

Microscopic thin-shell wormholes in magnetic Melvin universe

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Abstract We construct thin-shell wormholes in the magnetic Melvin universe. It is shown that in order to make a TSW in the Melvin spacetime the radius of the throat cannot be larger than $\frac{2}{B_0}$, in which B_0 is the magnetic field constant. We also analyze the stability of the constructed wormhole in terms of a linear perturbation around the equilibrium point. In our stability analysis we scan a full set of the Equation of States such as Linear Gas, Chaplygin Gas, Generalized Chaplygin Gas, Modified Generalized Chaplygin Gas, and Logarithmic Gas. Finally we extend our study to the wormhole solution in the unified Melvin and Bertotti–Robinson spacetime. In this extension we show that for some specific cases, the local energy density is partially positive but the total energy which supports the wormhole is positive.

1 Introduction

The magnetic Melvin universe (more appropriately the Bonnor–Melvin universe) [1–3] is sourced by a beam of magnetic field parallel to the z -axis in the Weyl coordinates $\{t, \rho, z, \varphi\}$. The metric depends only on the radial coordinate ρ which makes a typical case of cylindrical symmetry. It is a regular, non-black hole solution of the Einstein–Maxwell equations. The behavior of the magnetic field is $B(\rho) \sim \rho$ (for $\rho \rightarrow 0$) and $B(\rho) \sim \frac{1}{\rho^3}$ (for $\rho \rightarrow \infty$). At radial infinity the magnetic field vanishes but spacetime is not flat. On the symmetry axis ($\rho = 0$) the magnetic field vanishes; since the behavior is the same for $0 \leq |z| < \infty$ the Melvin spacetime is not asymptotically flat also for $|z| \rightarrow \infty$. The magnetic field can be assumed strong enough to warp spacetime to the extent that it produces possible wormholes. Strong magnetic fields are available in magnetars (i.e. $B \sim 10^{15}G$), while our Earth’s magnetic field is $B_{\text{Earth}} \sim 0.5G$), pulsars

and other objects. Since creation of strong magnetic fields can be at our disposal in a laboratory—at least in very short time intervals—it is natural to raise the question whether wormholes can be produced in a magnetized superconducting environment. *From this reasoning we aim to construct a thin-shell wormhole (TSW) in a magnetic Melvin universe.* The method is an art of spacetime tailoring, i.e. cutting and pasting at a throat region under well-defined mathematical junction conditions. Some related papers can be found in [4–17] for spherically symmetric bulk and in [18–24] for cylindrically symmetric cases. The TSW is threaded by exotic matter, which is taken for granted, and our principal aim is to search for the stability criteria for such a wormhole. Two cylindrically symmetric Melvin universes are glued at a hypersurface radius $\rho = a = \text{constant}$, which is endowed with surface energy-momentum to provide necessary support against the gravitational collapse. It turns out that in the Melvin spacetime the radial flare-out condition, i.e. $\frac{dg_{\varphi\varphi}}{da} > 0$ is satisfied for a restricted radial distance, which makes a small scale wormhole. Specifically, this amounts to a throat radius $\rho = a < \frac{2}{|B_0|}$, so that for high magnetic fields the throat radius can be made arbitrarily small. This can be dubbed as a microscopic wormhole. As stated recently such small wormholes may host the quantum Einstein–Podolsky–Rosen (EPR) pair [25]. The throat is linearly perturbed in the radial distance and the resulting perturbation equation is obtained. The problem is reduced to a one-dimensional particle problem whose oscillatory behavior for an effective potential $V(a)$ about the equilibrium point is provided by $V''(a_0) > 0$. Given the Equation of State (EoS) on the hypersurface we plot the parametric stability condition $V''(a_0) > 0$ to determine the possible stable regions. Our samples of EoS consist of a Linear gas, various forms of Chaplygin gas and a Logarithmic gas. We consider TSW also in the recently found Melvin–Bertotti–Robinson magnetic universe [26]. In the Bertotti–Robinson limit the wormhole is supported by total positive energy for any finite extension in the axial direction. For infinite extension the total energy

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reduces to zero, at least better than the total negative classical energy.

The organization of the paper is as follows. The construction of TSW from the magnetic Melvin spacetime is introduced in Sect. 2. Stability of the TSW is discussed in Sect. 3. Section 4 discusses the consequences of small velocity perturbations. Section 5 considers TSW in Melvin–Bertotti–Robinson spacetime and the Conclusion in Sect. 6 completes the paper.

2 Thin-shell wormhole in Melvin geometry

Let us start with the Melvin magnetic universe spacetime [1–3] in its axially symmetric form

$$ds^2 = U(\rho) \left(-dt^2 + d\rho^2 + dz^2 \right) + \frac{\rho^2}{U(\rho)} d\varphi^2 \tag{1}$$

in which

$$U(\rho) = \left(1 + \frac{B_0^2}{4} \rho^2 \right)^2 \tag{2}$$

where B_0 denotes the magnetic field constant. The Maxwell field two-form, however, is given by

$$F = \frac{\rho B_0}{U(\rho)} d\rho \wedge d\varphi. \tag{3}$$

We note that the Melvin solution in Einstein–Maxwell theory does not represent a black hole solution. The solution is regular everywhere as seen from the Ricci scalar and Ricci sequence

$$R = 0$$

$$R_{\mu\nu} R^{\mu\nu} = \frac{4B_0^4}{U(\rho)^8} \tag{4}$$

as well as the Kretschmann scalar

$$\mathcal{K} = \frac{4B_0^4 (3B_0^4 \rho^4 - 24B_0^2 \rho^2 + 80)}{U(\rho)^8}. \tag{5}$$

In [27], the general conditions which should be satisfied to have cylindrical wormhole possible are discussed. In brief, while the stronger condition implies that $\sqrt{g_{\varphi\varphi}}$ should take its minimum value at the throat, the weaker condition states that $\sqrt{g_{\varphi\varphi}g_{zz}}$ should be minimum at the throat. The stronger and weaker conditions are called *radial flare-out and areal flare-out conditions* respectively [28–30]. As we shall see in the sequel, in the case of TSW $\sqrt{g_{\varphi\varphi}}$ and $\sqrt{g_{\varphi\varphi}g_{zz}}$ should only be increasing function at the throat in *radial flare-out and areal flare-out conditions*. In the case of the Melvin spacetime,

$$\sqrt{g_{\varphi\varphi}} = \frac{\rho}{1 + \frac{B_0^2}{4} \rho^2} \tag{6}$$

and

$$\sqrt{g_{\varphi\varphi}g_{zz}} = \rho. \tag{7}$$

One easily finds that *areal flare-out condition* is trivially satisfied and the *radial flare-out condition* requires $\rho < \frac{2}{B_0}$.

Following Visser [4,31], from the bulk spacetime (1) we cut two non-asymptotically flat copies \mathcal{M}^\pm from a radius $\rho = a$ with $a > 0$ and then we glue them at a hypersurface $\Sigma = \Sigma^\pm$ which is defined as $\mathcal{H}(\rho) = \rho - a(\tau) = 0$. In this way the resultant manifold is complete. At hypersurface Σ the induced line element is given by

$$ds^2 = -d\tau^2 + U(a) dz^2 + \frac{a^2}{U(a)} d\varphi^2 \tag{8}$$

in which

$$-1 = U(a) \left(-\dot{\tau}^2 + \dot{\rho}^2 \right) \tag{9}$$

where a dot stands for derivative with respect to the proper time τ on the hypersurface Σ . The Israel junction conditions which are the Einstein equations on the junction hypersurface read ($8\pi G = 1$)

$$k_i^j - k\delta_i^j = -S_i^j, \tag{10}$$

in which $k_i^j = K_i^{j(+)} - K_i^{j(-)}$, $k = tr(k_i^j)$ and

$$K_{ij}^{(\pm)} = -n_\gamma^{(\pm)} \left(\frac{\partial^2 x^\gamma}{\partial X^i \partial X^j} + \Gamma_{\alpha\beta}^\gamma \frac{\partial x^\alpha}{\partial X^i} \frac{\partial x^\beta}{\partial X^j} \right)_\Sigma \tag{11}$$

is the extrinsic curvature. Also the normal unit vector is defined as

$$n_\gamma^{(\pm)} = \left(\pm \left| g^{\alpha\beta} \frac{\partial \mathcal{H}}{\partial x^\alpha} \frac{\partial \mathcal{H}}{\partial x^\beta} \right|^{-1/2} \frac{\partial \mathcal{H}}{\partial x^\gamma} \right)_\Sigma \tag{12}$$

and $S_i^j = \text{diag}(-\sigma, P_z, P_\varphi)$ is the energy-momentum tensor on Σ . Explicitly we find

$$n_\gamma^{(\pm)} = \pm \left(-\dot{a}U(a), U(a)\sqrt{\Delta}, 0, 0 \right)_\Sigma, \tag{13}$$

in which $\Delta = \frac{1}{U(a)} + \dot{a}^2$. The non-zero components of the extrinsic curvature are found as

$$K_\tau^{\tau(\pm)} = \pm \frac{1}{\sqrt{\Delta}} \left(\ddot{a} + \frac{U'}{U} \dot{a}^2 + \frac{U'}{2U^2} \right) \tag{14}$$

$$K_z^{z(\pm)} = \pm \frac{U'}{2U} \sqrt{\Delta}, \tag{15}$$

and

$$K_\varphi^{\varphi(\pm)} = \pm \left(\frac{1}{a} - \frac{U'}{2U} \right) \sqrt{\Delta}, \tag{16}$$

in which prime implies $\frac{\partial}{\partial a}$. Imposing the junction conditions [32–36] we find the components of the energy-momentum tensor on the shell which are expressed as

$$\sigma = -\frac{2}{a}\sqrt{\Delta} \tag{17}$$

$$P_z = \frac{2\ddot{a} + \frac{2U'}{U}\dot{a}^2 + \frac{U'}{U^2}}{\sqrt{\Delta}} + \left(\frac{2}{a} - \frac{U'}{U}\right)\sqrt{\Delta}, \tag{18}$$

and

$$P_\phi = \frac{2\ddot{a} + \frac{2U'}{U}\dot{a}^2 + \frac{U'}{U^2}}{\sqrt{\Delta}} + \frac{U'}{U}\sqrt{\Delta}. \tag{19}$$

Having the energy density on the shell, one may find the total exotic matter which supports the wormhole per unit z by

$$\Omega = 2\pi aU(a)\sigma, \tag{20}$$

which is clearly exotic.

3 Stability of the thin-shell wormhole against a linear perturbation

Recently, we have generalized the stability of TSWs in cylindrical symmetric bulks in [37]. Here we apply the same method to the TSWs in Melvin universe. Similar to the spherical symmetric TSW, we start with the energy conservation identity on the shell which implies

$$\begin{aligned} \left(aS_{;j}^{ij} = \right) \frac{d}{d\tau} (a\sigma) + \left[\frac{aU'}{2U} (P_z - P_\phi) + P_\phi \right] \frac{da}{d\tau} \\ = \frac{da}{d\tau} \frac{U'}{U} \left(4 - a \frac{U'}{U} \right) \sqrt{\Delta}. \end{aligned} \tag{21}$$

As we have shown in previous section the expressions given for surface energy density σ and surface pressures P_z and P_ϕ are for a dynamic wormhole. This means that if there exists an equilibrium radius for the throat radius, say $a = a_0$, at this point $\dot{a}_0 = 0$ and $\ddot{a}_0 = 0$ and consequently the form of the surface energy density and pressure reduce to the static forms as

$$\sigma_0 = -\frac{2}{a_0\sqrt{U_0}} \tag{22}$$

$$P_{z0} = \frac{2}{a_0\sqrt{U_0}} \tag{23}$$

and

$$P_{\phi 0} = 2\frac{U'_0}{U_0\sqrt{U_0}}. \tag{24}$$

Let us assume that after the perturbation the surface pressures are a general function of σ which may be written as

$$P_z = \Psi(\sigma) \tag{25}$$

and

$$P_\phi = \Phi(\sigma) \tag{26}$$

such that at the throat i.e. $a = a_0$, $\Psi(\sigma_0) = P_{z0}$ and $\Phi(\sigma_0) = P_{\phi 0}$. From (17) one finds a one-dimensional type equation of motion for the throat

$$\dot{a}^2 + V(a) = 0, \tag{27}$$

in which $V(a)$ is given by

$$V(a) = \frac{1}{U} - \left(\frac{a\sigma}{2}\right)^2. \tag{28}$$

Using the energy conservation identity (21), one finds

$$\begin{aligned} (a\sigma)' = - \left[\frac{aU'}{2U} (\Psi(\sigma) - \Phi(\sigma)) + \Phi(\sigma) \right] \\ + \frac{U'}{U} \left(4 - a \frac{U'}{U} \right) \sqrt{\Delta}, \end{aligned} \tag{29}$$

which helps us to show that $V'(a_0) = 0$ and

$$\begin{aligned} V''_0 = \frac{(2U_0 + a_0U'_0) [U'_0(\Phi'_0 - \Psi'_0)a_0 - 2U_0\Phi'_0]}{2U_0^3a_0^2} \\ - \frac{U_0^2(2U'_0 - 4a_0U''_0) + U_0(2U''_0U'_0a_0^2 + 7a_0U_0'^2) - 3a_0^2U_0'^3}{2U_0^4a_0}. \end{aligned} \tag{30}$$

Note that a subscript zero means that the corresponding quantity is evaluated at the equilibrium radius i.e., $a = a_0$. We also note that a prime denotes derivative with respect to its argument, for instance $\Psi'_0 = \frac{\partial\Psi}{\partial\sigma}|_{\sigma=\sigma_0}$ while $U'_0 = \frac{\partial U}{\partial a}|_{a=a_0}$. Now, if we expand the equation of motion of the throat about $a = a_0$ we find (up to second order)

$$\ddot{x} + \omega^2x \approx 0 \tag{31}$$

in which $x = a - a_0$ and $\omega^2 = \frac{1}{2}V''(a_0)$. This equation describes the motion of a harmonic oscillator provided $\omega^2 > 0$ which is the case of stability. If $\omega^2 < 0$ it implies that after the perturbation an exponential form fails to return back to its equilibrium point and therefore the wormhole is called unstable.

To draw conclusions as to the stability of the TSW in Melvin magnetic space we should examine the sign of $V''(a_0)$, and in any region where $V''(a_0) > 0$ the wormhole is stable and in contrast if $V''(a_0) < 0$ we conclude that the wormhole is unstable. From Eq. (30), we observe that this issue is identified with a, U_0, U'_0, U''_0 together with Φ'_0 and Ψ'_0 . Since the form of $U(a)$ is known, in order to examine the stability of the wormhole one should choose a specific EoS, i.e. $\Psi(\sigma)$ and $\Phi(\sigma)$. In the following section we shall consider the well-known cases of EoS which have been introduced in the literature. For each case we determine whether the TSW is stable or not.

3.1 Specific EoS

As we have already mentioned, in this chapter we go through the details of some specific EoS and the stability of the corresponding TSW.

3.1.1 Linear gas (LG)

Our first choice of the EoS is a LG in which $\Psi'(\sigma) = \beta_1$ and $\Phi'(\sigma) = \beta_2$ with β_1 and β_2 , two constant parameters related to the speed of sound in z and φ directions. We also find the form of $\Psi(\sigma)$ and $\Phi(\sigma)$ which are

$$\Psi(\sigma) = \beta_1\sigma + \Psi_0 \tag{32}$$

and

$$\Phi(\sigma) = \beta_2\sigma + \Phi_0 \tag{33}$$

with Ψ_0 and Φ_0 as integration constants. We impose $\Psi(\sigma_0) = P_{z0}$ and $\Phi(\sigma_0) = P_{\varphi 0}$, which yields

$$\Psi_0 = P_{z0} - \beta_1\sigma_0 \tag{34}$$

and

$$\Phi_0 = P_{\varphi 0} - \beta_2\sigma_0. \tag{35}$$

In the case with $\beta_1 = \beta_2 = \beta$, we find that Ψ and Φ are related as

$$\Psi - \Phi = P_{z0} - P_{\varphi 0}, \tag{36}$$

but in general they are independent. In Fig. 1 we consider $\beta_1 = \beta_2 = \beta$ and the resulting stable region with $V_0'' > 0$ is displayed.

3.1.2 Chaplygin gas (CG)

Our second choice of the EoS is a CG. The form of Ψ' and Φ' are given by

$$\Psi' = \frac{\beta_1}{\sigma^2} \quad \text{and} \quad \Phi' = \frac{\beta_2}{\sigma^2} \tag{37}$$

in which β_1 and β_2 are two new positive constants. Furthermore, one finds

$$\Psi(\sigma) = -\frac{\beta_1}{\sigma} + \Psi_0 \tag{38}$$

and

$$\Phi(\sigma) = -\frac{\beta_2}{\sigma} + \Phi_0 \tag{39}$$

in which as before Ψ_0 and Φ_0 are two integration constants. Imposing the equilibrium conditions $\Psi(\sigma_0) = P_{z0}$ and $\Phi(\sigma_0) = P_{\varphi 0}$ we find

$$\Psi_0 = P_{z0} + \frac{\beta_1}{\sigma_0} \tag{40}$$

and

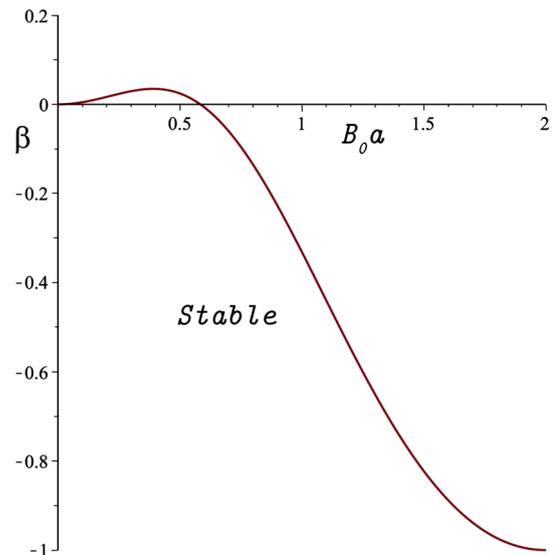


Fig. 1 Stability of TSW supported by LG in terms of $a_0 B_0$ and $\beta = \beta_1 = \beta_2$. We note that the upper bound of $a_0 B_0$ is chosen to be 2. This let $\frac{a^2}{f(a)}$ to remain an increasing function with respect to a . This condition is needed to have a TSW possible in CS spacetime [26]

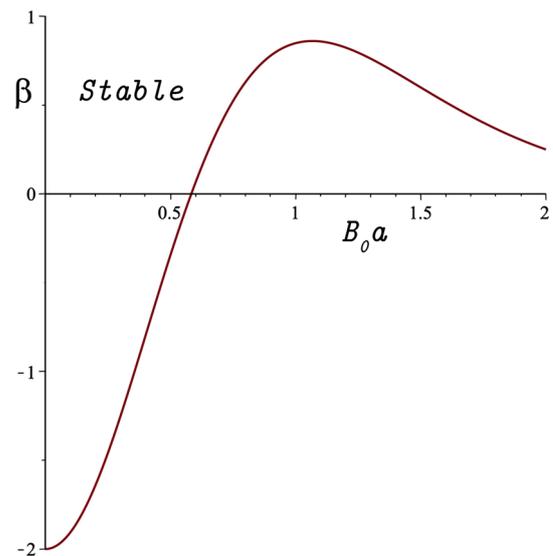


Fig. 2 Stability of TSW supported by CG in terms of $a_0 B_0$ and $\beta = \beta_1 = \beta_2$

$$\Phi_0 = P_{\varphi 0} + \frac{\beta_2}{\sigma_0}. \tag{41}$$

In Fig. 2 we plot the stability region of the TSW in terms of $\beta_1 = \beta_2 = \beta$ and $B_0 a$. We note that setting $\beta_1 = \beta_2 = \beta$ makes Ψ and Φ dependent as in the LG case i.e., (36), but in general they are independent.

3.1.3 Generalized Chaplygin gas (GCG)

After CG in this part we consider a GCG EoS which is defined as

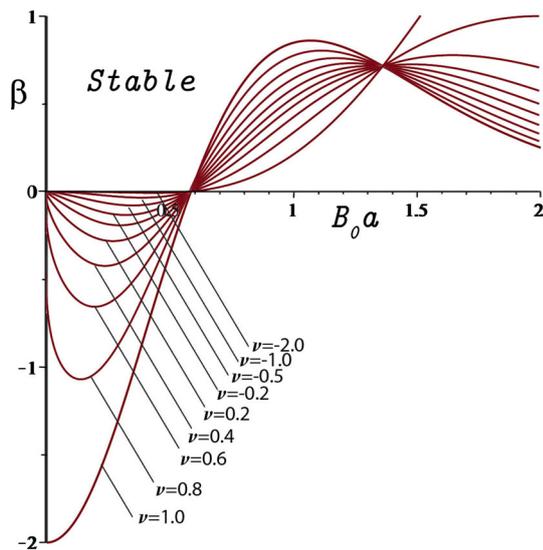


Fig. 3 Stability of TSW supported by GCG in terms of $a_0 B_0$ and $\beta = \beta_1 = \beta_2$ with various value of ν . The stable region is noted

$$\Psi' = \frac{\beta_1}{\sigma |\sigma|^\nu} \quad \text{and} \quad \Phi' = \frac{\beta_2}{\sigma |\sigma|^\nu} \tag{42}$$

and consequently

$$\Psi(\sigma) = -\frac{\beta_1}{\nu |\sigma|^\nu} + \Psi_0 \tag{43}$$

and

$$\Phi(\sigma) = -\frac{\beta_2}{\nu |\sigma|^\nu} + \Phi_0. \tag{44}$$

As before β_1 and β_2 are two new positive constants, $0 < \nu \leq 1$ and Ψ_0 and Φ_0 are integration constants. If we set $\beta_1 = \beta_2 = \beta$ again Ψ and Φ are not independent as Eq. (36). The equilibrium conditions imply

$$\Psi_0 = P_{z0} + \frac{\beta_1}{\nu |\sigma_0|^\nu} \tag{45}$$

while

$$\Phi_0 = P_{\varphi 0} + \frac{\beta_2}{\nu |\sigma_0|^\nu}. \tag{46}$$

In Fig. 3 we show the effect of the additional freedom i.e., ν in the stability of the corresponding TSW. We note that although in the standard definition of the GCG one has to consider $0 < \nu \leq 1$ in our figure we also considered beyond this limit.

3.1.4 Modified generalized Chaplygin gas (MGCG)

Another step toward further generalization is to combine the LG and the GCG. This is called MGCG and the form of the EoS may be written as

$$\Psi' = \xi_1 + \frac{\beta_1}{\sigma |\sigma|^\nu} \quad \text{and} \quad \Phi' = \xi_2 + \frac{\beta_2}{\sigma |\sigma|^\nu}. \tag{47}$$

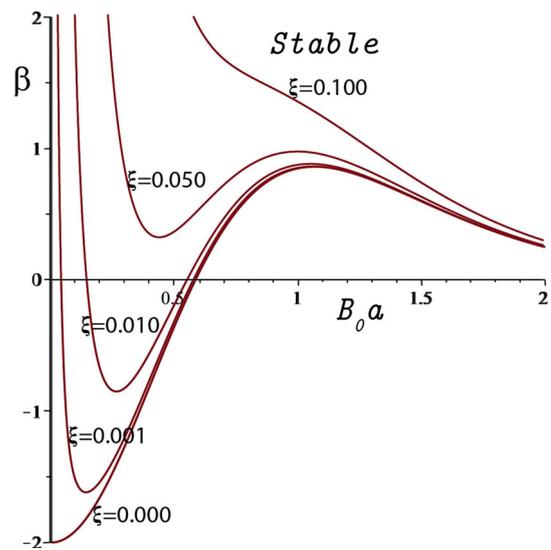


Fig. 4 Stability of TSW supported by MGCG in terms of $a_0 B_0$ and $\beta = \beta_1 = \beta_2$. The different curves are for different values of $\xi = \xi_1 = \xi_2$ and ν is chosen to be $\nu = 1$

Herein, $\beta_1 > 0, \beta_2 > 0, \xi_1$ and ξ_2 are constants and $0 < \nu \leq 1$. The form of Ψ and Φ can be found as

$$\Psi(\sigma) = \xi_1 \sigma - \frac{\beta_1}{\nu |\sigma|^\nu} + \Psi_0 \tag{48}$$

and

$$\Phi(\sigma) = \xi_2 \sigma - \frac{\beta_2}{\nu |\sigma|^\nu} + \Phi_0. \tag{49}$$

As before Ψ_0 and Φ_0 are integration constants which can be identified by imposing similar equilibrium conditions i.e., $\Psi(\sigma_0) = P_{z0}$ and $\Phi(\sigma_0) = P_{\varphi 0}$. After that we find

$$\Psi_0 = P_{z0} + \frac{\beta_1}{\nu |\sigma_0|^\nu} - \xi_1 \sigma_0 \tag{50}$$

and

$$\Phi_0 = P_{\varphi 0} + \frac{\beta_2}{\nu |\sigma_0|^\nu} - \xi_2 \sigma_0. \tag{51}$$

In Fig. 4 we plot the stability region of the TSW supported by the MGCG with additional arrangements as $\xi_1 = \xi_2 = \xi$ and $\beta_1 = \beta_2 = \beta$. We again comment that these make Ψ and Φ dependent, while in general they are independent. In Fig. 4 specifically we show the effect of the additional freedom to the GCG, i.e., ξ in a frame of β and $B_0 a$.

3.1.5 Logarithmic gas (LogG)

Finally we consider the LogG with

$$\Psi' = -\frac{\beta_1}{\sigma} \quad \text{and} \quad \Phi' = -\frac{\beta_2}{\sigma} \tag{52}$$

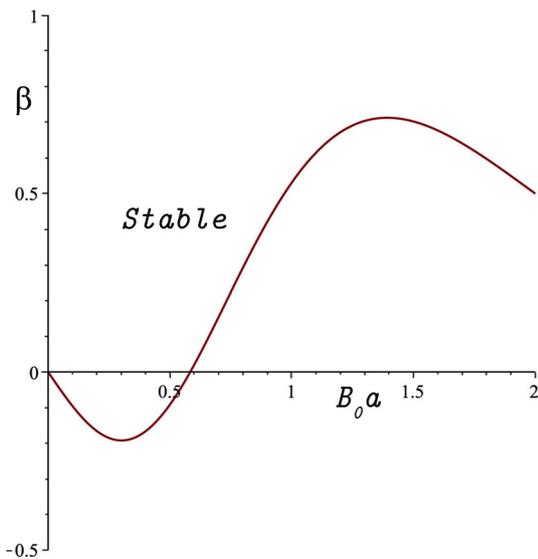


Fig. 5 Stability of TSW supported by LogG in terms of $a_0 B_0$ and $\beta = \beta_1 = \beta_2$. We note that the upper bound of $a_0 B_0$ is chosen to be 2

where $\beta_1 > 0$ and $\beta_2 > 0$ are two positive constants. The EoS are given by

$$\Psi = -\beta_1 \ln \left| \frac{\sigma}{\sigma_0} \right| + \Psi_0 \quad \text{and} \quad \Phi = -\beta_2 \ln \left| \frac{\sigma}{\sigma_0} \right| + \Phi_0 \quad (53)$$

in which the $\beta_1 \ln |\sigma_0| + \Psi_0$ and $\beta_2 \ln |\sigma_0| + \Phi_0$ are integration constants. Imposing the equilibrium conditions one finds $\Psi_0 = P_{z0}$ and $\Phi_0 = P_{\varphi 0}$. In Fig. 5 we plot the stability region in terms of $\beta_1 = \beta_2 = \beta$ versus $B_0 a$.

4 Small velocity perturbation

In the previous chapter we have considered a linear perturbation around the equilibrium point of the throat. As we have considered above, the EoS of the fluid on the thin shell after the perturbation had no relation with its equilibrium state. However, by setting $\beta_1 = \beta_2$ in our analysis in previous chapter, implicitly we accepted that $\Psi - \Phi = P_z - P_\varphi$ does not change in time, a restriction that is physically acceptable.

In this chapter we consider the EoS of the TSW after the perturbation same as its equilibrium point. This in fact means that the time evolution of the throat is slow enough that any intermediate step between the initial point and a certain final point can be considered as another equilibrium point (or static). Quantitatively it means that $\frac{P_z}{\sigma} = -1$ (same as $\frac{P_{z0}}{\sigma_0} = -1$) and $\frac{P_\varphi}{\sigma} = -a \frac{U'}{U}$ (same as $\frac{P_{\varphi 0}}{\sigma_0} = -a_0 \frac{U'_0}{U_0}$) and consequently, from (17), (18) and (19), we find a single second order differential equation which may be written as

$$2\ddot{a} + \frac{U'}{U} \dot{a}^2 = 0. \quad (54)$$

This equation gives the exact motion of the throat after the perturbation. (We note once more that the process of time evolution is considered with small velocity). This equation can be integrated to obtain

$$\dot{a} = \dot{a}_0 \sqrt{\frac{U_0}{U}}. \quad (55)$$

A second integration with the exact form of U yields

$$a \left(1 + \frac{B_0^2}{12} a^2 \right) = a_0 \left(1 + \frac{B_0^2}{12} a_0^2 \right) + \dot{a}_0 \sqrt{U_0} (\tau - \tau_0). \quad (56)$$

The motion of the throat is under a negative force per unit mass which is position and velocity dependent. As is clear from the expression of \dot{a} , the magnitude of velocity is always positive and it never vanishes. This means that the motion of the throat is not oscillatory but builds up in the same direction after perturbation. Also from (56) we see that in proper time if $\dot{a}_0 > 0$, a goes to infinity and when $\dot{a}_0 < 0$, a goes to zero. In both cases the particle-like motion does not return to its initial position $a = a_0$. These mean that the TSW is not stable under small velocity perturbations.

5 TSW in unified Bertotti–Robinson and Melvin spacetimes

Recently two of us found a new solution to the Einstein–Maxwell equations which represents unified Bertotti–Robinson and Melvin spacetimes [26] whose line element is given by

$$ds^2 = -e^{2u} dt^2 + e^{-2u} \left[e^{2\kappa} (d\rho^2 + dz^2) + \rho^2 d\varphi^2 \right] \quad (57)$$

where

$$e^u = F = \lambda_0 \left[\sqrt{\rho^2 + z^2} \cosh \left(\frac{B_0}{\lambda_0} \ln \rho \right) - z \sinh \left(\frac{B_0}{\lambda_0} \ln \rho \right) \right] \quad (58)$$

and

$$e^\kappa = \frac{F^2}{(\rho^2 + z^2)} \left[\frac{\rho^{1 + \frac{B_0}{2\lambda_0}}}{z + \sqrt{\rho^2 + z^2}} \right]^{\frac{2B_0}{\lambda_0}}. \quad (59)$$

Herein λ_0 and B_0 are two essential parameters of the spacetime which are related to the magnetic field of the system and the topology of the spacetime. The magnetic potential of the spacetime is given by

$$A_\mu = \Phi(\rho, z) \delta_\mu^\varphi \quad (60)$$

in which

$$\Phi_\rho(\rho, z) = \rho e^{-2u} \psi_z \quad (61)$$

and

$$\Phi_z(\rho, z) = -\rho e^{-2u} \psi_\rho \tag{62}$$

with

$$\psi = \lambda_0 \left[\sqrt{\rho^2 + z^2} \right] + B_0 z. \tag{63}$$

The standard method of making TSW implies that $\mathcal{H}(\rho) = \rho - a(\tau) = 0$ is the timelike hypersurface where the throat is located and the line element on the shell reads

$$ds^2 = -d\tau^2 + e^{-2u(a,z)} \left[e^{2\kappa(a,z)} dz^2 + a^2 d\varphi^2 \right]. \tag{64}$$

The 4-vector normal to the shell is found to be

$$n_\gamma^{(\pm)} = \pm \left(-\dot{a} e^\kappa, e^{2(\kappa-u)} \sqrt{\Delta}, 0, 0 \right)_\Sigma,$$

with $\Delta = (e^{2(u-\kappa)} + \dot{a}^2)$ and the non-zero elements of the extrinsic curvature tensor become

$$K_\tau^{\tau(\pm)} = \pm \left[\frac{\ddot{a} + (\kappa' - u') \dot{a}^2}{\sqrt{\Delta}} + u' \sqrt{\Delta} \right], \tag{65}$$

$$K_z^{z(\pm)} = \mp (u' - \kappa') \sqrt{\Delta}, \tag{66}$$

and

$$K_\varphi^{\varphi(\pm)} = \mp \left(u' - \frac{1}{a} \right) \sqrt{\Delta}. \tag{67}$$

Upon the Israel junction conditions, one finds

$$\sigma = 2\sqrt{\Delta} \left(2u' - \kappa' - \frac{1}{a} \right), \tag{68}$$

$$P_z = 2 \left[\frac{\ddot{a} + (\kappa' - u') \dot{a}^2}{\sqrt{\Delta}} + \frac{1}{a} \sqrt{\Delta} \right] \tag{69}$$

and

$$P_\varphi = 2 \left[\frac{\ddot{a} + (\kappa' - u') \dot{a}^2}{\sqrt{\Delta}} + \kappa' \sqrt{\Delta} \right]. \tag{70}$$

The results given above can be used to find the σ_0 , P_{z0} and $P_{\varphi 0}$ at the equilibrium radius $a = a_0$ i.e.,

$$\sigma_0 = 2e^{(u-\kappa)} \left(2u' - \kappa' - \frac{1}{a} \right) \Big|_{a=a_0}, \tag{71}$$

$$P_{z0} = \frac{2}{a} e^{(u-\kappa)} \Big|_{a=a_0} \tag{72}$$

and

$$P_{\varphi 0} = 2\kappa' e^{(u-\kappa)} \Big|_{a=a_0}. \tag{73}$$

Next, we use the exact form of κ and u to find the energy density of the shell, which can be written as

$$\sigma_0 = \frac{2a_0}{a_0^2 + z^2} - \frac{(\epsilon + 1)^2}{a_0} + \frac{2\epsilon a_0}{\sqrt{a_0^2 + z^2} \left(z + \sqrt{a_0^2 + z^2} \right)} \tag{74}$$

in which $\epsilon = \frac{B_0}{\lambda_0}$. To analyze the sign of σ_0 we introduce $\zeta = \frac{z}{a_0}$ and rewrite the latter equation as

$$a_0 \sigma = - (1 + \epsilon)^2 + \frac{2 \left(\zeta + (1 + \epsilon) \sqrt{1 + \zeta^2} \right)}{(1 + \zeta^2) \left(\zeta + \sqrt{1 + \zeta^2} \right)}. \tag{75}$$

One of the interesting cases is when we set $\epsilon = -1$, which yields

$$a_0 \sigma_0 = \frac{2\zeta}{(1 + \zeta^2) \left(\zeta + \sqrt{1 + \zeta^2} \right)}. \tag{76}$$

This is positive for $\zeta > 0$ ($z > 0$), negative for $\zeta < 0$ ($z < 0$) and zero for $\zeta = 0$ ($z = 0$). Another interesting case is when we set $\epsilon = 0$ which is the BR limit of the general solution (57–59). In this setting we find

$$a_0 \sigma_0 = \frac{2}{1 + \zeta^2} - 1 \tag{77}$$

which is positive for $|\zeta| < 1$. In Fig. 6 we plot the region on which $a_0 \sigma_0 \geq 0$ in terms of ϵ and ζ . To find the total energy of the shell we use

$$\Omega = \int_0^{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} \sigma_0 \delta(\rho - a_0) \sqrt{-g} d\rho dz d\varphi \tag{78}$$

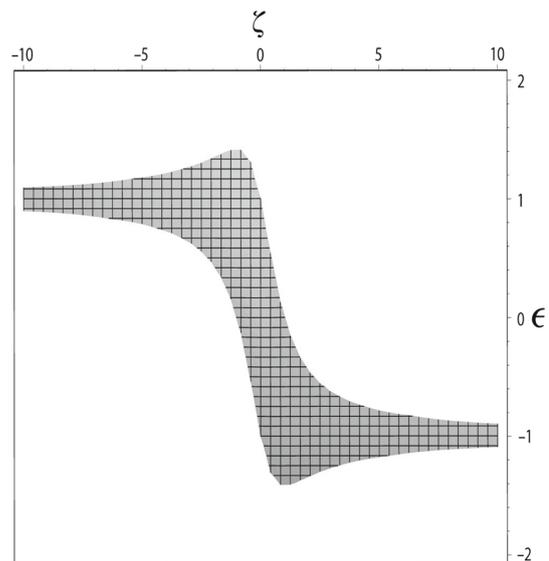


Fig. 6 $a_0 \sigma_0$ versus ϵ and ζ . The shaded region in the region on which $a_0 \sigma_0$ is positive

which after some manipulation becomes

$$\Omega = 2\pi \int_{-\infty}^{+\infty} \sigma_0 a_0 e^{2(\kappa_0 - u_0)} dz, \quad (79)$$

in which $\kappa_0 = \kappa|_{a=a_0}$ and $u_0 = u|_{a=a_0}$. Upon some further manipulation we arrive at

$$\frac{\Omega}{2\pi \lambda_0^2 a_0^{2\epsilon^2 - 1}} = \int_{-\infty}^{\infty} \left[\frac{2(\zeta + (1+\epsilon)\sqrt{1+\zeta^2})}{(1+\zeta^2)^3 (\zeta + \sqrt{1+\zeta^2})} - \frac{(1+\epsilon)^2}{(1+\zeta^2)^2} \right] \times \frac{(\sqrt{1+\zeta^2} \cosh(\epsilon \ln a_0) - \zeta \sinh(\epsilon \ln a_0))^2}{(\zeta + \sqrt{1+\zeta^2})^{4\epsilon}} d\zeta. \quad (80)$$

Although this integral cannot be evaluated explicitly for arbitrary ϵ at least for $\epsilon = 0$ it gives

$$\Omega = \lim_{R \rightarrow \infty} \frac{4\pi \lambda_0^2 R}{a_0 (1 + R^2)}, \quad (81)$$

which is positive. Obviously this limit (i.e. $\epsilon = 0$) corresponds to the Bertotti–Robinson limit of the general solution in which for $R < \infty$ construction of a TSW with a positive total energy becomes possible.

6 Conclusion

A large class of stable TSW solutions is found by employing the magnetic Melvin universe through the cut-and-paste technique. The Melvin spacetime is a typical cylindrically symmetric, regular solution of the Einstein–Maxwell equations. Herein the throat radius of the TSW is confined by a strong magnetic field; for this reason we phrase them as *microscopic wormholes*. Being regular its construction can be achieved by a finite energy. It has recently been suggested that the mysterious EPR particles may be connected through a wormhole [38]. From this point of view the magnetic Melvin wormhole may be instrumental to test such a claim. We have applied radial, linear perturbation to the throat radius of the TSW in search for stability regions. In such perturbations we observed that the initial radial speed must be chosen zero in order to attain a stable TSW. Different perturbations may cause collapse of the wormhole. As the material on the throat we have adopted various equations of states, ranging from an ordinary linear/logarithmic gas to a Chaplygin gas. The repulsive support derived from such sources gives life to the TSW against the gravitational collapse. Besides pure Melvin case we have also considered TSW in the magnetic universe

of unified Melvin and Bertotti–Robinson spacetimes. The pure Bertotti–Robinson TSW has positive total energy for each finite axial length ($R < \infty$). The energy becomes zero when the cut-off length $R \rightarrow \infty$.

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