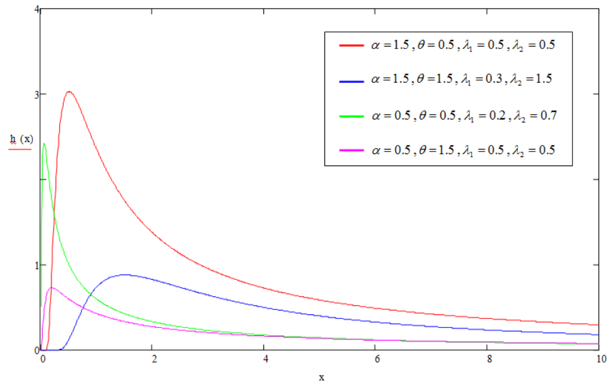


Fig. 4 The Hazard function of the Cubic Transmuted Fréchet distribution for different values of the parameters

2.6 Rényi Entropy

Entropy is used to measure the variation of the uncertainty of the random variable X. If X has the probability distribution function $f(\cdot)$ Rényi entropy [12] defined by

$$H_\delta(x) = \frac{1}{1-\delta}\int_0^\infty [\,f(x)]^\delta\,dx. \qquad (18)$$

Theorem 2.4.The Rényi entropy of a random variable $X \sim CTFD\ (\theta,\alpha)$, is given by

$$H_\delta(x) = \frac{1}{1-\delta}\alpha^\delta\theta^{\delta(\alpha-1)+1}\left\{\sum_{i=0}^{\delta}\sum_{j=0}^{i}\binom{\delta}{i}\binom{i}{j}(1+\lambda_1)^{\delta-i}\big(2(\lambda_2-\lambda_1)\big)^j(-3\lambda_2)^{i-j}\right.$$
$$\left.(\delta+2i-j)^{\frac{\alpha+1}{\alpha}(1-\delta)-1}\Gamma\left(1-\frac{\alpha+1}{\alpha}(1-\delta)\right)\right\} \qquad (19)$$

Proof: Suppose X has the pdf in (6). Then, one can calculate

$$\int_0^\infty [\,f(x)]^\delta\,dx = \int_0^\infty\left[\alpha\theta^\alpha x^{-\alpha-1}e^{-(\frac{x}{\theta})^{-\alpha}}\left[(1+\lambda_1)+2(\lambda_2-\lambda_1)e^{-2(\frac{x}{\theta})^{-\alpha}}-3\lambda_2 e^{-3(\frac{x}{\theta})^{-\alpha}}\right]\right]^\delta dx$$
$$= \int_0^\infty \alpha^\delta\theta^{\delta\alpha}x^{-\delta(\alpha+1)}e^{-\delta(\frac{x}{\theta})^{-\alpha}}\ \sum_{i=0}^{\delta}\binom{\delta}{i}(1+\lambda_1)^{\delta-i}\left(2(\lambda_2-\lambda_1)e^{-2(\frac{x}{\theta})^{-\alpha}}-3\lambda_2 e^{-3(\frac{x}{\theta})^{-\alpha}}\right)^i\ dx$$
$$= \alpha^\delta\theta^{\delta\alpha}\left\{\sum_{i=0}^{\delta}\sum_{j=0}^{i}\binom{\delta}{i}\binom{i}{j}(1+\lambda_1)^{\delta-i}\big(2(\lambda_2-\lambda_1)\big)^j(-3\lambda_2)^{i-j}\ \int_0^\infty x^{-\delta(\alpha+1)}e^{-(\delta+2i-j)(\frac{x}{\theta})^{-\alpha}}\ dx\right\} \quad (20)$$

Citation: Rania M. Shalabi, 2020. Cubic Transmuted Fréchet Distribution. Journal of Applied Sciences Research., 16(1): 9-19. DOI: 10.22587/jasr.2020.16.1.2

Using the transformation $y = (\delta + 2i - j)\left(\frac{x}{\theta}\right)^{-\alpha}$ $and\ \delta \geq 1$

Then $dy = \frac{-\alpha}{\theta}(\delta + 2i - j)\left(\frac{x}{\theta}\right)^{-\alpha-1}\ dx$ and $0 < y < \infty$

With substitution with this transformation in (20) using Gamma integration;

$$\int_0^\infty [f(x)]^\delta\, dx = \alpha^\delta \theta^{\delta(\alpha-1)+1}\left\{\sum_{i=0}^{\delta}\sum_{j=0}^{i}\binom{\delta}{i}\binom{i}{j}(1+\lambda_1)^{\delta-i}\big(2(\lambda_2-\lambda_1)\big)^j(-3\lambda_2)^{i-j}\right.$$
$$\left.(\delta+2i-j)^{\frac{\alpha+1}{\alpha}(1-\delta)-1}\Gamma\left(1-\frac{\alpha+1}{\alpha}(1-\delta)\right)\right\} \tag{21}$$

The Rényi entropy of the $CTFD\ (\theta, \alpha)$distribution can be obtained as

$$H_\delta(x) = \frac{1}{1-\delta}\alpha^\delta\theta^{\delta(\alpha-1)+1}\left\{\sum_{i=0}^{\delta}\sum_{j=0}^{i}\binom{\delta}{i}\binom{i}{j}(1+\lambda_1)^{\delta-i}\big(2(\lambda_2-\lambda_1)\big)^j(-3\lambda_2)^{i-j}\right.$$
$$\left.(\delta+2i-j)^{\frac{\alpha+1}{\alpha}(1-\delta)-1}\Gamma\left(1-\frac{\alpha+1}{\alpha}(1-\delta)\right)\right\}$$

2.7. Shannon Entropy

The Shannon entropy [13] of a random variable $X \sim CTFD\ (\theta, \alpha)$, is given by

$$\begin{aligned}H_\delta(x) &= -E(\log[f(x)])\\ &= \left[1+\frac{1}{2}\lambda_2+\frac{5}{6}\lambda_1\right] - (\log(\alpha\,\theta^\alpha)\\ &\quad -\int_0^\infty \left\{\alpha\theta^\alpha x^{-\alpha-1}e^{-\left(\frac{x}{\theta}\right)^{-\alpha}}\left[(1+\lambda_1)+2(\lambda_2-\lambda_1)e^{-2\left(\frac{x}{\theta}\right)^{-\alpha}}-3\lambda_2 e^{-3\left(\frac{x}{\theta}\right)^{-\alpha}}\right]\right.\\ &\quad\left.\times\ log\left(x^{-\alpha-1}\left[(1+\lambda_1)+2(\lambda_2-\lambda_1)e^{-2\left(\frac{x}{\theta}\right)^{-\alpha}}-3\lambda_2 e^{-3\left(\frac{x}{\theta}\right)^{-\alpha}}\right]\right)\right\} dx\end{aligned} \tag{22}$$

and the integral in (22) can be obtained numerically.

3. ORDER STATISTICS

In statistics, the k^{th} order statistic of a statistical sample is equal to its k^{th} smallest value. Together with rank statistics, order statistics are among the most fundamental tools in non-parametric statistics and inference. For a sample of size n, the n^{th} order statistic (or largest order statistic) is the maximum, that is, $X_{(n)} = max\,(X_1, X_2, \dots, X_n)$, and the smallest order statistic) is the minimum, that is, $X_{(1)} = max\,(X_1, X_2, \dots, X_n)$.

$The\ Range\ (X_1, X_2, \dots, X_n) = X_{(n)} - X_{(1)}$

If $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ denotes the order statistics of a random sample $X_1, X_2, \dots, X_n$ from a continuous population with cdf $F_X(x)$ and PDF $f_X(x)$. Then the PDF of $X_{(k)}$ is given by

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!\,(n-k)!}f_X(x)[F_X(x)]^{k-1}[1-F_X(x)]^{n-k} \tag{23}$$

Since $0 < [1-F_X(x)]^{n-k} < 1$, then,

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!\,(n-k)!}f(x)\sum_{i=0}^{\infty}\binom{n-k}{i}(-1)^i\,[f_X(x)]^{n-k-i} \tag{24}$$

for $k = 1, 2, \dots, n$. The pdf of the k^{th} order statistic for $CTFD\ (\theta, \alpha)$ is given by

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!\,(n-k)!}\left(\alpha\theta^\alpha x^{-\alpha-1}e^{-\left(\frac{x}{\theta}\right)^{-\alpha}}\left[(1+\lambda_1)+2(\lambda_2-\lambda_1)e^{-2\left(\frac{x}{\theta}\right)^{-\alpha}}-3\lambda_2 e^{-3\left(\frac{x}{\theta}\right)^{-\alpha}}\right]\right)$$
$$\times\sum_{i=0}^{\infty}\binom{n-k}{i}(-1)^i\left[(1+\lambda_1)e^{-\left(\frac{x}{\theta}\right)^{-\alpha}}+(\lambda_2-\lambda_1)e^{-2\left(\frac{x}{\theta}\right)^{-\alpha}}-\lambda_2 e^{-3\left(\frac{x}{\theta}\right)^{-\alpha}}\right]^{n-k-i} \tag{25}$$

Therefore, the PDF of the largest order statistic $X_{(n)}$is given by

$$f_{X_{(n)}}(x) = n\alpha\theta^\alpha x^{-\alpha-1}e^{-\left(\frac{x}{\theta}\right)^{-\alpha}}\left[(1+\lambda_1)+2(\lambda_2-\lambda_1)e^{-2\left(\frac{x}{\theta}\right)^{-\alpha}}-3\lambda_2 e^{-3\left(\frac{x}{\theta}\right)^{-\alpha}}\right]$$

Citation: Rania M. Shalabi, 2020. Cubic Transmuted Fréchet Distribution. Journal of Applied Sciences Research., 16(1): 9-19. DOI: 10.22587/jasr.2020.16.1.2

$$\times\left[(1+\lambda_1)e^{-(\frac{x}{\theta})^{-\alpha}}+(\lambda_2-\lambda_1)e^{-2(\frac{x}{\theta})^{-\alpha}}-\lambda_2 e^{-3(\frac{x}{\theta})^{-\alpha}}\right]^{n-1} \quad (26)$$

and the PDF of the smallest order statistic $X_{(1)}$ is given by

$$f_{X_{(1)}}(x)=n\alpha\theta^{\alpha}x^{-\alpha-1}e^{-\left(\frac{x}{\theta}\right)^{-\alpha}}\left[(1+\lambda_1)+2(\lambda_2-\lambda_1)e^{-2\left(\frac{x}{\theta}\right)^{-\alpha}}-3\lambda_2 e^{-3\left(\frac{x}{\theta}\right)^{-\alpha}}\right]$$
$$\times\left[1-(1+\lambda_1)e^{-(\frac{x}{\theta})^{-\alpha}}-(\lambda_2-\lambda_1)e^{-2(\frac{x}{\theta})^{-\alpha}}+\lambda_2 e^{-3(\frac{x}{\theta})^{-\alpha}}\right]^{n-1} \quad (27)$$

Note that $\lambda_2=\lambda_1=0$, the pdf of the rth order statistic for $FD\ (\theta,\alpha)$, as follows

$$f_{X_{(k)}}(x)=\frac{n!}{(k-1)!\,(n-k)!}\left(\alpha\theta^{\alpha}x^{-\alpha-1}e^{-(\frac{x}{\theta})^{-\alpha}}\right)\ \sum_{i=0}^{\infty}\binom{n-k}{i}(-1)^i\left[e^{-(\frac{x}{\theta})^{-\alpha}}\ \right]^{n-k-i} \quad (28)$$

The rth order moment of $X_{(k)}$ for $CTFD\ (\theta,\alpha)$is obtained by using

$$E(x_{(k)}^{r})=\int_0^{\infty}x_{(k)}^{r}\,f_{X_{(k)}}(x)\,dx. \quad (29)$$

where $f_{X_{(k)}}(x)$ is presented in (23).

4. DIFFERENT ESTIMATION METHODS

4.1 Maximum Likelihood Estimation (MLE)

Assume $x_1,\dots,x_n$ be a random sample of size n from $CTFD\,(\theta,\alpha)$ then the likelihood function can be written as

$$L(\theta,\alpha,\lambda_1,\lambda_2)=\prod_{i=1}^{n}\left[\alpha\theta^{\alpha}x^{-\alpha-1}e^{-(\frac{x}{\theta})^{-\alpha}}\left\{(1+\lambda_1)+2(\lambda_2-\lambda_1)e^{-2(\frac{x}{\theta})^{-\alpha}}-3\lambda_2 e^{-3(\frac{x}{\theta})^{-\alpha}}\right\}\right] \quad (30)$$

By accumulation taking logarithm of equation (30) , and the log- likelihood function can be written as

$$l(\theta,\alpha,\lambda_1,\lambda_2)=nln(\alpha)+n\alpha\ln(\theta)-(\alpha+1)\sum_{i=1}^{n}\ln(x_i)-\sum_{i=1}^{n}(\frac{x_i}{\theta})^{-\alpha}+\sum_{i=1}^{n}ln(\dot{x}_\iota) \quad (31)$$

Where $\dot{x}_\iota=1+\lambda_1+2(\lambda_2-\lambda_1)e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}}-3\lambda_2 e^{-2\left(\frac{x_i}{\theta}\right)^{-\alpha}}$

Maximizing $l(\theta,\alpha,\lambda_1,\lambda_2)$with respect to $\theta,\alpha,\lambda_1$and λ_2 respectively , obtain the following system with non-linear equations

$$\frac{\partial l}{\partial\theta}=\frac{n\alpha}{\theta}-\alpha\,\theta^{\alpha-1}\sum_{i=1}^{n}(x_i)^{-\alpha}\ +\sum_{i=1}^{n}\left\{\frac{\theta^{\alpha-1}x_i{}^{-\alpha}e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}}(2\lambda_1-2\lambda_2+6\lambda_2 e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}})}{1+\lambda_1+2(\lambda_2-\lambda_1)e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}}-3\lambda_2 e^{-2\left(\frac{x_i}{\theta}\right)^{-\alpha}}}\right\}=0 \quad (32)$$

$$\frac{dl}{d\alpha}=\frac{n}{\alpha}+n\ln(\theta)-\sum_{i=1}^{n}\ln(x_i)-\sum_{i=1}^{n}\left\{\left(\frac{x_i}{\theta}\right)^{-\alpha}\ln(\frac{x_i}{\theta})\right\}$$
$$+\sum_{i=1}^{n}\left\{\frac{\left(\frac{x_i}{\theta}\right)^{-\alpha}ln\left(\frac{x_i}{\theta}\right)e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}}(6\lambda_2 e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}}-2\lambda_2+2\lambda_1)}{1+\lambda_1+2(\lambda_2-\lambda_1)e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}}-3\lambda_2 e^{-2\left(\frac{x_i}{\theta}\right)^{-\alpha}}}\right\}=0 \quad (33)$$

$$\frac{dl}{d\lambda_1}=\sum_{i=1}^{n}\left\{\frac{1-2e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}}}{1+\lambda_1+2(\lambda_2-\lambda_1)e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}}-3\lambda_2 e^{-2\left(\frac{x_i}{\theta}\right)^{-\alpha}}}\right\}=0 \quad (34)$$

$$\frac{dl}{d\lambda_2}=\sum_{i=1}^{n}\left\{\frac{2e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}}-3e^{-2\left(\frac{x_i}{\theta}\right)^{-\alpha}}}{1+\lambda_1+2(\lambda_2-\lambda_1)e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}}-3\lambda_2 e^{-2\left(\frac{x_i}{\theta}\right)^{-\alpha}}}\right\}=0 \quad (35)$$

To get the estimates of $\hat{\theta}_{MLE},\hat{\alpha}_{MLE},\hat{\lambda}_{1_{MLE}}$ and $\hat{\lambda}_{2_{MLE}}$, the solution of this system should be obtained numerically.

4.2 Bootstrap (BtSp) Method

Let $\hat{\mathrm{F}}$ be the empirical distribution, putting $\frac{1}{\mathrm{n}}$ on each of the observed values $\mathrm{x_i}$, $\mathrm{i}=1{,}2,\dots,\mathrm{n}$. A BtSp sample is defined to be a random sample of size n drawn with replacement from $\hat{\mathrm{F}}$, say $X^*=x_1^*,x_2^*,\dots,x_n^*$ and

$$\hat{\mathrm{F}}\sim(x_1^*,x_2^*,\dots,x_n^*) \quad (36)$$

Citation: Rania M. Shalabi, 2020. Cubic Transmuted Fréchet Distribution. Journal of Applied Sciences Research., 16(1): 9-19. DOI: 10.22587/jasr.2020.16.1.2

the star indicates that X^* is not the actual data set X, but rather a randomized, or resampled , version of X. The corresponding to a BtSp data set X^* is a BtSp replication of $\hat{\mu} = s(X)$, say

$$\widehat{\mu^*} = s(X^*) \tag{37}$$

The quantity $s(X^*)$ is the result of applying the same function s(X) toX^* as was applied to X, and the BtSp estimate of $se_{\hat{F}}(\hat{\mu})$ is defined by $se_{\hat{F}}(\widehat{\mu^*})$ where $se_{\hat{F}}(\hat{\mu})$ is the standard error of $\hat{\mu}$ for data sets of size n randomly sampled from $\hat{F}$.

4.3 Least Squares (LS) Method

Let $x_1, \ldots, x_n$ be a random sample of size n from $CTFD\ (\theta, \alpha)$ in increasing order. By considering the associated order statistics $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$. The LS estimates can be obtained by minimizing the following expression

$$S(\theta,\alpha,\lambda_1,\lambda_2) = \sum_{i=1}^{n}\left\{F(x_i) - E(F(x_{(i)})\right\}^2$$

$$= \sum_{i=1}^{n}\left\{(1+\lambda_1)e^{-(\frac{x_i}{\theta})^{-\alpha}} + (\lambda_2-\lambda_1)e^{-2(\frac{x_i}{\theta})^{-\alpha}} - \lambda_2 e^{-3(\frac{x_i}{\theta})^{-\alpha}} - \frac{i}{n+1}\right\}^2 \tag{38}$$

Minimizing $S(\theta,\alpha,\lambda_1,\lambda_2)$ with respect to $\theta, \alpha, \lambda_1$and λ_2, obtain the following system with non-linear equations:

$$\frac{\partial S}{\partial \theta} = \sum_{i=1}^{n}\left\{(1+\lambda_1)e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}} + (\lambda_2-\lambda_1)e^{-2\left(\frac{x_i}{\theta}\right)^{-\alpha}} - \lambda_2 e^{-3\left(\frac{x_i}{\theta}\right)^{-\alpha}} - \frac{i}{n+1}\right\}$$
$$\times\left\{\left(\frac{x_i}{\theta}\right)^{-\alpha}\ln\left(\frac{x_i}{\theta}\right)e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}}\left[(1+\lambda_1) + 2(\lambda_2-\lambda_1)e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}} - 3\lambda_2 e^{-2(\frac{x_i}{\theta})^{-\alpha}}\right]\right\} = 0 \tag{39}$$

$$\frac{\partial S}{\partial \alpha} = \sum_{i=1}^{n}\left\{(1+\lambda_1)e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}} + (\lambda_2-\lambda_1)e^{-2\left(\frac{x_i}{\theta}\right)^{-\alpha}} - \lambda_2 e^{-3\left(\frac{x_i}{\theta}\right)^{-\alpha}} - \frac{i}{n+1}\right\}$$
$$\times\left\{\frac{\alpha}{\theta}\left(\frac{x_i}{\theta}\right)^{-\alpha}e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}}\left[3\lambda_2 e^{-2(\frac{x_i}{\theta})^{-\alpha}} - 2(\lambda_2-\lambda_1)e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}} - (1+\lambda_1)\right]\right\} = 0 \tag{40}$$

$$\frac{dS}{d\lambda_1} = \sum_{i=1}^{n}\left\{(1+\lambda_1)e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}} + (\lambda_2-\lambda_1)e^{-2\left(\frac{x_i}{\theta}\right)^{-\alpha}} - \lambda_2 e^{-3\left(\frac{x_i}{\theta}\right)^{-\alpha}} - \frac{i}{n+1}\right\} \times \left\{e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}} - e^{-2(\frac{x_i}{\theta})^{-\alpha}}\right\} = 0 \tag{41}$$

$$\frac{dS}{d\lambda_2} = \sum_{i=1}^{n}\left\{(1+\lambda_1)e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}} + (\lambda_2-\lambda_1)e^{-2\left(\frac{x_i}{\theta}\right)^{-\alpha}} - \lambda_2 e^{-3\left(\frac{x_i}{\theta}\right)^{-\alpha}} - \frac{i}{n+1}\right\} \times \left\{e^{-2\left(\frac{x_i}{\theta}\right)^{-\alpha}} - e^{-3(\frac{x_i}{\theta})^{-\alpha}}\right\} = 0 \tag{42}$$

To get the estimates of $\hat{\theta}_{LS}, \hat{\alpha}_{LS}, \hat{\lambda}_{1_{LS}}$ *and* $\hat{\lambda}_{2_{LS}}$, the solution of this system should be obtained numerically.

4.4 Weighted Least Squares (WLS) Method

Let $x_1, \ldots, x_n$ be a random sample of size n from $CTFD\ (\theta, \alpha, \lambda_1, \lambda_2)$ in increasing order. By considering the associated order statistics $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$. The WLS estimates can be obtained by minimizing the following expression

$$W(\theta,\alpha,\lambda_1,\lambda_2) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left\{F(x_i) - E(F(x_{(i)})\right\}^2$$

$$= \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left\{(1+\lambda_1)e^{-(\frac{x_i}{\theta})^{-\alpha}} + (\lambda_2-\lambda_1)e^{-2(\frac{x_i}{\theta})^{-\alpha}} - \lambda_2 e^{-3(\frac{x_i}{\theta})^{-\alpha}} - \frac{i}{n+1}\right\}^2 \tag{43}$$

Minimizing $S(\theta,\alpha,\lambda_1,\lambda_2)$ with respect to $\theta, \alpha, \lambda_1$and λ_2, have the following system with non-linear equations:

$$\frac{\partial S}{\partial \theta} = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left\{(1+\lambda_1)e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}} + (\lambda_2-\lambda_1)e^{-2\left(\frac{x_i}{\theta}\right)^{-\alpha}} - \lambda_2 e^{-3\left(\frac{x_i}{\theta}\right)^{-\alpha}} - \frac{i}{n+1}\right\}$$
$$\left\{\left(\frac{x_i}{\theta}\right)^{-\alpha}\ln\left(\frac{x_i}{\theta}\right)e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}}\left[(1+\lambda_1) + 2(\lambda_2-\lambda_1)e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}} - 3\lambda_2 e^{-2(\frac{x_i}{\theta})^{-\alpha}}\right]\right\} = 0 \tag{44}$$

$$\frac{\partial S}{\partial \alpha} = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left\{(1+\lambda_1)e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}} + (\lambda_2-\lambda_1)e^{-2\left(\frac{x_i}{\theta}\right)^{-\alpha}} - \lambda_2 e^{-3\left(\frac{x_i}{\theta}\right)^{-\alpha}} - \frac{i}{n+1}\right\}$$
$$\left\{\frac{\alpha}{\theta}\left(\frac{x_i}{\theta}\right)^{-\alpha}e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}}\left[3\lambda_2 e^{-2(\frac{x_i}{\theta})^{-\alpha}} - 2(\lambda_2-\lambda_1)e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}} - (1+\lambda_1)\right]\right\} = 0 \tag{45}$$

$$\frac{dS}{d\lambda_1} = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left\{(1+\lambda_1)e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}} + (\lambda_2-\lambda_1)e^{-2\left(\frac{x_i}{\theta}\right)^{-\alpha}} - \lambda_2 e^{-3\left(\frac{x_i}{\theta}\right)^{-\alpha}} - \frac{i}{n+1}\right\}$$
$$\left\{e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}} - e^{-2(\frac{x_i}{\theta})^{-\alpha}}\right\} = 0 \tag{46}$$

$$\frac{dS}{d\lambda_2}=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left\{(1+\lambda_1)e^{-\left(\frac{x_i}{\theta}\right)^{-\alpha}}+(\lambda_2-\lambda_1)e^{-2\left(\frac{x_i}{\theta}\right)^{-\alpha}}-\lambda_2 e^{-3\left(\frac{x_i}{\theta}\right)^{-\alpha}}-\frac{i}{n+1}\right\}$$
$$\left\{e^{-2\left(\frac{x_i}{\theta}\right)^{-\alpha}}-e^{-3\left(\frac{x_i}{\theta}\right)^{-\alpha}}\right\}=0 \quad (47)$$

To get the estimates of $\hat{\theta}_{WLS}, \hat{\alpha}_{WLS}, \hat{\lambda}_{1_{WLS}}$ *and* $\hat{\lambda}_{2_{WLS}}$, the solution of this system should be obtained numerically.

APPLICATIONS

In this section, the Cubic Transmuted Fréchet Distribution (CTFD) is applied on two data sets as follows;
Data Set 1; Carbon Fibers Data: An uncensored data set consisting of 100 observations on breaking stress of carbon fibers (in Gba):
0.92, 0.928, 0.997, 0.9971, 1.061, 1.117, 1.162, 1.183, 1.187, 1.192, 1.196, 1.213, 1.215, 1.2199, 1.22, 1.224, 1.225, 1.228, 1.237, 1.24, 1.244, 1.259, 1.261, 1.263, 1.276, 1.31, 1.321, 1.329, 1.331, 1.337, 1.351, 1.359, 1.388, 1.408, 1.449, 1.4497, 1.45, 1.459, 1.471, 1.475, 1.477, 1.48, 1.489, 1.501, 1.507, 1.515, 1.53, 1.5304, 1.533, 1.544, 1.5443, 1.552, 1.556, 1.562, 1.566, 1.585, 1.586, 1.599, 1.602, 1.614, 1.616, 1.617, 1.628, 1.684, 1.711, 1.718, 1.733, 1.738, 1.743, 1.759, 1.777, 1.794, 1.799, 1.806, 1.814, 1.816, 1.828, 1.83, 1.884, 1.892, 1.944, 1.972, 1.984, 1.987, 2.02, 2.0304, 2.029, 2.035, 2.037, 2.043, 2.046, 2.059, 2.111, 2.165, 2.686, 2.778, 2.972, 3.504, 3.863, 5.306.
A data set from a study on breaking stress of carbon fibers (in Gba) is considered, which has been studied by Nichols and Padgett [14]. The lifetimes are times until the break of the fibers.

Data Set 2: This data set is generated data to simulate the strengths of glass fibers. The data set is: 1.014, 1.081, 1.082, 1.185, 1.223, 1.248, 1.267, 1.271, 1.272, 1.275, 1.276, 1.278, 1.286, 1.288, 1.292, 1.304, 1.306, 1.355, 1.361, 1.364, 1.379, 1.409, 1.426, 1.459, 1.46, 1.476, 1.481, 1.484, 1.501, 1.506, 1.524, 1.526, 1.535, 1.541, 1.568, 1.579, 1.581, 1.591, 1.593, 1.602, 1.666, 1.67, 1.684, 1.691, 1.704, 1.731, 1.735, 1.747, 1.748, 1.757, 1.800, 1.806, 1.867, 1.876, 1.878, 1.91, 1.916, 1.972, 2.012, 2.456, 2.592, 3.197, 4.121.
This data set 2 has presented by Mahmoud and Mandouh [8].

For the sake of comparison and using these data sets, three alternative distributions: CTFD model, the Fréchet distribution (FD) and transmuted Fréchet distribution (TFD) have been compared. The estimated values of the model parameters along with corresponding standard errors are presented in Tables 1 and 4 for selected models using the MLE method.

In Tables 2 and 5, the goodness of fit of the Cubic Transmuted Fréchet Distribution CTFD model , the Fréchet distribution (FD) and Transmuted Fréchet Distribution (TFD) has been introduced using four different comparison measures used as selection criteria and they are: $-2\times$ log likelihood (-2log), Akaike's information criterion (AIC), corrected Akaike's information criterion (AICC), and Bayesian information criterion (BIC). The calculated values of these statistics (the smaller the better) all reveal that the Cubic Transmuted Fréchet Distribution CTFD is the most appropriate model according to four different criteria. It must to be noted that, in these criteria, the number of estimated parameters has a huge impact.

In Tables 3 and 6, Bootstrap, the least-squares and the weighted least-squares methods are used to estimate the parameters for the Cubic Transmuted Fréchet Distribution CTFD model.

Figures 5, 6, 7 and 8 show the estimated *pdf* and *cdf* for each method of estimation mentioned in the research to estimate the parameters of CTFD distribution, MLS, the method of Bootstrap (**BtSp**), the least-squares (LS) and the wieghted least-squares (WLS), by drawing the distribution of each method with the relative frequency distribution for each data set. These Figures showed the goodness-of-fit is better with both the estimation method using MLE and BS rather than the method of estimation using the LS and WLS.

Table 1 *MLE's* of the parameters and respective *SE's* for various distributions for Data Set 1

Distribution	Parameter	Estimate	SE
CTFD	α	2.512	0.175
	θ	0.911	0.047
	λ_1	0.326	0.013
	λ_2	0.611	0.016
TFD	α	2.911	0.183
	θ	0.761	0.053
	λ	0.26	0.042
FD	α	3.071	0.197
	θ	0.746	0.104

Table 2 Goodness-of fit statistics using the selection criteria values for Data Set 1

Distribution	-2log	AIC	AICC	BIC
CTFD	244.089	252.089	3.075	262.51
TFD	295.646	303.647	3.617	314.067
FD	304.017	312.017	3.705	322.438

Table 3 BtSp, LS and WLS Estimates for Data Set 1

Citation: Rania M. Shalabi, 2020. Cubic Transmuted Fréchet Distribution. Journal of Applied Sciences Research., 16(1): 9-19. DOI: 10.22587/jasr.2020.16.1.2

Distribution	Parameter	BtSp	LS	WLS
CTFD	α	3.437	3.334	3.681
	θ	1.575	2.267	2.263
	λ_1	0.752	0.162	-0.166
	λ_2	0.08	0.012	-1

Table 4 *MLE*'s of the parameters and respective *SE*'s for various distributions for Data Set 2

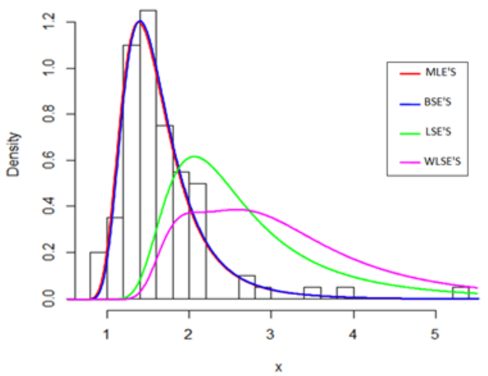
Distribution	Parameter	Estimate	SE
CTFD	α	5.184	0.385
	θ	2.654	0.111
	λ_1	-0.489	0.654
	λ_2	0.555	0.278
TFD	α	4.933	0.361
	θ	2.836	0.111
	λ	0.344	0.364
FD	α	5.234	0.826
	θ	2.995	0.141

Table 5 Goodness-of fit statistics using the selection criteria values for Data Set 2

Distribution	$-2\log$	AIC	AICC	BIC
CTFD	117.245	125.245	125.935	133.817
TFD	118.435	124.435	124.845	130.864
FD	135.466	139.466	139.666	143.752

Table 6 BtSp, LS and WLS Estimates for Data Set 2

Distribution	Parameter	BtSp	LS	WLS
CTFD	α	5.125	2.919	3.995
	θ	2.756	3.647	3.03
	λ_1	-0.185	0.208	0.491
	λ_2	0.586	0.199	0.509



Fi g. 5 Histogram for *pdf* of the fitted models for data set 1

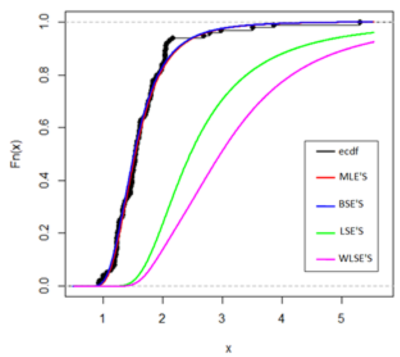


Fig. 6 Empirical *cdf* of the fitted models for data set 1

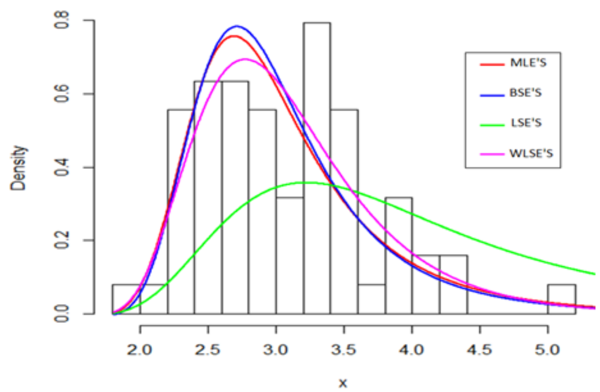


Fig. 7 Histogram for *pdf* of the fitted models for data set 2

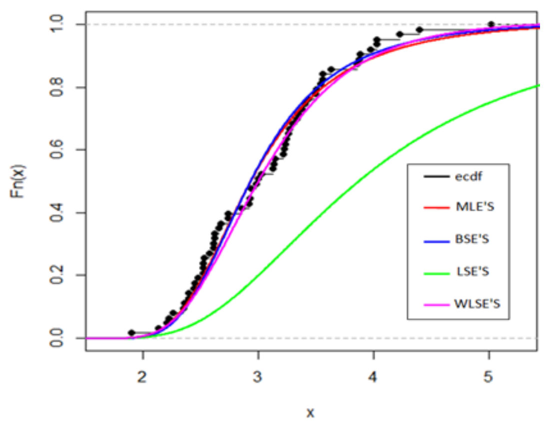


Fig. 8 Empirical *cdf* of the fitted models for data set 2

5. CONCLUSION

In this paper, the Cubic Transmuted Fréchet Distribution has been introduced. This distribution is high elastic to treat the complex data. Some statistical properties of the Cubic Transmuted Fréchet Distribution (CTFD) are discussed including moments, moment generating function, characteristic function, quantile function, reliability function and Shannon entropy. In order to estimate the model parameters, the maximum likelihood estimation technique, Bootstrap (BtSp), the least-squares (LS) and the weighted least-squares (WLS) methods are considered to estimate the parameters for CTFD model. It showed the goodness-of-fit is better with both the estimation method using MLE and BtSp rather than the method of estimation using the LS and WLS.
The Cubic Transmuted Fréchet Distribution has been applied on two real applications and the obtained results showed that the Cubic Transmuted Fréchet distribution offers better appropriate than Fréchet and transmuted Fréchet distributions for the applied data sets.

CONFLICT OF INTEREST

The author declares no conflicts of interest.

ACKNOWLEDGMENT

The author thanks the anonymous referees for their constructive criticism and valuable suggestions.

REFERENCES

[1] M. Fréchet, “Sur la loi de probabilité de l’écart maximum,” in *Annales de la societe Polonaise de Mathematique*, 1928.
[2] S. Nadarajah and S. Kotz, “Sociological models based on Fréchet random variables,” *Qual. Quant.*, vol. 42, no. 1, pp. 89–95, 2008.
[3] A. Zaharim, A. M. Razali, R. Z. Abidin, and K. Sopian, “Fitting of statistical distributions to wind speed data in Malaysia,” *Eur. J. Sci. Res.*, vol. 26, no. 1, pp. 6–12, 2009.
[4] M. Mubarak, “Parameter estimation based on the Frechet progressive type II censored data with binomial removals,” *Int. J. Qual. Stat. Reliab.*, vol. 2012, 2011.
[5] W. Barreto-Souza, G. M. Cordeiro, and A. B. Simas, “Some results for beta Fréchet distribution,” *Commun. Stat. Methods*, vol. 40, no. 5, pp. 798–811, 2011.
[6] A. Z. Afify, H. M. Yousof, G. M. Cordeiro, E. M. M. Ortega, and Z. M. Nofal, “The Weibull Fréchet distribution and its applications,” *J. Appl. Stat.*, vol. 43, no. 14, pp. 2608–2626, 2016.
[7] M. H. Tahir, G. M. Cordeiro, A. Alzaatreh, M. Mansoor, and M. Zubair, “The logistic-X family of distributions and its applications,” *Commun. Stat. Methods*, vol. 45, no. 24, pp. 7326–7349, 2016.
[8] M. R. Mahmoud and R. M. Mandouh, “On the transmuted Fréchet distribution,” *J. Appl. Sci. Res.*, vol. 9, no. 10, pp. 5553–5561, 2013.
[9] W. T. Shaw and I. R. Buckley, “The alchemy of probability distributions: Beyond gram-charlier & cornish-fisher expansions, and skew-normal or kurtotic-normal distributions,” *Submitt. Feb*, vol. 7, p. 64, 2007.
[10] T. Geetha and T. Poongothai, “Transmuted Fréchet Distribution for growth hormone during acute sleep deprivation,” *J. Eng. Res. Appl.*, vol. 6, no. 11, pp. 16–18, 2016.
[11] M. M. Rahman, B. Al-Zahrani, and M. Q. Shahbaz, “A general transmuted family of distributions,” *Pakistan J. Stat. Oper. Res.*, vol. 14, no. 2, pp. 451–469, 2018.
[12] A. Rényi, “On measures of entropy and information,” in *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics*, 1961.
[13] C. E. Shannon, “A mathematical theory of communication,” *Bell Syst. Tech. J.*, vol. 27, no. 3, pp. 379–423, 1948.
[14] M. D. Nichols and W. J. Padgett, “A bootstrap control chart for Weibull percentiles,” *Qual. Reliab. Eng. Int.*, vol. 22, no. 2, pp. 141–151, 2006.