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# Generalized fractal Jensen and Jensen–Mercer inequalities for harmonic convex function with applications

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## Abstract

In this paper, we present a generalized Jensen-type inequality for generalized harmonically convex function on the fractal sets, and a generalized Jensen–Mercer inequality involving local fractional integrals is obtained. Moreover, we establish some generalized Jensen–Mercer-type local fractional integral inequalities for harmonically convex function. Also, we obtain some generalized related results using these inequalities on the fractal space. Finally, we give applications of generalized means and probability density function.

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## 1 Introduction

In different fields of pure and applied mathematics, the convexity of functions has been used. Several new classes of convex functions and convex sets have been introduced and studied. Many researchers have derived the variety of new inequalities associated with these new classes of convex functions [1–3]. The harmonic set was introduced by Shi et al. [4]. It must be noted that the weighted harmonic mean is used to give a concept of the harmonic set. It has applications in the theory of electrical circuits and other fields of sciences. Harmonic convex functions are defined by using the weighted harmonic means, which have appeared as a significant and major generalization of convex functions. Several properties of harmonic convex functions have been investigated by İşcan [5], Dragomir [6], and Farid et al. in [7].

The definition of convex function is as follows.

**Definition 1** A function  $\Psi : I \subset \mathbb{R} \rightarrow \mathbb{R}$  is said to be convex function on  $I$  ( $\Psi \in K(I)$ ) if

$$\Psi(\vartheta \zeta_1 + (1 - \vartheta)\zeta_2) \leq \vartheta \Psi(\zeta_1) + (1 - \vartheta)\Psi(\zeta_2)$$

holds for all  $\zeta_1, \zeta_2 \in I$  and  $\vartheta \in [0, 1]$ .

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Jensen's inequality (J. I) is the best-known result in the literature. The generalizations and improvements of Jensen's inequality have been a topic of supreme interest for researchers during the last few decades as evident from a large number of publications on the topic (see [8–10]). This inequality has also been used in various areas of science and technology to solve several problems, such as engineering, mathematical statistics, financial economics, and computer science. For some recent results, see [11–13].

**Theorem 1** ([1]) *If  $\Psi \in K(I)$ , then*

$$\Psi\left(\sum_{i=1}^s \gamma_i a_i\right) \leq \sum_{i=1}^s \gamma_i \Psi(a_i)$$

*for all  $a_i \in I$  and  $\gamma_i \in [0, 1]$  ( $i = 1, 2, \dots, s$ ) with  $\sum_{i=1}^s \gamma_i = 1$ .*

In [14], Mercer gave a new variant of (J. I) as follows.

**Theorem 2** *If  $\Psi \in K(I)$  on the interval  $[m, M]$ , then*

$$\Psi\left(M + m - \sum_{i=1}^s \gamma_i a_i\right) \leq \Psi(M) + \Psi(m) - \sum_{i=1}^s \gamma_i \Psi(a_i)$$

*for all  $a_i \in [m, M]$  and  $\gamma_i \in [0, 1]$  ( $i = 1, 2, \dots, s$ ) with  $\sum_{i=1}^s \gamma_i = 1$ .*

In [5], Iscan gave the definition of harmonic convexity as follows.

**Definition 2** A function  $\Psi : I \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$  is said to be harmonically convex on  $I$  ( $\Psi \in HK(I)$ ) if

$$\Psi\left(\frac{\zeta_1 \zeta_2}{\vartheta \zeta_1 + (1 - \vartheta) \zeta_2}\right) \leq \vartheta \Psi(\zeta_2) + (1 - \vartheta) \Psi(\zeta_1)$$

holds for all  $\zeta_1, \zeta_2 \in I$ ,  $\zeta_1 < \zeta_2$ , and  $\vartheta \in [0, 1]$ .

Dragomir established (J. I) for  $\Psi \in HK(I)$  in [15] as follows.

**Theorem 3** *Let  $I \subseteq (0, \infty)$  be an interval and  $\Psi \in HK(I)$ , then*

$$\Psi\left(\frac{1}{\sum_{i=1}^s \frac{\gamma_i}{a_i}}\right) \leq \sum_{i=1}^s \gamma_i \Psi(a_i) \quad (1)$$

*for all  $a_i \in I$  and  $\gamma_i \in [0, 1]$ , ( $i = 1, 2, \dots, s$ ) with  $\sum_{i=1}^s \gamma_i = 1$ .*

Very recently, Baloch et al. [16] presented a variant of (J. I) in the Mercer sense for  $\Psi \in HK(I)$  as follows.

**Theorem 4** If  $\Psi \in HK(I)$  on the interval  $I = [m, M] \subseteq (0, \infty)$ , then the inequality

$$\Psi\left(\frac{1}{\frac{1}{M} + \frac{1}{m} - \sum_{i=1}^s \frac{\gamma_i}{a_i}}\right) \leq \Psi(M) + \Psi(m) - \sum_{i=1}^s \gamma_i \Psi(a_i)$$

holds for all  $a_i \in [m, M]$  and  $\gamma_i \in [0, 1]$ ,  $(i = 1, 2, \dots, s)$  with  $\sum_{i=1}^s \gamma_i = 1$ .

Fractional calculus and local fractional calculus are very powerful tools in applied mathematics [17–21]. Yang [22] stated the definition of local fractional calculus. The local fractional calculus is used to deal with various non-differentiable problems that arise in a complex real-world phenomenon system. Local fractional functional analysis is a fully new area of mathematics and also a totally new view of the mathematical world. The theory of fractional calculus has played an important role in various fields of applied and pure sciences, for example, electricity, mechanics, biology, economics, chemistry, notably control theory, image processing, etc. The local fractional calculus is extremely practical and comprehensive in science and engineering for its real-world models (see [23]). Mandelbrot defined a fractal set as the one whose Hausdorff dimension strictly exceeds the topological dimension [24]. Many researchers contemplated the properties of a function on the fractal space and built numerous sorts of fractional calculus by utilizing distinctive approaches, see [25, 26]. Mo et al. [27] defined the generalized convex function on fractal sets  $\mathbb{R}^q$  ( $0 < q \leq 1$ ) of real numbers and established generalized Jensen's and Hermite–Hadamard's inequalities for a generalized convex function in the concept of local fractional calculus. In (2017) Sun [28] introduced the concept of harmonic convex function on fractal sets  $\mathbb{R}^q$  ( $0 < q \leq 1$ ) of real numbers and gave some Hermite–Hadamard inequalities for a generalized harmonic function ( $\Psi \in GHK_q(I)$ ). In 2018, authors [29] worked on fractal integral inequalities for  $\Psi \in GHK(I)$ . Sun [30] in 2019 worked on generalization of some inequalities for ( $\Psi \in GHK_q(I)$ ) on the fractal space. Recently, in 2020, Iftikhar et al. [31] gave several new Newton-type inequalities in a local fractional calculus scheme. Some recent results for a generalized harmonic convex function can be seen in [32].

## 2 Preliminaries

Using the Gao–Yang–Kang concept, recall the set  $\mathbb{R}^q$  to classify the definitions of local fractional derivative, local fractional integral (see [22]), and so on.

The theory of Yang's fractional sets [22] can be stated as follows.

For  $0 < q \leq 1$ , the  $q$ -type set of element set is given below:

$\mathbb{Z}^q$ : The set defined as the  $q$ -type set of integer is  $\{0^q, \pm 1^q, \pm 2^q, \dots, \pm n^q, \dots\}$ .

$\mathbb{Q}^q$ : The set defined as the  $q$ -type set of rational numbers is  $\{m^q = (r/s)^q : r, s \in \mathbb{Z}, s \neq 0\}$ .

$\mathbb{J}^q$ : The set defined as the  $q$ -type set of irrational numbers is  $\{m^q \neq (r/s)^q : r, s \in \mathbb{Z}, s \neq 0\}$ .

$\mathbb{R}^q$ : The set defined as the  $q$ -type set of real numbers is  $\mathbb{R}^q = \mathbb{Q}^q \cup \mathbb{J}^q$ .

The following operations hold for  $r^q$ ,  $s^q$ , and  $t^q$  belonging to the set  $\mathbb{R}^q$  of real line numbers:

- (i)  $r^q + s^q$  and  $r^q s^q$  belong to the set  $\mathbb{R}^q$ ;
- (ii)  $r^q + s^q = s^q + r^q = (r + s)^q = (s + r)^q$ ;
- (iii)  $r^q + (s^q + t^q) = (r + s)^q + t^q$ ;
- (iv)  $r^q s^q = s^q r^q = (rs)^q = (sr)^q$ ;

- (v)  $r^\varrho(s^\varrho t^\varrho) = (r^\varrho s^\varrho)t^\varrho$ ;
- (vi)  $r^\varrho(s^\varrho + t^\varrho) = r^\varrho s^\varrho + r^\varrho t^\varrho$ ;
- (vii)  $r^\varrho + 0^\varrho = 0^\varrho + r^\varrho = r^\varrho$  and  $r^\varrho 1^\varrho = 1^\varrho r^\varrho = r^\varrho$ .

Let us recall some basics of local fractional calculus on  $\mathbb{R}^\varrho$ .

**Definition 3** ([22]) A non-differentiable function  $\Psi : \mathbb{R} \rightarrow \mathbb{R}^\varrho$ ,  $\zeta \rightarrow \Psi(\zeta)$  at  $\zeta_0$  is continuous in a local fractional sense if, for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$|\Psi(\zeta) - \Psi(\zeta_0)| < \epsilon^\varrho$$

holds for  $|\zeta - \zeta_0| < \delta$ , for all  $\epsilon, \delta \in \mathbb{R}$ . We denote  $\Psi(\zeta) \in C_\varrho(m, \mathfrak{M})$  if  $\Psi(\zeta)$  is local fractional continuous on the interval  $(m, \mathfrak{M})$ .

**Definition 4** ([22]) The definition of local fractional derivative of  $\Psi(\zeta)$  of order  $\varrho$  at  $\zeta = \zeta_0$  is

$$\Psi^{(\varrho)}(\zeta_0) = \left. \frac{d^\varrho \Psi(\zeta)}{d\zeta^\varrho} \right|_{\zeta=\zeta_0} = \lim_{\zeta \rightarrow \zeta_0} \frac{\Delta^\varrho(\Psi(\zeta) - \Psi(\zeta_0))}{(\zeta - \zeta_0)^\varrho},$$

where  $\Delta^\varrho(\Psi(\zeta) - \Psi(\zeta_0)) \cong \Gamma(1 + \varrho)(\Psi(\zeta) - \Psi(\zeta_0))$ .

If for any  $\zeta \in I \subseteq \mathbb{R}$  there exists  $\Psi^{((p+1)\varrho)}(\zeta) = \overbrace{D_\zeta^\varrho \cdots D_\zeta^\varrho}^{p+1 \text{ times}} \Psi(\zeta)$ , then we say  $\Psi \in D_{(p+1)\varrho}(I)$ , where  $p = 0, 1, 2, \dots$ .

**Definition 5** ([22]) If  $\Psi \in C_\varrho[m, \mathfrak{M}]$ , then the local fractional integral of  $\Psi(\zeta)$  of order  $\varrho$  is

$$\begin{aligned} {}_m I_{\mathfrak{M}}^{(\varrho)} \Psi(\zeta) &= \frac{1}{\Gamma(1 + \varrho)} \int_m^{\mathfrak{M}} \Psi(w) (dw)^\varrho \\ &= \frac{1}{\Gamma(1 + \varrho)} \lim_{\Delta w \rightarrow 0} \sum_{s=0}^{M-1} \Psi(w_s) (\Delta w_s)^\varrho, \end{aligned}$$

with  $\Delta w_s = w_{s+1} - w_s$  and  $\Delta w = \max \{\Delta w_s | s = 1, 2, \dots, M-1\}$ , where  $[w_s, w_{s+1}]$ ,  $s = 0, 1, \dots, M-1$ , and  $w_0 = m < w_1 < \cdots < w_i < \cdots < w_{M-1} < w_M = \mathfrak{M}$  is a partition of the interval  $[m, \mathfrak{M}]$ .

Here, it implies that  ${}_m I_{\mathfrak{M}}^{(\varrho)} \Psi(\zeta) = 0$  if  $m = \mathfrak{M}$  and  ${}_m I_{\mathfrak{M}}^{(\varrho)} \Psi(\zeta) = -{}_{\mathfrak{M}} I_m^{(\varrho)} \Psi(\zeta)$  if  $m < \mathfrak{M}$ . If for any  $\zeta \in [m, \mathfrak{M}]$  there exists  ${}_m I_{\mathfrak{M}}^{(\varrho)} \Psi(\zeta)$ , then we denote  $\Psi(\zeta) \in I_\zeta^{(\varrho)}[m, \mathfrak{M}]$ .

**Lemma 1** ([22])

(i) If  $\Psi(\zeta) = \Upsilon^{(\varrho)}(\zeta) \in C_\varrho[m, \mathfrak{M}]$ , then we have

$${}_m I_{\mathfrak{M}}^\varrho \Psi(\zeta) = \Upsilon(\mathfrak{M}) - \Upsilon(m).$$

(ii) If  $\Psi(\zeta), \Upsilon(\zeta) \in D_\varrho[m, \mathfrak{M}]$  and  $\Psi^{(\varrho)}(\zeta), \Upsilon^{(\varrho)}(\zeta) \in C_\varrho[m, \mathfrak{M}]$ , then we have

$${}_m I_{\mathfrak{M}}^\varrho \Psi(\zeta) \Upsilon^{(\varrho)}(\zeta) = \Psi(\zeta) \Upsilon(\zeta) \Big|_m^{\mathfrak{M}} - {}_m I_{\mathfrak{M}}^\varrho \Psi^{(\varrho)}(\zeta) \Upsilon(\zeta).$$

**Lemma 2** ([22])

$$\frac{d^{\varrho} \zeta^{\varsigma \varrho}}{d\zeta^{\varrho}} = \frac{\Gamma(1 + \varsigma \varrho)}{\Gamma(1 + (\varsigma - 1)\varrho)} \zeta^{(\varsigma-1)\varrho};$$

and

$$\frac{1}{\Gamma(1 + \varrho)} \int_{\mathfrak{m}}^{\mathfrak{M}} \zeta^{\varsigma \varrho} (d\zeta)^{\varrho} = \frac{\Gamma(1 + \varsigma \varrho)}{\Gamma(1 + (\varsigma + 1)\varrho)} (\mathfrak{M}^{(\varsigma+1)\varrho} - \mathfrak{m}^{(\varsigma+1)\varrho}),$$

while  $\varsigma \in \mathbb{R}$ .

The definition of a generalized harmonically convex function on fractal sets is as follows.

**Definition 6** ([28]) A function  $\Psi : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}^{\varrho}$  is said to be a generalized harmonic convex function on  $I$  i.e.  $\Psi \in GHK_{\varrho}(I)$  if

$$\Psi\left(\frac{\zeta_1 \zeta_2}{\vartheta \zeta_1 + (1 - \vartheta) \zeta_2}\right) \leq (1 - \vartheta)^{\varrho} \Psi(\zeta_1) + \vartheta^{\varrho} \Psi(\zeta_2) \quad (2)$$

holds for all  $\zeta_1, \zeta_2 \in I$ ,  $\zeta_1 < \zeta_2$ , and  $\vartheta \in [0, 1]$ .

The main aim of this paper is to present the generalized Jensen inequality (**G. J. I**) and generalized Jensen–Mercer inequality (**G. J. M. I**) for the class of functions  $GHK_{\varrho}(I)$  on the fractal space. Moreover, we establish an improvement and generalization of some Jensen–Mercer-type inequalities for harmonically convex function via local fractional integrals. Also, we obtain some generalized related results on a fractal space. Finally, we present some resulting applications to special means and probability density function.

### 3 Generalized Jensen's and Jensen–Mercer inequalities for $GHK_{\varrho}(I)$ in the fractal sense

In this section, we first present the (**G. J. I**) and establish (**G. J. M. I**) for  $\Psi \in GHK_{\varrho}(I)$  via fractional integrals. In order to prove (**G. J. M. I**), we need the main identity later in this section.

**Theorem 5** (**G. J. I**) Let  $I \subseteq (0, \infty)$  be an interval and  $\Psi \in GHK_{\varrho}(I)$ , then

$$\Psi\left(\frac{1}{\sum_{i=1}^s \frac{\gamma_i}{a_i}}\right) \leq \sum_{i=1}^s \gamma_i^{\varrho} \Psi(a_i) \quad (3)$$

holds for all  $a_i \in I$  and  $\gamma_i \in [0, 1]$ ,  $(i = 1, 2, \dots, s)$  with  $\sum_{i=1}^s \gamma_i = 1$ .

*Proof* The inequality is clearly true for  $s = 2$ . Suppose that for  $s = k$  the inequality is also true. Then, for any  $a_1, a_2, \dots, a_k \in I$  and  $\omega_i > 0$ ,  $i = 1, 2, \dots, k$  with  $\sum_{i=1}^k \omega_i = 1$ , we have

$$\Psi\left(\frac{1}{\sum_{i=1}^k \frac{\omega_i}{a_i}}\right) \leq \sum_{i=1}^k \omega_i^{\varrho} \Psi(a_i).$$

If  $a_1, a_2, \dots, a_k, a_{(k+1)} \in I$  and  $\gamma_i > 0$  for  $i = 1, 2, \dots, k+1$  with  $\sum_{i=1}^{k+1} \gamma_i = 1$ , then one sets up

$$\omega_i = \frac{\gamma_i}{1 - \gamma_{k+1}}$$

for all  $i = 1, 2, \dots, k$ . It is easy to see that  $\sum_{i=1}^k \omega_i = 1$ . Thus,

$$\begin{aligned} \Psi\left(\frac{1}{\sum_{i=1}^{k+1} \frac{\gamma_i}{a_i}}\right) &= \Psi\left(\frac{1}{(1 - \gamma_{k+1}) \frac{\frac{\gamma_1}{a_1} + \frac{\gamma_2}{a_2} + \dots + \frac{\gamma_k}{a_k}} + \frac{\gamma_{k+1}}{a_{k+1}}}\right) \\ &\leq (1 - \gamma_{k+1})^\varrho \Psi\left(\frac{1 - \gamma_{k+1}}{\frac{\gamma_1}{a_1} + \frac{\gamma_2}{a_2} + \dots + \frac{\gamma_k}{a_k}}\right) + \gamma_{k+1}^\varrho \Psi(a_{k+1}) \\ &= (1 - \gamma_{k+1})^\varrho \Psi\left(\frac{1}{\omega_1 \frac{1}{a_1} + \omega_2 \frac{1}{a_2} + \dots + \omega_k \frac{1}{a_k}}\right) + \gamma_{k+1}^\varrho \Psi(a_{k+1}) \\ &\leq (1 - \gamma_{k+1})^\varrho [\omega_1^\varrho \Psi(a_1) + \omega_2^\varrho \Psi(a_2) + \dots + \omega_k^\varrho \Psi(a_k)] + \gamma_{k+1}^\varrho \Psi(a_{k+1}) \\ &= (1 - \gamma_{k+1})^\varrho \left[ \left(\frac{\gamma_1}{1 - \gamma_{k+1}}\right)^\varrho \Psi(a_1) + \dots + \left(\frac{\gamma_k}{1 - \gamma_{k+1}}\right)^\varrho \Psi(a_k) \right] \\ &\quad + \gamma_{k+1}^\varrho \Psi(a_{k+1}) \\ &= \sum_{i=1}^{k+1} \gamma_i^\varrho \Psi(a_i). \end{aligned}$$

So, the mathematical induction gives the proof of Theorem 5.  $\square$

**Remark 1** If we take  $\varrho = 1$  in Theorem 5, then it gives inequality (1) proved by Dragomir.

The main lemma for  $\Psi \in GHK_\varrho(I)$  pertaining local fractional integrals is as follows.

**Lemma 3** Let  $I = [m, \mathfrak{M}] \subseteq (0, \infty)$  be an interval,  $\{a_i\}_{i=1}^s \in I$  be a finite positive increasing sequence, and  $\Psi \in GHK_\varrho(I)$  on the interval  $[m, \mathfrak{M}]$ , then

$$\Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{m} - \frac{1}{a_i}}\right) \leq \Psi(\mathfrak{M}) + \Psi(m) - \Psi(a_i) \quad (4)$$

holds for all  $1 \leq i \leq s$ .

*Proof* Let  $\frac{1}{b_i} = \frac{1}{\mathfrak{M}} + \frac{1}{m} - \frac{1}{a_i}$ . Then  $\frac{1}{b_i} + \frac{1}{a_i} = \frac{1}{m} + \frac{1}{\mathfrak{M}}$  so that the pairs  $m, \mathfrak{M}$  and  $a_i, b_i$  possess the same harmonic mean. Since that is the case, there exists  $\vartheta$  such that

$$\begin{aligned} a_i &= \frac{m\mathfrak{M}}{\vartheta m + (1 - \vartheta)\mathfrak{M}}, \\ b_i &= \frac{m\mathfrak{M}}{(1 - \vartheta)m + \vartheta\mathfrak{M}}, \end{aligned}$$

where  $0 \leq \vartheta \leq 1$  and  $1 \leq \iota \leq s$ . Hence, applying generalized harmonic convexity of  $\Psi$ , we get

$$\begin{aligned}\Psi(b_i) &= \Psi\left(\frac{m\mathfrak{M}}{(1-\vartheta)m + \vartheta\mathfrak{M}}\right) \\ &\leq \vartheta^\varrho \Psi(m) + (1-\vartheta)^\varrho \Psi(\mathfrak{M}) \\ &= \Psi(\mathfrak{M}) + \Psi(m) - [(1-\vartheta)^\varrho \Psi(m) + \vartheta^\varrho \Psi(\mathfrak{M})] \\ &\leq \Psi(\mathfrak{M}) + \Psi(m) - \Psi\left(\frac{m\mathfrak{M}}{\vartheta m + (1-\vartheta)\mathfrak{M}}\right) \\ &= \Psi(\mathfrak{M}) + \Psi(m) - \Psi(a_i)\end{aligned}$$

and  $\frac{1}{b_i} = \frac{1}{\mathfrak{M}} + \frac{1}{m} + \frac{1}{a_i}$ , which concludes the proof of the lemma.  $\square$

By using the (G. J. I), (G. J. M. I) for  $\Psi \in GHK_\varrho(I)$  can be represented in a local fractional integral as follows.

**Theorem 6** ((G. J. M. I)) *Let  $I = [m, \mathfrak{M}] \subseteq (0, \infty)$  be an interval and  $\Psi \in GHK_\varrho(I)$ , then the inequality*

$$\Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{m} - \sum_{i=1}^s \frac{\gamma_i}{a_i}}\right) \leq \Psi(\mathfrak{M}) + \Psi(m) - \sum_{i=1}^s \gamma_i^\varrho \Psi(a_i) \quad (5)$$

*holds for any finite positive sequence  $\{a_i\}_{i=1}^s \in I$  and  $\gamma_i \in [0, 1]$  ( $i = 1, 2, \dots, s$ ) with  $\sum_{i=1}^s \gamma_i = 1$ .*

*Proof* It follows from Theorem 5 and Lemma 3 together with the generalized harmonic convexity of  $\Psi$  on the interval  $[m, \mathfrak{M}]$  that

$$\begin{aligned}\Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{m} - \sum_{i=1}^s \frac{\gamma_i}{a_i}}\right) &= \Psi\left(\frac{1}{\sum_{i=1}^s \gamma_i \left(\frac{1}{\mathfrak{M}} + \frac{1}{m} - \frac{1}{a_i}\right)}\right) \\ &\leq \sum_{i=1}^s \gamma_i^\varrho \Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{m} - \frac{1}{a_i}}\right) \\ &\leq \sum_{i=1}^s \gamma_i^\varrho \Psi(\mathfrak{M}) + \sum_{i=1}^s \gamma_i^\varrho \Psi(m) - \sum_{i=1}^s \gamma_i^\varrho \Psi(a_i) \\ &= \Psi(\mathfrak{M}) + \Psi(m) - \sum_{i=1}^s \gamma_i^\varrho \Psi(a_i),\end{aligned}$$

and this concludes the proof.  $\square$

**Remark 2** If we take  $\varrho = 1$  in Theorem 6, then it gives [[16], Theorem 2.4] and also see Theorem 4.

#### 4 Improvement and generalization of some (G. J. M)-type inequalities for local fractional integrals

Based on Lemma 3, some (G. J. M)-type inequalities can be represented pertaining local fractional integral forms as follows.

**Theorem 7** Assume that  $\Psi \in GHK_{\varrho}(I)$  on the interval  $I = [m, M] \subset (0, \infty)$ , then the following inequality for local fractional integrals holds:

$$\begin{aligned} \Psi\left(\frac{1}{\frac{1}{M} + \frac{1}{m} - \frac{1}{a}}\right) &\leq \sum_{i=1}^s \gamma_i^{\varrho} \Psi\left(\frac{1}{\frac{1}{M} + \frac{1}{m} - (\frac{1-\vartheta}{a} + \frac{\vartheta}{a_i})}\right) \\ &\leq \Psi(M) + \Psi(m) - \sum_{i=1}^s \gamma_i^{\varrho} \Psi(a_i) \end{aligned} \quad (6)$$

for all  $a_i \in [m, M]$  and  $\gamma_i \in [0, 1]$  ( $i = 1, 2, \dots, s$ ) with  $\sum_{i=1}^s \gamma_i = 1$ , where  $\frac{1}{a} = \sum_{i=1}^s \gamma_i \frac{1}{a_i}$ .

*Proof* Since  $\Psi \in GHK_{\varrho}(I)$ , we have

$$\begin{aligned} \sum_{i=1}^s \gamma_i^{\varrho} \Psi\left(\frac{1}{\frac{1}{M} + \frac{1}{m} - (\frac{1-\vartheta}{a} + \frac{\vartheta}{a_i})}\right) &\geq \Psi\left(\frac{1}{\sum_{i=1}^s \gamma_i (\frac{1}{M} + \frac{1}{m} - (\frac{1-\vartheta}{a} + \frac{\vartheta}{a_i}))}\right) \\ &= \Psi\left(\frac{1}{\frac{1}{M} + \frac{1}{m} - \frac{1}{a} + \frac{\vartheta}{a} - \sum_{i=1}^s \gamma_i \frac{\vartheta}{a_i}}\right) \\ &= \Psi\left(\frac{1}{\frac{1}{M} + \frac{1}{m} - \frac{1}{a}}\right). \end{aligned} \quad (7)$$

On the other hand,

$$\begin{aligned} &\sum_{i=1}^s \gamma_i^{\varrho} \Psi\left(\frac{1}{\frac{1}{M} + \frac{1}{m} - (\frac{1-\vartheta}{a} + \frac{\vartheta}{a_i})}\right) \\ &= \sum_{i=1}^s \gamma_i^{\varrho} \Psi\left(\frac{1}{(1-\vartheta)(\frac{1}{M} + \frac{1}{m} - \frac{1}{a}) + \vartheta(\frac{1}{M} + \frac{1}{m} - \frac{1}{a_i})}\right) \\ &\leq \sum_{i=1}^s \gamma_i^{\varrho} \left[ (1-\vartheta)^{\varrho} \Psi\left(\frac{1}{\frac{1}{M} + \frac{1}{m} - \frac{1}{a}}\right) + \vartheta^{\varrho} \Psi\left(\frac{1}{\frac{1}{M} + \frac{1}{m} - \frac{1}{a_i}}\right) \right] \\ &\leq \sum_{i=1}^s \gamma_i^{\varrho} \left[ (1-\vartheta)^{\varrho} (\Psi(M) + \Psi(m) - \sum_{j=1}^s \gamma_j^{\varrho} \Psi(a_j)) \right. \\ &\quad \left. + \vartheta^{\varrho} (\Psi(M) + \Psi(m) - \Psi(a_i)) \right] \\ &= \Psi(M) + \Psi(m) - \sum_{i=1}^s \gamma_i^{\varrho} \Psi(a_i). \end{aligned} \quad (8)$$

Combining inequalities (7) and (8), we get (6).  $\square$

**Remark 3** If we take  $\varrho = 1$  in Theorem 7, then it gives [[33], Theorem 2.3].



**Corollary 1** Let all assumptions of Theorem 7 hold and  $\Psi(x) \in I_x^{(\varrho)}[m, M]$ , then

$$\begin{aligned} \Psi\left(\frac{1}{\frac{1}{M} + \frac{1}{m} - \frac{1}{a}}\right) &\leq \sum_{i=1}^s \gamma_i^{\varrho} \frac{\Gamma(1+\varrho)}{\left(\frac{1}{a_i} - \frac{1}{a}\right)^{\varrho}} \frac{1}{\frac{1}{M} + \frac{1}{m} - \frac{1}{a}} I_{\frac{1}{\frac{1}{M} + \frac{1}{m} - \frac{1}{a}}}^{(\varrho)} \frac{\Psi(x)}{x^{2\varrho}} \\ &\leq \Psi(M) + \Psi(m) - \sum_{i=1}^s \gamma_i^{\varrho} \Psi(a_i). \end{aligned} \quad (9)$$

*Proof* Multiplying by  $\frac{1}{\Gamma(1+\varrho)}$  on both sides of (6) and then integrating w.r.t  $\vartheta$  over  $[0, 1]$ , we get (9). Here, we used the fact

$$\begin{aligned} &\frac{1}{\Gamma(1+\varrho)} \int_0^1 \Psi\left(\frac{1}{\frac{1}{M} + \frac{1}{m} - \left(\frac{1-\vartheta}{a} + \frac{\vartheta}{a_i}\right)}\right) (d\vartheta)^{\varrho} \\ &= \frac{1}{\Gamma(1+\varrho)} \int_0^1 \Psi\left(\frac{1}{(1-\vartheta)\Psi\left(\frac{1}{M} + \frac{1}{m} - \frac{1}{a}\right) + \vartheta\Psi\left(\frac{1}{M} + \frac{1}{m} - \frac{1}{a_i}\right)}\right) (d\vartheta)^{\varrho} \\ &= \frac{1}{\left(\frac{1}{a_i} - \frac{1}{a}\right)^{\varrho}} \left[ \frac{1}{\Gamma(1+\varrho)} \int_{\frac{1}{\frac{1}{M} + \frac{1}{m} - \frac{1}{a}}}^{\frac{1}{\frac{1}{M} + \frac{1}{m} - \frac{1}{a_i}}} \frac{\Psi(t)}{t^{2\varrho}} (dt)^{\varrho} \right] \\ &= \frac{1}{\left(\frac{1}{a_i} - \frac{1}{a}\right)^{\varrho}} \frac{1}{\frac{1}{M} + \frac{1}{m} - \frac{1}{a}} I_{\frac{1}{\frac{1}{M} + \frac{1}{m} - \frac{1}{a}}}^{(\varrho)} \frac{\Psi(x)}{x^{2\varrho}}. \end{aligned} \quad \square$$

**Remark 4** If we take  $\varrho = 1$  in Corollary 1, then it gives [[33], Corollary 2.4] and if put  $\varrho = 1$ ,  $s = 2$ ,  $\gamma_1 = \gamma_2 = \frac{1}{2}$ ,  $a_1 = a$ , and  $a_2 = b$  in Corollary 1, then it gives [[33], Theorem 2.1].

Now, we present a more precise estimate in the following theorem.

**Theorem 8** Assume that  $\Psi \in GHK_{\varrho}(I)$  on the interval  $I = [m, M] \subset (0, \infty)$ , then the following inequality for local fractional integrals holds:

$$\begin{aligned} \Psi\left(\frac{1}{\frac{1}{M} + \frac{1}{m} - \frac{1}{a}}\right) &\leq \sum_{i=1}^s \gamma_i^{\varrho} \Psi\left(\frac{1}{\frac{1}{M} + \frac{1}{m} - \frac{1}{2}\left(\frac{1}{a} + \frac{1}{a_i}\right)}\right) \\ &\leq \sum_{i=1}^s \gamma_i^{\varrho} \frac{\Gamma(1+\varrho)}{\left(\frac{1}{a_i} - \frac{1}{a}\right)^{\varrho}} \frac{1}{\frac{1}{M} + \frac{1}{m} - \frac{1}{a}} I_{\frac{1}{\frac{1}{M} + \frac{1}{m} - \frac{1}{a}}}^{(\varrho)} \frac{\Psi(x)}{x^{2\varrho}} \\ &\leq \Psi(M) + \Psi(m) - \sum_{i=1}^s \gamma_i^{\varrho} \Psi(a_i) \end{aligned} \quad (10)$$

for all  $a_i \in [m, M]$  and  $\gamma_i \in [0, 1]$  ( $i = 1, 2, \dots, s$ ) with  $\sum_{i=1}^s \gamma_i = 1$ , where  $\frac{1}{a} = \sum_{i=1}^s \gamma_i \frac{1}{a_i}$ .

*Proof* As  $\Psi \in GHK_{\varrho}(I)$ , then for any  $a_1, b_1 \in [m, M]$  and  $\vartheta \in [0, 1]$ , we have

$$\begin{aligned} \Psi\left(\frac{2a_1b_1}{a_1 + b_1}\right) &= \Psi\left(\frac{1}{\frac{1}{2}\left(\frac{\vartheta}{a_1} + \frac{1-\vartheta}{b_1} + \frac{1-\vartheta}{a_1} + \frac{\vartheta}{b_1}\right)}\right) \\ &\leq \frac{1}{2^{\varrho}} \left[ \Psi\left(\frac{1}{\frac{\vartheta}{a_1} + \frac{1-\vartheta}{b_1}}\right) + \Psi\left(\frac{1}{\frac{1-\vartheta}{a_1} + \frac{\vartheta}{b_1}}\right) \right] \end{aligned}$$

$$\leq \frac{\Psi(a_1) + \Psi(b_1)}{2^q}.$$

Writing  $a_1 = \frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - \frac{1}{a}}$  and  $b_1 = \frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - \frac{1}{b}}$  for  $a, b \in [\mathfrak{m}, \mathfrak{M}]$ , we get

$$\begin{aligned} \Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - \frac{a+b}{2ab}}\right) &\leq \frac{1}{2^q} \left[ \Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - (\frac{\vartheta}{a} + \frac{1-\vartheta}{b})}\right) + \Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - (\frac{1-\vartheta}{a} + \frac{\vartheta}{b})}\right) \right] \\ &\leq \frac{1}{2^q} \left[ \Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - \frac{1}{a}}\right) + \Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - \frac{1}{b}}\right) \right]. \end{aligned}$$

Multiplying by  $\frac{1}{\Gamma(1+q)}$  on both sides of the above equation, then integrating w.r.t  $\vartheta$  over  $[0, 1]$  and using fact that

$$\begin{aligned} \frac{1}{\Gamma(1+q)} \int_0^1 \Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - (\frac{\vartheta}{a} + \frac{1-\vartheta}{b})}\right) (d\vartheta)^q \\ = \frac{1}{\Gamma(1+q)} \int_0^1 \Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - (\frac{1-\vartheta}{a} + \frac{\vartheta}{b})}\right) (d\vartheta)^q, \end{aligned}$$

we infer that

$$\begin{aligned} \frac{1}{\Gamma(1+q)} \Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - \frac{a+b}{2ab}}\right) &\leq \frac{1}{\Gamma(1+q)} \int_0^1 \Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - (\frac{\vartheta}{a} + \frac{1-\vartheta}{b})}\right) (d\vartheta)^q \\ &\leq \frac{1}{2^q \Gamma(1+q)} \left[ \Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - \frac{1}{a}}\right) + \Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - \frac{1}{b}}\right) \right]. \end{aligned}$$

Since  $\frac{1}{a_i}, \frac{1}{a} \in [\mathfrak{m}, \mathfrak{M}]$ , we can write

$$\begin{aligned} \Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - \frac{1}{2}(\frac{1}{a} + \frac{1}{a_i})}\right) &\leq \frac{\Gamma(1+q)}{(\frac{1}{a_i} - \frac{1}{a})^q} \frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - \frac{1}{a}} I^{(q)}_{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - \frac{1}{a_i}} \frac{\Psi(x)}{x^{2q}} \\ &\leq \frac{1}{2^q} \left[ \Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - \frac{1}{a}}\right) + \Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - \frac{1}{a_i}}\right) \right], \end{aligned}$$

due to

$$\frac{1}{\Gamma(1+q)} \int_0^1 \Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - (\frac{\vartheta}{a} + \frac{1-\vartheta}{a_i})}\right) (d\vartheta)^q = \frac{1}{(\frac{1}{a_i} - \frac{1}{a})^q} \frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - \frac{1}{a}} I^{(q)}_{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - \frac{1}{a_i}} \frac{\Psi(x)}{x^{2q}}.$$

Multiplying by  $\gamma_i^q > 0$  ( $i = 1, \dots, s$ ) and summing over  $i$  from 1 to  $s$ , we may deduce

$$\begin{aligned} \sum_{i=1}^s \gamma_i^q \Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - \frac{1}{2}(\frac{1}{a} + \frac{1}{a_i})}\right) \\ \leq \sum_{i=1}^s \frac{\gamma_i^q \Gamma(1+q)}{(\frac{1}{a_i} - \frac{1}{a})^q} \frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - \frac{1}{a}} I^{(q)}_{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - \frac{1}{a_i}} \frac{\Psi(x)}{x^{2q}} \\ \leq \frac{1}{2^q} \left[ \Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - \frac{1}{a}}\right) + \sum_{i=1}^s \gamma_i^q \Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - \frac{1}{a_i}}\right) \right]. \end{aligned} \quad (11)$$

On the other hand, by (G. J. I) for  $\Psi \in GHK_{\varrho}(I)$

$$\begin{aligned}\Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - \frac{1}{a}}\right) &= \Psi\left(\frac{1}{\sum_{i=1}^s \gamma_i \left(\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - \frac{1}{2}\left(\frac{1}{a} + \frac{1}{a_i}\right)\right)}\right) \\ &\leq \sum_{i=1}^s \gamma_i^{\varrho} \Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - \frac{1}{2}\left(\frac{1}{a} + \frac{1}{a_i}\right)}\right),\end{aligned}\quad (12)$$

and by Lemma 3 and (G. J. M. I) for  $\Psi \in GHK_{\varrho}(I)$

$$\begin{aligned}&\frac{1}{2^{\varrho}} \left[ \Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - \frac{1}{a}}\right) + \sum_{i=1}^s \gamma_i^{\varrho} \Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - \frac{1}{a_i}}\right) \right] \\ &\leq \frac{1}{2^{\varrho}} \left[ \Psi(\mathfrak{M}) + \Psi(\mathfrak{m}) - \sum_{j=1}^s \gamma_j^{\varrho} \Psi(a_j) + \Psi(\mathfrak{M}) + \Psi(\mathfrak{m}) - \sum_{i=1}^s \gamma_i^{\varrho} \Psi(a_i) \right] \\ &= \Psi(\mathfrak{M}) + \Psi(\mathfrak{m}) - \sum_{i=1}^s \gamma_i^{\varrho} \Psi(a_i).\end{aligned}\quad (13)$$

Combining (11), (12), and (13), we get (10).  $\square$

**Remark 5** If we take  $\varrho = 1$  in Theorem 8, then it gives [[33], Theorem 2.6].

**Remark 6** Let all the assumptions of Theorem 8 hold, then

$$\begin{aligned}\Psi\left(\frac{1}{\sum_{i=1}^s \frac{\gamma_i}{a_i}}\right) &\leq \sum_{i=1}^s \gamma_i^{\varrho} \Psi\left(\frac{2}{\sum_{i=1}^s \frac{\gamma_i}{a_i} + \frac{1}{a_i}}\right) \\ &\leq \sum_{i=1}^s \frac{\gamma_i^{\varrho} \Gamma(1+\varrho)}{\left(\sum_{i=1}^s \frac{\gamma_i}{a_i} - \frac{1}{a_i}\right)^{\varrho}} \frac{1}{\sum_{i=1}^s \frac{\gamma_i}{a_i}} I_{a_i}^{(\varrho)} \frac{\Psi(x)}{x^{2\varrho}} \\ &\leq \sum_{i=1}^s \gamma_i^{\varrho} \Psi(a_i).\end{aligned}$$

## 5 Related results

In this section, we present the following related results for local fractional integrals.

**Theorem 9** If  $\Psi : [c, d] \subseteq (0, \infty) \longrightarrow \mathbb{R}^{\varrho}$  such that  $\Psi \in GHK_{\varrho}(I)$  and  $a_1, a_2, \dots, a_s \in [c, d]$ , then

$$\begin{aligned}&\Psi\left(\frac{2a_1a_2}{a_1+a_2}\right) + \dots + \Psi\left(\frac{2a_{s-1}a_s}{a_{s-1}+a_s}\right) + \Psi\left(\frac{2a_s a_1}{a_s+a_1}\right) \\ &\leq \Psi(a_1) + \Psi(a_s) + \frac{(s-1)^{\varrho}}{s^{\varrho}} \sum_{i=1}^s \Psi(a_i) - \Psi\left(\frac{1}{\frac{1}{a_1} + \frac{1}{a_s} - \sum_{i=1}^s \frac{1}{a_i}}\right).\end{aligned}\quad (14)$$

*Proof* From the generalized harmonic convexity of  $\Psi$ , we get

$$\begin{aligned} & \Psi\left(\frac{2a_1a_2}{a_1+a_2}\right) + \cdots + \Psi\left(\frac{2a_{s-1}a_s}{a_{s-1}+a_s}\right) + \Psi\left(\frac{2a_s a_1}{a_s+a_1}\right) \\ & \leq \frac{\Psi(a_1) + \Psi(a_2)}{2^q} + \cdots + \frac{\Psi(a_{s-1}) + \Psi(a_s)}{2^q} + \frac{\Psi(a_s) + \Psi(a_1)}{2^q} \\ & = \Psi(a_1) + \cdots + \Psi(a_s) \\ & = \sum_{i=1}^s \Psi(a_i). \end{aligned} \quad (15)$$

It is obvious that

$$\begin{aligned} \sum_{i=1}^s \Psi(a_i) &= \frac{(s-1)^q}{s^q} \sum_{i=1}^s \Psi(a_i) + \frac{1}{s^q} \sum_{i=1}^s \Psi(a_i) \\ &= \Psi(a_1) + \Psi(a_s) + \frac{(s-1)^q}{s^q} \sum_{i=1}^s \Psi(a_i) - \left[ \Psi(a_1) + \Psi(a_s) \right. \\ & \quad \left. - \sum_{i=1}^s \left(\frac{1}{s}\right)^q \Psi(a_i) \right] \\ &\leq \Psi(a_1) + \Psi(a_s) + \frac{(s-1)^q}{s^q} \sum_{i=1}^s \Psi(a_i) - \Psi\left(\frac{1}{\frac{1}{a_1} + \frac{1}{a_s} - \sum_{i=1}^s \frac{1}{a_i}}\right). \end{aligned} \quad (16)$$

Combining (15) and (16), we get (14).  $\square$

**Remark 7** If we take  $q = 1$  in Theorem 9, then it reduces to Theorem 3.3 (see [16]).

**Theorem 10** If  $\Psi : [c, d] \subseteq (0, \infty) \rightarrow \mathbb{R}^q$  such that  $\Psi \in GHK_q(I)$  and  $a_1, a_2, \dots, a_s \in [c, d]$ , then

$$\sum_{i=1}^s \Psi(b_i) \leq \Psi(a_1) + \Psi(a_s) + \frac{(s-1)^q}{s^q} \sum_{i=1}^s \Psi(a_i) - \Psi\left(\frac{1}{\frac{1}{a_1} + \frac{1}{a_s} - \sum_{i=1}^s \frac{1}{a_i}}\right), \quad (17)$$

where  $b_i = \frac{1}{\frac{s-1}{s} \frac{1}{q} + \frac{1}{s} \frac{1}{a_i}}$  and  $q = \frac{1}{\frac{1}{s} \sum_{i=1}^s \frac{1}{a_i}}$ .

*Proof* Using (G, J, I) for  $\Psi \in GHK_q(I)$ , we have

$$\begin{aligned} \sum_{i=1}^s \Psi(b_i) &= \Psi(b_1) + \cdots + \Psi(b_n) \\ &= \Psi\left(\frac{1}{\frac{s-1}{s} \frac{1}{q} + \frac{1}{s} \frac{1}{a_1}}\right) + \cdots + \Psi\left(\frac{1}{\frac{s-1}{s} \frac{1}{q} + \frac{1}{s} \frac{1}{a_s}}\right) \\ &\leq \left[\left(\frac{1}{s}\right)^q \Psi(q) + \left(\frac{s-1}{s}\right)^q \Psi(a_1)\right] + \cdots + \left[\left(\frac{1}{s}\right)^q \Psi(q) + \left(\frac{s-1}{s}\right)^q \Psi(a_s)\right] \end{aligned}$$

$$\begin{aligned}
&= \Psi(\varrho) + \left(\frac{s-1}{s}\right)^{\varrho} \sum_{i=1}^s \Psi(a_i) \\
&= \Psi\left(\frac{1}{\frac{1}{s} \sum_{i=1}^s \frac{1}{a_i}}\right) + \left(\frac{s-1}{s}\right)^{\varrho} \sum_{i=1}^s \Psi(a_i) \\
&\leq \left(\frac{1}{s}\right)^{\varrho} \sum_{i=1}^s \Psi(a_i) + \left(\frac{s-1}{s}\right)^{\varrho} \sum_{i=1}^s \Psi(a_i).
\end{aligned} \tag{18}$$

Combining (16) and (18), we get (17).  $\square$

**Remark 8** If we take  $\varrho = 1$  in Theorem 10, then it reduces to Theorem 3.4 (see [16]).

**Theorem 11** Let  $I = [m, \mathfrak{M}] \subseteq (0, \infty)$  be an interval. If  $\Psi \in GHK_{\varrho}(I)$  on the interval  $I$  with  $m < \mathfrak{M}$  and for all  $\zeta_1, \zeta_2 \in [m, \mathfrak{M}]$ . If  $\Psi(x) \in I_x^{(\varrho)}[m, \mathfrak{M}]$ , then

$$\begin{aligned}
\Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{m} - \frac{\zeta_1 + \zeta_2}{2\zeta_1\zeta_2}}\right) &\leq \Psi(\mathfrak{M}) + \Psi(m) - \frac{\zeta_1^{\varrho}\zeta_2^{\varrho}}{(\zeta_2 - \zeta_1)^{\varrho}} \Gamma(1 + \varrho) I_{\zeta_1}^{(\varrho)} \frac{\Psi(x)}{x^{2\varrho}} \\
&\leq \Psi(\mathfrak{M}) + \Psi(m) - \Psi\left(\frac{2\zeta_1\zeta_2}{\zeta_1 + \zeta_2}\right).
\end{aligned} \tag{19}$$

**Proof** Since  $\Psi \in GHK_{\varrho}(I)$  on the interval  $I = [m, \mathfrak{M}]$ , setting  $\vartheta = \frac{1}{2}$  in inequality (2), we have

$$\Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{m} - \frac{\zeta_1 + \zeta_2}{2\zeta_1\zeta_2}}\right) \leq \Psi(\mathfrak{M}) + \Psi(m) - \frac{\Psi(\bar{\zeta}_1) + \Psi(\bar{\zeta}_2)}{2^{\varrho}}$$

for all  $\bar{\zeta}_1, \bar{\zeta}_2 \in [m, \mathfrak{M}]$ . Choosing  $\bar{\zeta}_1 = \frac{\zeta_1\zeta_2}{\vartheta\zeta_2 + (1-\vartheta)\zeta_1}$ ,  $\bar{\zeta}_2 = \frac{\zeta_1\zeta_2}{\vartheta\zeta_1 + (1-\vartheta)\zeta_2}$  for all  $\zeta_1, \zeta_2 \in [m, \mathfrak{M}]$  and  $\vartheta \in [0, 1]$ , we obtain

$$\Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{m} - \frac{\zeta_1 + \zeta_2}{2\zeta_1\zeta_2}}\right) \leq \Psi(\mathfrak{M}) + \Psi(m) - \frac{\Psi\left(\frac{\zeta_1\zeta_2}{\vartheta\zeta_2 + (1-\vartheta)\zeta_1}\right) + \Psi\left(\frac{\zeta_1\zeta_2}{\vartheta\zeta_1 + (1-\vartheta)\zeta_2}\right)}{2^{\varrho}}.$$

Multiplying by  $\frac{1}{\Gamma(1+\varrho)}$  on both sides of the above equation and then integrating w.r.t  $\vartheta$  over  $[0, 1]$ , we have

$$\begin{aligned}
&\frac{1}{\Gamma(1+\varrho)} \Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{m} - \frac{\zeta_1 + \zeta_2}{2\zeta_1\zeta_2}}\right) \\
&\leq \frac{1}{\Gamma(1+\varrho)} [\Psi(\mathfrak{M}) + \Psi(m)] - \frac{1}{2^{\varrho}} \left[ \frac{1}{\Gamma(1+\varrho)} \int_0^1 \Psi\left(\frac{\zeta_1\zeta_2}{\vartheta\zeta_2 + (1-\vartheta)\zeta_1}\right) (d\vartheta)^{\varrho} \right. \\
&\quad \left. + \frac{1}{\Gamma(1+\varrho)} \int_0^1 \Psi\left(\frac{\zeta_1\zeta_2}{\vartheta\zeta_1 + (1-\vartheta)\zeta_2}\right) (d\vartheta)^{\varrho} \right] \\
&= \frac{1}{\Gamma(1+\varrho)} [\Psi(\mathfrak{M}) + \Psi(m)] - \frac{1}{2^{\varrho}} \left( \frac{\zeta_1\zeta_2}{\zeta_2 - \zeta_1} \right)^{\varrho} \left[ \frac{1}{\Gamma(1+\varrho)} \int_{\zeta_1}^{\zeta_2} \frac{\Psi(x)}{x^{2\varrho}} (dx)^{\varrho} \right. \\
&\quad \left. + \frac{1}{\Gamma(1+\varrho)} \int_{\zeta_1}^{\zeta_2} \frac{\Psi(y)}{y^{2\varrho}} (dy)^{\varrho} \right]
\end{aligned}$$

$$= \frac{1}{\Gamma(1+\varrho)} [\Psi(\mathfrak{M}) + \Psi(\mathfrak{m})] - \frac{\zeta_1^\varrho \zeta_2^\varrho}{(\zeta_2 - \zeta_1)^\varrho} {}_{\zeta_1} I_{\zeta_2}^{(\varrho)} \frac{\Psi(x)}{x^{2\varrho}},$$

that is,

$$\Psi\left(\frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - \frac{\zeta_1 + \zeta_2}{2\zeta_1\zeta_2}}\right) \leq \Psi(\mathfrak{M}) + \Psi(\mathfrak{m}) - \frac{\zeta_1^\varrho \zeta_2^\varrho}{(\zeta_2 - \zeta_1)^\varrho} \Gamma(1+\varrho) {}_{\zeta_1} I_{\zeta_2}^{(\varrho)} \frac{\Psi(x)}{x^{2\varrho}}, \quad (20)$$

and so the first inequality of (19) is proved. If  $\Psi \in GHK_\varrho(I)$ , then for  $\vartheta \in [0, 1]$  we have

$$\begin{aligned} \Psi\left(\frac{2\zeta_1\zeta_2}{\zeta_1 + \zeta_2}\right) &= \Psi\left(\frac{1}{\frac{1}{2}\left(\frac{\vartheta}{\zeta_1} + \frac{(1-\vartheta)}{\zeta_2} + \frac{(1-\vartheta)}{\zeta_1} + \frac{\vartheta}{\zeta_2}\right)}\right) \\ &\leq \frac{1}{2^\varrho} \left[ \Psi\left(\frac{\zeta_1\zeta_2}{\vartheta\zeta_2 + (1-\vartheta)\zeta_1}\right) + \Psi\left(\frac{\zeta_1\zeta_2}{\vartheta\zeta_1 + (1-\vartheta)\zeta_2}\right) \right]. \end{aligned}$$

Multiplying by  $\frac{1}{\Gamma(1+\varrho)}$  on both sides of the above equation and then integrating w.r.t  $\vartheta$  over  $[0, 1]$ , we get

$$\Psi\left(\frac{2\zeta_1\zeta_2}{\zeta_1 + \zeta_2}\right) \leq \frac{\zeta_1^\varrho \zeta_2^\varrho}{(\zeta_2 - \zeta_1)^\varrho} \Gamma(1+\varrho) {}_{\zeta_1} I_{\zeta_2}^{(\varrho)} \frac{\Psi(x)}{x^{2\varrho}},$$

and then

$$-\Psi\left(\frac{2\zeta_1\zeta_2}{\zeta_1 + \zeta_2}\right) \geq -\frac{\zeta_1^\varrho \zeta_2^\varrho}{(\zeta_2 - \zeta_1)^\varrho} \Gamma(1+\varrho) {}_{\zeta_1} I_{\zeta_2}^{(\varrho)} \frac{\Psi(x)}{x^{2\varrho}}.$$

Adding  $\Psi(\mathfrak{M}) + \Psi(\mathfrak{m})$  to both sides of the above equation, we have

$$\Psi(\mathfrak{M}) + \Psi(\mathfrak{m}) - \Psi\left(\frac{2\zeta_1\zeta_2}{\zeta_1 + \zeta_2}\right) \geq \Psi(\mathfrak{M}) + \Psi(\mathfrak{m}) - \frac{\zeta_1^\varrho \zeta_2^\varrho}{(\zeta_2 - \zeta_1)^\varrho} \Gamma(1+\varrho) {}_{\zeta_1} I_{\zeta_2}^{(\varrho)} \frac{\Psi(x)}{x^{2\varrho}}. \quad (21)$$

Combining (20) and (21), we get the second inequality of (19).  $\square$

**Remark 9** If we take  $\varrho = 1$  in Theorem 11, then it gives [[16], Theorem 3.5].

## 6 Application to special means

Let  $\zeta_1 < \zeta_2$  and  $\zeta_1, \zeta_2 \in \mathbb{R}^\varrho$  considering the following  $\varrho$ -type special means.

(1) The generalized arithmetic mean:

$$A_\varrho(\zeta_1, \zeta_2) = \frac{\zeta_1^\varrho + \zeta_2^\varrho}{2^\varrho}.$$

(2) The generalized  $\xi$ -logarithmic mean:

$$L_{\xi\varrho}(\zeta_1, \zeta_2) = \left[ \frac{\Gamma(1+\xi\varrho)}{\Gamma(1+(\xi+1)\varrho)} \frac{\zeta_2^{(\xi+1)\varrho} - \zeta_1^{(\xi+1)\varrho}}{(\zeta_2 - \zeta_1)^\varrho} \right]^{1/\xi},$$

where  $\xi \in \mathbb{Z} \setminus \{-1, 0\}$ ,  $\zeta_1, \zeta_2 \in \mathbb{R}$ ,  $\zeta_1 \neq \zeta_2$ .

(3) The generalized geometric mean:

$$G_\varrho(\zeta_1, \zeta_2) = (\zeta_1^\varrho \zeta_2^\varrho)^{\frac{1}{2}}.$$

(4) The generalized harmonic mean:

$$H_{\varrho}(\zeta_1, \zeta_2) = \frac{(2\zeta_1\zeta_2)^{\varrho}}{\zeta_1^{\varrho} + \zeta_2^{\varrho}}.$$

Consider the mapping  $\Psi : (0, \infty) \rightarrow \mathbb{R}^{\varrho}$ ,  $\Psi(x) = \frac{\Gamma(1+k\varrho)}{\Gamma(1+(k+1)\varrho)} x^{(k+1)\varrho}$ ,  $x > 0$ ,  $k \geq 1$ . Then  $|\Psi^{\varrho}(x)|^q = x^{kq\varrho}$ , where  $q \geq 1$  is a generalized harmonic convex function on  $(0, \infty)$ .

**Proposition 1** For  $\mathfrak{m}, \mathfrak{M}, \zeta_1, \zeta_2 \in \mathbb{R}$ ,  $\zeta_1, \zeta_2 \in [\mathfrak{m}, \mathfrak{M}]$ ,  $0 < \mathfrak{m} < \mathfrak{M}$ , and  $k > 1$ , we have

$$\left[ \frac{1}{2^{\varrho} A_{\varrho}(\mathfrak{m}^{-1}, \mathfrak{M}^{-1}) - H_{\varrho}^{-1}(\zeta_1, \zeta_2)} \right]^{k+1} \leq 2^{\varrho} A_{\varrho}(\mathfrak{m}^{k+1}, \mathfrak{M}^{k+1}) - A_{\varrho}(\zeta_1^{k+1}, \zeta_2^{k+1}); \quad (22)$$

and

$$\begin{aligned} & \left( \frac{1}{2^{\varrho} A_{\varrho}(\mathfrak{m}^{-1}, \mathfrak{M}^{-1}) - H_{\varrho}^{-1}(\zeta_1, \zeta_2)} \right)^{k+1} \\ & \leq \Gamma(1+\varrho) L_{-(k+1)\varrho}^{-(k+1)} (2A(\mathfrak{m}^{-1}, \mathfrak{M}^{-1}) - \zeta_1^{-1}, 2A(\mathfrak{m}^{-1}, \mathfrak{M}^{-1}) - \zeta_2^{-1}) \\ & \leq 2^{\varrho} A_{\varrho}(\mathfrak{m}^{k+1}, \mathfrak{M}^{k+1}) - A_{\varrho}(\zeta_1^{k+1}, \zeta_2^{k+1}). \end{aligned} \quad (23)$$

*Proof* Let  $s = 2$ ,  $\gamma_1 = \gamma_2 = \frac{1}{2}$ ,  $a_1 = \zeta_1$ ,  $a_2 = \zeta_2$ , and  $\Psi(x) = \frac{\Gamma(1+k\varrho)}{\Gamma(1+(k+1)\varrho)} x^{(k+1)\varrho}$ , where  $x > 0$ ,  $k > 1$  in Theorem 6 and Corollary 1 lead to the desired inequality (22) and (23) immediately respectively.  $\square$

## 7 Applications to probability density functions (P.D.F)

Consider a random variable  $\chi$  with generalized P.D.F  $\tau : [\mathfrak{m}, \mathfrak{M}] \rightarrow [0^{\varrho}, 1^{\varrho}]$  i.e.  $\tau \in GHK_{\varrho}([\mathfrak{m}, \mathfrak{M}])$  possessing the cumulative distribution function

$$P_{\varrho}(\chi \leq z) = F_{\varrho}(z) = \frac{1}{\Gamma(1+\varrho)} \int_{\mathfrak{m}}^z \tau(\varsigma)(d\varsigma)^{\varrho}.$$

The generalized expectation

$$E_{\varrho} = \frac{1}{\Gamma(1+\varrho)} \int_{\mathfrak{m}}^{\mathfrak{M}} \varsigma^{\varrho} \tau(\varsigma)(d\varsigma)^{\varrho}.$$

It is easy to see that

$$E_{\varrho}(\chi) = \mathfrak{M}^{\varrho} - \frac{1}{\Gamma(1+\varrho)} \int_{\mathfrak{m}}^{\mathfrak{M}} F_{\varrho}(\varsigma)(d\varsigma)^{\varrho}.$$

The generalized expectation  $E_{p\varrho}$  is a  $p$ -moment,  $p \in \mathbb{R} \setminus \{-1, 0\}$  is

$$E_{p\varrho} = \frac{1}{\Gamma(1+\varrho)} \int_{\mathfrak{m}}^{\mathfrak{M}} \varsigma^{p\varrho} \tau(\varsigma)(d\varsigma)^{\varrho}.$$

Define the generalized  $(-\varrho)$ -moment by

$$E_{-\varrho} = \frac{1}{\Gamma(1+\varrho)} \int_{\mathfrak{m}}^{\mathfrak{M}} \frac{1}{\varsigma^{\varrho}} \tau(\varsigma)(d\varsigma)^{\varrho}.$$

For more details, see [34, 35].

The following two propositions can be obtained by using the cumulative distribution function and generalized  $(-\varrho)$ -moment of the random variable  $\chi$  from our results associated with Sects. 3 and 4.

**Proposition 2** *Let  $s = 2$ ,  $\gamma_1 = \gamma_2 = \frac{1}{2}$ ,  $a_1 = \zeta_1$ , and  $a_2 = \zeta_2$  in Theorem 6, we have*

$$P_{\varrho}\left(\chi \leq \frac{1}{\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{m}} - \frac{\zeta_1 + \zeta_2}{2\zeta_1\zeta_2}}\right) \leq P_{\varrho}(\chi \leq \mathfrak{M}) + P_{\varrho}(\chi \leq \mathfrak{m}) - \frac{P_{\varrho}(\chi \leq \zeta_1) + P_{\varrho}(\chi \leq \zeta_2)}{2^{\varrho}}.$$

**Proposition 3** *Let  $s = 2$ ,  $\gamma_1 = \gamma_2 = \frac{1}{2}$ ,  $a_1 = \mathfrak{m}$ , and  $a_2 = \mathfrak{M}$  in Corollary 1, we have*

$$\begin{aligned} P_{\varrho}\left(\chi \leq \frac{2\mathfrak{m}\mathfrak{M}}{\mathfrak{m} + \mathfrak{M}}\right) &\leq \frac{\mathfrak{m}^{\varrho}\mathfrak{M}^{\varrho}}{(\mathfrak{M} - \mathfrak{m})^{\varrho}} \frac{\Gamma(1 + \varrho)\Gamma(1 - 2\varrho)}{\Gamma(1 - \varrho)} \left[ \left(\frac{1}{\mathfrak{M}}\right)^{\varrho} - E_{-\varrho}(\chi) \right] \\ &\leq \frac{P_{\varrho}(\chi \leq \mathfrak{M}) + P_{\varrho}(\chi \leq \mathfrak{m})}{2^{\varrho}}. \end{aligned}$$

## 8 Conclusion

In this paper, for the first time, we introduce Jensen's inequality for harmonic convex functions by using local fractional calculus. As a result we also introduce a variant of generalized Jensen–Mercer inequality in the fractal sense. Then we present the main lemma involving local fractional integrals. By using this main lemma, we establish generalized Jensen–Mercer-type inequalities on fractal sets  $\mathbb{R}^{\varrho}$  ( $0 < \varrho \leq 1$ ). Moreover, an improvement and generalization of some results related to the class of generalized harmonically convex function via local fractional integrals are established. Using these inequalities, we obtain some generalized related results on the fractal space. Lastly, some applications to some special means of real numbers and probability density function are established. It is quite open to investigate Jensen and Jensen-type inequalities for other classes of generalized convex functions in the fractal sense. Since there is a massive body of literature about refinements, converses, and reverses of such Jensen and Jensen-type inequalities, so their generalized variants using local fractional calculus can be revealed also. One of the important directions is to give fractal Jensen–Mercer–Ostrowski type inequalities and give applications to continuous random variables and improve quadrature rules. Since local fractional calculus is quite effective from an application point of view, a bridge between inequality theory and their applications can be established, which may eventually generate optimal solutions.

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### Availability of data and materials

The data used to support the findings of this study are included within the article.

## Declarations

### Competing interests

The authors declare that they have no competing interests.



### Authors' contributions

All authors contributed equally to this article. All authors read and approved the final manuscript.

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