

# Physical model of dimensional regularization

Jonathan F. Schonfeld <sup>a</sup>

Lexington, MA 02420, USA

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**Abstract** We explicitly construct fractals of dimension  $4-\varepsilon$  on which dimensional regularization approximates scalar-field-only quantum-field theory amplitudes. The construction does not require fractals to be Lorentz-invariant in any sense, and we argue that there probably is no Lorentz-invariant fractal of dimension greater than 2. We derive dimensional regularization's power-law screening first for fractals obtained by removing voids from 3-dimensional Euclidean space. The derivation applies techniques from elementary dielectric theory. Surprisingly, fractal geometry by itself does not guarantee the appropriate power-law behavior; boundary conditions at fractal voids also play an important role. We then extend the derivation to 4-dimensional Minkowski space. We comment on generalization to non-scalar fields, and speculate about implications for quantum gravity.

## 1 Introduction

Is “dimension deficit” really the correct physical meaning of the parameter  $\varepsilon$  in dimensional regularization? The only way to prove it is by explicitly constructing a fractal spacetime on which dimensional regularization approximates quantum-field amplitudes. Introducing such a construction for scalar-only quantum-field theories is the purpose of this paper.

Ideally, this is the first step in a longer research program aimed at extending this construction to non-scalar fields. Even if that does not materialize, the scalar construction should be of interest in its own right, as it casts a fresh light on the foundations of dimensional regularization, one of the cornerstones of modern quantum-field theory. After all, other schemes such as Pauli–Villars and lattice regularization have well-defined physical meanings that enable scientists to benefit from intuition established in a variety of other domains. Why not dimensional regularization?

<sup>a</sup>e-mail: [jschonfeld@aya.yale.edu](mailto:jschonfeld@aya.yale.edu)

The remainder of this paper is organized as follows. Section 2 presents essential concepts and argumentation about dimensional regularization and fractals, and sets the stage for the constructions and derivations that follow. Section 3 derives the power-law screening characteristic of dimensional regularization for propagators on fractals defined by removing voids from 3-dimensional Euclidean space, in order to establish basic intuition and lines of argumentation. The results of Sect. 3 apply techniques from elementary dielectric theory. Surprisingly, fractal geometry by itself does not guarantee the power-law behavior required for dimensional regularization; boundary conditions at fractal voids also play an important role. Section 4 extends Sect. 3 to fractals in 4-dimensional Minkowski space. Note that in Sect. 4 the fractals themselves are not Lorentz invariant, but that is alright because (see below) anisotropy in fractal power-law scaling appears to have no impact on dimensional regularization for small  $\varepsilon$  (we will in fact argue that there is no such thing as a relativistically invariant fractal with dimension greater than 2). Section 5 contains a discussion of weaknesses in our reasoning, as well as prospects for generalization to non-scalar fields, and speculation about implications for quantum gravity.

## 2 Preliminaries about dimensional regularization and fractals

Dimensional regularization [1] for scalar fields amounts to changing the momentum-space volume element  $d^4 p$  in Feynman diagrams to  $|p/\mu|^{-\varepsilon} d^4 p$ , where  $|p|$  is the Minkowski norm of momentum  $p$ ,  $\varepsilon$  is positive and  $\mu$  is a fixed scale. The important thing is that the multiplier behaves like a fractional power of the scale factor as  $p$  scales to infinity along any fixed direction. That is what turns logarithmic divergences in  $p$  into poles in  $\varepsilon$ . This has the same effect as multiplying the scalar propagator (instead of the integration volume) in momentum space by  $|p/\mu|^{-\delta}$ , where  $\delta = \varepsilon/2$ , because the only diver-

gent loops have two scalar propagators. For this reason, the goal of the fractal constructions that follow is to show that scalar propagators in fractal spacetimes exhibit screening of the form  $|p/\mu|^{-\delta}$  in momentum space for large momentum  $p$  or, as appropriate,  $|x\mu|^{+\delta}$  in position space for small position  $x$  and non-negative  $\delta$  (more precisely, when quantum amplitudes are defined by path integrals over random fractals [see below] obtained by removing voids from linear spacetime, then the quantum amplitudes, when ensemble-averaged over random fractals, numerically correspond to Feynman diagrams in the underlying integer-dimensional linear spacetime with propagators screened as described above).

There is a hitch, however. In Sect. 4 we shall find ourselves dealing with fractals that are not themselves Lorentz-invariant. This means that we will really show that propagators in position space scale as  $|x\mu f(\Omega)|^{+\delta}$  at short distance or  $|g(\Omega)p/\mu|^{-\delta}$  at large momentum, where  $\Omega$  is solid angle in four dimensions and  $f$  or  $g$  is some function that is nonzero almost everywhere. But that is alright as far as Lorentz-invariance of quantum amplitudes is concerned, because the function  $f$  has no *material* impact on dimensional regularization for small dimension deficit: Small  $\varepsilon$  (or  $\delta$ ) ensures that integration over  $\Omega$  does not diverge; and ignoring terms of order  $\varepsilon$  and higher ensures that the only quantitative effect that  $g$  has on Feynman integrals is to modify the effective value of  $\mu$ , because the  $O(\varepsilon)$  term in  $g(\Omega)^{-\varepsilon}$  can only manifest itself by multiplying the  $1/\varepsilon$  linear-scale divergence by the integral of  $\ln[g(\Omega)]$  over all solid angles.

The constructions in this paper focus on random “take-away” fractals. Randomness exempts us from the complications of accidental crystallographic symmetries. For the purposes of this paper, a random “take-away” fractal is a set formed by the following recursive procedure: Start with a linear space of integer dimension  $D$ , and a reference void of volume  $V$ . Distribute points randomly throughout space with arbitrary density  $\rho$ , and, centered at each such point, remove a copy of the reference void. Call this the zeroth iteration. Now choose an arbitrary scale factor  $\xi > 1$  and define the  $k$ th iteration inductively as follows:

- Distribute points randomly with density  $\rho\xi^{Dk}$  throughout whatever part of the Euclidean space has not been removed by preceding iterations.
- Centered at each such point, remove from the  $k - 1$ 'st iteration a copy of the reference void linearly scaled by factor  $\xi^{-k}$ .

In the limit of infinite  $k$ , what is left has fractal dimension  $D + \ln(1 - \rho V) / \ln \xi$  [2]. The factor  $(1 - \rho V)$  is the volumetric proportion of iteration  $k - 1$  removed by iteration  $k$ , and the ratio of logarithms is clearly minus a dimension deficit for physical  $\rho V < 1$ . (The geometry of the reference void can have its own probability distribution, but this is more

generality than we require. Also, for small  $\rho V$  it is unimportant that voids removed at iteration  $k$  might overlap with one another or with voids removed at earlier iterations.)

A *relativistic* fractal would have a Lorentz-invariant reference void. But such a void – bounded by hyperbolic spheres – would have infinite  $V$ . Interestingly, if the starting linear space had dimension  $D = 2$ , a fractal could still be defined. For example, consider as a reference void the space between a forward light cone and a mass shell. Basically this is a central lobe flaring into two wings whose thickness falls like one over the distance from the vertex. A random distribution of such voids produces a fractal after just a single iteration, because there are  $\rho 2\pi r dr$  void centers at distance  $r$  from any reference point, and their wing width falls like  $1/r$  (basically,  $1/r$  is analogous to  $V\xi^{-Dk}$  in the fundamental fractal definition, and  $\rho 2\pi r$  is analogous to  $\rho\xi^{Dk}$ ). For arbitrary  $D$  the number of void centers at distance  $r$  is proportional to  $\rho r^{D-1} dr$  but wing (really a cup lip) width still falls like  $1/r$ , so the two factors match only for  $D = 2$  and we conclude that there probably are no relativistic fractals in higher dimensions. This reinforces the importance of our earlier discussion of Lorentz invariance vs. nontrivial  $f$  or  $g$ . Perhaps this uniqueness of  $D = 2$  is behind the tendency of renormalization-group flows to converge to 2-dimensional fractals in reduced models of gravity [3] (Alternatively, one might construct a relativistic fractal by replacing the idea of a single infinite-volume Lorentz-invariant reference void with an appropriately weighted ensemble of voids created by applying Lorentz boosts to a single finite-volume “seed” void. But one is driven to the same conclusion about  $D = 2$  because the Lorentz-invariant measure on the set of all boosts has itself infinite total weight).

### 3 Propagator on fractal derived from Euclidean 3-space

As indicated above, the narrow mathematical objective of this paper is to derive power-law screening at small distances or large momenta for wave-equation propagators in 4-dimensional Minkowski space limited by a fractal distribution of voids. To make the thought process as clear as possible, we build to this objective with three cases of successively increasing sophistication. The last case is Minkowski space.

For the first case, consider recovering the  $\ln r$  Green function for potential theory in two dimensions by limiting three dimensions to the space between two closely separated parallel planes. In school we encounter the problem of a point charge between two parallel conducting planes, but because the infinite sequence of image charges involves alternating signs, the potential does not approach  $\ln r$  for vanishing plane separation [4]. If instead of conducting planes – i.e. constant-value Dirichlet boundary condition – we impose the other canonical potential-theory boundary condition – zero-

normal-derivative Neumann – the image charges are in the same locations but all have identical sign. So they add coherently to produce  $\ln r$  for vanishingly small plane separation. Naively, the coefficient of  $\ln r$  diverges as  $q/a$ , where  $q$  is the original point charge and  $a$  is plane separation, but  $a$  cancels out because the 2D Green function is meant to be integrated over the limiting plane, while in 3D it is to be integrated over the space between the converging planes, and that volume is proportional to  $a$ . This sets a pattern for the cases that follow: invocation of Neumann boundary conditions modulated by vanishing volume between voids.

For the second case, consider the Green function at short distance for potential theory in  $D = 3$ -dimensional Euclidean space limited by a fractal distribution of spherical voids. If the spheres are small, the field around each primarily induces an electrostatic dipole [5] with polarizability  $\gamma_k$  for the spheres of iteration  $k$ . According to dielectric theory [5], these spheres collectively amplify or shield a distant charge by a factor

$$\Phi_k = \left[ 1 + \frac{4\pi\rho\xi^{3k}\gamma_k}{1 - \frac{4\pi}{3}\rho\xi^{3k}\gamma_k} \right]^{-1} \tag{1}$$

Each iteration of the fractal process multiplies the Green function (potential) of a point charge by this factor in the space between spheres, *but only for iterations whose spheres are smaller than the distance to the point charge*, since larger spheres do not fit. At the same time, each iteration also multiplies the point-charge potential by a factor of  $(1 - \rho V)$  for integration volume regardless of sphere size. In other words, the Green function for point charge  $q$  becomes

$$-\frac{q}{r} \prod_{k=0}^{\infty} (1 - \rho V) \Phi_k \prod_{l=0}^{l_{\max}} \Phi_l^{-1} \tag{2}$$

where  $l_{\max}$  is the highest iteration whose spheres are larger than or equal to  $r$ . For the infinite product to be well defined,  $(1 - \rho V)\Phi_k$  must be unity. Thus we discover that spheres have to come in a mix of boundary conditions so that on average

$$\frac{4\pi}{3}\rho\xi^{3k}\gamma_k = \frac{-\rho V}{3 - \rho V} \tag{3}$$

It is elementary to show that polarizability for a spherical void at iteration  $k$  is  $3V/4\pi\xi^{3k}$  for Dirichlet boundary conditions and  $-3V/8\pi\xi^{3k}$  for Neumann. So Eq. (3) says that for every iteration the voids must be a mix of  $2(4 - \rho V)/3(3 - \rho V)$  Neumann and  $(1 - \rho V)/3(3 - \rho V)$  Dirichlet. As a result, Eq. (2) reduces to

$$-\frac{q}{r} (1 - \rho V)^{l_{\max}} \tag{4}$$

Since  $l_{\max}$  satisfies  $r \sim$  radius of iteration- $l_{\max}$  sphere, proportional to  $V^{1/3}/\xi^{l_{\max}}$ , expression (4) amounts to power-law screening of the form

$$\left( \frac{r}{V^{1/3}} \right)^{-\ln(1-\rho V)/\ln \xi} \tag{5}$$

The exponent in Eq. (5) is the dimension deficit.

#### 4 Propagator on fractal derived from Minkowski 4-space

In Minkowski space, we must step away from fractals defined by spherical voids because the wave equation – rather than Poisson’s equation – prevails. The 4-space wave equation is governed by initial conditions on 3-dimensional space and boundary conditions on 2-dimensional walls, in contrast with the 4-space Poisson equation, which would require conditions on the entirety of arbitrarily shaped 3-dimensional boundaries. For this reason we now assume cylindrical voids, parallel to the time axis and 3-dimensionally spherical in cross-section (or that we are in a Lorentz frame in which the voids look that way). The fractal is now the distribution of cross-section 3-spheres in position space;  $\rho$  and  $V$  now refer directly to that distribution (Voids parallel to the time axis also guarantees time-translation invariance and therefore Hamiltonian quantum dynamics and unitarity).

As before, we want to demonstrate that the fractal has the effect of multiplying the Lorentz-invariant free-space propagator by an expression similar to (5). We can confine the demonstration to the vicinity of the light cone, since that is the only region where the free-space propagator,  $-q(|x|^2 + ie)^{-1}$  with infinitesimal  $e$ , really matters. (Also, we assume the scalar field is massless because we are only interested in short distances.) Near the light cone, propagation past a 4-cylinder looks like a plane wave passing a polarizable 3-sphere. And as long as the width of the plane-wave pulse  $\gg$  sphere separation, the basic logic of the dielectric model in Sect. 3 still applies, leading again to the multiplier (5), because scattered fields in the near field exactly reproduce statically induced dipoles (see for example [5, Sec. 9.2]).

If plane-wave pulse-width not  $\gg$  sphere separation, then presumably scattered waves from nearby spheres are unable to add coherently, in which case one cannot include the factor  $\Phi_k$  for that separation. In this way position-space power-law screening (5) is augmented by an extra momentum-space factor

$$\left( \frac{\omega}{\rho^{1/3}} \right)^{+\ln(1-\rho V)/\ln \xi} \tag{6}$$

where  $\omega$  is the source frequency.

## 5 Discussion

By focusing on scalar fields, we have begun a longer-term attempt to provide an explicit physical basis for dimensional regularization. In this paper, dimensional regularization emerges as a considerable idealization: It ignores non-unity  $f(\Omega)$  or  $g(\Omega)$  for small  $\varepsilon$ ; and fractal screening (Eq. (4)) is really stepwise, not literally a smooth power law (although perhaps the steps can be eliminated by defining the fractal in the limit of vanishing  $\ln \xi$  and  $\rho V$  with finite ratio). These non-idealizations clearly depend on details of how the underlying fractal is defined.

We readily acknowledge weaknesses in our reasoning. In particular, it hinges on various approximations and idealizations, including a reliance on spherical voids or cross sections, dipole-only responses, and multiplicatively iterative dielectric calculations.

Generalization to non-scalar fields is by no means ensured, since they involve not just power-law screening but also non-trivial component index structure and constraints related to gauge invariance.

But if the fractal construction really does generalize to all types of fields (and if it also generalizes to curved geometries), then one can speculate that literally setting spacetime's dimension to  $4 - \varepsilon$  might render gravity's renormalizability a non-issue without unification with other forces or as-yet unobserved symmetries (assuming nothing discontinuous but essential happens at  $\varepsilon = 0$ ). Such a scenario has some numerical plausibility: Consider quantum corrections to the Einstein–Hilbert Lagrangian  $(1/2\kappa^2)(-g)^{1/2}R$ , where  $\kappa^2 = 8\pi G/c^4$  is proportional to Newton's constant;  $g$  is the determinant of the metric tensor; and  $R$  is the Ricci scalar, essentially a sum of curvature components. Assume gravity is minimally coupled to matter fields but not otherwise unified with matter. The simplest induced nonrenormalizable interactions, from coupling to a massless scalar field

[6] or massless photons [7], are in one loop and proportional to  $(1/\varepsilon)(-g)^{1/2}Q$ , where  $Q$  is quadratic in curvature components. Dimensionally, the generic proportionality constant can only be a geometric-combinatoric number times  $L_P^2/2\kappa^2$ , where  $L_P$  is the Planck length. The tightest “fifth force” observational bound [8] on  $R + a_2 R^2$  extensions of the Einstein–Hilbert action (i.e.  $Q = R^2$ ) is  $a_2 < 4 \times 10^{-9} \text{ m}^2$ , suggesting  $\varepsilon = L_P^2/a_2 > 10^{-61}$  (ignoring geometric and combinatoric factors), easily small enough to have escaped observation. This echoes an earlier suggestion [9] that gravity's non-renormalizability could be mitigated with a self-similar distribution of virtual black holes.

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