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Stability criteria for nonlinear Volterra integro-dynamic matrix Sylvester systems on measure chains

Sreenivasulu Ayyalappagari^{1*}  and Venkata Appa Rao Bhogapurapu²

*Correspondence:

sreenu5408@gmail.com

¹Department of Mathematics,
Koneru Lakshmaiah Education
Foundation and Malla Reddy
Institute of Technology and Science,
Dhulapally, Secunderabad, 500100,
Telangana, India

Full list of author information is
available at the end of the article

Abstract

In this paper, we establish sufficient conditions for various stability aspects of a nonlinear Volterra integro-dynamic matrix Sylvester system on time scales. We convert the nonlinear Volterra integro-dynamic matrix Sylvester system on time scale to an equivalent nonlinear Volterra integro-dynamic system on time scale using vectorization operator. Sufficient conditions are obtained to this system for stability, asymptotic stability, exponential stability, and strong stability. The obtained results include various stability aspects of the matrix Sylvester systems in continuous and discrete models.

Keywords: Nonlinear Volterra integro-dynamic; Stability; Timescales

1 Introduction

The applications of differential equations in science and engineering problems are well known. Difference equations are the discrete counterpart of differential equations. The applicability of difference equations is gaining an important role in computer science, control theory, image processing, digital filter design, numerical analysis, and finite element techniques. Agarwal [1] presented several theoretical methods and applications of difference equations. The Ψ -stability for difference equations is studied in [2].

In real life, applications are simple and efficient for integral equations. The existence of solutions for nonlinear quadratic integral equations was obtained in [3]. The solution of Fredholm integral equations via fixed point on extending b-metric spaces was introduced in [4]. The solutions of Volterra integral equations were obtained using numerical scheme in d-metric spaces [5], fixed point theory [6], and hybrid contractions [7]. Recently, fractional differential equations have been applied in many areas of science and engineering. Many authors [8–11] studied different types of fractional differential equations. Kim [12] studied semi-linear problems with a non-symmetric linear part and a nonlinear part of monotone type in real Hilbert spaces. Nguyen Duc Phuonga [13] proved that the regularized solution satisfies the conditions of a well-posed problem in the sense of Hadamard and also used the modified quasi-boundary method to deal with the inverse source problem, a well-posed method.

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Time scales theory was introduced by Hilger [14, 15] to unify discrete and continuous cases in a well-organized structure. For an excellent introduction to the calculus and dynamic equations on time scales, one can refer to [1, 16–18]. In [19], the authors presented basic qualitative and quantitative results for solutions to nonlinear dynamic equations on time scales with applications to economic modeling. The authors [20, 21] studied stability of dynamic equations on time scales. The solution of Volterra integro-dynamic equations on time scales using variation of parameter was obtained in [22] and the qualitative and quantitative results were obtained in [23]. In [24], authors established the existence of Ψ -bounded solutions for a system of linear dynamic equations on time scales. Recently, Agarwal [25] described the connection between the F-contraction mappings on metric-like spaces to integral equations on time scales.

The matrix differential systems are generalizations of a system of differential equations. The fundamental matrix of a system consists of linearly independent solutions. The transition matrix of a system is obtained from fundamental matrix. Dacunha [26] studied transition matrix and generalized matrix exponential via the Peano–Baker series. Murty et al. [27–30] studied matrix Lyapunov systems. The controllability for a fuzzy dynamical matrix Lyapunov system was studied in [31]. In [32], authors established the necessary and sufficient conditions for controllability and observability of the Sylvester matrix dynamical system on time scales. In this paper, we deal with a nonlinear Volterra integro-dynamic matrix Sylvester system on time scales and establish the conditions for stability, asymptotic stability, exponential stability, and strong stability.

$$\begin{cases} X^\Delta(t) = A(t)X(t) + X(t)B(t) + \mu(t)A(t)X(t)B(t) \\ \quad + \int_{t_0}^t (L_1(t, X(s))X(s) + X(s)L_2(t, X(s)))\Delta s, \\ X(t_0) = X_0, \end{cases} \quad (1)$$

where $X(t)$ is an $n \times n$ matrix, Here, $A(t), B(t), L_1(t)$, and $L_2(t)$ are $n \times n, n \times n, n \times n$ and $n \times m$ are rd-continuous matrices respectively, $X^\Delta(t)$ is the generalized delta derivative of X , and $\mu(t)$ is a graininess function.

2 Preliminaries

Definition 2.1 ([16]) A time scale \mathbb{T} is a closed subset of \mathbb{R} . By an interval we mean the intersection of the given interval with a time scale. For $t < \sup \mathbb{T}$ and $r > \inf \mathbb{T}$, define the forward jump operator σ and backward jump operator ρ respectively by $\sigma(t) = \inf\{s \in T : s > t\} \in \mathbb{T}$, $\rho(t) = \sup\{s \in T : s < r\} \in \mathbb{T}$, $\forall t, r \in \mathbb{T}$ if $\sigma(t) = t$, t is said to be right-dense (otherwise t is said to be right-scattered) and $\rho(r) = r$, r is said to be left-dense (otherwise r is said to be left-scattered). The graininess function $\mu(t) : \mathbb{T} \rightarrow [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$.

Definition 2.2 ([16]) For $x : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in T$ (if $t = \sup \mathbb{T}$, assume that t is not left-scattered), define the delta derivative of $x(t)$, represented by $x^\Delta(\tau)$, to be a number, with the property that, for any $\epsilon > 0$, \exists in the neighborhood U of t such that

$$| [x(\sigma(t)) - x(s)] - x^\Delta(t)[\sigma(\tau) - s] | \leq \epsilon |\sigma(t) - s|, \quad \forall s \in U.$$

If x is delta differentiable for every $t \in T$, we say that $x : \mathbb{T} \rightarrow \mathbb{R}$ is delta differentiable on T .

Definition 2.3 ([16]) A function h is known as rd-continuous provided that it is continuous at right-dense points in T , and finite limit at left-dense points, and the set of rd-continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$. The set of functions $C_{rd}^1(\mathbb{T}, \mathbb{R})$ includes the functions h whose derivative is in $C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.4 ([16]) For $a, t \in \mathbb{T}$ and a function $h \in C_{rd}(\mathbb{T}, \mathbb{R})$, delta-integral is defined to be

$$\int_a^t h(s) \Delta s = H(t) - H(a),$$

where $H \in C_{rd}^1(\mathbb{T}, \mathbb{R})$ is anti-derivative of h , i.e., $H^\Delta(t) = h$.

Theorem 2.1 ([16]) Assume that $h : \mathbb{T} \rightarrow \mathbb{R}$ is a function, and let $t \in \mathbb{T}^k$. Then we have the following:

- i. If t is right-dense, then h is differentiable at t iff the limit $\lim_{t \rightarrow s} \frac{h(t) - h(s)}{t - s}$ exists as a finite number. In this case $h^\Delta(t) = \lim_{t \rightarrow s} \frac{h(t) - h(s)}{t - s}$.
- ii. If h is differentiable at t , then $h(\sigma(t)) = h(t) + \mu h^\Delta(t)$.
- iii. If h is continuous at t and t is right-scattered, then f is differentiable at t with $h^\Delta(t) = \frac{h(t) - h(s)}{\mu(t)}$.
- iv. If h is continuous at t , then f is differentiable at t .

Definition 2.5 ([32]) Let two right-dense continuous matrices be A_1 and A_2 on \mathbb{T} , on time scales \mathbb{T} . Therefore,

$$A_1^\Delta(t) \otimes A_2(t) + A_1(\sigma(t)) \otimes A_2^\Delta(t) = (A_1 \otimes A_2)^\Delta(t).$$

Now, we are applying the Vec operator to the delta-differentiable Volterra integro-dynamic matrix Sylvester system (1) by using Kronecker product (KP) properties, we obtain

$$\begin{cases} z^\Delta(t) = P(t)z(t) + \int_0^t G(t, s, z(s)) \Delta s, \\ z(t_0) = z_0, \end{cases} \quad (2)$$

which is piecewise continuous on $\Omega := \{(t, s) \in \mathbb{T}_0 \times \mathbb{T}_0 : t_0 \leq s \leq t < \infty\}$, and $P(t) = [(B^* \otimes I) + (I \otimes A) + \mu(t)(B^* \times A)]$ of order $n^2 \times n^2$, $G(t, s, z(s)) = [(L_2^* \otimes I) + (I \otimes L_1)]$ is an $n^2 \times n^2$ matrix function. $z(t) = \text{Vec } X(t)$.

Definition 2.6 ([16]) For $P \in \mathcal{R}$, the generalized exponential function is defined as

$$e_P(t, s) = \exp\left(\int_s^t \xi_\mu(t)(P(\tau)) \Delta \tau\right), \quad t, s \in \mathbb{T},$$

where $\xi_\mu(t)(P(\tau))$ is the cylinder transformation given by

$$\xi_\mu(t)(P(\tau)) = \begin{cases} \frac{\log(1 + \mu(\tau)P)}{\mu(\tau)} & \text{if } \mu(\tau) \neq 0, \\ P & \text{if } \mu(\tau) = 0. \end{cases}$$

Let the complete metric space with the distance (metric) d be defined by $d((u_1, v_1), (u_2, v_2)) = \sqrt{(v_1 - v_2)^2 + (u_1 - u_2)^2}$, $\forall (u_1, v_1), (u_2, v_2) \in \mathbb{T}_1 \times \mathbb{T}_2$.

Here, $\mathbb{T}_1 \times \mathbb{T}_2 = \{(u, v) : u \in \mathbb{T}_1, v \in \mathbb{T}_2\}$ are two given time scales.

A function $f : \mathbb{T}_1 \times \mathbb{T}_2 \rightarrow \mathbb{R}$ is called continuous at $(u, v) \in \mathbb{T}_1 \times \mathbb{T}_2$, $\forall \epsilon > 0, \exists \delta > 0$ such that $\|f(u, v) - f(u_0, v_0)\| < \epsilon$, $\forall (u_0, v_0) \in \mathbb{T}_1 \times \mathbb{T}_2$ satisfying $d((u, v), (u_0, v_0)) < \delta$.

Lemma 2.1 ([18])

i. If ψ is rd-continuous and nonnegative, thus

$$1 + \int_s^t \psi(u) \Delta u \leq e_{\psi}(t, s) \leq \exp\left(\int_s^t \psi(u) \Delta u\right) \quad \forall t \geq s.$$

ii. For a nonnegative ψ with $-\psi \in \mathcal{R}^+$, it becomes

$$1 - \int_s^t \psi(u) \Delta u \leq e_{-\psi}(t, s) \leq \exp\left(-\int_s^t \psi(u) \Delta u\right) \quad \forall t \geq s.$$

Theorem 2.2 ([26]) Let $f(u, v)$ be a real finite-valued function whose domain is the cartesian product $S_1 \times S_2$. Suppose that $f(u, v)$ is continuous in u at $u=a$ for all v in S_2 , and continuous in v at $v=b$ uniformly for u in S_1 , then $f(u, v)$ is continuous in (u, v) at (a, b) .

Let rd-continuous functions $f(u, v)$ on $\mathbb{T}_1 \times \mathbb{T}_2$ have the following properties:

- f is rd-continuous in v for fixed u ;
- f is rd-continuous in u for fixed v ;
- If u_0 and v_0 are both left-dense, thus the limit of $f(u, v)$ exists (finite) as (u, v) approaches (u_0, v_0) along any path in $(u, v) \in \mathbb{T}_1 \times \mathbb{T}_2 : u < u_0, v < v_0$;
- If $(u_0, v_0) \in \mathbb{T}_1 \times \mathbb{T}_2$ with u_0 maximal or right-dense and v_0 maximal or right-dense. Thus, f is continuous at (u_0, v_0) .

3 Stability

In this section, when we say the zero solution of (2) we mean the zero solution of (2) with $z_0 = 0$.

Definition 3.1 The zero solution of (2) is stable. If for every $\epsilon > 0$ there exists $\delta > 0$ such that, for any solution $z(t)$ of (2), the inequality $\|z_0\| < \delta \implies \|z\| < \epsilon \quad \forall t \in \mathbb{T}_0$.

We assume that the zero solution of the following system

$$\begin{cases} x^\Delta(t) = P(t)x(t), \\ x(t_0) = x_0 \end{cases} \quad (3)$$

is stable. From Theorem 2.1 [20] we suppose that there exists $\eta > 0$ such that

$$\|\psi_p(t, s)\| \leq \eta, \quad \forall s \in [t_0, t]_{\mathbb{T}}, \quad (4)$$

where $\psi_p(t, s)$ is a fundamental matrix of (3).

Next, we impose a condition on $G(t, s, z)$ for proving that the zero solution of (2) is stable.

We assume the following condition:

H1: $\exists \alpha > 0$ so that $\|G(t, s, z)\| \leq Q(t, s)\|z\|$ with $Q(t, s)$ rd-continuous for $s \in [t_0, t]_{\mathbb{T}}$ and $\|z\| < \alpha$.

Theorem 3.1 *Assume that equation (4) and (H1) are satisfied, and there exists a positive constant $L > 0$ such that*

$$\int_{t_0}^{\infty} \int_{t_0}^s Q(s, u) \Delta s \Delta u < L. \quad (5)$$

Then the zero solution of (2) is stable.

Proof. For any $0 < \epsilon < \alpha$, let $\delta(\epsilon) < \frac{\epsilon}{\eta e^{\eta L}}$ and $\|z(t)\| < \delta(\epsilon)$.

Assume that there exists $t_1 \in \mathbb{T}_0$ such that $\|z(t_0)\| = \epsilon$ and $\|z(t_0)\| < \epsilon$ on $[t_0, t_1]_{\mathbb{T}}$. By using the variation of parameters formula, we get

$$\begin{aligned} \|z(t)\| &< \|\psi_P(t, t_0)\| \|z_0\| + \int_{t_0}^t \|\psi_P(t, \sigma(s))\| \int_{t_0}^s Q(s, u) \|z(u)\| \Delta u \Delta s \\ &\leq \eta \delta(\epsilon) + \eta \int_{t_0}^t \int_{t_0}^s Q(s, u) \|z(u)\| \Delta u \Delta s \end{aligned}$$

for $t \in [t_0, t_1]_{\mathbb{T}}$.

Let $r(t) = \sup_{s \in [t_0, t_1]_{\mathbb{T}}} \|z(s)\|$, and according to

$$r(t) \leq \eta \delta(\epsilon) + \eta \int_{t_0}^t \int_{t_0}^s Q(s, u) r(u) \Delta u \Delta,$$

from Gronwal's inequality ([17], Theorem 6.4) and Lemma 2.1, we get

$$\begin{aligned} \|z(t)\| &\leq r(t) \\ &\leq \eta \delta(\epsilon) \exp \left(\int_{t_0}^t \log \left(\frac{1 + \mu(s) \eta \int_{t_0}^s Q(s, u) \Delta u}{\mu(s)} \right) \Delta s \right) \\ &\leq \eta \delta(\epsilon) \exp \left(\int_{t_0}^t \int_{t_0}^s \eta Q(s, u) \Delta u \Delta s \right) \\ &\leq \eta \delta(\epsilon) e^{\eta L} < \epsilon, \quad \forall t \in [t_0, t_1]_{\mathbb{T}}. \end{aligned}$$

Thus, $\|z(t_1)\| < \epsilon$. Therefore, it is a contradiction. Then the zero solution of (2) is stable. \square

In place of (H1) assume that

$\hat{H}1$: $\|G(t, s, z)\| \leq Q(t, s)\|z\|$ with $Q(t, s)$ rd-continuous for $s \in [t_0, t]_{\mathbb{T}}$ and $\|z\| \in \mathbb{R}^{n^2}$.

Corollary. If equations (4), (5) and ($\hat{H}1$) are satisfied, then the solutions of system (3) are bounded.

4 Asymptotic stability

Suppose that there exists a constant $\beta > 0$ such that

$$\int_{t_0}^t \|\psi_P(t, \sigma(s))\| \Delta s < \beta \quad (6)$$

for all $t \in \mathbb{T}_0$ with $t \geq \sigma(t_0)$ (which is similar to the results of [20]).

The fundamental matrix ψ_P has the following property:

$$\psi_P(t, \sigma(s)) \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (7)$$

Definition 4.1 The zero solution of (2) is asymptotically stable. If it is stable and attractive (i.e., if for any solution $z(t)$ of (3) there exists $\delta_0 \geq 0$ such that $\|z_0\| < \delta_0 \implies \|z\| \rightarrow 0$ as $t \rightarrow \infty$).

Theorem 4.1 Assume that condition (H1) and (6) are satisfied and

$$\sup_{t \in \mathbb{T}_0} \int_{t_0}^t Q(t, s) \Delta s < \frac{1}{\beta}. \quad (8)$$

Moreover, assume that

$$\lim_{s \rightarrow \infty} \int_{t_0}^s Q(t, s) \Delta s = 0, \quad \forall t \in \mathbb{T}_0. \quad (9)$$

Then the zero solution of (2) is asymptotically stable.

Proof. First, we have to prove the stability of the zero solution of (2). From (8), there exists a positive constant γ such that

$$0 < \gamma < \frac{1}{\beta} \quad \text{and} \quad \sup_{t \in \mathbb{T}_0} \int_{t_0}^t Q(t, s) \Delta s \leq \gamma. \quad (10)$$

From (7), there exists a positive constant M such that

$$\|\psi_P(t, t_0)\| \leq M, \quad \forall t \in \mathbb{T}_0. \quad (11)$$

For any $0 < \epsilon < \alpha$ and t_0 , let $\delta(\epsilon) < \min\{\frac{(1-\gamma\beta)\epsilon}{M}, \epsilon\}$.

Consider the solution $z(t)$ of (2) such that $\|z_0\| < \delta$. Assume that there exists $t_1 \in \mathbb{T}_0$ such that $\|z(t_1)\| = \epsilon$ and $\|z(t)\| < \epsilon$ on $[t_0, t]_{\mathbb{T}_0}$. By using the variation of parameters formula [17], we get

$$\begin{aligned} \|z(t)\| &\leq \|\psi_P(t, t_0)\| \|z_0\| + \int_{t_0}^t \|\psi_P(t, \sigma(s))\| \int_{t_0}^s Q(s, u) \|z(u)\| \Delta u \Delta s \\ &< N\delta + \epsilon \int_{t_0}^t \|\psi_P(t, \sigma(s))\| \int_{t_0}^s Q(s, u) \Delta u \Delta s \\ &< (1 - \gamma\beta)\epsilon + \epsilon\beta\gamma = \epsilon, \quad \forall t \in [t_0, t]_{\mathbb{T}}. \end{aligned}$$

Therefore, $\|z(t_1)\| < \epsilon$, hence, it is a contradiction. Thus, the zero solution of (2) is stable.

Next, we prove that the zero solution of (2) is attractive. Let $\epsilon = 1, \exists \delta_0 < \delta(1) < 1$ such that $\|z_0\| < \delta_0$ indicates that

$$\|z(t)\| < \min(\alpha, 1), \quad \forall t \in \mathbb{T}_0. \quad (12)$$

Assume that there exists z_0 with $\|z_0\| < \delta_0$ such that the solution $z(t)$ of (2) satisfies

$$\limsup_{t \rightarrow \infty} \|z(t)\| = \lambda > 0. \quad (13)$$

From (10), $\gamma\beta < 1$, and there exists a constant θ such that $\gamma\beta < \theta < 1$. From (13) there exists $t_1 \in \mathbb{T}_0$ such that

$$\|z(u)\| \leq \frac{\lambda}{\theta}, \quad \forall u \in [t_1, \infty)_{\mathbb{T}}, \quad (14)$$

and from (9) there exists $T \in [t_1, \infty)_{\mathbb{T}}$ such that

$$\int_{t_0}^{t_1} Q(s, u) \Delta u < \frac{(\theta - \gamma\beta)\lambda}{2\theta\beta}, \quad \forall s \in [T, \infty)_{\mathbb{T}}. \quad (15)$$

Then

$$\begin{aligned} \|z(t)\| &\leq \|\psi_P(t, t_0)\| \delta_0 + \int_{t_0}^t \|\psi_P(t, \sigma(s))\| \int_{t_0}^s Q(s, u) \|z(u)\| \Delta u \Delta s \\ &\leq \|\psi_P(t, t_0)\| \delta_0 + \|\psi_P(t, t_0)\| \int_{t_0}^T \|\psi_P(t_0, \sigma(s))\| \int_{t_0}^s Q(s, u) \|z(u)\| \Delta u \Delta s \\ &\quad + \int_T^t \|\psi_P(t, \sigma(s))\| \int_{t_0}^{t_1} Q(s, u) \|z(u)\| \Delta u \Delta s \\ &\quad + \int_T^t \|\psi_P(t, \sigma(s))\| \int_{t_1}^s Q(s, u) \|z(u)\| \Delta u \Delta s. \end{aligned}$$

From (6), (12), and (15), we get

$$\int_T^t \|\psi_P(t, \sigma(s))\| \int_{t_0}^{t_1} Q(s, u) \|z(u)\| \Delta u \Delta s \leq \frac{(\theta - \gamma\beta)\lambda}{2\theta\beta}.$$

Furthermore, using (6), (10), and (14), according to

$$\int_T^t \|\psi_P(t, \sigma(s))\| \int_{t_1}^s Q(s, u) \|z(u)\| \Delta u \Delta s \leq \frac{\gamma\beta\lambda}{\theta}.$$

Then

$$\begin{aligned} \|z(t)\| &\leq \|\psi_P(t, t_0)\| \delta_0 + \|\psi_P(t, t_0)\| \int_{t_0}^T \|\psi_P(t_0, \sigma(s))\| \\ &\quad \times \int_{t_0}^s Q(s, u) \|z(u)\| \Delta u \Delta s + \frac{(\theta + \gamma\beta)\lambda}{2\theta}. \end{aligned}$$

Since $\|\psi_P(t, t_0)\| \rightarrow 0$ as $t \rightarrow \infty$, by (7) we get $\lambda + \frac{(\theta + \gamma\beta)\lambda}{2\theta}$, and then $\lambda < \lambda$. Therefore, it is a contradiction. Thus, the zero solution of (2) is attractive. Hence, the zero solution of (2) is asymptotically stable. \square

5 Exponential asymptotic stability

Definition 5.1 The zero solution of (2) is exponentially asymptotically stable, then there exists $\eta > 0$, and for every $\epsilon > 0$ there exists $\delta > 0$ such that, for any solution $z(t)$ of (2), $\|z_0\| < \delta \implies \|z\| < \epsilon e_{-\eta}(t, t_0), \forall t \in \mathbb{T}_0$.

Suppose that there exists $L, \eta > 0$ with $-\eta \in \mathcal{R}^+(\mathbb{T}, \mathbb{R})$ such that

$$\|\psi_P(t, s)\| \leq M e_{-\nu}(t, s), \quad \forall s \in [t_0, t]_{\mathbb{T}}. \quad (16)$$

(This result is useful for proving that the zero solution of (3) is exponentially asymptotically stable.)

Theorem 5.1 Let conditions (H1) and (16) be satisfied, and there exists a positive constant ν such that

$$\sup_{t \in \mathbb{T}_0} \int_{t_0}^t e_{-\nu}(s, \sigma(t)) Q(t, s) \Delta s < \frac{\eta}{L}. \quad (17)$$

Then the zero solution of (2) is exponentially asymptotically stable.

Proof By using (16) $\forall t \in \mathbb{T}_0$ and $\|z_0\| < \frac{\epsilon}{L}$, we get

$$\begin{aligned} \|z(t)\| &\leq \|\psi_P(t, t_0)\| \|z_0\| + \int_{t_0}^t \|\psi_P(t, \sigma(s))\| \int_{t_0}^s Q(s, u) \|z(u)\| \Delta u \Delta s \\ &\leq L e_{-\eta}(t, t_0) \|z_0\| + L \int_{t_0}^t e_{-\eta}(t, \sigma(s)) \int_{t_0}^s Q(s, u) \|z(u)\| \Delta u \Delta s. \end{aligned} \quad (18)$$

There exists a positive constant $a < b$ and ϵ with $-a, -\epsilon \in \mathcal{R}^+(\mathbb{T}, \mathbb{R})$ such that $-\eta = -a \oplus -\epsilon$, and

$$\sup_{t \in \mathbb{T}_0} \int_{t_0}^t e_{-a}(s, \sigma(t)) Q(t, s) \Delta s < \frac{\epsilon}{L}.$$

Multiplying by $e_{-a}(s, \sigma(t))$ on both sides of (18), we get

$$\begin{aligned} e_{-a}(t_0, t) \|z(t)\| &\leq L e_{-\epsilon}(t, t_0) \|z_0\| + L \int_{t_0}^t e_{-a}(t_0, \sigma(s)) e_{-\epsilon}(t, \sigma(s)) \\ &\quad \times \int_{t_0}^s Q(s, u) \|z(u)\| \Delta u \Delta s \\ &= L e_{-\epsilon}(t, t_0) \|z_0\| + L \int_{t_0}^t e_{-\epsilon}(t, \sigma(s)) \\ &\quad \times \int_{t_0}^s e_{-a}(u, \sigma(s)) Q(s, u) e_{-a}(t_0, u) \|z(u)\| \Delta u \Delta s. \end{aligned}$$

If we define $r(t) = \sup_{s \in [t_0, t]_{\mathbb{T}}} e_{-a}(t_0, t) \|z(s)\|$, consequently

$$\begin{aligned} e_{-a}(t_0, t) \|z(t)\| &\leq L e_{-\epsilon}(t, t_0) \|z_0\| + L r(t) \int_{t_0}^t e_{-\epsilon}(t, \sigma(s)) \\ &\quad \times \int_{t_0}^s e_{-a}(u, \sigma(s)) Q(s, u) \Delta u \Delta s \\ &\leq L e_{-\epsilon}(t, t_0) \|z_0\| + \epsilon r(t) \int_{t_0}^t e_{-\epsilon}(t, \sigma(s)) \Delta s. \end{aligned}$$

By using [Theorem 2.39, [17]], we have

$$e_{-a}(t_0, t) \|z(t)\| \leq L e_{-\epsilon}(t, t_0) \|z_0\| + \{1 - e_{-\epsilon}(t, t_0)\} r(t). \quad (19)$$

Now we consider two cases:

Case (i): If $e_{-a}(t_0, s) \|z(s)\| \leq e_{-a}(t_0, t) \|z(t)\|$ for any $s \in [t_0, t]_{\mathbb{T}}$. So, we get $r(t) = e_{-a}(t_0, t) \|z(t)\|$. Then from (19) we get

$$r(t) \leq L e_{-\epsilon}(t_0, t) \|z_0\| \{1 - e_{-\epsilon}(t, t_0)\} r(t).$$

Then $r(t) \leq L \|z_0\|, \forall t \in \mathbb{T}_0$. Thus, $r(t) = e_{-a}(t_0, t) \|z(t)\| \implies \|z(t)\| \leq L e_{-a}(t_0, t) \|z_0\|, \forall t \in \mathbb{T}_0$.

Case (ii): There exists $s \in [t_0, t]_{\mathbb{T}}$ such that $e_{-a}(t_0, s) \|z(s)\| > e_{-a}(t_0, t) \|z(t)\|$. Then there exists $t_1 \in [t_0, t]_{\mathbb{T}}$ such that $r(t) = e_{-a}(t_0, t) \|z(t_1)\|$. Then from (19) we get

$$r(t_1) = e_{-a}(t_0, t_1) \|z(t_1)\| \leq L e_{-\epsilon}(t_1, t_0) \|z_0\| \{1 - e_{-\epsilon}(t_0, t_1)\} r(t_1).$$

Then $r(t_1) \leq L \|z_0\|, \forall t_1 \in \mathbb{T}_0$. Thus, $r(t_1) \geq e_{-a}(t_0, t) \|z(t)\| \implies \|z(t)\| \leq L e_{-a}(t, t_0) \|z_0\|, \forall t \in \mathbb{T}_0$.

Therefore, from case (i) and case (ii), the zero solution of (2) is exponentially asymptotically stable. \square

6 Strong stability

Definition 6.1 The zero solution of (2) is known as strongly stable if for all $\epsilon > 0 \exists \delta > 0$ such that, for any solution $z(t)$ of (2), the inequality $t_1 \in \mathbb{T}_0$ and $\|z(t_0)\| < \delta \implies \|z(t)\| < \epsilon, \forall t \leq t_0 \in \mathbb{T}_0$.

Theorem 6.1 ([21], Theorem 4.3) Let $\psi_P(t, s)$ be a fundamental matrix for (3). Then the zero solution of (3) is strongly stable on \mathbb{T}_0 if and only if there exists a positive constant G such that

$$\|\psi_P(t, t_0) \psi_P^{-1}(s, t_0)\| \leq G, \quad \forall t_0 \leq s \leq t < \infty,$$

or equivalently,

$$\|\psi_P(t, t_0)\| \leq G \quad \text{and} \quad \|\psi_P^{-1}(s, t_0)\| \leq G, \quad \forall t \in \mathbb{T}_0.$$

We need the following assumptions:

K_1 : There exist a continuous function $\varphi : \mathbb{T}_0 \rightarrow (0, \infty)$ and constants $q_1 \geq 1, G_1 > 0$ such that

$$\int_{t_0}^t (\varphi(s) \|\psi_P(t, t_0) \psi_P^{-1}(s, t_0)\|)^{q_1} \Delta s \leq G_1, \quad \forall t \in \mathbb{T}_0.$$

K_2 : There exist a continuous function $\varphi : \mathbb{T}_0 \rightarrow (0, \infty)$ and constants $q_2 \geq 1, G_2 > 0$ such that

$$\int_{t_0}^t (\varphi(s) \|\psi_P(t, t_0) \psi_P^{-1}(s, t_0)\|)^{q_2} \Delta s \leq G_2, \quad \forall t \in \mathbb{T}_0.$$

K_3 : There exist a continuous function $\varphi : \mathbb{T}_0 \rightarrow (0, \infty)$ and constants $q_3 \geq 1, G_3 > 0$ such that

$$\int_{t_0}^t (\varphi(s) \|\psi_P(t, t_0) \psi_P^{-1}(s, t_0)\|)^{q_3} \Delta s \leq G_3, \quad \forall t \in \mathbb{T}_0.$$

K_4 : There exist a continuous function $\varphi : \mathbb{T}_0 \rightarrow (0, \infty)$ and constants $q_4 \geq 1, G_4 > 0$ such that

$$\int_{t_0}^t (\varphi(s) \|\psi_P(t, t_0) \psi_P^{-1}(s, t_0)\|)^{q_4} \Delta s \leq G_4, \quad \forall t \in \mathbb{T}_0.$$

Theorem 6.2 Assume that the fundamental matrix $\psi_P(t, t_0)$ satisfies one of the following assumptions:

Q_1 : K_1 and K_2 are true;

Q_2 : K_1 and K_4 are true;

Q_3 : K_2 and K_3 are true;

Q_4 : K_3 and K_4 are true.

Thus, the zero solution of (3) is strongly stable on \mathbb{T}_0 .

Proof. We show that $\psi_P(t, t_0)$ and $\psi_P^{-1}(t, t_0)$ are bounded on \mathbb{T}_0 . First, we consider case Q_2 . For this we show that $\psi_P(t, t_0)$ is bounded on \mathbb{T}_0 .

Consider

$$r(t) = \varphi^{q_1}(t) (\psi_P(t, t_0))^{-q_1}, \quad \forall t \in \mathbb{T}_0.$$

From the identity

$$\begin{aligned} \left(\int_{t_0}^t r(s) \Delta s \right) \psi_P(t, t_0) &= \int_{t_0}^t (\varphi(s) \psi_P(t, t_0) \psi_P^{-1}(s, t_0)) \\ &\quad \times (r(s) (\varphi(s))^{-1} \psi_P(s, t_0)) \Delta s, \quad \forall t \in \mathbb{T}_0. \end{aligned}$$

Consequently,

$$\begin{aligned} \left(\int_{t_0}^t r(s) \Delta s \right) \|\psi_P(t, t_0)\| &\leq \int_{t_0}^t (\varphi(s) \|\psi_P(t, t_0) \psi_P^{-1}(s, t_0)\|) \\ &\quad \times (r(s) (\varphi(s))^{-1} \|\psi_P(s, t_0)\|) \Delta s, \quad \forall t \in \mathbb{T}_0. \end{aligned} \quad (20)$$

If $q_1 = 1$, we get $r(s)(\varphi(s))^{-1} \|\psi_P(s, t_0)\| = 1$. From (20) and hypothesis K_1 , we have

$$\left(\int_{t_0}^t r(s) \Delta s \right) \|\psi_P(t, t_0)\| \leq \int_{t_0}^t (\varphi(s) \|\psi_P(t, t_0) \psi_P^{-1}(s, t_0)\|) \Delta s, \quad \forall t \in \mathbb{T}_0.$$

If $q_1 > 1$, set $r_1 = \frac{q_1}{q_1-1}$ such that $r(s)(\varphi(s))^{-1} \|\psi_P(s, t_0)\| = (r(s))^{\frac{1}{r_1}}$. From (20), we obtain

$$\left(\int_{t_0}^t r(s) \Delta s \right) \varphi(t) (r(t))^{\frac{1}{q_1}} \leq \int_{t_0}^t (\varphi(s) \|\psi_P(t, t_0) \psi_P^{-1}(s, t_0)\|) (r(s))^{\frac{1}{r_1}} \Delta s, \quad \forall t \in \mathbb{T}_0.$$

Using Holder's inequality [17], we obtain

$$\begin{aligned} \left(\int_{t_0}^t r(s) \Delta s \right) \varphi(t) (r(t))^{\frac{1}{q_1}} &\leq \left(\int_{t_0}^t (\varphi(s) \|\psi_P(t, t_0) \psi_P^{-1}(s, t_0)\|)^{q_1} \Delta s \right)^{\frac{1}{q_1}} \\ &\quad \times \left(\int_{t_0}^t (r(s)) \Delta s \right)^{\frac{1}{r_1}}, \quad \forall t \in \mathbb{T}_0. \end{aligned}$$

Next, by using assumptions K_1 , we have

$$\begin{aligned} \left(\int_{t_0}^t r(s) \Delta s \right)^{\frac{1}{q_1}} \varphi(t) (r(t))^{\frac{1}{p_1}} &\leq G_1^{\frac{1}{q_1}}, \quad \forall t \in \mathbb{T}_0. \\ \int_{t_0}^t r(s) \Delta s \|\psi_P(t, t_0)\|^{q_1} &\leq G_1, \quad \forall t \in \mathbb{T}_0. \end{aligned}$$

Then, for $q_1 \geq 1$, the function $\|\psi_P(t, t_0)\|$ satisfies the inequality

$$\|\psi_P(t, t_0)\|^{q_1} \leq G_1^{\frac{1}{q_1}} \left(\int_{t_0}^t r(s) \Delta s \right)^{\frac{-1}{q_1}}, \quad \forall t \in \mathbb{T}_0.$$

Let $R(t) = \int_{t_0}^t r(s) \Delta s, \forall t \in \mathbb{T}_0$, so we obtain

$$\|\psi_P(t, t_0)\|^{q_1} \leq G_1^{\frac{1}{q_1}} (R(t))^{\frac{-1}{q_1}}, \quad \forall t \in \mathbb{T}_0.$$

Note $R^\Delta(t) = r(t) \geq G_1^{-1} (\varphi(t))^{q_1} R(t), \forall t \in \mathbb{T}_0$.

Consequently, there exists a constant L_1 such that

$$\|\psi_P(t, t_0)\| \leq L_1, \quad \forall t \in \mathbb{T}_0.$$

Next, we prove that $\psi_P^{-1}(t, t_0)$ is bounded on \mathbb{T}_0 .

Consider

$$r(t) = \varphi^{p_4}(t) \|\psi_P^{-1}(t, t_0)\|^{q_4}, \quad t \in \mathbb{T}_0.$$

From the identity

$$\left(\int_{t_0}^t r(s) \Delta s \right) \psi_P^{-1}(t, t_0) = \int_{t_0}^t (r(s) (\varphi(t))^{-1} \psi_P^{-1}(s, t_0))$$

$$\times (\varphi(s)\psi_P(s, t_0)\psi_P^{-1}(t, t_0))\Delta s, \quad \forall t \in \mathbb{T}_0.$$

Consequently,

$$\begin{aligned} \left(\int_{t_0}^t r(s)\Delta s \right) \|\psi_P^{-1}(t, t_0)\| &\leq \int_{t_0}^t (r(s)(\varphi(t))^{-1} \|\psi_P^{-1}(s, t_0)\|) \\ &\quad \times (\varphi(s)\|\psi_P(s, t_0)\psi_P^{-1}(t, t_0)\|)\Delta s, \quad \forall t \in \mathbb{T}_0. \end{aligned}$$

If $q_4 = 1$, we get $r(s)(\varphi(t))^{-1} \|\psi_P^{-1}(s, t_0)\| = 1$. Using assumptions G_4 , we get

$$\left(\int_{t_0}^t r(s)\Delta s \right) \|\psi_P^{-1}(t, t_0)\| \leq \int_{t_0}^t (\varphi(s)\|\psi_P(s, t_0)\psi_P^{-1}(t, t_0)\|)\Delta s, \quad \forall t \in \mathbb{T}_0.$$

If $q_4 > 1$, set $r_4 = \frac{q_4}{q_4-1}$ such that $r(s)(\varphi(t))^{-1} \|\psi_P^{-1}(s, t_0)\| = (r(s))^{\frac{1}{r_4}}$. Consequently,

$$\left(\int_{t_0}^t r(s)\Delta s \right) \|\psi_P^{-1}(t, t_0)\| \leq \int_{t_0}^t (\varphi(s)\|\psi_P(s, t_0)\psi_P^{-1}(t, t_0)\|)(r(s))^{\frac{1}{r_4}}\Delta s, \quad \forall t \in \mathbb{T}_0.$$

Using Holder's inequality leads to

$$\begin{aligned} &\left(\int_{t_0}^t r(s)\Delta s \right) \|\psi_P^{-1}(t, t_0)\| \\ &\leq \left(\int_{t_0}^t (\varphi(s)\|\psi_P(s, t_0)\psi_P^{-1}(t, t_0)\|)^{q_4}\Delta s \right)^{\frac{1}{q_4}} (r(s))^{\frac{1}{r_4}}, \quad \forall t \in \mathbb{T}_0. \end{aligned}$$

Now, by using assumption K_4 , we obtain

$$\left(\int_{t_0}^t r(s)\Delta s \right) \|\psi_P^{-1}(t, t_0)\| \leq \left(\int_{t_0}^t r(s)\Delta s \right)^{\frac{1}{r_4}} (K_4)^{\frac{1}{q_4}}, \quad \forall t \in \mathbb{T}_0.$$

or

$$\left(\int_{t_0}^t r(s)\Delta s \right)^{\frac{1}{q_4}} \|\psi_P(t, t_0)\| \leq (K_4)^{\frac{1}{q_4}}, \quad \forall t \in \mathbb{T}_0.$$

Then, for $q_4 \geq 1$, the function $\|\psi_P^{-1}(t, t_0)\|$ satisfies the inequality

$$\|\psi_P^{-1}(t, t_0)\| \leq (K_4)^{\frac{1}{q_4}} \left(\int_{t_0}^t r(s)\Delta s \right)^{\frac{-1}{q_4}}, \quad \forall t \in \mathbb{T}_0.$$

Let $R(t) = \int_{t_0}^t r(s)\Delta s, \forall t \in \mathbb{T}_0$, we obtain

$$\|\psi_P^{-1}(t, t_0)\| \leq (G_4)^{\frac{1}{q_4}} (R(t))^{\frac{-1}{q_4}}, \quad \forall t \in \mathbb{T}_0.$$

Note $R^\Delta(t) = r(t) \geq G_4^{-1}(\varphi(t))^{q_4} R(t), \forall t \in \mathbb{T}_0$. Then there exists a constant L_2 such that

$$\|\psi_P^{-1}(t, t_0)\| \leq L_2, \quad \forall t \in \mathbb{T}_0.$$

Hence, the result holds by using Theorem 6.1. Similarly, the theorem is valid for remaining cases Q_1 , Q_3 , or Q_4 . \square

Theorem 6.3 *Suppose that the function G is continuous and satisfies the following assumption:*

$$\|G(t, s, z) - G(t, s, x)\| \leq h(t, s)\|z - x\|. \quad (21)$$

For $t_0 \leq s \leq t < \infty$ and $\forall z, x \in \mathbb{R}^{n^2}$, there exists a function $\beta \in \mathcal{R}^+(\mathbb{T}_0, \mathbb{R})$ such that

$$\sup_{t \in \mathbb{T}_0} \frac{1}{e_\beta(t, t_0)} \int_{t_0}^t \|\psi_P(t, \sigma(s))\| \int_{t_0}^s h(s, u) e_\beta(u, t_0) \Delta u \Delta s < 1 \quad (22)$$

and

$$\sup_{t \in \mathbb{T}_0} \frac{1}{e_\beta(t, t_0)} \|\psi_P(t, t_0)\| < \infty, \quad (23)$$

where h is an rd-continuous nonnegative function on $C = \{(t, s) : t_0 \leq s \leq t < \infty\}$. Then there exists a unique solution of (2).

Proof We consider the space of continuous function $Q(\mathbb{T}_0; \mathbb{R}^{n^2})$ with

$$\sup_{t \in \mathbb{T}_0} \frac{\|z(t)\|}{e_\beta(t, t_0)} < \infty,$$

and we denote this space by $Q_\beta(\mathbb{T}_0; \mathbb{R}^{n^2})$. We couple the linear space $Q_\beta(\mathbb{T}_0; \mathbb{R}^{n^2})$ with a metric, namely

$$C_\beta^\infty = \sup_{t \in \mathbb{T}_0} \frac{\|z(t) - x(t)\|}{e_\beta(t, t_0)}.$$

It is easy to understand that $Q(\mathbb{T}_0; \mathbb{R}^{n^2})$ (coupled with the norm $(\|z\|_\beta^\infty = \sup_{t \in \mathbb{T}_0} \frac{\|z(t)\|}{e_\beta(t, t_0)})$) is a Banach space ([15], Lemma 4.1).

Consider the operator T from $Q_\beta(\mathbb{T}_0; \mathbb{R}^{n^2})$ to $Q_\beta(\mathbb{T}_0; \mathbb{R}^{n^2})$, given by

$$Tz(t) = \psi_P(t, t_0)z(t_0) + \int_{t_0}^t \|\psi_P(t, \sigma(s))\| \int_{t_0}^s G(s, u, z) \Delta u \Delta s, \quad (24)$$

and note

$$\begin{aligned} \|Tz\|_\beta^\infty &= \sup_{t \in \mathbb{T}} \frac{1}{e_\beta(t, t_0)} \left\| \psi_P(t, t_0)z(t_0) + \int_{t_0}^t \|\psi_P(t, \sigma(s))\| \int_{t_0}^s G(s, u, z) \Delta u \Delta s \right\| \\ &\leq \sup_{t \in \mathbb{T}} \frac{1}{e_\beta(t, t_0)} \left(\|\psi_P(t, t_0)z(t_0)\| + \int_{t_0}^t \|\psi_P(t, \sigma(s))\| \int_{t_0}^s \|G(s, u, z)\| \Delta u \Delta s \right) \\ &\leq \sup_{t \in \mathbb{T}} \frac{1}{e_\beta(t, t_0)} \|\psi_P(t, t_0)\| \|z(t_0)\| + \sup_{t \in \mathbb{T}} \frac{1}{e_\beta(t, t_0)} \\ &\quad \times \int_{t_0}^t \|\psi_P(t, \sigma(s))\| \int_{t_0}^s \|G(s, u, z)\| \Delta u \Delta s. \end{aligned}$$

By using (21), (22), and (23), we have

$$\begin{aligned}
 \|Tz\|_\beta^\infty &\leq \sup_{t \in \mathbb{T}} \frac{1}{e_\beta(t, t_0)} \|\psi_P(t, t_0)\| \|z(t_0)\| \\
 &\quad + \sup_{t \in \mathbb{T}} \frac{1}{e_\beta(t, t_0)} \int_{t_0}^t \|\psi_P(t, \sigma(s))\| \int_{t_0}^s h(s, u) \|z(u)\| \Delta u \Delta s \\
 &\leq \sup_{t \in \mathbb{T}} \frac{1}{e_\beta(t, t_0)} \|\psi_P(t, t_0)\| \|z(t_0)\| + \sup_{t \in \mathbb{T}} \frac{1}{e_\beta(t, t_0)} \\
 &\quad \times \int_{t_0}^t \|\psi_P(t, \sigma(s))\| \int_{t_0}^s h(s, u) e_\beta(u, t_0) \frac{\|z(u)\|}{e_\beta(u, t_0)} \Delta u \Delta s \\
 &\leq \sup_{t \in \mathbb{T}} \frac{1}{e_\beta(t, t_0)} \|\psi_P(t, t_0)\| \|z(t_0)\| + \|z\|_\beta^\infty \sup_{t \in \mathbb{T}} \frac{1}{e_\beta(t, t_0)} \\
 &\quad \times \int_{t_0}^t \|\psi_P(t, \sigma(s))\| \int_{t_0}^s h(s, u) e_\beta(u, t_0) \Delta u \Delta s.
 \end{aligned}$$

Also,

$$\begin{aligned}
 \|Tz - Tx\|_\beta^\infty &\leq \sup_{t \in \mathbb{T}} \frac{1}{e_\beta(t, t_0)} \left\| \int_{t_0}^t \psi_P(t, \sigma(s)) \int_{t_0}^s G(s, u, z) - G(s, u, x) \Delta u \Delta s \right\| \\
 &\leq \sup_{t \in \mathbb{T}} \frac{1}{e_\beta(t, t_0)} \int_{t_0}^t \|\psi_P(t, \sigma(s))\| \int_{t_0}^s \|G(s, u, z) - G(s, u, x)\| \Delta u \Delta s \\
 &\leq \sup_{t \in \mathbb{T}} \frac{1}{e_\beta(t, t_0)} \int_{t_0}^t \|\psi_P(t, \sigma(s))\| \int_{t_0}^s h(s, u) \|z(u) - x(u)\| \Delta u \Delta s \\
 &\leq \|z - x\|_\beta^\infty \sup_{t \in \mathbb{T}} \frac{1}{e_\beta(t, t_0)} \int_{t_0}^t \|\psi_P(t, \sigma(s))\| \int_{t_0}^s h(s, u) e_\beta(u, t_0) \Delta u \Delta s.
 \end{aligned}$$

Hence, T is a contraction. Then there exists a unique solution of system (2) by the Banach fixed points theorem (22), (23). \square

Theorem 6.4 Suppose that the function G is continuous and satisfies the following assumption:

$$\|G(t, s, z) - G(t, s, x)\| \leq h(t, s) \|z - x\| \quad (25)$$

for $t_0 \leq s \leq t < \infty$ and $\forall z, x \in \mathbb{R}^{n^2}$ such that

$$\sup_{t \in \mathbb{T}_0} \frac{1}{e_\beta(t, t_0)} \int_{t_0}^t \|\psi_P(t, \sigma(s))\| \int_{t_0}^s h(s, u) \Delta u \Delta s < 1 \quad (26)$$

and

$$N = \sup_{t \in \mathbb{T}_0} \|\psi_P(t, t_0)\| < \infty, \quad (27)$$

where h is an rd-continuous nonnegative function on $C = \{(t, s) : t_0 \leq s \leq t < \infty\}$. Thus, the zero unique solution of (2) is strongly stable on \mathbb{T}_0 .

Proof For $t_0 \in \mathbb{T}_0$, equation (24) becomes

$$\begin{aligned}
 Tz(t) &= \psi_P(t, t_0)z(t_0) + \int_{t_0}^t \psi_P(t, \sigma(s)) \int_{t_0}^s G(s, u, z) \Delta u \Delta s \\
 &= \psi_P(t, t_1)\psi_P(t_1, t_0)z(t_0) + \int_{t_0}^{t_1} \psi_P(t, \sigma(s)) \int_{t_0}^s G(s, u, z) \Delta u \Delta s \\
 &\quad + \int_{t_1}^t \psi_P(t, \sigma(s)) \int_{t_0}^s G(s, u, z) \Delta u \Delta s \\
 &= \psi_P(t, t_1)\psi_P(t_1, t_0)z(t_0) + \int_{t_0}^{t_1} \psi_P(t, t_1)\psi_P(t_1, \sigma(s)) \int_{t_0}^s G(s, u, z) \Delta u \Delta s \\
 &\quad + \int_{t_1}^t \psi_P(t, \sigma(s)) \int_{t_0}^s G(s, u, z) \Delta u \Delta s \\
 &= \psi_P(t, t_1)Tz(t_1) + \int_{t_1}^t \psi_P(t, \sigma(s)) \int_{t_0}^s G(s, u, z) \Delta u \Delta s.
 \end{aligned}$$

By applying Theorem (6.3), we deduce that there exists a unique solution $z(t)$ of (2) on \mathbb{T}_0 such that

$$\begin{aligned}
 \|z(t)\| &= \|\psi_P(t, t_1)\| \|z(t_1)\| + \int_{t_1}^t \|\psi_P(t, \sigma(s))\| \int_{t_0}^s \|G(s, u, z)\| \Delta u \Delta s \\
 &\leq \sup_{t \in [t_1, \infty)_{\mathbb{T}}} \int_{t_1}^t \|\psi_P(t, \sigma(s))\| \int_{t_0}^s \|G(s, u, z)\| \Delta u \Delta s \\
 &\quad + \sup_{t \in [t_1, \infty)_{\mathbb{T}}} \|\psi_P(t, t_1)\| \|z(t_1)\|.
 \end{aligned}$$

Now, by using (25), we get

$$\begin{aligned}
 \|z(t)\| &= \sup_{t \in \mathbb{T}_0} \|z(t)\| \leq \|z(t_1)\| \sup_{t \in [t_1, \infty)_{\mathbb{T}}} \|\psi_P(t, t_1)\| \\
 &\quad + \sup_{w \in [t_1, \infty)_{\mathbb{T}}} \|z(w)\| \sup_{t \in [t_1, \infty)_{\mathbb{T}}} \int_{t_1}^t \|\psi_P(t, \sigma(s))\| \int_{t_0}^s h(s, u) \Delta u \Delta s \quad (28)
 \end{aligned}$$

for $t \leq t_0 \in \mathbb{T}_0$. Let $\epsilon > 0$ be an arbitrary constant, and let $\delta(\epsilon) = \frac{\epsilon(1-A)}{M}$ such that

$$\|z(t_1)\| < \delta(\epsilon), \quad (29)$$

where $A = \sup_{t \in \mathbb{T}_0} \int_{t_1}^t \|\psi_P(t, \sigma(s))\| \int_{t_0}^s h(s, u) \Delta u \Delta s$.

Now (28), it becomes

$$\begin{aligned}
 \sup_{t \in [t_1, \infty)_{\mathbb{T}}} \|z(t)\| (1-A) &\leq M \|z(t_1)\|, \\
 \sup_{t \in [t_1, \infty)_{\mathbb{T}}} \|z(t)\| &\leq \epsilon.
 \end{aligned}$$

Hence we proved the zero solution of (2) is strongly stable on \mathbb{T}_0 . \square

7 Conclusion

The matrix Sylvester type systems that appear in mathematical physics and Volterra integro dynamical systems play an important role in many optimization and engineering problems. Recently, times scale calculus and dynamic equations on time scales have attracted much interest of many researchers due to their main feature of generalization and unification of continuous and discrete models. In this paper, we focus our attention on establishing the stability criteria for a nonlinear Volterra integro-dynamic matrix Sylvester system on time scales. The properties of Kronecker product of matrices are used to obtain the results. First, we convert the nonlinear Volterra integro-dynamic matrix Sylvester system on time scales into its equivalent vector dynamic system on time scales with the help of vectorization operator. Later, we obtain sufficient conditions for the stability, asymptotic stability, exponential stability, and strong stability for this system. The proposed results are valid for both continuous and discrete versions of Volterra integro-dynamic matrix Sylvester systems.

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Author details

¹Department of Mathematics, Koneru Lakshmaiah Education Foundation and Malla Reddy Institute of Technology and Science, Dhulapally, Secunderabad, 500100, Telangana, India. ²Department of Mathematics, Koneru Lakshmaiah Education Foundation, Vaddeswaram, Guntur, 522502, Andhra Pradesh, India.

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