CrossMark

Emergent universe in the braneworld scenario

Y. Heydarzade[1,a], **H. Hadi**[1], **F. Darabi**[1,3,b], **A. Sheykhi**[2,c]

[1] Department of Physics, Azarbaijan Shahid Madani University, Tabriz, Iran
[2] Physics Department and Biruni Observatory, College of Sciences, Shiraz University, Shiraz 71454, Iran
[3] Research Institute for Astronomy and Astrophysics of Maragha (RIAAM), Maragha 55134-441, Iran

Abstract According to Padmanabhan's proposal, the difference between the surface degrees of freedom and the bulk degrees of freedom in a region of space may result in the acceleration of Universe expansion through the relation $\Delta V / \Delta t = N_{\text{sur}} - N_{\text{bulk}}$ where N_{bulk} and N_{sur} are referred to the degrees of freedom related to the matter and energy content inside the bulk and surface area, respectively (Padmanabhan, arXiv:1206.4916v1, 2012). In this paper, we study the dynamical effect of the extrinsic geometrical embedding of an arbitrary four-dimensional brane in a higher-dimensional bulk space and investigate the corresponding degrees of freedom. Considering the modification of the Friedmann equations arising from a general braneworld scenario, we obtain a correction term in Padmanabhan's relation, denoting the number of degrees of freedom related to the extrinsic geometry of the brane embedded in higher-dimensional spacetime as $\Delta V / \Delta t = N_{\text{sur}} - N_{\text{bulk}} - N_{\text{extr}}$ where N_{extr} is for the degree of freedom related to the extrinsic geometry of the brane, while N_{sur} and N_{bulk} are defined as before. Finally, we study the validity of the first and second laws of thermodynamics for this general braneworld scenario in the state of thermal equilibrium and in the presence of confined matter fields to the brane with the induced geometric matter fields.

1 Introduction

Recent research supports the idea that the gravitational field equations can be derived in the same way as the equations of an emergent phenomenon like fluid mechanics or elasticity are obtained [1–8]. According to the emergent gravity paradigm the gravitational field equations can be derived from thermodynamic principles [1,9]. In this way, Padmanabhan has treated the Einstein field equations as emergent,

[a] e-mail: heydarzade@azaruniv.edu

[b] e-mail: f.darabi@azaruniv.edu

[c] e-mail: asheykhi@shirazu.ac.ir

while the existence of a spacetime manifold, its metric, and its curvature have been assumed [10]. In a cosmological context, it has been argued that the accelerated expansion of the Universe can be derived from the difference between the surface and bulk degrees of freedom through the relation $\Delta V / \Delta t = N_{\text{sur}} - N_{\text{bulk}}$, in which N_{bulk} and N_{sur} are for the degrees of freedom related to matter and energy content (or dark matter and dark energy) inside the bulk and surface area, respectively [11]. In order to explain the present accelerated expansion of the Universe, which is in agreement with a different data set or observational data [13–21], different models have been proposed. One of these models is the dark energy model, which admits that the universe is dominated by a dark fluid with negative pressure. However, there are several dark energy models, such as dynamical dark energy [22–24], quintessence [25–29], and k-essence [30], for a review the reader is referred to [31]. Also, the LCDM model or the concordance model is a particular case of dark energy that is parameterized by a cosmological constant Λ with the equation of state parameter equal minus one, i.e. $p = -\rho$. The strong energy condition, i.e. $\rho + 3p > 0$, is violated by the dark energy because of the requirement of the positive acceleration of the Universe through the second Friedmann equation $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$. Another approach lies in the framework of modified gravity theories which describe the present acceleration of the Universe such as $f(R)$ gravity [32–36], $f(T)$ gravity [37–41], Hořava–Lifshitz gravity [42–44], Gauss–Bonnet gravity [45–48], Weyl gravity [49,50], Lovelock gravity [51,52], massive gravity [53–56], and braneworld scenarios [57–60]. In these models, there is no need for the introduction of an ad hoc component, usually called dark energy with unusual features. In these models, some additional terms are considered in the gravitational Lagrangian, which lead to the modification of the gravitational theory resulting in an effective dark energy sector with a geometrical origin. One can also find such models in the low energy limit of heterotic string theory [61]. All of these mod-

els admit a series of conditions coming from various laws of physics such as thermodynamics laws [62] or astrophysical data.

To explain the structure of spacetime and its relation with thermodynamics of the system, one can refer to the four laws of black hole mechanics which are derived from the classical Einstein field equations. These four laws are analogous to those of thermodynamics [63]. By the discovery of quantum Hawking radiation [64] it turns out that this analogy is an identity. By deriving the Einstein field equation from the relation of entropy and horizon area together with the thermodynamic law of $Q = T\,dS$, which connects the heat Q, the entropy S, and the temperature T, Jacobson showed that classical general relativity is a kind of thermodynamics where the surface gravity is a temperature [4]. The generalized second law of thermodynamics is specially investigated in different modified gravity models. For example, we can refer to the investigations devoted to the study of the generalized second law (GSL) of thermodynamics in $f(T)$ gravity, models in which two types of horizons are used to check the validity of the generalized second law of thermodynamics with corrected entropies [65]. One can also find that in the state of thermal equilibrium, in the Kaluza–Klein universe which is composed of dark matter and dark energy, the validity of the laws of thermodynamics are true [66]. The investigations of the deep connection between gravity and thermodynamics have been widely considered in the cosmological context where it has been shown that in the form of the first law of thermodynamics on the apparent horizon, the differential form of the Friedmann equation in the FRW universe can be written [1,67–81]. The GSL in an accelerating universe related to the apparent horizon has been considered in [82–84]. It was discussed in [83] that, in contrast to the case of the apparent horizon, the general second law of thermodynamics breaks down in the case of a universe enveloped by the event horizon with the usual definitions of entropy and temperature. This study reveals that from the thermodynamical point of view, in an accelerating universe with spatial curvature, the apparent horizon is a physical boundary. Also, the general expression of temperature at the apparent horizon of the FRW universe allows one to show that the GSL holds in Einstein, Gauss–Bonnet, and more general Lovelock gravity [85,86]. Also, the GSL of thermodynamics in the framework of braneworld scenarios is studied in [87,88]. One can find other studies on the GSL of thermodynamics in [89–100].

In particle physics, warped product geometries, well known as Randall–Sundrum models, are very important [101,102]. In these models, it is imagined that our real world is a higher-dimensional universe described by a warped geometry with Z_2 symmetry. The standard gauge interactions are confined to the four-dimensional brane embedded in a higher-dimensional bulk space where gravitons are propagating through the extra dimensions. More specifically, our universe is assumed to be a five-dimensional anti-de Sitter space where the elementary particles, except for the gravitons, are localized on a $(3+1)$-dimensional brane or branes.

In this paper, we consider a general braneworld model which provides a geometrical origin for dark energy or accelerating expansion of the Universe [103]. Considering the modification of Friedmann equations resulting from this general braneworld scenario, we obtain a correction term on Padmanabhan's relation. This paper is organized as follows: In Sect. 2, we introduce general geometrical setup of the braneworld. In Sect. 3, this braneworld model is studied under the Israel–Darmois–Lanczos junction condition, which provides the Z_2 symmetry, and the corresponding number of degrees of freedom related to the extrinsic geometry of such a brane model is obtained. In Sect. 4, we find the correction term to the Padmanabhan relation in our general braneworld model which does not have any specific junction condition. In Sect. 5, we explore the thermodynamics of such a general brane model. At last, in Sect. 6, we present our concluding remarks.

2 General geometrical setup of the braneworld

The effective Einstein–Hilbert action for the $4D$ spacetime $(\mathcal{M}_4, g)$ embedded in an nD bulk space $(\mathcal{M}_n, \mathcal{G})$ can be written

$$I_{\text{EH}} = \frac{1}{2\alpha_*} \int d^n x \sqrt{-\mathcal{G}}\,\mathcal{R} + \int_\Sigma d^4 x \sqrt{-g}\,\mathcal{L}_m, \tag{1}$$

where α_*, $\mathcal{R}$ and $\mathcal{L}_m$ are, respectively, as gravitational coupling constant in the bulk space, the bulk Ricci scalar, and the Lagrangian density of the matter fields confined to the brane. Variation of the action (1) with respect to the bulk metric $\mathcal{G}_{AB}(A, B = 0, \ldots, n-1)$ leads to the following field equations for the bulk space:

$$G_{AB} = \alpha_* S_{AB}, \tag{2}$$

where S_{AB} is

$$S_{AB} = T_{AB} + \frac{1}{2}\mathcal{V}\mathcal{G}_{AB}, \tag{3}$$

and $T_{AB} = -2\frac{\delta \mathcal{L}_m}{\delta g_{AB}} + g_{AB}\mathcal{L}_m$ is the energy-momentum tensor of the matter fields confined to the four-dimensional brane through the action of the confining potential $\mathcal{V}$. The confining potential $\mathcal{V}$ satisfies three general conditions: (I) it has a deep minimum on the original non-perturbed brane (we will discuss the original non-perturbed and perturbed geometry in the following), (II) it depends only on extra coordinates, and (III) it preserves the gauge group related to the subgroup of the isometry group of the bulk space [105,106]. Using the confining potential $\mathcal{V}$ the matter fields are exactly localized

on the brane and one obtains

$$\alpha_* S_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad S_{\mu a} = 0, \quad S_{ab} = 0, \tag{4}$$

where $\mu, \nu = 0, \dots, 3$ and $a, b = 4, \dots, n - 1$ labels the number of four-dimensional brane and bulk extra dimensions, respectively, and $T_{\mu\nu}$ is the confined matter source on the four-dimensional brane. This is the so-called "confinement hypothesis".

Now, it is worth to have a brief discussion on the bulk and brane energy scales and their corresponding gravitational coupling constants. The bulk space gravitational coupling constant α_* is

$$\alpha_* = 8\pi G_n = \frac{8\pi}{M_n^{n-2}}, \tag{5}$$

where G_n is equivalently known as the bulk gravitational constant and M_n is the fundamental energy scale of the bulk space. In the usual four-dimensional spacetime, we have $G_4 = G = M_{\text{Pl}}^{-2}$ where G is the Newtonian gravitational constant. In the static weak field limit of the Einstein field equations, one obtains the n-dimensional Poisson equation for the gravitational potential, which admits the following solution:

$$V(r) \sim \frac{\alpha_*}{r^{n-3}}, \tag{6}$$

where by assuming that the length scale of the extra dimensions is denoted L, this potential behaves as $V(r) \sim r^{3-n}$, on scales with size $r \lesssim L$, and depends on the number of dimensions of the spacetime n. On the other hand, for the scales larger than L, the potential $V(r)$ behaves as $V(r) \sim L^{4-n} r^{-1}$ [107]. For $n = 4$, we recover the Newtonian four-dimensional gravitational potential $V(r) \sim r^{-1}$. This means that the Newtonian gravitational constant G or the usual Planck scale M_{Pl} are effective coupling constants, which describe gravity on scales much larger than the length scale of extra dimensions, and they are proportional to the bulk fundamental energy scale M_n via

$$M_{\text{Pl}}^2 = M_n^{n-2} L^{n-4}, \tag{7}$$

where L^{n-4} denotes the volume of the extra-dimensional space. This relation indicates that for an extra-dimensional volume which is about the Planck scale, i.e. $L \sim M_{\text{Pl}}^{-1}$, we have $M_n \sim M_{\text{Pl}}$. But for a volume which is significantly above the Planck scale, we find that the fundamental energy scale of the bulk space M_n is much smaller than the four-dimensional effective energy scale, $M_{\text{Pl}} \sim 10^{19}$ GeV.

In order to obtain the effective Einstein field equation induced on the brane, we consider the following geometrical setup. Suppose that the $4D$ background manifold $\mathcal{M}_4$ is isometrically embedded in a n-dimensional bulk $\mathcal{M}_n$ by a differential map $\mathcal{Y}^A : \mathcal{M}_4 \longrightarrow \mathcal{M}_n$ such that

$$\mathcal{G}_{AB} \mathcal{Y}_{,\mu}^A \mathcal{Y}_{,\nu}^B = \bar{g}_{\mu\nu}, \mathcal{G}_{AB} \mathcal{Y}_{,\mu}^A \bar{\mathcal{N}}_a^B = 0, \mathcal{G}_{AB} \bar{\mathcal{N}}_a^A \bar{\mathcal{N}}_b^B = g_{ab}, \tag{8}$$

where $\mathcal{G}_{AB}$ ($\bar{g}_{\mu\nu}$) is the metric of the bulk (brane) space $\mathcal{M}_n(\mathcal{M}_4)$, $\{\mathcal{Y}^A\}$ ($\{x^\mu\}$) is the basis of the bulk (brane), $\bar{\mathcal{N}}_a^A$ are $(n-4)$ normal unit vectors orthogonal to the brane and $g_{ab} = \epsilon \delta_{ab}$ in which $\epsilon = \pm 1$ corresponds to two possible signatures for each extra dimension of the bulk space. A perturbation of the background manifold $\mathcal{M}_4$ in a sufficiently small neighborhood of the brane along an arbitrary transverse direction ξ^a is given by

$$\mathcal{Z}^A(x^\mu, \xi^a) = \mathcal{Y}^A(x^\mu) + (\mathcal{L}_\xi \mathcal{Y}(x^\mu))^A, \tag{9}$$

where $\mathcal{L}_\xi$ represents the Lie derivative along ξ^a denoting the non-compact extra dimensions. The presence of a tangent component of the vector ξ along the brane can cause some difficulties because it can induce some undesirable coordinate gauges. But it was shown that in the theory of geometric perturbations, it is quite possible to choose this vector to be orthogonal to the background manifolds [108]. Then, choosing the extra dimensions ξ^a to be orthogonal to the brane ensures us the gauge independency [109] and having perturbations of the geometrical embedding along the orthogonal extra directions $\mathcal{N}_a^A$. Thus, the local coordinates of the perturbed brane will be

$$\mathcal{Z}_{,\mu}^A(x^\nu, \xi^a) = \mathcal{Y}_{,\mu}^A(x^\nu) + \xi^a \mathcal{N}_{a,\mu}^A,$$
$$\mathcal{Z}_{,a}^A(x^\nu, \xi^a) = \mathcal{N}_a^A. \tag{10}$$

Equation (9) implies that, since the vectors $\mathcal{N}^A$ depend only on the local coordinates x^μ, $\mathcal{N}^A = \mathcal{N}^A(x^\mu)$, they will not propagate along the extra dimensions which can be shown as

$$\mathcal{N}_a^A = \bar{\mathcal{N}}_a^A + \xi^b \left[\bar{\mathcal{N}}_a^A, \bar{\mathcal{N}}_b^A \right] = \bar{\mathcal{N}}_a^A. \tag{11}$$

The above assumptions lead to the embedding equations of the perturbed geometry as

$$\mathcal{G}_{AB} \mathcal{Z}_{,\mu}^A \mathcal{Z}_{,\nu}^B = g_{\mu\nu}, \mathcal{G}_{AB} \mathcal{Z}_{,\mu}^A \mathcal{N}_a^B$$
$$= g_{\mu a}, \mathcal{G}_{AB} \mathcal{N}_a^A \mathcal{N}_b^B = g_{ab}, \tag{12}$$

where, by setting $\mathcal{N}_a^A = \delta_a^A$, the metric of the bulk space $\mathcal{G}_{AB}$ in the Gaussian frame and in the vicinity of $\mathcal{M}_4$ takes the form of

$$\mathcal{G}_{AB} = \begin{pmatrix} g_{\mu\nu} + A_{\mu c} A_\nu^c & A_{\mu a} \\ A_{\nu b} & g_{ab} \end{pmatrix}. \tag{13}$$

Then the line element of the bulk space will have the following form:

$$\mathrm{d}S^2 = \mathcal{G}_{AB} \mathrm{d}\mathcal{Z}^A \mathrm{d}\mathcal{Z}^B = g_{\mu\nu}(x^\alpha, \xi^a) \mathrm{d}x^\mu \mathrm{d}x^\nu$$
$$+ g_{ab} \mathrm{d}\xi^a \mathrm{d}\xi^b, \tag{14}$$

where

$$g_{\mu\nu} = \bar{g}_{\mu\nu} - 2\xi^a \bar{K}_{\mu\nu a} + \xi^a \xi^b \bar{g}^{\alpha\beta} \bar{K}_{\mu\alpha a} \bar{K}_{\nu\beta b} \tag{15}$$

is the metric of the perturbed brane, while $\bar{g}_{\mu\nu}$ is the metric of original non-perturbed brane (the first fundamental form) and

$$\bar{K}_{\mu\nu a} = -\mathcal{G}_{AB}\mathcal{Y}^A_{,\mu}\mathcal{N}^B_{a;\nu} = -\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial \xi^a} \tag{16}$$

is the extrinsic curvature of the original brane (the second fundamental form). In the following, we will use the notation $A_{\mu c} = \xi^d A_{\mu cd}$ where

$$A_{\mu cd} = \mathcal{G}_{AB}\mathcal{N}^A_{d;\mu}\mathcal{N}^B_c = \bar{A}_{\mu cd}, \tag{17}$$

which represents the twisting vector fields (the normal fundamental form). For any fixed extra dimension ξ^a, we have a new perturbed brane and we can define an extrinsic curvature similar to the original one by

$$\tilde{K}_{\mu\nu a} = -\mathcal{G}_{AB}\mathcal{Z}^A_{,\mu}\mathcal{N}^B_{a;\nu}$$
$$= \bar{K}_{\mu\nu a} - \xi^b\left(\bar{K}_{\mu\gamma a}\bar{K}^\gamma_{\nu b} + A_{\mu ca}A^c_{\ b\nu}\right). \tag{18}$$

Note that the definitions (13), (15), and (18) require the extrinsic curvature of the perturbed brane to be

$$\tilde{K}_{\mu\nu a} = -\frac{1}{2}\frac{\partial \mathcal{G}_{\mu\nu}}{\partial \xi^a}. \tag{19}$$

In a geometric setup, the presence of gauge fields $A_{\mu a}$ tilts the embedded family of sub-manifolds with respect to the normal vector $\mathcal{N}^A$. According to our construction, although the original brane is orthogonal to the normal vector $\mathcal{N}^A$, Eq. (12) implies that this will not be true for the deformed geometry. Hence, we change the embedding coordinates in the following form:

$$\mathcal{X}^A_{,\mu} = \mathcal{Z}^A_{,\mu} - g^{ab}\mathcal{N}^A_a A_{b\mu}, \tag{20}$$

where the coordinates $\mathcal{X}^A$ describe a new family of embedded manifolds whose members always will be orthogonal to the normal vector $\mathcal{N}^A$. In this coordinate system, the embedding equations of the perturbed brane will be similar to the original one, described by the relations given in Eq. (8), so that the coordinates $\mathcal{Y}^A$ are replaced by the new coordinates $\mathcal{X}^A$. This geometrical embedding of the new local coordinates will be suitable for obtaining the induced Einstein field equations on the brane. In these coordinates, the extrinsic curvature of a perturbed brane is given by

$$K_{\mu\nu a} = -\mathcal{G}_{AB}\mathcal{X}^A_{,\mu}\mathcal{N}^B_{a;\nu} = \bar{K}_{\mu\nu a} - \xi^b\bar{K}_{\mu\gamma a}\bar{K}^\gamma_{\nu b}$$
$$= -\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial \xi^a}, \tag{21}$$

which is known as the generalized York relation and shows the propagation of the extrinsic curvature because of the propagation of the metric in the direction of extra dimensions in the bulk space. The Gauss–Codazzi equations for the components of the Riemann tensor of the bulk space in the embed-

ding vielbein $\{\mathcal{X}^A_{,\alpha}, \mathcal{N}^A_a\}$ will be

$$R_{\alpha\beta\gamma\delta} = 2g^{ab}K_{\alpha[\gamma a}K_{\delta]\beta b} + \mathcal{R}_{ABCD}\mathcal{X}^A_{,\alpha}\mathcal{X}^B_{,\beta}\mathcal{X}^C_{,\gamma}\mathcal{X}^D_{,\delta}, \tag{22}$$

$$K_{\alpha[\gamma c;\delta]} = g^{ab}A_{[\gamma ac}K_{\delta]\alpha b} + \mathcal{R}_{ABCD}\mathcal{X}^A_{,\alpha}\mathcal{N}^B_c\mathcal{X}^C_{,\gamma}\mathcal{X}^D_{,\delta}, \tag{23}$$

where $\mathcal{R}_{ABCD}$ and $R_{\alpha\beta\gamma\delta}$ are the Riemann tensors of the bulk and the perturbed brane, respectively [110]. Then one can find the Ricci tensor by contracting the Gauss equation (22), thus

$$R_{\mu\nu} = (K_{\mu\alpha c}K_\nu^{\ \alpha c} - K_c K_{\mu\nu}^{\ c}) + \mathcal{R}_{AB}\mathcal{X}^A_{,\mu}\mathcal{X}^B_{,\nu}$$
$$-g^{ab}\mathcal{R}_{ABCD}\mathcal{N}^A_a\mathcal{X}^B_{,\mu}\mathcal{X}^C_{,\nu}\mathcal{N}^D_b, \tag{24}$$

where a subsequent contraction will give the Ricci scalar:

$$R = \mathcal{R} + (K \circ K - K_a K^a) - 2g^{ab}\mathcal{R}_{AB}\mathcal{N}^A_a\mathcal{N}^B_b$$
$$+g^{ad}g^{bc}\mathcal{R}_{ABCD}\mathcal{N}^A_a\mathcal{N}^B_b\mathcal{N}^C_c\mathcal{N}^D_d, \tag{25}$$

where use has been made of the notations $K \circ K \equiv K_{a\mu\nu}K^{a\mu\nu}$ and $K_a \equiv g^{\mu\nu}K_{a\mu\nu}$. Consequently, using Eqs. (24) and (25), the relation between the Einstein tensors of the bulk and the brane can be obtained:

$$G_{AB}\mathcal{X}^A_{,\mu}\mathcal{X}^B_{,\nu} = G_{\mu\nu} + \lambda g_{\mu\nu} - Q_{\mu\nu} - g^{ab}\mathcal{R}_{AB}\mathcal{N}^A_a\mathcal{N}^B_b g_{\mu\nu}$$
$$+g^{ab}\mathcal{R}_{ABCD}\mathcal{N}^A_a\mathcal{X}^B_\mu\mathcal{X}^C_\nu\mathcal{N}^D_b, \tag{26}$$

where G_{AB}, $G_{\mu\nu}$ are the Einstein tensors of the bulk and brane, respectively, and

$$Q_{\mu\nu} = \frac{1}{\epsilon}\left(g^{ab}(K_{a\mu}^{\ \gamma}K_{\gamma\nu b} - K_a K_{\mu\nu b})\right.$$
$$\left. -\frac{1}{2}(K \circ K - K_a K^a)g_{\mu\nu}\right), \tag{27}$$

where $K_a = g^{\mu\nu}K_{a\mu\nu}$ and $K \circ K = K^{a\mu\nu}K_{a\mu\nu}$. From the definition of $Q_{\mu\nu}$, it is an independent conserved geometrical quantity, i.e. $\nabla_\mu Q^{\mu\nu} = 0$ [103].

Using the decomposition of the Riemann tensor of the bulk space into the Weyl curvature tensor, the Ricci tensor, and the scalar curvature,

$$\mathcal{R}_{ABCD} = C_{ABCD} - \frac{2}{n-2}\left(\mathcal{G}_{B[D}\mathcal{R}_{C]A} - \mathcal{G}_{A[D}\mathcal{R}_{C]B}\right)$$
$$-\frac{2}{(n-1)(n-2)}\mathcal{R}(\mathcal{G}_{A[D}\mathcal{R}_{C]B}), \tag{28}$$

we obtain the induced 4D Einstein equation on the brane,

$$G_{\mu\nu} = G_{AB}\mathcal{X}^A_{,\mu}\mathcal{X}^B_{,\nu} + Q_{\mu\nu} - \mathcal{E}_{\mu\nu} + \frac{n-3}{n-2}g^{ab}\mathcal{R}_{AB}\mathcal{N}^A_a\mathcal{N}^B_b g_{\mu\nu}$$
$$-\frac{n-4}{n-2}\mathcal{R}_{AB}\mathcal{X}^A_{,\mu}\mathcal{X}^B_{,\nu} + \frac{n-4}{(n-1)(n-2)}\mathcal{R}g_{\mu\nu}, \tag{29}$$

where $\mathcal{E}_{\mu\nu} = g^{ab}C_{ABCD}\mathcal{X}^A_{,\mu}\mathcal{N}^B_a\mathcal{N}^C_b\mathcal{X}^D_{,\nu}$ is the electric part of the Weyl tensor $\mathcal{C}_{ABCD}$ of the bulk space. From the brane point of view, the electric part of the Weyl tensor describes a traceless matter, called dark radiation or Weyl matter. For a constant curvature bulk space, we have $\mathcal{E}_{\mu\nu} = 0$.

Then the induced Einstein equation, in a constant curvature and Ricci flat bulk ($\mathcal{E}_{\mu\nu} = \mathcal{R}_{AB} = 0$), will take the following form:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} + Q_{\mu\nu}, \tag{30}$$

where $T_{\mu\nu}$ is the confined matter source on the brane and $Q_{\mu\nu}$ is a pure geometrical energy-momentum source. We also assume that the spacetime on the brane is isotropic and homogeneous and so we have a Friedmann–Robertson–Walker (FRW) metric on the brane,

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right), \tag{31}$$

where $a(t)$ is the cosmic scale factor, $k = +1, -1$, and 0 corresponds to the closed, open, and flat universes, respectively, and $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$. The confined matter source on the brane $T_{\mu\nu}$ can be considered in the perfect fluid form in co-moving coordinates, thus

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \tag{32}$$

where $u_\alpha = \delta_\alpha^0$ is the 4-velocity vector of the fluid, and ρ and p are the energy density and isotropic pressure, respectively. For the metric (31), the components of the extrinsic curvature tensor can be obtained by using the Codazzi equation,

$$K_{00} = -\frac{1}{\dot{a}}\frac{d}{dt}\left(\frac{b}{a}\right),$$
$$K_{ij} = \frac{b}{a^2} g_{ij}, \quad i, j = 1, 2, 3, \tag{33}$$

where a dot denotes the derivative with respect to the cosmic time t, and $b = b(t)$ is an arbitrary function of time [103,111]. Then, by defining the parameters $h(t) = \dot{b}/b$ and $H(t) = \dot{a}/a$, the components of $Q_{\mu\nu}$ represented by (27) take the form of

$$Q_{00} = \frac{1}{\epsilon}\frac{3b^2}{a^4},$$
$$Q_{ij} = -\frac{1}{\epsilon}\frac{b^2}{a^4}\left(\frac{2h}{H} - 1\right) g_{ij}. \tag{34}$$

Similar to the confined matter field source on the brane $T_{\mu\nu}$, the geometric energy-momentum tensor $Q_{\mu\nu}$ can be identified as

$$Q_{\mu\nu} = (\rho_{\text{extr}} + p_{\text{extr}})u_\mu u_\nu + p_{\text{extr}} g_{\mu\nu}, \tag{35}$$

where the ρ_{extr} and p_{extr} denote the "extrinsic geometric energy density" and "extrinsic geometric pressure", respectively (the suffix "extr" stands for "extrinsic") [103]. Then, using Eqs. (34) and (35) we obtain

$$\rho_{\text{extr}} = \frac{1}{\epsilon}\frac{3b^2}{a^4},$$
$$p_{\text{extr}} = -\frac{1}{\epsilon}\frac{b^2}{a^4}\left(\frac{2h}{H} - 1\right). \tag{36}$$

Using Eqs. (30), (34), and (36) and separating the space and time components we arrive at

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = 4\pi G(\rho - p) + \frac{1}{\epsilon}\frac{b^2}{a^4}\frac{1}{\dot{a}b}\frac{d}{dt}(ab) \tag{37}$$

and

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{1}{\epsilon}\frac{b^2}{a^2}\frac{1}{\dot{a}b}\frac{d}{dt}\left(\frac{b}{a}\right). \tag{38}$$

Eliminating the $\ddot{a}$ terms gives the following modified Friedmann equation on the brane:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{1}{\epsilon}\frac{b^2}{a^4}, \tag{39}$$

which possesses a modified term arising from the extrinsic geometry of the brane in the bulk space. In the next sections, we will study this braneworld modification to the Friedmann equations in more detail.

3 The brane model with junction conditions

Using the Israel–Darmois–Lanczos junction condition which exactly provides the Z_2 symmetry[1] (mirror symmetry) [104] (see [103] for a brief review), one can obtain the extrinsic curvature tensor component of the original non-perturbed brane in terms of the confined matter sources on brane as $k_{11} = b(t) = -\alpha_*^2\rho a^2$ [103]. Then the modified Friedmann equations (37), (38), and (39) will take the forms

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{1}{\epsilon}\alpha_*^4\rho^2 \tag{40}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{1}{\epsilon}\alpha_*^4\rho^2, \tag{41}$$

which shows the cosmology to be ρ^2 dependent [112].

Now, we intend to obtain modification of the basic law governing the emergence of space due to the difference between the degrees of freedom in the framework of this model. Using relation $\ddot{a}/a = \dot{H} + H^2$, Eq. (41), can be written as

$$\dot{H} + H^2 = -\frac{4\pi G}{3}(\rho + 3p) + \frac{1}{\epsilon}\alpha_*^4\rho^2. \tag{42}$$

---

[1] The Z_2 symmetry means that when you approach the brane from one side and go through it in the bulk, you face the same bulk having a reversed normal unit vector to the brane, i.e. $\mathcal{N}^a \to -\mathcal{N}^a$. Indeed, in the presence of Z_2 symmetry, the original non-perturbed brane located at $\xi^a = 0$ acts as a mirror for all objects that feel the extra dimensions. The Z_2 symmetry governs any perturbation of the original brane leading to a mirror perturbation on the other side of the brane [108].

Multiplying Eq. (42) by $-4\pi H^{-4}$, we get

$$-4\pi\frac{\dot{H}}{H^4} = \frac{4\pi}{H^2} + \frac{4\pi G}{3}\frac{(\rho+3p)}{H^4}4\pi - 4\pi\frac{\alpha_*^4\rho^2}{\epsilon H^4}. \quad (43)$$

Assuming $V = 4\pi H^{-3}/3$ to be the volume of the sphere on the brane with Hubble radius H^{-1}, we have

$$\frac{dV}{dt} = -4\pi\frac{\dot{H}}{H^4} = \frac{4\pi}{H^2} + \frac{16\pi^2 G}{3}\frac{(\rho+3p)}{H^4} - 4\pi\frac{\alpha_*^4\rho^2}{\epsilon H^4}. \quad (44)$$

On the other hand, according to Padmanabhan's idea, the number of degrees of freedom on the spherical surface of Hubble radius H^{-1} is given by [11]

$$N_{\text{sur}} = \frac{A}{L_p^2} = \frac{4\pi}{L_p^2 H^2}, \quad (45)$$

where L_p is the Planck length and $A = 4\pi H^{-2}$ represents the area of the Hubble horizon. Using the area law $S = A/4L_p^2$, as the saturation of Bekenstein limit [12], we can write[2]

$$N_{\text{sur}} = 4S. \quad (46)$$

Also, the bulk degrees of freedom obey the equipartition law of energy,

$$N_{\text{bulk}} = \frac{2|E|}{k_B T}, \quad (47)$$

where E, k_B, and T are the energy inside of the bulk, the Boltzmann constant, and the temperature of the bulk, respectively. In the following, we use the units of $k_B = c = \hbar = G = L_p = 1$ for simplicity. We also assume the temperature associated with the Hubble horizon is the Hawking temperature, $T = H/2\pi$, and the energy contained inside the Hubble volume in Planck units $V = 4\pi/3H^3$ is the Komar energy,

$$E_{\text{Komar}} = |(\rho+3p)|V. \quad (48)$$

The novel idea of Padmanabhan is that the cosmic expansion, conceptually equivalent to the emergence of space, is being driven toward holographic equipartition, and the basic law governing the emergence of space must relate the emergence of space to the difference between the number of degrees of freedom in the holographic surface and the ones in the emerged bulk [11]. Using Eqs. (47) and (48), the bulk degrees of freedom may be obtained:

$$N_{\text{bulk}} = -\frac{16\pi^2}{3}\frac{(\rho+3p)}{H^4}, \quad (49)$$

where it is assumed that $\rho + 3p < 0$. Thus, Eq. (44) can be written as

$$\frac{dV}{dt} = N_{\text{sur}} - N_{\text{bulk}} - N_{\text{extr}}, \quad (50)$$

where

$$N_{\text{extr}} = \left(\frac{3}{2\pi\epsilon}\right)\frac{\alpha_*^4\rho^2 V}{T}, \quad (51)$$

appears as the number of degrees of freedom corresponding to the extrinsic geometry of the embedded brane in a higher-dimensional spacetime. Indeed, there are three modes of degrees of freedom, the surface degrees of freedom, the bulk degrees of freedom and the ones that are related to the extrinsic geometry of the embedded brane. Since N_{extr} represents the number of degrees of freedom, it must be positive. Equation (51) shows that the positiveness of N_{extr} requires $\epsilon = +1$, representing a spacelike extra dimension. Therefore, it turns out that by applying the Israel–Darmois–Lanczos junction condition, the timelike extra dimension will be ruled out. This indicates that unlike the other braneworld scenarios where there is no essential requirement for ϵ being positive, in the present scenario the positiveness of ϵ is imposed by Eq. (51). Generally, the braneworlds with different extrinsic geometries have different cosmological evolutions. It is seen that in this scheme, by applying the Israel–Darmois–Lanczos junction condition, the number of degrees of freedom depends on the bulk space energy scale α_*, the signature of the extra dimensions $\epsilon = +1$, and the confined matter density ρ as well as the volume V and horizon temperature T.

At the end of this section, we remark that the presence of the quadratic energy density in the Friedmann equations which was initially anticipated as a possible solution to the observed accelerated expansion of the Universe, was shown to be incompatible with the big-bang nucleosynthesis [113–115]. Also, it is shown that this quadratic ρ term can constrain the high energy inflationary regimes in comparison with the observational SDSS/2DF/WMAP data [116–119]. In order to reconcile the above-mentioned braneworld scenario with the Z_2 symmetry or the Israel–Darmois–Lanczos condition with the observational data, one may propose that this scenario should be modified; see [120,121]. Therefore, in the following section we will study the case of a general braneworld embedding procedure without any simplifying junction condition or Z_2 symmetry.

---

[2] The Bekenstein limit is an upper limit on the entropy or information that can be contained within a given finite region of space which has a finite amount of energy. It implies that the information necessary to perfectly describe a system must be finite if the region of space and the energy is finite.

4 The general braneworld model without any specific junction condition

We consider the geometric quantity (35) with the barotropic equation of state,

$$p_{\text{extr}} = \omega_{\text{extr}} \rho_{\text{extr}}, \tag{52}$$

where ω_{extr} is the geometric equation of state parameter and it generally can be a function of time. Using Eqs. (36) and (52), we obtain the following equation for $b(t)$:

$$\frac{\dot{b}}{b} = \frac{1}{2} (1 - 3\omega_{\text{extr}}) \frac{\dot{a}}{a}, \tag{53}$$

where ω_{extr} is an unknown function. In general, solving the above equation is impossible unless the functional form of ω_{extr} is given. Let us consider the simple case where $\omega_{\text{extr}} =$ constant. In this case Eq. (53) can be solved immediately as

$$b = b_0 \left(\frac{a}{a_0} \right)^{\frac{1}{2}(1 - 3\omega_{\text{extr}})}, \tag{54}$$

where $a_0 = a(t_0)$ is the scale factor of the Universe at the present time and b_0 is an integration constant representing the curvature warp of the Universe at the present time. Substituting the solution (54) into Eq. (34) gives the components of the geometric quantity in terms of b_0, a_0, and $a(t)$ as

$$Q_{00} = \frac{1}{\epsilon} \frac{3b_0^2}{a_0^{1-3\omega_{\text{extr}}}} a^{-3(1+\omega_{\text{extr}})},$$

$$Q_{ij} = \frac{1}{\epsilon} \omega_{\text{extr}} \frac{3b_0^2}{a_0^{1-3\omega_{\text{extr}}}} a^{-3(1+\omega_{\text{extr}})} g_{ij}, \tag{55}$$

and consequently using Eq. (36) we get

$$\rho_{\text{extr}} = \frac{1}{\epsilon} \frac{3b_0^2}{a_0^{1-\omega_{\text{extr}}}} a^{-3(1+\omega_{\text{extr}})},$$

$$p_{\text{extr}} = \frac{1}{\epsilon} \omega_{\text{extr}} \frac{3b_0^2}{a_0^{1-3\omega_{\text{extr}}}} a^{-3(1+\omega_{\text{extr}})}. \tag{56}$$

Then, using Eqs. (56) and (35), the induced Einstein equation on the brane (30) gives the following equation for the confined energy density:

$$\rho = 3 \left(\frac{\dot{a}}{a} \right)^2 + \frac{3k}{a^2} - \frac{1}{\epsilon} \frac{3b_0^2}{a_0^{1-3\omega_{\text{extr}}}} a^{-3(1+\omega_{\text{extr}})}. \tag{57}$$

Note that we have not included the cosmological constant because it is possible to construct a geometrical origin for the dark energy in a general geometrical embedding scheme with a brane possessing an extrinsic curvature, to recover the acceleration of the Universe [103,122]. The generalization to the case that the cosmological constant is not zero is trivial. Similarly, the confined isotropic pressure component can be obtained from Eqs. (30), (35), and (56),

$$p = -2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a} \right)^2 - \frac{k}{a^2} - \frac{1}{\epsilon} \frac{3b_0^2 \omega_{\text{extr}}}{a_0^{1-3\omega_{\text{extr}}}} a^{-3(1+\omega_{\text{extr}})}. \tag{58}$$

Combining these equations leads to the following equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p) - \frac{1}{3\epsilon} \frac{1}{a^4} b_0^2 (1 + 3\omega_{\text{extr}}) \left(\frac{a}{a_0} \right)^{1-3\omega_{\text{extr}}}. \tag{59}$$

Using the same procedure as in the previous section, we obtain

$$\frac{\mathrm{d}V}{\mathrm{d}t} = N_{\text{sur}} - N_{\text{bulk}} - N_{\text{extr}}, \tag{60}$$

where the number of degrees of freedom related to the extrinsic geometry of spacetime has the general form

$$N_{\text{extr}} = 4\pi H^{-3} \frac{\frac{1}{3\epsilon} \frac{1}{a^4} b_0^2 (1 + 3\omega_{\text{extr}}) \left(\frac{a}{a_0} \right)^{1-3\omega_{extr}}}{2\pi \frac{H}{2\pi}}$$

$$= \frac{V}{2\pi T} \frac{1}{\epsilon a^4} b_0^2 (1 + 3\omega_{\text{extr}}) \left(\frac{a}{a_0} \right)^{1-3\omega_{\text{extr}}}. \tag{61}$$

For the case of a general geometric embedding scheme, the number of degrees of freedom related to the geometric embedding state of the brane in a higher-dimensional bulk beside the scale factor of the Universe a, depend on the signature of extra dimensions ϵ, the volume V, the horizon temperature T, and the warp factor of the Universe b_0 as well as the equation of state parameter of the geometric fluid ω_{extr}. If the curvature warp of the universe b_0 vanishes, all of the extrinsic curvature components will also vanish, and the braneworld will behave just like a trivial plane. In this case, the number of degrees of freedom corresponding to the extrinsic curvature vanishes and we recover the original Padmanabhan relation, $\frac{\mathrm{d}V}{\mathrm{d}t} = N_{\text{sur}} - N_{\text{bulk}}$.

Another interesting result is that when the geometric equation of state parameter becomes $\omega_{\text{extr}} = -1/3$, the corresponding degrees of freedom also vanishes. Moreover, there are two possibilities for satisfying the positivity of N_{extr}. The first possibility is $\omega_{\text{extr}} > -1/3$ with $\epsilon = +1$, which indicates a spacelike extra dimension. For this case, all of the known energy conditions, such as the weak, null, strong, and dominant energy conditions, are satisfied for the geometric fluid. The second possibility is $\omega_{\text{extr}} < -1/3$ with $\epsilon = -1$, which accounts for a timelike extra dimension. In this case, the energy conditions may be violated by the geometric fluid. It is worth mentioning that, according to (56), satisfying the weak energy condition requires $\epsilon = +1$, which is the same result as coming from the positiveness of N_{extr}.

Our universe is not pure de Sitter, but we know that it evolves toward an asymptotically de Sitter phase. For the purpose of reaching holographic equipartition we need to have $\mathrm{d}V/\mathrm{d}t \to 0$ in Eq. (60), which leads to $N_{\text{sur}} = N_{\text{bulk}} +$

N_{extr}. In order to understand the prominent feature of N_{extr} it is better to look at Eq. (60) without this term. Following the discussion of Padmanabhan, one can consider N_{bulk} to consist of two terms, one related to the normal matter with $\rho + 3p > 0$ and the other one related to the dark energy with $\rho + 3p < 0$ [11]. Therefore, it is possible to divide the degrees of freedom of the bulk into two terms, one coming from the degrees of freedom of dark energy leading to acceleration and the other one coming from the degrees of freedom of normal matter leading to deceleration. Then Eq. (60) takes the form of $\frac{dV}{dt} = N_{\text{sur}} + N_m - N_{\text{de}}$. Thus, it is seen that a universe without a dark energy component has no hope of reaching the holographic equipartition [10].

We may list the important results of the present study as follows.

- We can avoid the term N_{de}, namely dark energy or the cosmological constant, which has been proposed by Padmanabhan. In our general setup, dark energy has a completely geometrical origin [103]. In fact, the geometrical component denoted by N_{extr} plays the role of N_{de} as proposed by Padmanabhan. Similarly, we can better understand Eq. (60) if we separate out the matter component resulting in deceleration from the geometrical component resulting in acceleration. For the sake of simplicity, we will assume that the universe has just two components, normal matter with $\rho + 3p > 0$ and an effective (geometric) matter with $\rho_{\text{extr}} + 3p_{\text{extr}} < 0$. By our consideration, Eq. (60) can be expressed in an equivalent form as

$$\frac{dV}{dt} = N_{\text{sur}} + N_m - N_{\text{extr}}, \tag{62}$$

where $N_{\text{sur}}, N_m, N_{\text{extr}}$ are positive with

$$(N_m - N_{\text{extr}}) = \left(\frac{2V}{K_B T}\right)[(\rho + 3p) + |\rho_{\text{extr}} + 3p_{\text{extr}}|]. \tag{63}$$

It is seen that the holographic equipartition condition with asymptotically vanishing emergence of space $(dV/dt \rightarrow 0)$ can be satisfied only if the Universe possesses N_{extr}. Equivalently, the existence of a geometric term (due to the embedding of the brane) is required for the asymptotic holographic equipartition, which leads the cosmos finding its equilibrium. In the presence of N_{extr}, the emergence of space will lead to N_{extr} dominating over N_m with the universe experiencing accelerated expansion due to dark energy. Asymptotically, N_{extr} will approach N_{sur} and $dV/dt \rightarrow 0$ in a de Sitter universe.

- We can keep the term N_{de} as the dark energy degrees of freedom in the bulk and consider the new term N_{extr} as a geometric contribution to the term N_{de}. In this regard,

we can write the following equation:

$$\frac{dV}{dt} = N_{\text{sur}} + N_m - (N_{\text{de}} + N_{\text{extr}}). \tag{64}$$

The holographic equipartition condition is satisfied if $N_{\text{sur}} + N_m = (N_{\text{de}} + N_{\text{extr}})$. Moreover, we have

$$\begin{aligned}
&[N_m - (N_{\text{de}} + N_{\text{extr}})] \\
&= \left(\frac{2V}{K_B T}\right)[(\rho + 3p) + |\rho_{de} + 3p_{\text{de}}| \\
&\quad + |\rho_{\text{extr}} + 3p_{\text{extr}}|].
\end{aligned} \tag{65}$$

In the presence of $(N_{\text{de}} + N_{\text{extr}})$, the emergence of space will lead to $(N_{\text{de}} + N_{\text{extr}})$ dominating over N_m with the universe experiencing accelerated expansion due to dark energy and geometric embedding. Asymptotically, $(N_{\text{de}} + N_{\text{extr}})$ will approach N_{sur} and $dV/dt \rightarrow 0$ in a de Sitter universe.

5 Thermodynamics of a general braneworld scenario

The first law of thermodynamics for apparent horizon[3] reads [72]

$$-dE = TdS, \tag{66}$$

where T is the time-dependent temperature of a thermal heat bath as is perceived by an observer at $r = 0$, dE is the change in the mass of the matter present on the observer's side of the horizon, and dS is the increase in the horizon entropy. In this section, we confirm the validity of the first law of thermodynamics for apparent horizon in the presence of the additional terms due to the extrinsic curvature of a braneworld model and then we study the second law of thermodynamics with the assumption that in the apparent horizon the spacetime shows thermodynamical behavior. We also show that the second law of thermodynamics for an apparent horizon is always satisfied for an expanding universe.

The entropy of system is obtained as the sum of the surface entropy and the internal entropy. The internal entropy includes the entropy related to the ordinary matter fields localized on the brane and the geometric entropy corresponding to the induced geometric matter. In this section, we consider a general embedding scheme without any specific junction condition. For a flat universe, $k = 0$, we consider a perfect fluid form for the geometric fluid $Q_{\mu\nu}$ as in Eq. (35) and using Eqs. (56), (57), (59), and the conservation equation,

$$G^{\mu\nu}_{\ ;\mu} = (8\pi T^{\mu\nu} + Q^{\mu\nu})_{;\mu} = 0, \tag{67}$$

---

[3] At each hypersurface of constant time, the apparent horizon of an observer located at $r = 0$ is defined as the sphere whose orthogonal ingoing future-directed light-rays have vanishing expansion.

we obtain the three following equations:

$$\dot{\rho} + \dot{\rho}_{\text{extr}} + 3H(\rho + p + \rho_{\text{extr}} + p_{\text{extr}}) = 0, \qquad (68)$$

$$\dot{H} + H^2 = -\frac{4\pi}{3}(\rho + 3p + \rho_{\text{extr}} + 3p_{\text{extr}}), \qquad (69)$$

$$H^2 = \frac{8\pi}{3}\rho + \frac{8\pi}{3}\rho_{\text{extr}}. \qquad (70)$$

As is seen from the above equations, there are two kinds of matter sources for these equations. The first one is the normal matter ρ confined on the brane and the second one is the induced geometric matter ρ_{extr}. In order to obtain the entropy expression associated with the normal and geometric matter, we consider the variation of their corresponding energies, dE_{A_m} and $dE_{A_{extr}}$, which are achievable by the energy crossing formula on the apparent horizon as [72]

$$-dE_{A_m} = 4\pi R^2 T_{\mu\nu} K^\mu K^\nu dt = 4\pi R^2 (\rho + p) dt, \qquad (71)$$

$$-dE_{A_{extr}} = 4\pi R^2 Q_{\mu\nu} K^\mu K^\nu dt = 4\pi R^2 (\rho_{\text{extr}} + p_{\text{extr}}) dt, \qquad (72)$$

where we have used the subscript A to denote that the quantities are on the "Apparent horizon". In order to achieve the total energy crossing formula, we should consider Eqs. (71) and (72) together with $dE_A = dE_{A_m} + dE_{A_{extr}}$. Then we obtain

$$\begin{aligned} -dE_A &= 4\pi R^2 (T_{\mu\nu} + Q_{\mu\nu}) K^\mu K^\nu dt \\ &= 4\pi R^2 (\rho + p + \rho_{\text{extr}} + p_{\text{extr}}) dt. \end{aligned} \qquad (73)$$

From Eq. (68) we have

$$\rho + p + \rho_{\text{extr}} + p_{\text{extr}} = -\frac{\dot{\rho} + \dot{\rho}_{\text{extr}}}{3H}. \qquad (74)$$

Also, the derivative of Eq. (70) will give

$$2\dot{H}H = \frac{8\pi}{3}(\dot{\rho} + \dot{\rho}_{\text{extr}}). \qquad (75)$$

Then, using Eqs. (74) and (75), we obtain

$$\rho + p + \rho_{\text{extr}} + p_{\text{extr}} = -\frac{2\dot{H}}{8\pi}. \qquad (76)$$

Inserting Eq. (76) into Eq. (73) and considering H^{-1} instead of R_A, the energy crossing term will take the form of

$$dE_A = H^{-2}\dot{H}dt. \qquad (77)$$

We note that the radius of apparent horizon is $R_A = (H^2 + \frac{k}{a^2})^{\frac{-1}{2}}$ where for the flat spatial geometry $k = 0$, the radius of the apparent horizon R_A coincides with the Hubble horizon, $R_A = \frac{1}{H}$. Thus, by taking into account the area law of the entropy, i.e. $S_A = A/4 = 4\pi H^{-2}/4$, we find

$$T_A dS_A = \frac{H}{2\pi} d\left(\frac{4\pi H^{-2}}{4}\right) = -H^{-2}\dot{H}dt, \qquad (78)$$

which denotes the total surface crossing entropy of the system. Thus, one can confirm the validity of the first law of thermodynamics using Eqs. (77) and (78) as

$$-dE_A = T_A dS_A, \qquad (79)$$

where $T_A = (2\pi R_A)^{-1}$. Using (78), we obtain

$$\dot{S}_A = -2\pi \dot{H} H^{-3}. \qquad (80)$$

On the other hand, we have the internal entropy S_I for the system, which is related to the volume inside the horizon (we have used the subscript I to denote the quantities "Inside the apparent horizon"). For the internal entropy, we have

$$T_I dS_I = P dV + dE_I, \qquad (81)$$

where $P = p + p_{\text{extr}}$ and $E_I = (\rho + \rho_{\text{extr}})V$. Variation of (81) results in

$$\begin{aligned} \dot{S}_I &= \frac{(\rho + \rho_{\text{extr}} + p + p_{\text{extr}})\dot{V} + V(\dot{\rho} + \dot{\rho}_{\text{extr}})}{T_I} \\ &= \dot{S}_m + \dot{S}_{\text{extr}}, \end{aligned} \qquad (82)$$

where we have divided it in the entropy corresponding to the normal matter S_m and the geometric matter S_{extr}. The extrinsic geometric entropy shows its effect through the induced geometric fluid on the brane by adding a new term to the total internal entropy. In the above equation, T_I is the temperature of the thermal system inside the horizon. We consider a thermal system which is bounded by an apparent horizon that has reached equilibrium with its internal volume. This assumption allows us to put $T_I = T_A$ [88]. By putting $V = \frac{4\pi}{3}H^{-3}$ and using Eqs. (75) and (76) in Eq. (82), we obtain

$$\dot{S}_m + \dot{S}_{\text{extr}} = 2\pi \left(\frac{\dot{H}^2}{H^5} + \frac{\dot{H}}{H^3}\right). \qquad (83)$$

Then the total derivative of the entropy by adding Eq. (83) to Eq. (80) becomes

$$\dot{S}_t = \dot{S}_m + \dot{S}_{\text{extr}} + \dot{S}_A = 2\pi \left(\frac{\dot{H}^2}{H^5} + \frac{\dot{H}}{H^3}\right) - 2\pi \dot{H} H^{-3}. \qquad (84)$$

According to the second law of thermodynamics, the entropy of the thermodynamical systems can never decrease. So, the derivative of the entropy with respect to time is always greater than zero, i.e., $\dot{S}_t \geq 0$, so we have

$$\dot{S}_t = 2\pi \dot{H}^2 H^{-5} \geq 0. \qquad (85)$$

For a universe which expands, $H > 0$, the above equation is always true, representing that the second law of thermodynamics always holds. Then the entropy of the universe always increases and it depends on the normal matter ρ and the induced geometric matter ρ_{extr}, and by looking at Eq. (70), we see that it is independent of the pressure profiles p and p_{extr}. Therefore, in our braneworld model, similar to the

Kaluza–Klein model with $\dot{S}_t = \frac{21\pi^2}{8G_5H^2} \geqslant 0$, the entropy of the universe on the apparent horizon always increases [66].

6 Conclusion

In this paper, we have addressed the following question: what is the dynamical effect of the extrinsic geometrical embedding of an arbitrary four-dimensional brane in a higher-dimensional bulk space, from Padmanabhan's point of view on the emergent Universe? We have shown that other than the surface degrees of freedom and the bulk degrees of freedom, there are new degrees of freedom related to the extrinsic geometry of the brane embedded in a higher-dimensional bulk space which may play a basic role in the cosmological evolution. Based on this scenario, we have corrected the Padmanabhan relation as $\Delta V/\Delta t = N_{\mathrm{sur}} - N_{\mathrm{bulk}} - N_{\mathrm{extr}}$ where N_{extr} accounts for the new degrees of freedom corresponding to the extrinsic geometry of the brane. This term has a contribution to the Padmanabhan relation such that it plays the role of a dark energy. In this regard, we have separately investigated the braneworld scenarios with and without specific junction conditions. Moreover, we have shown that, for the case of braneworlds with the Israel–Darmois–Lanczos junction condition, the number of degrees of freedom is determined by the bulk space energy scale α_*, the signature of the extra dimensions $\epsilon = +1$, and the confined matter density ρ as well as the volume V and horizon temperature T. In this case, because of the positivity of the number of degrees of freedom, N_{extr}, the possibility of having a timelike extra dimension is ruled out.

For the case of a general geometric embedding scheme, the number of degrees of freedom related to the geometric embedding state of the brane in a higher-dimensional bulk space, beside the scale factor of the Universe a, depends on the signature of extra dimensions $\epsilon = \pm 1$, the volume V, the horizon temperature T, and the warp factor of Universe b_0, as well as the equation of state parameter of the geometric fluid ω_{extr}. If the curvature warp of the Universe b_0 vanishes, all of the extrinsic curvature components vanish too, and the braneworld behaves just as a trivial plane. In this case, the corresponding number of degrees of freedom also vanishes and we recover the original Padmanabhan relation, $\frac{\mathrm{d}V}{\mathrm{d}t} = N_{\mathrm{sur}} - N_{\mathrm{bulk}}$. Also, when the geometric equation of state parameter will be $\omega_{\mathrm{extr}} = -1/3$, the corresponding degrees of freedom also vanish in this approach. Moreover, the positivity of the number of degrees of freedom requires $\omega_{\mathrm{extr}} > -1/3$ for a spacelike extra dimension, while we have $\omega_{\mathrm{extr}} < -1/3$ for a timelike extra dimension. For the former case, all of the known energy conditions, as the weak, null, strong, and dominant energy conditions, are satisfied by the geometric fluid. They may be violated by the geometric fluid

for the latter case. Then we investigated the thermodynamical aspects of this general braneworld model. We confirmed the validity of the first law of thermodynamics on an apparent horizon in the presence of the additional terms due to the extrinsic curvature of a braneworld model and then we studied the second law of thermodynamics with the assumption that on the apparent horizon the spacetime shows a thermodynamical behavior. We found that the second law of thermodynamics is always satisfied for an expanding universe. It should be noticed that the presence of thermal equilibrium fluctuations and quantum fluctuations can contribute as an entropy correction, and consequently the number of degrees of freedom will be corrected, correspondingly. This work is under our current study and will be reported in the near future.

References

1. T. Padmanabhan, Rep. Prog. Phys. **73**, 046901 (2010)
2. T. Padmanabhan, Lessons from classical gravity about the quantum structure of spacetime. J. Phys. Conf. Ser. **306**, 012001 (2011)
3. A.D. Shakharov, Sov. Phys. Dokl. **12**, 1040 (1968)
4. T. Jacobson, Phys. Rev. Lett. **75**, 1260 (1995)
5. G.E. Volovik, *The universe in a helium droplet* (Oxford University Press, 2003)
6. B.L. Hu, arXiv:1010.5837
7. C. Barcelo, S. Liberati, M. Visser, Living Rev. Rel. **8**, 12 (2005)
8. E. Verlinde, JHEP **1104**, 029 (2011)
9. T. Padmanabhan, Phys. Rev. D **81**, 124040 (2010)
10. T. Padmanabhan, Res. Astro. Astrophys. **12**, 891 (2012)
11. T. Padmanabhan (2012). arXiv:1206.4916v1
12. J.D. Bekenstein, Phys. Rev. D **23**, 287 (1981)
13. A.G. Riess et al., Astron. J. **116**, 1009 (1998)
14. S. Perlmutter et al., Nature (London) **391**, 51 (1998)
15. P.M. Garnavich et al., Astrophys. J. **509**, 74 (1998)
16. S. Perlmutter et al., Astrophys. J. **517**, 565 (1999)
17. M. Tegmark et al., SDSS. Phys. Rev. D **74**, 123507 (2006). arXiv:astro-ph/0608632
18. E. Komatsu et al., WMAP. Astrophys. J. Suppl. **180**, 330 (2009). arXiv:0803.0547
19. T.-C. Chang, U.-L. Pen, J.B. Peterson, P. McDonald, Phys. Rev. Lett. **100**, 091303 (2008)
20. H.J. Seo, D.J. Eisenstein, Astrophys. J. **598**, 720 (2003)
21. J.C. Mather et al., Astrophys. J. **354**, L37–L40 (1990)
22. U. Alam, V. Sahni, A.A. Starobinsky, JCAP **06**, 008 (2004)
23. C. Clarkson, M. Cortês, B. Bassett, JCAP **08**, 011 (2007)

24. A. Upadhye, M. Ishak, P.J. Steinhardt, Phys. Rev. D **72**, 063501 (2005)
25. B. Ratra, P.J.E. Peebles, Phys. Rev. D **37**, 3406 (1988)
26. C. Wetterich, Nucl. Phys. B **302**, 668 (1988)
27. A.R. Liddle, R.J. Scherrer, Phys. Rev. D **59**, 023509 (1999)
28. Z.K. Guo, N. Ohta, Y.Z. Zhang, Mod. Phys. Lett. A **22**, 883 (2007)
29. S. Dutta, E.N. Saridakis, R.J. Scherrer, Phys. Rev. D **79**, 103005 (2009)
30. C. Armendariz-Picon, V.F. Mukhanov, P.J. Steinhardt, Phys. Rev. D **63**, 103510 (2001)
31. E.J. Copeland, M. Sami, S. Tsujikawa, Int. J. Mod. Phys. D **15**, 1753 (2006)
32. A. De Felice, S. Tsujikawa, Living Rev. Rel. **13**, 3 (2010)
33. S. Capozziello, V.F. Cardone, A. Troisi, Phys. Rev. D **71**, 043503 (2005)
34. S. Nojiri, S.D. Odintsov, Phys. Rev. D **74**, 086005 (2006)
35. L. Amendola, R. Gannouji, D. Polarski, S. Tsujikawa, Phys. Rev. D **75**, 083504 (2007)
36. S. Capozziello, F. Darabi, D. Vernieri, Mod. Phys. Lett. A **26**, 65 (2011)
37. E.V. Linder, Phys. Rev. D **81**, 127301 (2010)
38. E.V. Linder, Phys. Rev. D **82**, 109902 (2010)
39. S.H. Chen, J.B. Dent, S. Dutta, E.N. Saridakis, Phys. Rev. D **83**, 023508 (2011)
40. K. Atazadeh, F. Darabi, Eur. Phys. J. C **72**, 2016 (2012)
41. A. Paliathanasis, S. Basilakos, E.N. Saridakis, S. Capozziello, K. Atazadeh, F. Darabi, M. Tsamparlis, Phys. Rev. D **89**(10), 104042 (2014)
42. P. Horava, JHEP **0903**, 020 (2009)
43. G. Calcagni, JHEP **0909**, 112 (2009)
44. E.N. Saridakis, Eur. Phys. J. C **67**, 229 (2010)
45. J.T. Wheeler, Nucl. Phys. B **268**, 737 (1986)
46. S. Nojiri, S.D. Odintsov, Phys. Lett. B **631**, 1 (2005)
47. A. De Felice, S. Tsujikawa, Phys. Lett. B **675**, 1 (2009)
48. A.R. Rastkar, M.R. Setare, F. Darabi, Astrophys. Space Sci. **337**, 487 (2012)
49. P.D. Mannheim, D. Kazanas, Astrophys. J. **342**, 635 (1989)
50. E.E. Flanagan, Phys. Rev. D **74**, 023002 (2006)
51. D. Lovelock, J. Math. Phys. **12**, 498 (1971)
52. N. Deruelle, L. Farina-Busto, Phys. Rev. D **41**, 3696 (1990)
53. C. de Rham, G. Gabadadze, A.J. Tolley, Phys. Rev. Lett. **106**, 231101 (2011)
54. K. Hinterbichler, Rev. Mod. Phys. **84**, 671 (2012)
55. C. de Rham, Living Rev. Rel. **17**, 7 (2014)
56. G. Leon, J. Saavedra, E.N. Saridakis, Class. Quantum Grav. **30**, 135001 (2013)
57. C. Deffayet, G. Dvali, G. Gabadadze, Phys. Rev. D **65**, 044023 (2002)
58. G.R. Dvali, G. Gabadadze, M. Porrati, Phys. Lett. B **485**, 208–214 (2000)
59. M., V. Sahni, Y. Shtanov, JCAP **0311**, 014 (2003)
60. M.D. Maia, E.M. Monte, J.M.F. Maia, Phys Lett. B **585**, 11 (2004)
61. M. Gasperini, G. Veneziano, Astropart. Phys. **1**, 317 (1993)
62. H. Mohseni, Shadjadi. Europhys. Lett. **92**, 50014 (2010)
63. J.M. Bardeen, B. Carter, S.W. Hawking, Commun. Math. Phys. **31**, 161 (1973)
64. S.W. Hawking, Commun. Math. Phys. **43**, 199 (1975)
65. M. Sharif, S. Rani, Eur. Phys. J. Plus, 128 (2013)
66. M. Sharif, R. Saleem, Mod. Phys. Lett. A **28**, 1350072 (2013)
67. M. Akbar, R.G. Cai, Phys. Rev. D **75**, 084003 (2007)
68. R.G. Cai, L.M. Cao, Phys. Rev. D **75**, 064008 (2007)
69. R.G. Cai, S.P. Kim, JHEP **02**, 050 (2005)
70. A.V. Frolov, L. Kofman, JCAP **05**, 009 (2003)
71. U.K. Danielsson, Phys. Rev. D **71**, 023516(2005)

72. R. Bousso, Phys. Rev. D **71**, 064024 (2005)
73. G. Calcagni, JHEP **09**, 060 (2005)
74. U.H. Danielsson, Phys. Rev. D **71**, 023516 (2005)
75. E. Verlinde, arXiv:hep-th/0008140v2
76. B. Wang, E. Abdalla, R.K. Su, Phys. Lett. B **503**, 394 (2001)
77. B. Wang, E. Abdalla, R.K. Su, Mod. Phys. Lett. A **17**, 23 (2002)
78. R.G. Cai, Y.S. Myung, Phys. Rev. D **67**, 124021 (2003)
79. R.G. Cai, L.M. Cao, Nucl. Phys. B **785**, 135 (2007)
80. A. Sheykhi, B. Wang, R.G. Cai, Nucl. Phys. **779**, 1 (2007)
81. A. Sheykhi, B. Wang, R.G. Cai, Phys. Rev. D **76**, 023515 (2007)
82. B. Wang, Y. Gong, E. Abdalla, Phys. Rev. D **74**, 083520 (2006)
83. J. Zhou, B. Wang, Y. Gong, E. Abdalla, Phys. Lett. B **652**, 86 (2007)
84. A. Sheykhi, Class. Quantum Grav. **27**, 025007 (2010)
85. M. Akbar, Chin. Phys. Lett. **25**, 4199 (2008)
86. M. Akbar, Int. J. Theor. Phys. **48**, 2665 (2009)
87. A. Sheykhi, B. Wang, Phys. Lett. B **678**, 434 (2009)
88. A. Sheykhi, B. Wang, Mod. Phys. Lett. A **25**, 1199 (2010)
89. G. Izquierdo, D. Pavón. Phys. Lett. B **633**, 420 (2006)
90. K. Karami, JCAP **01**, 015 (2010)
91. K. Karami, S. Ghaffari, Phys. Lett. B **685**, 115 (2010)
92. K. Karami, S. Ghaffari, Phys. Lett. B **688**, 125 (2010)
93. K. Karami, S. Ghaffari, M.M. Soltanzadeh, Astrophys. Space Sci. (2010). doi:10.1007/s10509-010-0434-7
94. E. Babichev, V. Dokuchaev, Y. Eroshenko, Phys. Rev. Lett. **93**, 021102 (2004)
95. M.D. Pollock, T.P. Singh, Class. Quantum Grav. **6**, 901 (1989)
96. P.C.W. Davies, Class. Quantum Grav. **4**, L225 (1987)
97. D. Izquierdo, Pavon. Phys. Lett. B **639**, 1 (2006)
98. H. Mohseni, Sadjadi. Phys. Rev. D **73**, 063525 (2006)
99. H. Mohseni, Sadjadi. Phys. Rev. D **76**, 104024 (2007)
100. H. Mohseni, Sadjadi. Phys. Lett. B **645**, 108 (2007)
101. L. Randall, R. Sundrum, Phys. Rev. Lett. **83**, 4690–4693 (1999)
102. L. Randall, R. Sundrum, Phys. Rev. Lett. **83**, 3370–3373 (1999)
103. M.D. Maia, E.M. Monte, J.M.F. Maia, J.S. Alcaniz, Class. Quantum Grav. **22**, 1623 (2005)
104. W. Israel, Nuovo Cimento **44B**, 4349 (1966). erratum **48B**, 463 (1967)
105. V.A. Rubakov, M.E. Shaposhnikov, Phys. Lett. B **125**, 136 (1983)
106. S. Jalalzadeh, H.R. Sepangi, Class. Quantum Grav. **22**, 2035 (2005)
107. R. Maartens, K. Koyama, Living Rev. Relativ. **13**, 5 (2010)
108. J. Nash, Ann. Math. **63**, 20 (1956)
109. S. Jalalzadeh, H.R. Sepangi, Class. Quantum Grav **22**, 2035 (2005)
110. L.P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, 1966)
111. M.D. Maia, W.L. Roque, Phys. Lett. A **139**, 121 (1989)
112. M.D. Maia, E.M. Monte, J.M.F. Maia, Int. J. Mod. Phys. A **17**(29), 4341 (2002)
113. P. Binetruy, C. Defiayet, D. Langlois, Nucl. Phys. B **565**, 269 (2000)
114. P. Binetruy et al., Phys. Lett. B **477**, 285 (2000)
115. J.M. Cline et al., Phys. Rev. Lett. **83**, 4245 (1999)
116. S. Tsujikawa, A.R. Liddle, JCAP **0403**, 001 (2004)
117. J.D. Bratt et al., Phys. Rev. D **70**, 063525 (2004)
118. M.D. Maia, arXiv: astro-ph/0404370
119. S. Tsujikawa, M. Sami, R. Maartens, Phys. Rev. D **70**, 063525 (2004)
120. B. Carter, J.P. Uzan, Nucl. Phys. B **606**, 45 (2001)
121. G. Dvali, G. Gabadadze, M. Porrati, Phys. Lett. B **485**, 208 (2000)
122. S. Jalalzadeh, T. Rostami, Int. J. Mod. Phys. D **24**, 1550027 (2015)