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An inertial iterative algorithm for generalized equilibrium problems and Bregman relatively nonexpansive mappings in Banach spaces

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Abstract

The aim of this paper is to introduce and study an inertial hybrid iterative method for solving generalized equilibrium problems involving Bregman relatively nonexpansive mappings in Banach spaces. We study the strong convergence for the proposed algorithm. Finally, we list some consequences and computational example to emphasize the efficiency and relevancy of main result.

MSC: 47H09; 47H05; 47J25

Keywords: Bregman relatively nonexpansive mapping; Fixed point problem; Generalized equilibrium problem; Inertial hybrid iterative method

1 Introduction

Throughout the paper, unless otherwise stated, let Y be a reflexive Banach space with Y^* its dual, let $K \neq \emptyset$ be a closed convex subset of Y and denote by $\mathbb{R}$ the set of real numbers. Consider the following generalized equilibrium problem (in short, GEP): Find $u_0 \in K$ such that

$$G(u_0, u) + b(u_0, u) - b(u_0, u_0) \geq 0, \quad \forall u \in K, \tag{1.1}$$

where $G, b : K \times K \to \mathbb{R}$ are bifunctions. We will write Sol(GEP(1.1)) for the solution of (1.1). If $b \equiv 0$, then GEP(1.1) reduces to the equilibrium problem (in short, EP): Find $u_0 \in K$ such that

$$G(u_0, u) \geq 0, \quad \forall u \in K. \tag{1.2}$$

It is known that equilibrium problems have a great impact and influence in the development of several topics of science and engineering. It turned out that many well-known problems could be fitted into the equilibrium problems. It has been shown that the theory of equilibrium problems provides a natural, novel, and unified framework for several problems arising in nonlinear analysis, optimization, economics, finance, game theory, and engineering. The equilibrium problems include many mathematical problems as particular cases, for example, mathematical programming problem, variational inclusion problem, variational inequality problem, complementary problem, saddle point problem, Nash equilibrium problem in noncooperative games, minimax inequality problem, minimization problem, and fixed point problem, see [1–3]. For example, if we set $G(u_0,u) = \langle Du_0, u - u_0\rangle$ and $b(u_0,u) = 0$, $\forall u_0$, $u \in K$, where $D : K \to Y^*$ is a nonlinear mapping, then EP(1.2) reduces to the classical variational inequality problem (in short, VIP): Find $u_0 \in K$ such that

$$\langle Du_0, u - u_0\rangle \geq 0, \quad \forall u \in K, \tag{1.3}$$

which was introduced by Hartmann and Stampacchia [4]. The solution set of VIP(1.3) is denoted by Sol(VIP(1.3)).

Korpelevich [5] originated the iterative method for VIP on Hilbert space H in 1976 as:

$$\begin{cases} u_0 \in C \subseteq H, \\ v_n = \operatorname{proj}_C(u_n - \sigma Du_n), \\ u_{n+1} = \operatorname{proj}_C(u_n - \sigma Dv_n), \quad n \geq 0, \end{cases}$$

where $\sigma > 0$, proj_C denotes the projection of H onto C, and D is a monotone and Lipschitz continuous mapping. This method is called the extragradient iterative method.

Nadezkhina et al. [6] proposed a hybrid extragradient algorithm involving nonexpansive mapping T on C and in 2006 studied the convergence analysis of

$$\begin{cases} u_0 \in C \subseteq H, \\ x_n = \operatorname{proj}_C(u_n - \sigma_n Du_n), \\ v_n = \alpha_n u_n + (1-\alpha_n)T\operatorname{proj}_C(u_n - \sigma_n Dx_n), \\ C_n = \{z \in C : \|v_n - z\|^2 \leq \|u_n - z\|^2\}, \\ D_n = \{z \in C : \langle u_n - z, u_0 - u_n\rangle \geq 0\}, \\ u_{n+1} = \operatorname{proj}_{C_n \cap D_n} u_0, \quad n \geq 0. \end{cases} \tag{1.4}$$

The iterative algorithm (1.4) has been extended by many authors; see [7–15] for details. The idea considered in [6] has been generalized in [16] from a Hilbert space to a Banach space Y as

$$\begin{cases} u_0 \in K \subseteq Y, \\ v_n = J^{-1}(\alpha_n Ju_n + (1-\alpha_n)JTu_n), \\ C_n = \{z \in K : \Psi(z, v_n) \leq \Psi(z, u_n)\}, \\ D_n = \{z \in K : \langle u_n - z, Ju_0 - Ju_n\rangle \geq 0\}, \\ u_{n+1} = \prod_{C_n \cap D_n} u_0, \end{cases}$$

where Π_K denotes the generalized projection of Y onto K, Ψ is the Lyapunov function such that $\Psi(u,v) = \|v\|^2 - 2\langle v, Ju\rangle + \|u\|^2$, $\forall u, v \in Y$, $J : Y \to 2^{Y^*}$ is the normalized duality mapping, with J^{-1} denoting its inverse. For further work, see [17–20].

In 1967, an important technique was discovered by Bregman [21] in the light of Bregman distance function. This technique is very useful not only in the design and interpretation the iterative method but also in solving optimization and feasibility problems and in approximating equilibria, FPP, VIP, etc.; for details, see [22–25].

In 2010, Reich [26] et al. introduced an iterative algorithm on Banach space involving maximal monotone operators. In the light of Bregman projection, there were various iterative algorithms studied by researchers in this field, see, for instance, [27–31].

In 2008, Mainge [32] developed an inertial Krasnosel'skiǐ–Mann algorithm as follows:

$$\begin{cases} t_n = u_n + \theta_n(u_n - u_{n-1}), \\ u_{n+1} = (1-\alpha_n)t_n + \alpha_n T t_n. \end{cases}$$

The convergence of such an algorithm has been analyzed by various researchers and illustrated its importance on data analysis and some imaging problems, see [33–41] for details. It is noticeable that the research on an inertial iterative algorithm is still unexplored on a Banach space.

Inspired by the work in [20, 30, 32], we establish an inertial hybrid iterative algorithm involving Bregman relatively nonexpansive mapping to find a common solution of GEP(1.1) and fixed point problem in a Banach space. Moreover, we analyze its convergence for our main result. At last, we list some consequences and a computational example to emphasize the efficiency and relevancy of the main result.

2 Preliminaries

Assume $g : Y \to (-\infty, +\infty]$ is a proper, convex, and lower semicontinuous mapping, and $g^* : Y^* \to (-\infty, +\infty]$, Fenchel conjugate of g, is defined as

$$g^*(u_0) = \sup\{\langle u_0, u\rangle - g(u) : u \in Y\}, \quad u_0 \in Y^*.$$

And for any $w \in \operatorname{int}(\operatorname{dom} g)$, the interior of the domain of g, and $u \in Y$, the right-hand derivative of g at w in the direction u is

$$g^0(w,u) = \lim_{\lambda \to 0^+} \frac{g(w+\lambda u) - g(w)}{\lambda}.$$

A mapping g is called Gateaux differentiable at w if the above limit exists. So, $g^0(w,u)$ agrees with $\nabla g(w)$, the value of the gradient of g at w. It is called Frechet differentiable at w, if the limit is attained uniformly in u, $\|u\| = 1$. It is called uniformly Frechet differentiable on $K \subseteq Y$, if the above limit is attained uniformly for $w \in K$ and $\|u\| = 1$.

The mapping g is called Legendre if the following hold [22]:

(i) $\operatorname{int}(\operatorname{dom} g) \neq \emptyset$, g is Gateaux differentiable on $\operatorname{int}(\operatorname{dom} g)$, $\operatorname{dom} \nabla g = \operatorname{int}(\operatorname{dom} g)$;

(ii) $\operatorname{int}(\operatorname{dom} g^*) \neq \emptyset$, g^* is Gateaux differentiable on $\operatorname{int}(\operatorname{dom} g^*)$, $\operatorname{dom} \nabla g^* = \operatorname{int}(\operatorname{dom} g^*)$.

We have the following [22]:

(i) g is Legendre iff g^* is Legendre mapping;

(ii) $(\partial g)^{-1} = \partial g^*$;

(iii) $\nabla g = (\nabla g^*)^{-1}$, $\operatorname{ran} \nabla g = \operatorname{dom} \nabla g^* = \operatorname{int}(\operatorname{dom} g^*)$, $\operatorname{ran} \nabla g^* = \operatorname{dom} \nabla g = \operatorname{int}(\operatorname{dom} g)$;

(iv) the mappings g and g^* are strictly convex on $\operatorname{int}(\operatorname{dom} g)$ and $\operatorname{int}(\operatorname{dom} g^*)$.

Definition 2.1 ([21]) Let $g : Y \to (-\infty, +\infty]$ be Gateaux differentiable and convex and $D_g : \mathrm{dom}g \times \mathrm{int}(\mathrm{dom}g) \to [0, +\infty)$ such that

$$D_g(u,w) = g(u) - g(w) - \langle \nabla g(w), u - w \rangle, \quad w \in \mathrm{int}(\mathrm{dom}g), u \in \mathrm{dom}g,$$

is known as Bregman distance with respect to g.

We have listed some important properties of D_g [42]: for $u, u_1, u_2 \in (\mathrm{dom}g)$ and $w_1, w_2 \in \mathrm{int}(\mathrm{dom}g)$,

(i) Two point identity:

$$D_g(w_1,w_2) + D_g(w_2,w_1) = \langle \nabla g(w_1) - \nabla g(w_2), w_1 - w_2 \rangle;$$

(ii) Three point identity:

$$D_g(u,w_1) + D_g(w_1,w_2) - D_g(u,w_2) = \langle \nabla g(w_2) - \nabla g(w_1), u - w_1 \rangle;$$

(iii) Four point identity:

$$D_g(u_1,w_1) - D_g(u_1,w_2) - D_g(u_2,w_1) + D_g(u_2,w_2) = \langle \nabla g(w_2) - \nabla g(w_1), u_1 - u_2 \rangle.$$

Definition 2.2 ([26, 28]) Let $T : K \to \mathrm{int}(\mathrm{dom}g)$ be a mapping and $F(T) = \{u \in K : Tu = u\}$, where $F(T)$ is the set of fixed points of T. Then

(i) a point $u_0 \in K$ is called an asymptotic fixed point if K contains a sequence $\{u_n\}$ with $u_n \rightharpoonup u_0$ such that $\lim_{n\to\infty} \|Tu_n - u_n\| = 0$. We denote by $\widehat{F}(T)$ the set of all asymptotic fixed points of T;

(ii) T is called Bregman quasinonexpansive if

$$F(T) \neq \emptyset; \qquad D_g(u_0, Tu) \leq D_g(u_0, u), \quad \forall u \in K, u_0 \in F(T);$$

(iii) T is called Bregman relatively nonexpansive if

$$F(T) = \widehat{F}(T) \neq \emptyset; \qquad D_g(u_0, Tu) \leq D_g(u_0, u), \quad \forall u \in K, u_0 \in F(T);$$

(iv) T is called Bregman firmly nonexpansive if $\forall u_1, u_2 \in K$,

$$\langle \nabla g(Tu_1) - \nabla g(Tu_2), Tu_1 - Tu_2 \rangle \leq \langle \nabla g(u_1) - \nabla g(u_2), Tu_1 - Tu_2 \rangle,$$

or, correspondingly,

$$\begin{aligned} &D_g(Tu_1, Tu_2) + D_g(Tu_2, Tu_1) + D_g(Tu_1, u_1) + D_g(Tu_2, u_2) \\ &\quad \leq D_g(Tu_1, u_2) + D_g(Tu_2, u_1). \end{aligned}$$

Example 2.1 ([29]) Let $A : Y \to 2^{Y^*}$ be a maximal monotone mapping and Y be a real reflexive Banach space. If $A^{-1}(0) \neq \emptyset$ and the Legendre function $g : Y \to (-\infty, +\infty]$ is bounded on bounded subsets of Y and uniformly Frechet differentiable then the resolvent with respect to A,

$$\mathrm{res}^g_A(u) = (\nabla g + A)^{-1} \circ \nabla g(u),$$

is a single-valued, closed, and Bregman relatively nonexpansive mapping from Y onto $D(A)$, and $F(\mathrm{res}^g_A) = A^{-1}(0)$.

Definition 2.3 ([21]) Let $g : Y \to (-\infty, +\infty]$ be a Gateaux differentiable and convex function. The Bregman projection of $w \in \mathrm{int}(\mathrm{dom}g)$ onto K, nonempty closed convex subset of $\mathrm{int}(\mathrm{dom}g)$, is a unique vector $\mathrm{proj}^g_K w \in K$ satisfies

$$D_g\bigl(\mathrm{proj}^g_K(w), w\bigr) = \inf\bigl\{D_g(u, w) : u \in K\bigr\}.$$

Remark 2.2 ([27])

(i) If Y is a smooth Banach space and $g(u) = \frac{1}{2}\|u\|^2$, $\forall u \in Y$, then $\mathrm{proj}^g_K(u)$ turns $\Pi_K(u)$, the generalized projection, [43] into

$$\Psi\bigl(\Pi_K(u), u\bigr) = \min_{v\in K} \Psi(v, u),$$

where Ψ is the Lyapunov function such that $\Psi(u, v) = \|v\|^2 - 2\langle v, Ju\rangle + \|u\|^2$, $\forall u, v \in Y$, $J : Y \to 2^{Y^*}$ is the normalized duality mapping;

(ii) If Y is a Hilbert space and $g(u) = \frac{1}{2}\|u\|^2$, $\forall u \in Y$, then $\mathrm{proj}^g_K(u)$ turns into metric projection.

Definition 2.4 ([23]) Let $g : Y \to (-\infty, +\infty]$ be a Gateaux differentiable and convex function. Then, g is called:

(i) totally convex at $w \in \mathrm{int}(\mathrm{dom}g)$ if its modulus of total convexity at u, i.e., the mapping $v_g : \mathrm{int}(\mathrm{dom}g) \times [0, +\infty) \to [0, +\infty)$ such that

$$v_g(w, s) = \inf\bigl\{D_g(v, w) : v \in \mathrm{dom}g, \|v - w\| = s\bigr\},$$

is positive for $s > 0$;

(ii) totally convex if it is totally convex at each point of $w \in \mathrm{int}(\mathrm{dom}g)$;

(iii) totally convex on bounded sets if $v_g : \mathrm{int}(\mathrm{dom}g) \times [0, +\infty) \to [0, +\infty)$ such that

$$v_g(B, s) = \inf\bigl\{v_g(w, s) : w \in B \cap \mathrm{dom}g\bigr\}.$$

Definition 2.5 ([23, 26]) A mapping $g : Y \to (-\infty, +\infty]$ is called:

(i) coercive if $\lim_{\|u\|\to+\infty} \frac{g(u)}{\|u\|} = +\infty$;

(ii) sequentially consistent if for any $\{u_n\}, \{v_n\} \subseteq Y$ with $\{u_n\}$ bounded,

$$\lim_{n\to\infty} D_g(v_n, u_n) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \|v_n - u_n\| = 0.$$

Lemma 2.3 ([24]) *If $g : Y \to (-\infty, +\infty]$ is a convex function with domain of at least two points. Then, g is sequentially consistent iff it is totally convex on bounded sets.*

Lemma 2.4 ([44]) *Let $g : Y \to (-\infty, +\infty]$ be uniformly Frechet differentiable and bounded on $K \subseteq Y$, a bounded set. Then, g is uniformly continuous on K and ∇g is uniformly continuous on K from the strong topology of Y to the strong topology of Y^*.*

Lemma 2.5 ([26]) *Let $g : Y \to (-\infty, +\infty]$ be a Gateaux differentiable and totally convex function. If $u_0 \in Y$ and $\{D_g(u_n, u_0)\}$ is bounded, then $\{u_n\}$ is also bounded.*

Lemma 2.6 ([24]) *Let $g : Y \to (-\infty, +\infty]$ be a Gateaux differentiable and totally convex function on* $\operatorname{int}(\operatorname{dom} g)$. *Let $w \in \operatorname{int}(\operatorname{dom} g)$ and $K \subseteq \operatorname{int}(\operatorname{dom} g)$, a nonempty closed convex set. If $v \in K$, then the following statements are equivalent*:

(i) *$v \in K$ is the Bregman projection of w onto K with respect to g, i.e., $v = \operatorname{proj}_K^g(w)$;*

(ii) *the vector v is the unique solution of the variational inequality*

$$\langle \nabla g(w) - \nabla g(v), v - u \rangle \geq 0, \quad \forall u \in K;$$

(iii) *the vector v is the unique solution of the inequality*

$$D_g(u, v) + D_g(v, w) \leq D_g(u, w), \quad \forall u \in K.$$

Lemma 2.7 ([28]) *Let $g : Y \to (-\infty, +\infty]$ be Legendre and $T : K \to K$ be a Bregman quasi-nonexpansive mapping with respect to g. Then, $F(T)$ is closed and convex.*

Lemma 2.8 ([26]) *Let $g : Y \to (-\infty, +\infty]$ be a Gateaux differentiable and totally convex function, $u_0 \in Y$ and $K \subseteq Y$, a nonempty closed convex set. Suppose that $\{u_n\}$ is bounded and any weak subsequential limit of $\{u_n\}$ belongs to K. If $D_g(u_n, u_0) \leq D_g(\operatorname{proj}_K^g u_0, u_0)$ then $\{u_n\}$ strongly converges to $\operatorname{proj}_K^g u_0$.*

Assumption 2.1 Let $G : K \times K \longrightarrow \mathbb{R}$ satisfy:

(i) $G(u, u) = 0$, $\forall u \in K$;

(ii) G is monotone, i.e., $G(u_1, u_2) + G(u_2, u_1) \leq 0$, $\forall u_1, u_2 \in K$;

(iii) for each $u_1, u_2, u_3 \in K$, $\limsup_{s \to 0} G(su_3 + (1 - s)u_1, u_2) \leq G(u_1, u_2)$;

(iv) for each $u \in K$, $v \to G(u, v)$ is convex and lower semicontinuous.

Assumption 2.2 Let $b : K \times K \to \mathbb{R}$ satisfy:

(i) b is skew-symmetric, i.e., $b(u_1, u_1) - b(u_1, u_2) - b(u_2, u_1) + b(u_2, u_2) \geq 0$, $\forall u_1, u_2 \in K$;

(ii) b is convex in the second argument;

(iii) b is continuous.

3 Resolvent operator

The resolvent of $G : K \times K \to \mathbb{R}$ with respect to b is the operator $\operatorname{res}_{G,b}^g : Y \to 2^K$ defined as follows:

$$\begin{aligned} \operatorname{res}_{G,b}^g(u) = \bigl\{ u_0 \in K : {} & G(u_0, v) + \langle \nabla g(u_0) - \nabla g(u), v - u_0 \rangle \\ & + b(u_0, v) - b(u_0, u_0) \geq 0, \forall v \in K \bigr\}, \quad \forall u \in Y. \end{aligned} \tag{3.1}$$

We obtain some properties of the resolvent operator $\operatorname{res}_{G,b}^g$.

Lemma 3.1 *Let $g : Y \to (-\infty, +\infty]$ be a Gateaux differentiable and coercive function. Let $G, b : K \times K \to \mathbb{R}$ fulfil Assumptions 2.1 and 2.2, respectively, and let $\mathrm{res}^g_{G,b} : Y \to 2^K$ be defined by (3.1). Then, the following hold:*

(i) $\mathrm{dom}(\mathrm{res}^g_{G,b}) = Y$;

(ii) $\mathrm{res}^g_{G,b}$ *is single-valued*;

(iii) $\mathrm{res}^g_{G,b}$ *is a Bregman firmly nonexpansive type mapping, that is*, $\forall u, v \in Y$,

$$\begin{aligned}&\bigl\langle \nabla g\bigl(\mathrm{res}^g_{G,b}u\bigr) - \nabla g\bigl(\mathrm{res}^g_{G,b}v\bigr), \mathrm{res}^g_{G,b}u - \mathrm{res}^g_{G,b}v\bigr\rangle \\ &\quad \le \bigl\langle \nabla g(u) - \nabla g(v), \mathrm{res}^g_{G,b}u - \mathrm{res}^g_{G,b}y\bigr\rangle;\end{aligned}$$

(iv) $F(\mathrm{res}^g_{G,b}) = \mathrm{Sol}(\mathrm{GEP}(1.1))$ *is closed and convex*;

(v) $D_g(q, \mathrm{res}^g_{G,b}u) + D_g(\mathrm{res}^g_{G,b}u, u) \le D_g(q, u)$, $\forall q \in F(\mathrm{res}^g_{G,b})$;

(vi) $\mathrm{res}^g_{G,b}$ *is Bregman quasinonexpansive*.

Proof (i) The proof follows the lines of the proof of Lemma 1 [30]. For the sake of completeness, we give the proof. First, we show that for any $\xi \in Y^*$, there exists $u \in C$ such that

$$G(u, v) + b(u, v) - b(u, u) + g(v) - g(u) - \langle \xi, v - u\rangle \ge 0 \tag{3.2}$$

for any $v \in K$. Since g is coercive, the function $h : Y \times Y \to (-\infty, +\infty]$ defined by

$$h(u, v) = h(v) - h(u) - \langle \xi, v - u\rangle$$

satisfies

$$\lim_{\|u-v\|\to+\infty} \frac{h(u, v)}{\|u - v\|} = -\infty,$$

for each fixed $v \in K$. Therefore, it follows from Theorem 1 in [1], together with Assumptions 2.1 and 2.2, that (3.2) holds. Now, we prove that (3.2) implies that

$$G(u, v) + b(u, v) - b(u, u) + \bigl\langle \nabla g(u), v - u\bigr\rangle - \langle \xi, v - u\rangle \ge 0 \tag{3.3}$$

for any $v \in K$. We know that (3.2) holds for $v = tu + (1 - t)\bar{v}$, where $\bar{v} \in K$ and $t \in (0, 1)$. Hence

$$\begin{aligned}&G\bigl(u, tu + (1 - t)\bar{v}\bigr) + b\bigl(u, tu + (1 - t)\bar{v}\bigr) - b(u, u) \\ &\quad + g\bigl(tu + (1 - t)\bar{v}\bigr) - g(u) - \bigl\langle \xi, tu + (1 - t)\bar{v} - u\bigr\rangle \ge 0, \quad \forall \bar{v} \in K.\end{aligned} \tag{3.4}$$

Since

$$g\bigl(tu + (1 - t)\bar{v}\bigr) - g(u) \le \bigl\langle \nabla g\bigl(tu + (1 - t)\bar{v}\bigr), tu + (1 - t)\bar{v} - u\bigr\rangle,$$

we get from (3.3), Assumption 2.1 (iv) and Assumption 2.2 (ii) that

$$\begin{aligned} &tG(u,u)+(1-t)G(u,\bar{v})+tb(u,u)+(1-t)b(u,\bar{v})-b(u,u)\\ &\quad+\bigl\langle\nabla g\bigl(tu+(1-t)\bar{v}\bigr),tu+(1-t)\bar{v}-u\bigr\rangle\\ &\quad-\bigl\langle\xi,tu+(1-t)\bar{v}-u\bigr\rangle\geq 0,\quad\forall\bar{v}\in K. \end{aligned}$$

From Assumption 2.1 (i), we have

$$\begin{aligned} &(1-t)G(u,\bar{v})+(1-t)b(u,\bar{v})-(1-t)b(u,u)\\ &\quad+\bigl\langle\nabla g\bigl(tu+(1-t)\bar{v}\bigr),(1-t)(\bar{v}-u)\bigr\rangle-\bigl\langle\xi,(1-t)(\bar{v}-u)\bigr\rangle\geq 0 \end{aligned}$$

and

$$(1-t)\bigl[G(u,\bar{v})+b(u,\bar{v})-b(u,u)+\bigl\langle\nabla g\bigl(tu+(1-t)\bar{v}\bigr),(\bar{v}-u)\bigr\rangle-\bigl\langle\xi,(\bar{v}-u)\bigr\rangle\bigr]\geq 0.$$

Therefore

$$G(u,\bar{v})+b(u,\bar{v})-b(u,u)+\bigl\langle\nabla g\bigl(tu+(1-t)\bar{v}\bigr),(\bar{v}-u)\bigr\rangle-\bigl\langle\xi,(\bar{v}-u)\bigr\rangle\geq 0,\quad\forall\bar{v}\in K.$$

Since g is a Gateaux differentiable function, it follows that ∇g is norm-to-weak* continuous. Therefore, letting $t\to 1_-$, we obtain that

$$G(u,\bar{v})+b(u,\bar{v})-b(u,u)+\bigl\langle\nabla g(u),(\bar{v}-u)\bigr\rangle-\bigl\langle\xi,(\bar{v}-u)\bigr\rangle\geq 0,\quad\forall\bar{v}\in K.$$

Hence, for any $u\in Y$, taking $\xi=\nabla g(u)$, we obtain $\bar{u}\in K$ such that

$$G(u,\bar{v})+b(u,\bar{v})-b(u,u)+\bigl\langle\nabla g(u),(\bar{v}-u)\bigr\rangle-\bigl\langle\nabla g(\bar{u}),(\bar{v}-u)\bigr\rangle\geq 0,\quad\forall\bar{v}\in K,$$

i.e.,

$$G(u,\bar{v})+b(u,\bar{v})-b(u,u)+\bigl\langle\nabla g(u)-\nabla g(\bar{u}),(\bar{v}-u)\bigr\rangle\geq 0,\quad\forall\bar{v}\in K,$$

that is, $u\in\operatorname{res}^g_{G,b}(u)$. Hence $\operatorname{dom}(\operatorname{res}^g_{G,b})=Y$.

(ii) For $u\in Y$, let $z_1,z_2\in F(\operatorname{res}^g_{G,b})$. Then $z_1,z_2\in K$ and hence

$$G(z_1,z_2)+\bigl\langle\nabla g(z_1)-\nabla g(u),z_2-z_1\bigr\rangle+b(z_1,z_2)-b(z_1,z_1)\geq 0$$

and

$$G(z_2,z_1)+\bigl\langle\nabla g(z_2)-\nabla g(u),z_1-z_2\bigr\rangle+b(z_2,z_1)-b(z_2,z_2)\geq 0.$$

Adding these two inequalities and using Assumption 2.1 (i), we have

$$\bigl\langle\nabla g(z_1)-\nabla g(z_2),z_2-z_1\bigr\rangle+b(z_1,z_2)-b(z_1,z_1)+b(z_2,z_1)-b(z_2,z_2)\geq 0.$$

Since b is skew-symmetric, we have

$$\langle \nabla g(z_1) - \nabla g(z_2), z_2 - z_1 \rangle \geq 0. \tag{3.5}$$

By interchanging the position of z_1 and z_2, we have

$$\langle \nabla g(z_2) - \nabla g(z_1), z_1 - z_2 \rangle \geq 0. \tag{3.6}$$

Adding (3.5) and (3.6), we have

$$2\langle \nabla g(z_1) - \nabla g(z_2), z_2 - z_1 \rangle \geq 0.$$

This implies that

$$\langle \nabla g(z_2) - \nabla g(z_1), z_2 - z_1 \rangle \leq 0. \tag{3.7}$$

Since g is convex and Gateaux differentiable, we have

$$\langle \nabla g(z_2) - \nabla g(z_1), z_2 - z_1 \rangle \geq 0. \tag{3.8}$$

By (3.7) and (3.8), we have

$$\langle \nabla g(z_2) - \nabla g(z_1), z_2 - z_1 \rangle = 0.$$

Since g is a Legendre function, $z_1 = z_2$. Hence, $\mathrm{res}^g_{G,b}$ is single-valued.

(iii) For $u, v \in K$, we have

$$\begin{aligned} &G(\mathrm{res}^g_{G,b}u, \mathrm{res}^g_{G,b}v) + \langle \nabla g(\mathrm{res}^g_{G,b}v) - \nabla g(v), \mathrm{res}^g_{G,b}u - \mathrm{res}^g_{G,b}v \rangle \\ &\quad + b(\mathrm{res}^g_{G,b}v, \mathrm{res}^g_{G,b}u) - b(\mathrm{res}^g_{G,b}v, \mathrm{res}^g_{G,b}v) \geq 0 \end{aligned}$$

and

$$\begin{aligned} &G(\mathrm{res}^g_{G,b}v, \mathrm{res}^g_{G,b}u) + \langle \nabla g(\mathrm{res}^g_{G,b}u) - \nabla g(u), \mathrm{res}^g_{G,b}v - \mathrm{res}^g_{G,b}u \rangle \\ &\quad + b(\mathrm{res}^g_{G,b}u, \mathrm{res}^g_{G,b}v) - b(\mathrm{res}^g_{G,b}u, \mathrm{res}^g_{G,b}u) \geq 0. \end{aligned}$$

Adding the above two inequalities, then using the skew symmetry of b and Assumption 2.1(ii), we have

$$\langle \nabla g(\mathrm{res}^g_{G,b}u) - \nabla g(u) - \nabla g(\mathrm{res}^g_{G,b}v) + \nabla g(v), \mathrm{res}^g_{G,b}v - \mathrm{res}^g_{G,b}u \rangle \geq 0,$$

hence

$$\begin{aligned} &\langle \nabla g(\mathrm{res}^g_{G,b}u) - \nabla g(\mathrm{res}^g_{G,b}v), \mathrm{res}^g_{G,b}(u) - \mathrm{res}^g_{G,b}(v) \rangle \\ &\quad \leq \langle \nabla g(u) - \nabla g(v), \mathrm{res}^g_{G,b}(u) - \mathrm{res}^g_{G,b}(v) \rangle. \end{aligned}$$

This means that $\mathrm{res}^g_{G,b}$ is a Bregman firmly nonexpansive mapping.

(iv) We now show that $F(\mathrm{res}^g_{G,b}) = \mathrm{Sol}(\mathrm{GEP}(1.1))$. We have

$$\begin{aligned} u \in F(\mathrm{res}^g_{G,b}) \quad &\Leftrightarrow \quad u \in \mathrm{res}^g_{G,b}(u) \\ &\Leftrightarrow \quad G(u,v) + \langle \nabla g(u) - \nabla g(u), v-u \rangle + b(u,v) - b(u,u) \geq 0, \quad \forall v \in K \\ &\Leftrightarrow \quad G(u,v) + b(u,v) - b(u,u) \geq 0, \quad \forall v \in K \\ &\Leftrightarrow \quad u \in \mathrm{Sol}(\mathrm{GEP}(1.1)). \end{aligned} \tag{3.9}$$

Further, since $\mathrm{res}^g_{G,b}$ is a Bregman firmly nonexpansive type mapping, it follows from [44, Lemma 1.3.1] that $F(\mathrm{res}^g_{G,b})$ is a closed and convex subset of K. Therefore, from (3.9), we obtain that $\mathrm{Sol}(\mathrm{GEP}(1.1)) = F(\mathrm{res}^g_{G,b})$ is closed and convex.

(v) Now, we prove that $\mathrm{res}^g_{G,b}$ is a Bregman quasinonexpansive mapping.

For $u,v \in K$, from (b), we have

$$\begin{aligned} &\langle \nabla g(\mathrm{res}^g_{G,b} u) - \nabla g(\mathrm{res}^g_{G,b} v), \mathrm{res}^g_{G,b}(u) - \mathrm{res}^g_{G,b}(v) \rangle \\ &\quad \leq \langle \nabla g(u) - \nabla g(v), \mathrm{res}^g_{G,b}(u) - \mathrm{res}^g_{G,b}(v) \rangle. \end{aligned}$$

Moreover, we have

$$\begin{aligned} &D_g(\mathrm{res}^g_{G,b}(u), \mathrm{res}^g_{G,b}(v)) + D_g(\mathrm{res}^g_{G,b}(v), \mathrm{res}^g_{G,b}(u)) \\ &\quad \leq D_g(\mathrm{res}^g_{G,b}(u), v) - D_g(\mathrm{res}^g_{G,b}(u), u) + D_g(\mathrm{res}^g_{G,b}(v), u) - D_g(\mathrm{res}^g_{G,b}(v), v). \end{aligned}$$

Taking $v = q \in F(\mathrm{res}^g_{G,b})$, we see that

$$\begin{aligned} &D_g(\mathrm{res}^g_{G,b}(u), q) + D_g(q, \mathrm{res}^g_{G,b}(u)) \\ &\quad \leq D_g(\mathrm{res}^g_{G,b}(u), q) - D_g(\mathrm{res}^g_{G,b}(u), u) + D_g(q,u) - D_g(q,q). \end{aligned}$$

Hence

$$D_g(q, \mathrm{res}^g_{G,b}(u)) + D_g(\mathrm{res}^g_{G,b}(u), u) \leq D_g(q,u). \tag{3.10}$$

(vi) Equation (3.10) implies that $\mathrm{res}^g_{G,b}$ is a Bregman quasinonexpansive mapping. □

4 Main result

We developed the strong convergence algorithm for the inertial iterative method to find the common solution of GEP(1.1) and FPP for a Bregman relatively nonexpansive mapping in a reflexive Banach space.

Theorem 4.1 *Let $K \subseteq Y$ with $K \subseteq \mathrm{int}(\mathrm{dom} g)$, where $g : Y \to (-\infty, +\infty]$ is a coercive Legendre function which is bounded, uniformly Frechet differentiable, and totally convex on bounded subsets of Y. Let $G, b : K \times K \to \mathbb{R}$ satisfy Assumptions 2.1 and 2.2, respectively. Let $T : K \to K$ be a Bregman relatively nonexpansive mapping. Let $\Omega = \mathrm{Sol}(\mathrm{GEP}(1.1)) \cap$*

$F(T) \neq \emptyset$. Let $\{x_n\}, \{z_n\} \subseteq K$ be generated by

$$\begin{cases} x_0 = x_{-1} \in K, \\ u_n = x_n + \theta_n(x_n - x_{n-1}), \\ v_n = \nabla g^*(\alpha_n \nabla g(u_n) + (1-\alpha_n)\nabla g(Tu_n)), \\ w_n = \nabla g^*(\beta_n \nabla g(Tu_n) + (1-\beta_n)\nabla g(v_n)), \\ z_n = \operatorname{res}^g_{G,b} w_n, \\ C_n = \{z \in C : D_g(z, z_n) \leq D_g(z, u_n)\}, \\ Q_n = \{z \in C : \langle \nabla g(x_0) - \nabla g(x_n), z - x_n \rangle \leq 0\}, \\ x_{n+1} = \operatorname{proj}^g_{C_n \cap Q_n} x_0, \quad \textit{for all } n \geq 0, \end{cases} \tag{4.1}$$

where $\{\theta_n\} \subseteq (0,1)$, $\{\alpha_n\}, \{\beta_n\} \subseteq [0,1]$ with $\lim_{n\to\infty} \alpha_n = 0$. Then, $\{x_n\}$ converges strongly to $\operatorname{proj}^g_\Omega x_0$.

Proof For convenience, we divide the proof into several steps:

Step I. Ω and $C_n \cap Q_n$ are closed and convex, $\forall n \geq 0$.

By Lemmas 2.7 and 3.1, Ω is closed and convex and therefore $\operatorname{proj}^g_\Omega x_0$ is well defined.

Obviously, Q_n is closed and convex. Further, we prove that C_n is closed and convex, $\forall n \geq 0$. We can easily show that C_n is closed and convex, $\forall n$. Thus, $C_n \cap Q_n$ is closed and convex, $\forall n \geq 0$.

Step II. $\Omega \subset C_n \cap Q_n$, $\forall n \geq 0$ and $\{x_n\}$ is well defined.

Let $p \in \Omega$, then

$$\begin{aligned} D_g(p, z_n) &= D_g\big(p, \operatorname{res}^g_{G,\phi} w_n\big) \\ &\leq D_g(p, w_n) \\ &= D_g\big(p, \nabla g^*\big(\beta_n \nabla g(Tu_n) + (1-\beta_n)\nabla g(v_n)\big) \\ &\leq \beta_n D_g(p, u_n) + (1-\beta_n) D_g(p, v_n), \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} D_g(p, v_n) &= D_g\big(p, \nabla g^*\big(\alpha_n \nabla g(u_n) + (1-\alpha_n)\nabla g(Tu_n)\big) \\ &\leq \alpha_n D_g(p, u_n) + (1-\alpha_n) D_g(p, u_n) \\ &= D_g(p, u_n). \end{aligned} \tag{4.3}$$

Substituting (4.3) into (4.2), we have

$$D_g(p, z_n) \leq D_g(p, u_n).$$

Thus, $p \in C_n$. Therefore, $\Omega \subset C_n$, $\forall n \geq 0$. Further, by induction we show that $\Omega \subset C_n \cap Q_n$, $n \geq 0$. As $Q_0 = C$, we have $\Omega \subset C_0 \cap Q_0$. Suppose that $\Omega \subset C_m \cap Q_m$, for some $m > 0$. Then, $\exists\, x_{m+1} \in C_m \cap Q_m$ such that $x_{m+1} = \operatorname{proj}^g_{C_m \cap Q_m} x_0$. From the definition of x_{m+1}, we get $\langle \nabla g(x_0) - \nabla g(x_{m+1}), x_{m+1} - z \rangle \geq 0$, $\forall z \in C_k \cap Q_m$. Since $\Omega \subset C_m \cap Q_m$, we have

$$\big\langle \nabla g(x_0) - \nabla g(x_{m+1}), p - x_{m+1} \big\rangle \leq 0, \quad \forall p \in \Omega$$

which implies $p \in Q_{m+1}$. Hence, $\Omega \subset C_{m+1} \cap Q_{m+1}$ implies $\Omega \subset C_n \cap Q_n$, $\forall n \geq 0$ and hence $x_{n+1} = \mathrm{proj}^g_{C_n \cap Q_n} x_0$ is well defined, $\forall n \geq 0$. Hence, $\{x_n\}$ is well defined.

Step III. The sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{z_n\}$, and $\{w_n\}$ are bounded.

Using the definition of Q_n, we get $x_n = \mathrm{proj}^g_{Q_n} x_0$. From the fact that $x_n = \mathrm{proj}^g_{Q_n} x_0$, and using Lemma 2.6 (iii), we obtain

$$\begin{aligned} D_g(x_n, x_0) &= D_g\big(\mathrm{proj}^g_{Q_n} x_0, x_0\big) \\ &\leq D_g(u, x_0) - D_g\big(u, \mathrm{proj}^g_{Q_n} x_0\big) \leq D_g(u, x_0), \quad \forall u \in \Omega \subset Q_n. \end{aligned} \tag{4.4}$$

This implies that $\{D_g(x_n, x_0)\}$ is bounded and hence $\{x_n\}$ is bounded by Lemma 2.5.

Now,

$$D_g(p, x_n) = D_g\big(p, \mathrm{proj}^g_{C_{n-1} \cap Q_{n-1}} x_0\big) \leq D_g(p, x_0) - D_g(x_n, x_0)$$

implies that $\{D_g(p, x_n)\}$ is bounded. Using $D_g(p, Tx_n) \leq D_g(p, x_n)$, $\forall p \in \Omega$, yields that $\{Tx_n\}$ is bounded. Therefore, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, and $\{z_n\}$ are bounded.

Step IV. $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$; $\lim_{n\to\infty} \|x_n - u_n\| = 0$; $\lim_{n\to\infty} \|z_n - u_n\| = 0$; $\lim_{n\to\infty} \|z_n - w_n\| = 0$; $\lim_{n\to\infty} \|u_n - w_n\| = 0$, and $\lim_{n\to\infty} \|u_n - Tu_n\| = 0$.

Since $x_{n+1} = \mathrm{proj}^g_{C_n \cap Q_n} x_0 \in Q_n$ and $x_n \in \mathrm{proj}^g_{Q_n} x_0$, we get

$$D_g(x_n, x_0) \leq D_g(x_{n+1}, x_0), \quad \forall n \geq 0,$$

which implies $\{D_g(x_n, x_0)\}$ is nondecreasing. By the boundedness of $\{D_g(x_n, x_0)\}$, $\lim_{n\to\infty} D_g(x_n, x_0)$ exists and is finite. Further,

$$\begin{aligned} D_g(x_{n+1}, x_n) &= D_g\big(x_{n+1}, \mathrm{proj}^g_{Q_n} x_0\big) \\ &\leq D_g(x_{n+1}, x_0) - D_g\big(\mathrm{proj}^g_{Q_n} x_0, x_0\big) \\ &= D_g(x_{n+1}, x_0) - D_g(x_n, x_0) \end{aligned}$$

which yields

$$\lim_{n\to\infty} D_g(x_{n+1}, x_n) = 0.$$

Using Lemma 2.3,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{4.5}$$

From the definition of u_n, $\|u_n - x_n\| = \|\theta_n(x_n - x_{n-1})\| \leq \|x_n - x_{n-1}\|$, which implies by (4.5) that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{4.6}$$

Since

$$\|u_n - x_{n+1}\| \leq \|u_n - x_n\| + \|x_n - x_{n+1}\|,$$

it follows from (4.5) and (4.6) that

$$\lim_{n\to\infty} \|u_n - x_{n+1}\| = 0. \tag{4.7}$$

Using Lemma 2.4 and the fact that g is uniformly Frechet differentiable, we get

$$\lim_{n\to\infty} \left|g(u_n) - g(x_{n+1})\right| = 0 \tag{4.8}$$

and

$$\lim_{n\to\infty} \left\|\nabla g(u_n) - \nabla g(x_{n+1})\right\| = 0.$$

By the definition of D_g, we get

$$D_g(x_{n+1}, u_n) = g(x_{n+1}) - g(u_n) - \left\langle \nabla g(u_n), x_{n+1} - u_n\right\rangle. \tag{4.9}$$

We have that ∇g is bounded on bounded subsets of Y because g is bounded on Y. Since g is uniformly Frechet differentiable, it is uniformly continuous on bounded subsets. Hence, by (4.7)–(4.9),

$$\lim_{n\to\infty} D_g(x_{n+1}, u_n) = 0. \tag{4.10}$$

As $x_{n+1} = \mathrm{proj}^g_{C_n\cap Q_n} x_0 \in C_n$, we have

$$D_g(x_{n+1}, z_n) \le D_g(x_{n+1}, u_n), \tag{4.11}$$

and hence by (4.10) and (4.11),

$$\lim_{n\to\infty} D_g(x_{n+1}, z_n) = 0.$$

Thanks to Lemma 2.3,

$$\lim_{n\to\infty} \|x_{n+1} - z_n\| = 0. \tag{4.12}$$

Taking into account

$$\|z_n - u_n\| \le \|z_n - x_{n+1}\| + \|x_{n+1} - u_n\|,$$

by (4.7) and (4.12), we get

$$\lim_{n\to\infty} \|z_n - u_n\| = 0. \tag{4.13}$$

By Lemma 2.4,

$$\lim_{n\to\infty} \left|g(z_n) - g(u_n)\right| = 0 \tag{4.14}$$

and

$$\lim_{n\to\infty} \left\| \nabla g(z_n) - \nabla g(u_n) \right\| = 0. \tag{4.15}$$

Next, we estimate the difference

$$\begin{aligned} &D_g(p, u_n) - D_g(p, z_n) \\ &\quad = g(p) - g(u_n) - \langle \nabla g(u_n), p - u_n \rangle - g(p) + g(z_n) + \langle \nabla g(z_n), p - z_n \rangle \\ &\quad = g(z_n) - g(u_n) + \langle \nabla g(z_n), p - z_n \rangle - \langle \nabla g(u_n), p - u_n \rangle \\ &\quad = g(z_n) - g(u_n) + \langle \nabla g(z_n), u_n - z_n \rangle + \langle \nabla g(z_n) - \nabla g(u_n), p - u_n \rangle. \end{aligned} \tag{4.16}$$

Since $\{z_n\}$, $\{u_n\}$, $\{\nabla g(z_n)\}$, and $\{\nabla g(u_n)\}$ are bounded, using (4.13)–(4.16), we get

$$\lim_{n\to\infty} \left| D_g(p, u_n) - D_g(p, z_n) \right| = 0. \tag{4.17}$$

Further, it follows from Lemma 3.1(v) that

$$\begin{aligned} D_g(z_n, w_n) &\le D_g(p, w_n) - D_g(p, z_n) \\ &\le D_g\big(p, \nabla g^*\big(\beta_n \nabla g(Tu_n) + (1 - \beta_n) \nabla g(v_n)\big)\big) - D_g(p, z_n) \\ &\le \beta_n D_g(p, Tu_n) + (1 - \beta_n) D_g(p, u_n) - D_g(p, z_n) \\ &\le D_g(p, u_n) - D_g(p, z_n). \end{aligned} \tag{4.18}$$

Since $\{D_g(p, u_n)\}$ and $\{D_g(p, z_n)\}$ are bounded, by (4.17) and (4.18),

$$\lim_{n\to\infty} D_g(z_n, w_n) = 0,$$

and hence

$$\lim_{n\to\infty} \|z_n - w_n\| = 0. \tag{4.19}$$

From (4.13) and (4.19), we get

$$\lim_{n\to\infty} \|u_n - w_n\| = 0. \tag{4.20}$$

Using uniform Frechet differentiability of g, Lemma 2.4, (4.19), and (4.20), we have

$$\lim_{n\to\infty} \left\| \nabla g(z_n) - \nabla g(w_n) \right\| = 0, \tag{4.21}$$

$$\lim_{n\to\infty} \left\| \nabla g(u_n) - \nabla g(w_n) \right\| = 0. \tag{4.22}$$

Note that

$$\begin{aligned}
&\bigl\|\nabla g(u_n)-\nabla g(w_n)\bigr\| \\
&\quad=\bigl\|\nabla g(u_n)-\nabla g\bigl(\nabla g^*\bigl(\beta_n\nabla g(Tu_n)+(1-\beta_n)\nabla g(v_n)\bigr)\bigr)\bigr\| \\
&\quad=\|\nabla g(u_n)-\beta_n\nabla g(Tu_n)-(1-\beta_n)\nabla g(v_n))\| \\
&\quad=\bigl\|\beta_n\bigl(\nabla g(u_n)-\nabla g(Tu_n)\bigr)+(1-\beta_n)\bigl(\nabla g(u_n)-\nabla g(v_n)\bigr)\bigr\| \\
&\quad=\bigl\|\beta_n\bigl(\nabla g(u_n)-\nabla g(Tu_n)\bigr) \\
&\qquad+(1-\beta_n)\bigl(\nabla g(u_n)-\nabla g\bigl(\nabla g^*\bigl(\alpha_n\nabla g(u_n)+(1-\alpha_n)\nabla g(Tu_n)\bigr)\bigr)\bigr)\bigr\| \\
&\quad=\bigl\|\beta_n\bigl(\nabla g(u_n)-\nabla g(Tu_n)\bigr)+(1-\beta_n)(1-\alpha_n)\bigl(\nabla g(u_n)-\nabla g(Tu_n)\bigr)\bigr\| \\
&\quad=\bigl[1-\alpha_n(1-\beta_n)\bigr]\bigl\|\nabla g(u_n)-\nabla g(Tu_n))\bigr\|.
\end{aligned} \tag{4.23}$$

By (4.22), (4.23), and using $\lim_{n\to\infty}\alpha_n=0$, we get

$$\lim_{n\to\infty}\bigl\|\nabla g(u_n)-\nabla g(Tu_n)\bigr\|=0. \tag{4.24}$$

Moreover, we have from (4.24) that

$$\lim_{n\to\infty}\|u_n-Tu_n\|=0. \tag{4.25}$$

Step V. $\bar{x}\in\Omega$.

First, we prove that $\bar{x}\in F(T)$. As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}\subseteq\{x_n\}$ such that $x_{n_k}\rightharpoonup\bar{x}\in K$ as $k\to\infty$. Due to (4.6), (4.13), (4.19), and (4.20), the sequences $\{x_n\}$, $\{u_n\}$, $\{w_n\}$, and $\{z_n\}$ have same asymptotic behavior and thus there exist subsequences $\{u_{n_k}\}$ of $\{u_n\}$, $\{w_{n_k}\}$ of $\{w_n\}$, and $\{z_{n_k}\}$ of $\{z_n\}$ such that $u_{n_k}\rightharpoonup\bar{x}$, $w_{n_k}\rightharpoonup\bar{x}$, and $z_{n_k}\rightharpoonup\bar{x}$ as $k\to\infty$. Using $u_{n_k}\rightharpoonup\bar{x}$ and (4.25) shows that

$$\lim_{k\to\infty}\|u_{n_k}-Tu_{n_k}\|=0.$$

By the definition of T, $\bar{x}\in\widehat{F}(T)=F(T)$.

Next, we prove that $\bar{x}\in\mathrm{Sol}(\mathrm{GEP}(1.1))$. As $z_n=\mathrm{res}^g_{G,b}w_n$, we have

$$G(z_{n_k},v)+\bigl\langle\nabla g(z_{n_k})-\nabla g(w_{n_k}),v-z_{n_k}\bigr\rangle+b(v,z_{n_k})-b(z_{n_k},z_{n_k})\geq 0,\quad\forall v\in C.$$

Using Assumption 2.1, we have

$$\bigl\langle\nabla g(z_{n_k})-\nabla g(w_{n_k}),v-z_{n_k}\bigr\rangle\geq G(v,z_{n_k})-b(v,z_{n_k})+b(z_{n_k},z_{n_k}),\quad\forall v\in C. \tag{4.26}$$

Using the definition of G, ϕ, (4.21), and letting $k\to\infty$ in (4.26), we obtain

$$0\geq G(v,\bar{x})-b(v,\bar{x})+b(\bar{x},\bar{x}).$$

Consider $v_s:=sv+(1-s)\bar{x}$, $\forall s\in(0,1]$ and $v\in K$. Then, $v_s\in K$ and hence

$$G(v_s,\bar{x})-b(v_s,\bar{x})+b(\bar{x},\bar{x})\leq 0.$$

Now,

$$\begin{aligned}0 &= G(v_s, v_s)\\ &\leq sG(v_s, v) + (1-s)G(v_s, \bar{x})\\ &\leq sG(v_s, v) + (1-s)\bigl[b(v_s, \bar{x}) - b(\bar{x}, \bar{x})\bigr]\\ &\leq sG(v_s, v) + (1-s)s\bigl[b(v, \bar{x}) - b(\bar{x}, \bar{x})\bigr].\end{aligned}$$

Thus we get

$$G(\bar{x}, v) + b(v, \bar{x}) - b(\bar{x}, \bar{x}) \geq 0, \quad \forall v \in K,$$

which implies $\bar{x} \in \mathrm{Sol}(\mathrm{GEP}(1.1))$. Therefore, $\bar{x} \in \Omega$.

Step VI. $x_n \to \bar{x} = \mathrm{proj}^g_\Omega x_0$.

Let $\tilde{u} = \mathrm{proj}^g_\Omega x_0$. As $\{x_n\}$ is weakly convergent, $x_{n+1} = \mathrm{proj}^g_\Omega x_0$ and $\mathrm{proj}^g_\Omega x_0 \in \Omega \subset C_n \cap Q_n$. By (4.4) we see that

$$D_g(x_{n+1}, x_0) \leq D_g\bigl(\mathrm{proj}^g_\Omega x_0, x_0\bigr).$$

Using Lemma 2.8, $\{x_n\}$ is strongly convergent to $\tilde{u} = \mathrm{proj}^g_\Omega x_0$. Hence, by the uniqueness of the limit, $\{x_n\}$ converges strongly to $\bar{x} = \mathrm{proj}^g_\Omega x_0$. □

5 Consequences

If $g(x) = \frac{1}{2}\|x\|^2$, $\forall x \in Y$, a Bregman relatively nonexpansive becomes a relatively nonexpansive mapping.

Corollary 5.1 *Let $G, b : K \times K \to \mathbb{R}$ be bifunctions satisfying Assumptions 2.1 and 2.2, respectively. Let T be a relatively nonexpansive mapping on K. Let $\Omega = \mathrm{Sol}(\mathrm{GEP}(1.1)) \cap F(T) \neq \emptyset$. Let $\{x_n\}, \{z_n\} \subseteq K$ be generated by*

$$\begin{cases} x_0 = x_{-1} \in K, \\ u_n = x_n + \theta_n(x_n - x_{n-1}), \\ v_n = J^{-1}(\alpha_n J(u_n) + (1-\alpha_n)J(Tu_n)), \\ w_n = J^{-1}(\beta_n J(Tu_n) + (1-\beta_n)J(v_n)), \\ z_n = \mathrm{res}_{G,b} w_n, \\ C_n = \{z \in K : \Psi(z, z_n) \leq \Psi(z, u_n)\}, \\ Q_n = \{z \in K : \langle J(x_0) - J(x_n), z - x_n\rangle \leq 0\}, \\ x_{n+1} = \prod_{C_n \cap Q_n} x_0, \quad \textit{for all } n \geq 0, \end{cases}$$

where Ψ is defined in Remark 2.2, $\{\theta_n\} \subseteq (0,1)$, $\{\alpha_n\}, \{\beta_n\} \subseteq [0,1]$ with $\lim_{n\to\infty} \alpha_n = 0$. Then, $\{x_n\}$ converges strongly to $\prod_\Omega x_0$.

Also, if $\mathrm{GEP}(1.1) = K$ then Theorem 4.1 can be rewritten as

Corollary 5.2 *Let T be a relatively nonexpansive mapping on K with $F(T) \neq \emptyset$. Let $\{x_n\}, \{z_n\} \subseteq K$ be generated by*

$$\begin{cases} x_0 = x_{-1} \in K, \\ u_n = x_n + \theta_n(x_n - x_{n-1}), \\ v_n = J^{-1}(\alpha_n J(u_n) + (1-\alpha_n)J(Tu_n)), \\ z_n = J^{-1}(\beta_n J(Tu_n) + (1-\beta_n)J(v_n)), \\ C_n = \{z \in C : \Psi(z, z_{n+1}) \leq \Psi(z, u_n)\}, \\ Q_n = \{z \in C : \langle J(x_0) - J(x_n), z - x_n\rangle \leq 0\}, \\ x_{n+1} = \prod_{C_n \cap Q_n} x_0, \quad \text{for all } n \geq 0, \end{cases}$$

where Ψ is defined in Remark 2.2, $\{\theta_n\} \subseteq (0,1)$, $\{\alpha_n\}, \{\beta_n\} \subseteq [0,1]$ with $\lim_{n\to\infty} \alpha_n = 0$. Then, $\{x_n\}$ converges strongly to $\prod_{F(T)} x_0$.

Moreover, if GEP(1.1) $= K$ then, using Example 2.1 for $A : E \to 2^{E^*}$, maximal monotone operator, we have

Corollary 5.3 *Let $K \subseteq Y$ with $K \subseteq \operatorname{int}(\operatorname{dom} g)$, where $g : Y \to (-\infty, +\infty]$ is a coercive Legendre function which is bounded, uniformly Frechet differentiable, and totally convex on bounded subsets of Y. Let $A : E \to 2^{E^*}$ be a maximal monotone operator with $A^{-1}(0) \neq \emptyset$. Let $\{x_n\}, \{z_n\} \subseteq K$ be generated by*

$$\begin{cases} x_0 = x_{-1} \in K, \\ u_n = x_n + \alpha_n(x_n - x_{n-1}), \\ v_n = \nabla g^*(\alpha_n \nabla g(u_n) + (1-\alpha_n)\nabla g(\operatorname{res}_A^g u_n)), \\ w_n = \nabla g^*(\beta_n \nabla g(\operatorname{res}_A^g u_n) + (1-\beta_n)\nabla g(v_n)), \\ z_n = \operatorname{res}_{G,b}^g w_n, \\ C_n = \{z \in C : D_g(z, z_n) \leq D_g(z, u_n)\}, \\ Q_n = \{z \in C : \langle \nabla g(x_0) - \nabla g(x_n), z - x_n\rangle \leq 0\}, \\ x_{n+1} = \operatorname{proj}_{C_n \cap Q_n}^g x_0, \quad \text{for all } n \geq 0, \end{cases}$$

where $\{\theta_n\} \subseteq (0,1)$, $\{\alpha_n\}, \{\beta_n\} \subseteq [0,1]$ with $\lim_{n\to\infty} \alpha_n = 0$. Then, $\{x_n\}$ converges strongly to $\operatorname{proj}_{A^{-1}(0)} x_0$.

6 Numerical example

Example 6.1 Let $Y = \mathbb{R}$, $K = [r_1, r_2]$, where $r_1, r_2 \in \mathbb{R}$ are arbitrary but fixed, and $g : \mathbb{R} \to \mathbb{R}$ with $g(u) = \frac{2}{3}u^2$. Obviously, $g : \mathbb{R} \to \mathbb{R}$ is a coercive Legendre function which is bounded, uniformly Frechet differentiable, and totally convex on bounded subsets of $\mathbb{R}$, and $\nabla g(u) = \frac{4}{3}u$. As $g^*(u^*) = \sup\{\langle u^*, u\rangle - g(u) : u \in \mathbb{R}\}$, we get $g^*(w) = \frac{3}{8}w^2$ and $\nabla g^*(w) = \frac{3}{4}w$. Let $G : K \times K \to \mathbb{R}$ with $G(u,v) = (u-1)(v-u), \forall u, v \in K$ and let $b : K \times K \to \mathbb{R}$ be such that $b(u,v) = uv$, $\forall u, v \in K$. Obviously, G and b satisfy Assumptions 2.1 and 2.2, respectively, and Sol(GEP(1.1)) $= \{\frac{1}{2}\} \neq \emptyset$. Let $T : K \to K$ be such that $Tx = \frac{x+1}{3}$. Clearly, T is a Bregman relatively nonexpansive mapping and $F(T) = \{\frac{1}{2}\}$. Thus, $\Omega = \{\frac{1}{2}\} \neq \emptyset$. Suppose $\{\alpha_n\} = \{\frac{1}{n^3}\}$,

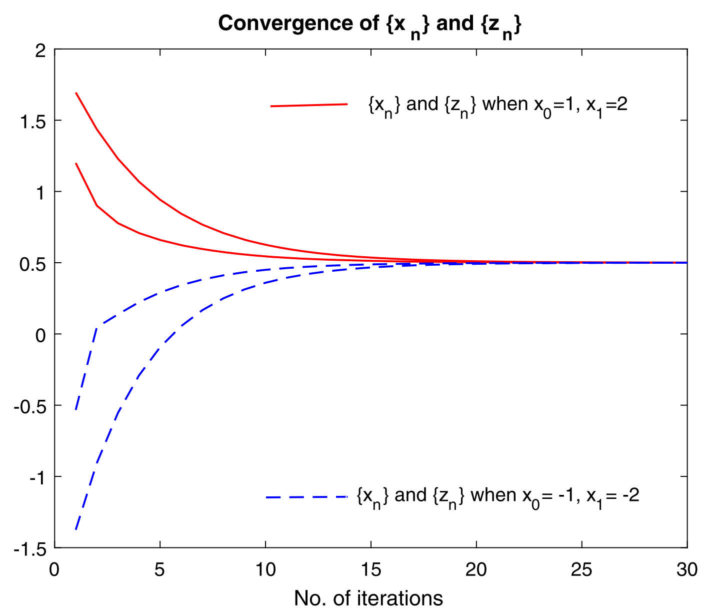

Figure 1 Convergence of sequences

Table 1 Values of x_n and z_n

No. of iterations	x_n $x_0 = 1, x_1 = 2$	z_n	x_n $x_0 = -1, x_1 = -2$	z_n
1	1.695000	1.200000	−1.374971	−0.533333
2	1.436308	0.900583	−0.906199	0.046971
3	1.229982	0.777508	−0.554620	0.137395
4	1.068285	0.707982	−0.290936	0.222908
5	0.942164	0.659502	−0.093173	0.290718
6	0.843949	0.623263	0.055150	0.342544
7	0.767517	0.595549	0.166392	0.381722
8	0.708055	0.574170	0.249823	0.411218
9	0.661802	0.557615	0.312396	0.433385
10	0.625829	0.544772	0.359326	0.450031
11	0.597852	0.534800	0.394524	0.462523
12	0.576094	0.527053	0.420922	0.471898
13	0.559174	0.521032	0.440721	0.478930
14	0.546016	0.516352	0.455570	0.484206
15	0.535783	0.512714	0.466707	0.488164
20	0.510175	0.503614	0.492188	0.497220
25	0.502893	0.501027	0.498235	0.499370
29	0.501058	0.500376	0.499522	0.499827
30	0.500823	0.500292	0.499670	0.499880

$\{\beta_n\} = \{\frac{1}{n^2}\}$ and $\theta_n = 0.6$. After simplification, the hybrid iterative scheme (4.1) becomes: given x_0, x_1,

$$u_n = x_n + \theta_n(x_n - x_{n-1}),$$

$$v_n = \alpha_n u_n + (1-\alpha_n)\left(\frac{u_n + 1}{3}\right),$$

$$w_n = \beta_n\left(\frac{u_n + 1}{3}\right) + (1-\beta_n)v_n; \qquad z_n = \frac{4w_n + 3}{10},$$

$$C_n = [e_n, \infty), \quad \text{where } e_n := \frac{z_n + u_n}{2}; \qquad Q_n = [x_n, \infty);$$
$$x_{n+1} = \mathrm{proj}^g_{C_n \cap Q_n} x_0, \quad \forall n \geq 1.$$

Then, there are unique sequences $\{x_n\}$ and $\{z_n\}$ obtained by (4.1) converging to $\bar{x} = \{\frac{1}{2}\} \in \Omega$.

7 Conclusion

The aim of this paper is to introduce and study an inertial hybrid iterative method to find the common solution of GEP and FPP for a Bregman relatively nonexpansive mapping in a Banach space. From a theoretical and application point of view, an inertial method via Bregman relatively nonexpansive mapping has great importance for data analysis and some imaging problems. It is worth mentioning that the convergence for inertial iterative methods in the setting of Banach spaces is still unexplored.

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Abbreviations
GEP, Generalized equilibrium problem; EP, Equilibrium problem; VIP, Variational inequality problem; FPP, Fixed point problem.

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Competing interests
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