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Existence of solutions for a class of nonlinear boundary value problems on the hexasilinane graph

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Abstract

Chemical graph theory is a field of mathematics that studies ramifications of chemical network interactions. Using the concept of star graphs, several investigators have looked into the solutions to certain boundary value problems. Their choice to utilize star graphs was based on including a common point connected to other nodes. Our aim is to expand the range of the method by incorporating the graph of hexasilinane compound, which has a chemical formula $\mathrm{H}_{12}\mathrm{Si}_6$. In this paper, we examine the existence of solutions to fractional boundary value problems on such graphs, where the fractional derivative is in the Caputo sense. Finally, we include an example to support our significant findings.

Keywords: Hexasilinane graph; Fractional calculus; Fixed points

1 Introduction

A subfield of mathematics known as chemical graph theory is concerned with the implications of the connectedness of chemical networks. Whether natural or synthetic, almost every chemical system may be represented by a chemical graph (i.e., molecular transformations in a chemical reaction). Moreover, the word "chemical" is used to highlight that, in contrast to graph theory, we may depend on scientific investigation of many ideas and theorems rather than exact mathematical proofs, which is a significant difference.

When it comes to graph theory, Lumer [1] was the first to use differential equation theory on graphs. With the use of ramification spaces and different operator specifications, he explored the solutions of extended evolution equations. Zavgorodnij [2] investigated linear differential equations in 1989 using a geometric network, with suggested boundary value problem solutions arranged at the network interior nodes. On the other hand, Gordeziani et al. [3] utilized the double-sweep method to obtain analytical results for differential equations, which they observed to be more productive on graphs;.

However, utilizing fixed point methods, a limited amount of studies on star graphs (see Fig. 1) associated with the solutions of boundary value problems has emerged in the particular research (see, e.g., [4, 5]).

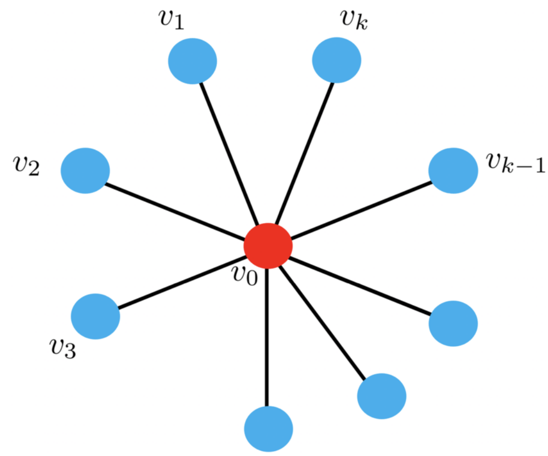

Figure 1 A picture of a star graph $\mathbb{G}^\star$ having k edges and a junction node v_0

In 2014, Graef et al. [4] used the idea of a star graph and proposed the existence of solutions to the following fractional differential equation:

$$\begin{cases} -{}^{\mathrm{RL}}\,\mathfrak{D}^{\ell} z_\gamma(s) = h_\gamma(s) L_\gamma(s, y(s)) \quad (s \in (0, \tilde{\varrho}_\gamma), \gamma = 1, 2), \\ z_1(0) = z_2(0) = 0, \qquad z_1(\tilde{\varrho}_1) = z_2(\tilde{\varrho}_2), \qquad {}^{\mathrm{RL}}\mathfrak{D}^{\nu} w_1(\tilde{\varrho}_1) + {}^{\mathrm{RL}}\,\mathfrak{D}^{\nu} w_2(\tilde{\varrho}_2) = 0, \end{cases} \tag{1.1}$$

where $\ell \in (1,2]$, $\nu \in (0,\ell)$ and $h_\gamma : [0, \tilde{\varrho}_\gamma] \to \mathbb{R}$ are continuous functions with $h_\gamma(s) \neq 0$ on $[0, \tilde{\varrho}_\nu]$, and $L_\gamma : [0, \tilde{\varrho}_\gamma] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Also, ${}^{\mathrm{RL}}\mathfrak{D}^{\ell}$ and ${}^{\mathrm{RL}}\mathfrak{D}^{\nu}$ represent the Riemann–Liouville fractional derivatives of orders ℓ and ν, respectively.

Mehandiratta et al. [5], extended the work of Graef et al. [4] by proposing the following fractional differential equation:

$$\begin{cases} \mathfrak{D}^{\ell} z_\gamma(s) = \mathcal{S}_\gamma(s, z_\gamma(s), \mathfrak{D}^{\nu} z_\gamma(s)) \quad (s \in (0, \tilde{\varrho}_\gamma), \gamma = 1, 2, \ldots, n), \\ z_\gamma(0) = 0, \qquad z_\gamma(\tilde{\varrho}_\gamma) \neq z_{\tilde{\gamma}}(\tilde{\varrho}_{\tilde{\gamma}}) \quad (\gamma \neq \tilde{\gamma}), \qquad \sum_{\gamma=1}^{n} z'_\gamma(\tilde{\varrho}_\gamma) = 0, \end{cases} \tag{1.2}$$

where $\ell \in (1,2]$, $\nu \in (0, \ell - 1]$, $\mathcal{S}_\gamma : [0, \tilde{\varrho}_\gamma] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, and $\mathfrak{D}^{\delta}$ denotes the Caputo fractional derivative of order $\delta \in \{\ell, \nu\}$.

Recently, Mophou et al. [6] investigated the solution of the following fractional Sturm–Liouville boundary value problems on a star graph:

$$\begin{cases} \mathfrak{D}^{\ell}_{b_\gamma^{-1}}(\beta^\gamma \mathcal{D}^{\ell}_{a^+} z^\gamma)(s) + q^\gamma(s) z^\gamma(s) = \mathcal{S}^\gamma(s), \quad s \in (a, b^\gamma), \gamma = 1, 2, \ldots, n, \\ I^{1-\ell}_{a^+} z^\gamma(a^+) = I^{1-\ell}_{a^+} z^\nu(a^+), \quad \ell \neq \nu = 1, 2, \ldots, n, \\ \sum_{\gamma=1}^{n} \beta^\gamma(a) \mathcal{D}^{\ell}_{a^+} z^\gamma(a^+) = 0, \\ I^{1-\ell}_{a^+} z^1(b_1^-) = 0, \\ I^{1-\ell}_{a^+} z^\gamma(b_\gamma^-) = \nu_\gamma, \quad \gamma = 2, 3, \ldots, p, \\ \beta^\gamma(b_\gamma) \mathcal{D}^{\ell}_{a^+} z^\gamma(b_\gamma^-) = \nu_\gamma, \quad \gamma = p, p+1, \ldots, n, \end{cases} \tag{1.3}$$

where $\mathcal{D}^{\ell}_{a^+}$ and $\mathfrak{D}^{\ell}_{b_\gamma^{-1}}$, $\gamma = 1, 2, \ldots, n$, are, respectively, the left Riemann–Liouville and right Caputo fractional derivatives of order $\ell \in (0,1)$, $I^{\ell}_{a^+}$ is the Riemann–Liouville fractional

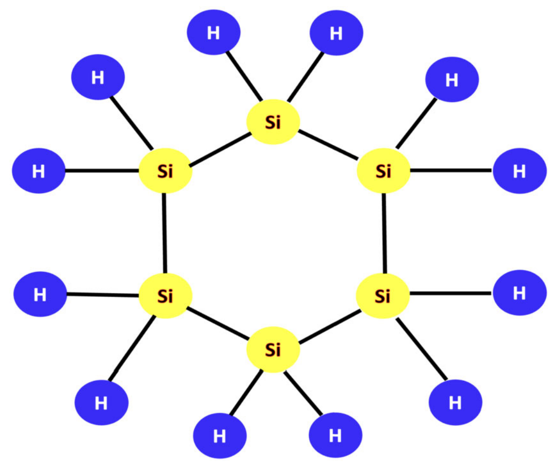

Figure 2 Chemical structure of hexasilinane compound $H_{12}Si_6$

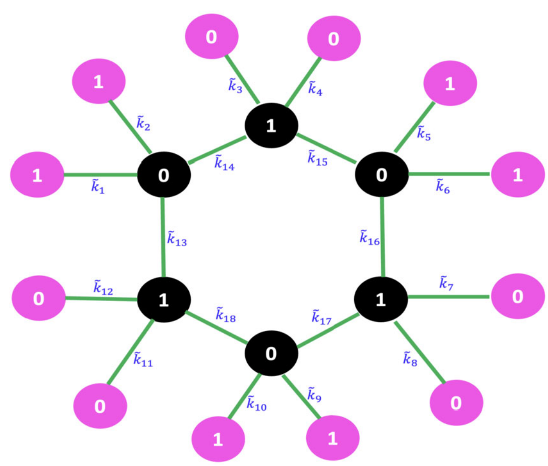

Figure 3 A hexasilinane compound graph with vertices 0 or 1

integral of order ℓ, the real functions β^γ and q^γ are defined on $[a, b_\gamma]$ ($\gamma = 1, 2, \dots, n$), the functions $\mathcal{S}^\gamma$ belong to $L^2(a, b_\gamma)$, $\gamma = 1, 2, \dots, n$, and the controls ν_γ, $\gamma = 1, 2, \dots, n$, are real variables.

For the recent research in this area, we refer to [7–9] and the references therein.

To extend the work presented in [4–6], we use the concept of hexasilinane graphs (see Fig. 2), which are more general than star graphs.

Moreover, the techniques employed in [4–6] are inadequate since the hexasilinane graphs contain more junction points than star graphs. As a result, we adopt a different method, in which the graph vertices are labeled by 0 or 1 with edge length $|\tilde{b}_k| = 1$ (see Fig. 3).

Here we examine the existence of solutions to the following system:

$$\begin{cases} \mathfrak{D}^{\ell} z_{\gamma}(s) = \mathcal{S}_{\gamma}(s, z_{\gamma}(s), \mathfrak{D}^{\nu} z_{\gamma}(s), z'_{\gamma}(s), z''_{\gamma}(s)) \quad (s \in [0,1]), \\ \varpi_1 z_{\gamma}(0) = \varpi_2 \mathfrak{D}^1 z_{\gamma}(1) + \varpi_3 \mathfrak{D}^2 z_{\gamma}(1), \\ \varpi_1 z_{\gamma}(1) = \varpi_2 \mathfrak{D}^{\ell-1} z_{\gamma}(1) + \varpi_3 \int_0^{\theta} \mathfrak{D}^{\ell-1} z_{\gamma}(\xi)\, d\xi, \end{cases} \tag{1.4}$$

where ϖ_p ($p = 1,2,3$) are real constants with $\varpi_p \neq 0$, $\theta \in (0,1)$, $\mathfrak{D}^{\ell}$ and $\mathfrak{D}^{\nu}$ denote the Caputo fractional derivatives of orders $2 < \ell \leq 3$ and $\nu \in (0,2)$, respectively. Also, $\mathcal{S}_{\gamma} : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuously differentiable function for $\gamma = 18$, where γ denotes the total number of edges of hexasilinane compound with $|\tilde{k}_{\gamma}| = 1$.

Our goal is to prove the existence of solutions to the suggested problem (1.4) by using appropriate fixed point theorems. Finally, we give an example to demonstrate the significance of our findings in light of the existing literature.

For the details about fixed point theory and its applications in different spaces, we refer to [10–18] and the references therein. Several new papers have recently been published dealing with the existence of solutions to nonlinear fractional differential equations (for details, see [19–24]).

2 Preliminaries

Definition 2.1 ([25]) The Caputo fractional derivative of order $\ell > 0$ for $\mathcal{S} \in C^{\xi}[0,+\infty)$ is given by

$$\mathfrak{D}^{\ell} \mathcal{S}(s) = \frac{1}{\Gamma(\xi - \ell)} \int_0^s (s-\theta)^{\xi-\ell-1} \mathcal{S}^{(\xi)}(\theta)\, d\theta \quad \bigl(\xi - 1 < \ell < \xi, \xi = [\ell] + 1\bigr),$$

where $[\ell]$ is the integer part of ℓ.

For $\ell > 0$, the general solution of $\mathfrak{D}^{\ell} z(s) = 0$ is given as

$$z(s) = b_0 + b_1 s + b_2 s^2 + \cdots + b_{n-1} s^{n-1},$$

and

$$\mathbb{I}^{\ell} \mathfrak{D}^{\ell} z(s) = z(s) + b_0 + b_1 s + b_2 s^2 + \cdots + b_{n-1} s^{n-1}$$

for $b_k \in \mathbb{R}$ and $k = 0,1,\ldots,n-1$.

Lemma 2.2 *Let $\phi_1, \phi_2, \ldots, \phi_{18}$ be continuous real-valued functions on $[0,1]$. Then $z^{\star}_{\gamma}$ is a solution of the problem*

$$\begin{cases} \mathfrak{D}^{\ell} z_{\gamma}(s) = \Upsilon_{\gamma}(s) \quad (s \in [0,1], \gamma = 1,2,\ldots,18), \\ \varpi_1 z_{\gamma}(0) = \varpi_2 \mathfrak{D}^1 z_{\gamma}(1) + \varpi_3 \mathfrak{D}^2 z_{\gamma}(1), \\ \varpi_1 z_{\gamma}(1) = \varpi_2 \mathfrak{D}^{\ell-1} z_{\gamma}(1) + \varpi_3 \int_0^{\theta} \mathfrak{D}^{\ell-1} z_{\gamma}(\xi)\, d\xi, \end{cases} \tag{2.1}$$

if and only if $z^{\star}_{\gamma}$ is a solution of the fractional integral equation

$$z_{\gamma}(s) = \int_0^s \frac{(s-\xi)^{\ell-1}}{\Gamma(\ell)} \Upsilon_{\gamma}(\xi)\, d\xi + \frac{1}{V_0}\left(\frac{\varpi_2}{\varpi_1} + s\right)\biggl\{\varpi_3 \int_0^{\theta} \int_0^{\xi} \Upsilon_{\gamma}(\tau)\, d\tau\, d\xi$$

$$+\varpi_2\int_0^1 \Upsilon_\gamma(\xi)\,d\xi - \varpi_1\int_0^1 \frac{(1-\xi)^{\ell-1}}{\Gamma(\ell)}\Upsilon_\gamma(\xi)\,d\xi\Bigg\} + \left(\frac{V_0-\varpi_2-\varpi_1 s}{\varpi_1 V_0}\right)$$
$$\times\left\{\varpi_2\int_0^1 \frac{(1-\xi)^{\ell-2}}{\Gamma(\ell-1)}\Upsilon_\gamma(\xi)\,d\xi + \varpi_3\int_0^1 \frac{(1-\xi)^{\ell-3}}{\Gamma(\ell-2)}\Upsilon_\gamma(\xi)\,d\xi\right\}, \tag{2.2}$$

where

$$V_0 = \left[\varpi_1 + \frac{\varpi_2(\Gamma(3-\ell)-1)}{\Gamma(3-\ell)} - \frac{\varpi_3\theta^{3-\ell}}{\Gamma(4-\ell)}\right] \neq 0.$$

Proof Let $z_\gamma^\star$ be a solution of (2.1), where $\gamma = 1,2,\dots,18$. Thus there are constants $b_0^{(\gamma)}, b_1^{(\gamma)} \in \mathbb{R}$ such that

$$z_\gamma^\star(s) = \int_0^s \frac{(s-\xi)^{\ell-1}}{\Gamma(\ell)}\Upsilon_\gamma(\xi)\,d\xi + b_0^{(\gamma)} + b_1^{(\gamma)}s. \tag{2.3}$$

Using the boundary conditions for (2.1), we have

$$\begin{aligned}
b_1^{(\gamma)} &= \frac{\varpi_3}{V_0}\left(\int_0^\theta\int_0^\xi \Upsilon_\gamma(\tau)\,d\tau\,d\xi - \int_0^1 \frac{(1-\xi)^{\ell-3}}{\Gamma(\ell-2)}\Upsilon_\gamma(\xi)\,d\xi\right) + \frac{\varpi_2}{V_0}\left(\int_0^1 \Upsilon_\gamma(\xi)\,d\xi\right.\\
&\quad\left. - \int_0^1 \frac{(1-\xi)^{\ell-2}}{\Gamma(\ell-1)}\Upsilon_\gamma(\xi)\,d\xi\right) - \frac{\varpi_1}{V_0}\int_0^1 \frac{(1-\xi)^{\ell-1}}{\Gamma(\ell)}\Upsilon_\gamma(\xi)\,d\xi,\\
b_0^{(\gamma)} &= \frac{\varpi_3}{\varpi_1}\left(\frac{\varpi_2}{V_0}\int_0^\theta\int_0^\xi \Upsilon_\gamma(\tau)\,d\tau\,d\xi + \left(1-\frac{\varpi_2}{V_0}\right)\int_0^1 \frac{(1-\xi)^{\ell-3}}{\Gamma(\ell-2)}\Upsilon_\gamma(\xi)\,d\xi\right)\\
&\quad + \frac{\varpi_2}{\varpi_1}\left(\frac{\varpi_2}{V_0}\int_0^1 \Upsilon_\gamma(\xi)\,d\xi + \left(1-\frac{\varpi_2}{V_0}\right)\int_0^1 \frac{(1-\xi)^{\ell-2}}{\Gamma(\ell-1)}\Upsilon_\gamma(\xi)\,d\xi\right)\\
&\quad - \frac{\varpi_2}{V_0}\int_0^1 \frac{(1-\xi)^{\ell-1}}{\Gamma(\ell)}\Upsilon_\gamma(\xi)\,d\xi.
\end{aligned}$$

Substituting the values of $b_0^{(\gamma)}$ and $b_1^{(\gamma)}$ into (2.3), we obtain the desired solution (2.2).

With regard to the contrary, when $z_\gamma^\star$ is a solution of (2.1), it is self-evident that $z_\gamma^\star$ is a solution of (2.2). □

We now present the fixed point theorems of Krasnoselskii and Schaefer.

Theorem 2.3 ([26]) *Let $\mathcal{V}$ be a closed, bounded, convex, and nonempty subset of a Banach space $\mathcal{U}$, and let $\mathcal{S}_1,\mathcal{S}_2 : \mathcal{V} \to \mathcal{U}$ be two operators such that $\mathcal{S}_1 k + \mathcal{S}_2 k' \in \mathcal{V}$ whenever $k,k' \in \mathcal{V}$. Suppose that $\mathcal{S}_1$ is compact and continuous and $\mathcal{S}_2$ is a contraction. Then $\mathcal{S}_1+\mathcal{S}_2$ has a fixed point.*

Theorem 2.4 ([26]) *Let $\mathcal{U}$ be a Banach space, and let $\mathcal{S} : \mathcal{U} \to \mathcal{U}$ be a completely continuous mapping. If the set $\{z \in \mathcal{U} : z = \vartheta\mathcal{S}z$ for some $\vartheta \in [0,1]\}$ is bounded, then $\mathcal{S}$ has at least one fixed point in $\mathcal{U}$.*

3 Main results

Throughout this paper, $\mathcal{U}_\gamma = \{z_\gamma : z_\gamma, \mathfrak{D}^\nu z_\gamma, z_\gamma', z_\gamma'' \in C[0,1]\}$ is a Banach space with norm

$$\|z_\gamma\|_{\mathcal{U}_\gamma} = \sup_{s\in[0,1]}\left|z_\gamma(s)\right| + \sup_{s\in[0,1]}\left|\mathfrak{D}^\nu z_\gamma(s)\right| + \sup_{s\in[0,1]}\left|z_\gamma'(s)\right| + \sup_{s\in[0,1]}\left|z_\gamma''(s)\right|$$

for $\gamma = 1, 2, \ldots, 18$. It is obvious that the product space $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_{18}$ is a Banach space with norm

$$\|z = (z_1, z_2, \ldots, z_{18})\|_{\mathcal{U}} = \sum_{\gamma=1}^{18} \|z_\gamma\|_{\mathcal{U}_\gamma}.$$

Referring to Lemma 2.2, we introduce the operator $\mathcal{S} : \mathcal{U} \to \mathcal{U}$ by

$$\mathcal{S}(z_1, z_2, \ldots, z_{18})(s) := \bigl(\mathcal{S}_1(z_1, z_2, \ldots, z_{18}), \ldots, \mathcal{S}_{18}(z_1, z_2, \ldots, z_{18})(s)\bigr), \tag{3.1}$$

where

$$\begin{aligned}
&\mathcal{S}_\gamma(z_1, z_2, \ldots, z_{18})(s) \\
&\quad= \int_0^s \frac{(s-\xi)^{\ell-1}}{\Gamma(\ell)} \mathcal{S}_\gamma\bigl(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\bigr)\, d\xi \\
&\qquad+ \frac{1}{V_0}\left(\frac{\varpi_2}{\varpi_1} + s\right) \\
&\qquad\times \left[\varpi_3 \int_0^\theta \int_0^\xi \mathcal{S}_\gamma\bigl(\tau, z_\gamma(\tau), \mathfrak{D}^\nu z_\gamma(\tau), z'_\gamma(\tau), z''_\gamma(\tau)\bigr)\, d\tau\, d\xi\right. \\
&\qquad+ \varpi_2 \int_0^1 \mathcal{S}_\gamma\bigl(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\bigr)\, d\xi \\
&\qquad\left.- \varpi_1 \int_0^1 \frac{(1-\xi)^{\ell-1}}{\Gamma(\ell)} \mathcal{S}_\gamma\bigl(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\bigr)\, d\xi\right] \\
&\qquad+ \frac{1}{\varpi_1 V_0}(V_0 - \varpi_2 - \varpi_1 s) \\
&\qquad\times \left[\varpi_2 \int_0^1 \frac{(1-\xi)^{\ell-2}}{\Gamma(\ell-1)} \mathcal{S}_\gamma\bigl(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\bigr)\, d\xi\right. \\
&\qquad\left.+ \varpi_3 \int_0^1 \frac{(1-\xi)^{\ell-3}}{\Gamma(\ell-2)} \mathcal{S}_\gamma\bigl(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\bigr)\, d\xi\right]
\end{aligned} \tag{3.2}$$

for $s \in [0, 1]$ and $z_\gamma \in \mathcal{U}_\gamma$.

To facilitate calculations, we use the following notation:

$$V_0 = \left[\varpi_1 + \frac{\varpi_2(\Gamma(3-\ell)-1)}{\Gamma(3-\ell)} - \frac{\varpi_3 \theta^{3-\ell}}{\Gamma(4-\ell)}\right] \neq 0, \tag{3.3}$$

$$V_1 = \left[|\varpi_1| + \frac{|\varpi_2||(\Gamma(3-\ell)-1)|}{\Gamma(3-\ell)} + \frac{|\varpi_3|}{\Gamma(4-\ell)}\right] \neq 0, \tag{3.4}$$

$$\begin{aligned}
\mathcal{I}_0^* &= \frac{1}{\Gamma(\ell+1)} + \left(\frac{|\varpi_2| + |\varpi_1|}{|\varpi_1| V_1}\right)\left(\frac{|\varpi_3|}{2} + |\varpi_2| + \frac{|\varpi_1|}{\Gamma(\ell+1)}\right) \\
&\quad+ \left(\frac{|V_1 - \varpi_2 - \varpi_1|}{|\varpi_1| V_1}\right)\left(\frac{|\varpi_2|}{\Gamma(\ell)} + \frac{|\varpi_3|}{\Gamma(\ell-1)}\right),
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
\mathcal{I}_1^* &= \frac{1}{\Gamma(\ell-\nu+1)} + \left(\frac{1}{V_1 \Gamma(2-\nu)}\right) \\
&\quad\times \left(\frac{|\varpi_3(2 + \Gamma(\ell-1))|}{2\Gamma(\ell-1)} + \frac{|\varpi_2(1 + \Gamma(\ell))|}{\Gamma(\ell)} + \frac{|\varpi_1|}{\Gamma(\ell+1)}\right),
\end{aligned} \tag{3.6}$$

$$\mathcal{I}_2^* = \frac{1}{\Gamma(\ell)} + \frac{1}{V_1}\left(\frac{|\varpi_3(2+\Gamma(\ell-1))|}{2\Gamma(\ell-1)} + \frac{|\varpi_2(1+\Gamma(\ell))|}{\Gamma(\ell)} + \frac{|\varpi_1|}{\Gamma(\ell+1)}\right), \tag{3.7}$$

$$\mathcal{I}_3^* = \frac{1}{\Gamma(\ell-1)}, \tag{3.8}$$

$$\begin{aligned}\mathcal{I}_4^* &= \left(\frac{|\varpi_2|+|\varpi_1|}{|\varpi_1|V_1}\right)\left(\frac{|\varpi_3|}{2} + |\varpi_2| + \frac{|\varpi_1|}{\Gamma(\ell+1)}\right) \\ &\quad + \left(\frac{|V_1-\varpi_2-\varpi_1|}{|\varpi_1|V_1}\right)\left(\frac{|\varpi_2|}{\Gamma(\ell)} + \frac{|\varpi_3|}{\Gamma(\ell-1)}\right),\end{aligned} \tag{3.9}$$

$$\mathcal{I}_5^* = \left(\frac{1}{V_1\Gamma(2-\nu)}\right)\left(\frac{|\varpi_3(2+\Gamma(\ell-1))|}{2\Gamma(\ell-1)} + \frac{|\varpi_2(1+\Gamma(\ell))|}{\Gamma(\ell)} + \frac{|\varpi_1|}{\Gamma(\ell+1)}\right), \tag{3.10}$$

$$\mathcal{I}_6^* = \frac{1}{V_1}\left(\frac{|\varpi_3(2+\Gamma(\ell-1))|}{2\Gamma(\ell-1)} + \frac{|\varpi_2(1+\Gamma(\ell))|}{\Gamma(\ell)} + \frac{|\varpi_1|}{\Gamma(\ell+1)}\right). \tag{3.11}$$

Theorem 3.1 *Let $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_{18} : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous functions, and let there exist constants $M_\gamma > 0$, $\gamma = 1, 2, \dots, 18$, satisfying*

$$\left|\mathcal{S}_\gamma(s, z_1, z_2, z_3, z_4)\right| \le M_\gamma$$

for all $z, z_1, z_2, z_3, z_4 \in \mathbb{R}$ and $s \in [0,1]$. Then problem (1.4) has a solution.

Proof It is obvious from (3.2) that the fixed points of the operator $\mathcal{S}$ given in (3.1) exist if and only if (1.4) has a solution. To prove this, we first show that $\mathcal{S}$ is completely continuous.

As $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_{18}$ are continuous, $\mathcal{S} : \mathcal{U} \to \mathcal{U}$ is continuous too. Let $\mathcal{V} \in \mathcal{U}$ be a bounded set, and let $z = (z_1, z_2, \dots, z_{18}) \in \mathcal{U}$. So for each $s \in [0,1]$, we have

$$\begin{aligned}&\left|(\mathcal{S}_\gamma z)(s)\right| \\ &\quad \le \int_0^s \frac{(s-\xi)^{\ell-1}}{\Gamma(\ell)} \left|\mathcal{S}_\gamma\left(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z_\gamma'(\xi), z_\gamma''(\xi)\right)\right| d\xi \\ &\qquad + \frac{1}{V_1}\left(\frac{|\varpi_2|}{|\varpi_1|} + s\right) \\ &\qquad \times \left[|\varpi_3| \int_0^\theta \int_0^\xi \left|\mathcal{S}_\gamma\left(\tau, z_\gamma(\tau), \mathfrak{D}^\nu z_\gamma(\tau), z_\gamma'(\tau), z_\gamma''(\tau)\right)\right| d\tau\, d\xi\right. \\ &\qquad + |\varpi_2| \int_0^1 \left|\mathcal{S}_\gamma\left(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z_\gamma'(\xi), z_\gamma''(\xi)\right)\right| d\xi \\ &\qquad \left. + |\varpi_1| \int_0^1 \frac{(1-\xi)^{\ell-1}}{\Gamma(\ell)} \left|\mathcal{S}_\gamma\left(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z_\gamma'(\xi), z_\gamma''(\xi)\right)\right| d\xi\right] \\ &\qquad + \frac{|V_1 - \varpi_2 - \varpi_1 s|}{|\varpi_1|V_1} \\ &\qquad \times \left[|\varpi_2| \int_0^1 \frac{(1-\xi)^{\ell-2}}{\Gamma(\ell-1)} \left|\mathcal{S}_\gamma\left(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z_\gamma'(\xi), z_\gamma''(\xi)\right)\right| d\xi\right. \\ &\qquad \left. + \varpi_3 \int_0^1 \frac{(1-\xi)^{\ell-3}}{\Gamma(\ell-2)} \left|\mathcal{S}_\gamma\left(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z_\gamma'(\xi), z_\gamma''(\xi)\right)\right| d\xi\right] \\ &\quad \le M_\gamma \mathcal{I}_0^*,\end{aligned}$$

where $\mathcal{I}_0^*$ is given in (3.5). Also,

$$\begin{aligned}
&\bigl|(\mathfrak{D}^\nu \mathcal{S}_\gamma z)(s)\bigr| \\
&\quad\le \int_0^s \frac{(s-\xi)^{\ell-\nu-1}}{\Gamma(\ell-\nu)}\bigl|\mathcal{S}_\gamma\bigl(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\bigr)\bigr|\,d\xi \\
&\qquad+\left(\frac{s^{1-\nu}}{V_1\Gamma(2-\nu)}\right) \\
&\qquad\times\Biggl[|\varpi_3|\int_0^\theta\int_0^\xi\bigl|\mathcal{S}_\gamma\bigl(\tau, z_\gamma(\tau), \mathfrak{D}^\nu z_\gamma(\tau), z'_\gamma(\tau), z''_\gamma(\tau)\bigr)\bigr|\,d\tau\,d\xi \\
&\qquad+|\varpi_2|\int_0^1\bigl|\mathcal{S}_\gamma\bigl(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\bigr)\bigr|\,d\xi \\
&\qquad+|\varpi_1|\int_0^1\frac{(1-\xi)^{\ell-1}}{\Gamma(\ell)}\bigl|\mathcal{S}_\gamma\bigl(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\bigr)\bigr|\,d\xi\Biggr] \\
&\qquad+\left(\frac{s^{1-\nu}}{V_1\Gamma(2-\nu)}\right) \\
&\qquad\times\Biggl[|\varpi_2|\int_0^1\frac{(1-\xi)^{\ell-2}}{\Gamma(\ell-1)}\bigl|\mathcal{S}_\gamma\bigl(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\bigr)\bigr|\,d\xi \\
&\qquad+|\varpi_3|\int_0^1\frac{(1-\xi)^{\ell-3}}{\Gamma(\ell-2)}\bigl|\mathcal{S}_\gamma\bigl(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\bigr)\bigr|\,d\xi\Biggr] \\
&\quad\le M_\gamma\mathcal{I}_1^*, \\
&\bigl|(\mathcal{S}'_\gamma z)(s)\bigr| \\
&\quad\le \int_0^s \frac{(s-\xi)^{\ell-2}}{\Gamma(\ell-1)}\bigl|\mathcal{S}_\gamma\bigl(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\bigr)\bigr|\,d\xi \\
&\qquad+\frac{1}{V_1}\Biggl[|\varpi_3|\int_0^\theta\int_0^\xi\bigl|\mathcal{S}_\gamma\bigl(\tau, z_\gamma(\tau), \mathfrak{D}^\nu z_\gamma(\tau), z'_\gamma(\tau), z''_\gamma(\tau)\bigr)\bigr|\,d\tau\,d\xi \\
&\qquad+|\varpi_2|\int_0^1\bigl|\mathcal{S}_\gamma\bigl(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\bigr)\bigr|\,d\xi \\
&\qquad+|\varpi_1|\int_0^1\frac{(1-\xi)^{\ell-1}}{\Gamma(\ell)}\bigl|\mathcal{S}_\gamma\bigl(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\bigr)\bigr|\,d\xi\Biggr] \\
&\qquad+\frac{1}{V_1}\Biggl[|\varpi_2|\int_0^1\frac{(1-\xi)^{\ell-2}}{\Gamma(\ell-1)}\bigl|\mathcal{S}_\gamma\bigl(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\bigr)\bigr|\,d\xi \\
&\qquad+|\varpi_3|\int_0^1\frac{(1-\xi)^{\ell-3}}{\Gamma(\ell-2)}\bigl|\mathcal{S}_\gamma\bigl(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\bigr)\bigr|\,d\xi\Biggr] \\
&\quad\le M_\gamma\mathcal{I}_2^*,
\end{aligned}$$

and

$$\begin{aligned}
\bigl|(\mathcal{S}''_\gamma z)(s)\bigr| &\le \int_0^s \frac{(s-\xi)^{\ell-3}}{\Gamma(\ell-2)}\bigl|\mathcal{S}_\gamma\bigl(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\bigr)\bigr|\,d\xi \\
&\le M_\gamma\mathcal{I}_3^*
\end{aligned}$$

for all $s \in [0,1]$, where $\mathcal{I}_1^*$–$\mathcal{I}_3^*$ are defined in (3.6)–(3.8), respectively. Therefore

$$\|(\mathcal{S}_\gamma z)(s)\|_{\mathcal{U}_\gamma} \leq M_\gamma(\mathcal{I}_0^* + \mathcal{I}_1^* + \mathcal{I}_2^* + \mathcal{I}_3^*).$$

Hence

$$\begin{aligned}\|(\mathcal{S}z)(s)\|_{\mathcal{U}} &= \sum_{\gamma=1}^{18} \|(\mathcal{S}_\gamma z)(s)\|_{\mathcal{U}_\gamma} \\ &\leq \sum_{\gamma=1}^{18} M_\gamma(\mathcal{I}_0^* + \mathcal{I}_1^* + \mathcal{I}_2^* + \mathcal{I}_3^*) \\ &< \infty,\end{aligned}$$

which reveals that $\mathcal{S}$ is uniformly bounded.

Now we have to show that $\mathcal{S}$ is equicontinuous. For this purpose, let $z = (z_1, z_2, \ldots, z_{18}) \in \mathcal{V}$ and $s_1, s_2 \in [0,1]$ with $s_1 < s_2$. Then we have

$$\begin{aligned}&\left|(\mathcal{S}_\gamma z)(s_2) - (\mathcal{S}_\gamma z)(s_1)\right| \\ &\quad\leq \int_0^{s_1} \frac{(s_2-\theta)^{\ell-1} - (s_1-\theta)^{\ell-1}}{\Gamma(\ell)} \\ &\qquad\times \left|\mathcal{S}_\gamma\left(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\right)\right| d\xi \\ &\qquad+ \int_{s_1}^{s_2} \frac{(s_2-\theta)^{\ell-1}}{\Gamma(\ell)} \left|\mathcal{S}_\gamma\left(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\right)\right| d\xi \\ &\qquad+ \left(\frac{s_2 - s_1}{V_1}\right) \\ &\qquad\times \left[|\varpi_3| \int_0^\theta \int_0^\xi \left|\mathcal{S}_\gamma\left(\tau, z_\gamma(\tau), \mathfrak{D}^\nu z_\gamma(\tau), z'_\gamma(\tau), z''_\gamma(\tau)\right)\right| d\tau\, d\xi\right. \\ &\qquad+ |\varpi_2| \int_0^1 \left|\mathcal{S}_\gamma\left(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\right)\right| d\xi \\ &\qquad\left.+ |\varpi_1| \int_0^1 \frac{(1-\xi)^{\ell-1}}{\Gamma(\ell)} \left|\mathcal{S}_\gamma\left(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\right)\right| d\xi\right] \\ &\qquad+ \left(\frac{s_2 - s_1}{V_1}\right) \\ &\qquad\times \left[|\varpi_2| \int_0^1 \frac{(1-\xi)^{\ell-2}}{\Gamma(\ell-1)} \left|\mathcal{S}_\gamma\left(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\right)\right| d\xi\right. \\ &\qquad\left.+ |\varpi_3| \int_0^1 \frac{(1-\xi)^{\ell-3}}{\Gamma(\ell-2)} \left|\mathcal{S}_\gamma\left(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\right)\right| d\xi\right].\end{aligned}$$

We can see that if $s_1 \to s_2$, then, independently, the right-hand side of the expression converges to zero. Also,

$$\lim_{s_1 \to s_2} \left|(\mathfrak{D}^\nu \mathcal{S}_\gamma z)(s_2) - (\mathfrak{D}^\nu \mathcal{S}_\gamma z)(s_1)\right| = 0,$$

$$\lim_{s_1 \to s_2} \left|(\mathcal{S}'_\gamma z)(s_2) - (\mathcal{S}'_\gamma z)(s_1)\right| = 0,$$

$$\lim_{s_1 \to s_2} \left| \left(\mathcal{S}_\gamma'' z\right)(s_2) - \left(\mathcal{S}_\gamma'' z\right)(s_1) \right| = 0.$$

As a result, $\|(\mathcal{S}z)(s_2) - (\mathcal{S}z)(s_1)\|_{\mathcal{U}} \to 0$ as $s_1 \to s_2$. This proves that $\mathcal{S}$ is equicontinuous on $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_{18}$. Now the Arzelà–Ascoli theorem implies the complete continuity of the operator.

Further, we define the subset Λ of $\mathcal{U}$ as

$$\Lambda := \left\{ (z_1, z_2, \ldots, z_{18}) \in \mathcal{U} : (z_1, z_2, \ldots, z_{18}) = \vartheta \mathcal{S}(z_1, z_2, \ldots, z_{18}), \vartheta \in (0,1) \right\}.$$

We will show that Λ is bounded. For this, let $(z_1, z_2, \ldots, z_{18}) \in \Lambda$. Then we can write

$$(z_1, z_2, \ldots, z_{18}) = \vartheta \mathcal{S}(z_1, z_2, \ldots, z_{18}),$$

and so

$$z_\gamma(s) = \vartheta \mathcal{S}_\gamma(z_1, z_2, \ldots, z_{18})$$

for all $s \in [0,1]$ and $\gamma = 1, 2, \ldots, 18$. Thus

$$\begin{aligned}
\left|z_\gamma(s)\right| \le{}& \vartheta \Biggl[\int_0^s \frac{(s-\xi)^{\ell-1}}{\Gamma(\ell)} \left| \mathcal{S}_\gamma\left(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z_\gamma'(\xi), z_\gamma''(\xi)\right)\right| d\xi \\
&+ \frac{1}{V_1}\left(\frac{|\varpi_2|}{|\varpi_1|} + s\right) \Biggl\{ |\varpi_3| \int_0^\theta \int_0^\xi \left| \mathcal{S}_\gamma\left(\tau, z_\gamma(\tau), \mathfrak{D}^\nu z_\gamma(\tau), z_\gamma'(\tau), z_\gamma''(\tau)\right)\right| d\tau\, d\xi \\
&+ |\varpi_2| \int_0^1 \left| \mathcal{S}_\gamma\left(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z_\gamma'(\xi), z_\gamma''(\xi)\right)\right| d\xi \\
&+ |\varpi_1| \int_0^1 \frac{(1-\xi)^{\ell-1}}{\Gamma(\ell)} \left| \mathcal{S}_\gamma\left(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z_\gamma'(\xi), z_\gamma''(\xi)\right)\right| d\xi \Biggr\} \\
&+ \frac{|V_1 - \varpi_2 - \varpi_1 s|}{|\varpi_1| V_1} \\
&\times \Biggl\{ |\varpi_2| \int_0^1 \frac{(1-\xi)^{\ell-2}}{\Gamma(\ell-1)} \left| \mathcal{S}_\gamma\left(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z_\gamma'(\xi), z_\gamma''(\xi)\right)\right| d\xi \\
&+ |\varpi_3| \int_0^1 \frac{(1-\xi)^{\ell-3}}{\Gamma(\ell-2)} \left| \mathcal{S}_\gamma\left(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z_\gamma'(\xi), z_\gamma''(\xi)\right)\right| d\xi \Biggr\} \Biggr] \\
\le{}& \vartheta M_\gamma \mathcal{I}_0^*,
\end{aligned}$$

and by similar computations we have

$$\begin{aligned}
&\left|\mathfrak{D}^\nu z_\gamma(s)\right| \le \vartheta M_\gamma \mathcal{I}_1^*, \\
&\left|z_\gamma'(s)\right| \le \vartheta M_\gamma \mathcal{I}_2^*, \\
&\left|z_\gamma''(s)\right| \le \vartheta M_\gamma \mathcal{I}_3^*,
\end{aligned}$$

where $\mathcal{I}_0^*$–$\mathcal{I}_3^*$ are given in (3.5)–(3.8). Hence

$$\|z\|_{\mathcal{U}} = \sum_{\gamma=1}^{18} \|z_\gamma\|_{\mathcal{U}_\gamma}$$

$$\begin{aligned}&\leq \vartheta \sum_{\gamma=1}^{18} M_\gamma \left(\mathcal{I}_0^* + \mathcal{I}_1^* + \mathcal{I}_2^* + \mathcal{I}_3^*\right)\\&< \infty,\end{aligned}$$

which shows the boundedness of Λ. Now using Theorem 2.4 and Lemma 2.2, we see that $\mathcal{S}$ has a fixed point in $\mathcal{U}$. This demonstrates that (1.4) does indeed have a solution. □

We will now examine the solution of problem (1.4) by applying various conditions.

Theorem 3.2 *Suppose that $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_{18} : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions and that there exist bounded continuous functions $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_{18} : [0,1] \to \mathbb{R}$, $\mathcal{Z}_1, \mathcal{Z}_2, \dots, \mathcal{Z}_{18} : [0,1] \to [0,\infty)$ and nondecreasing continuous functions $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_{18} : [0,1] \to [0,\infty)$ such that*

$$\left|\mathcal{S}_\gamma(s, z_1, z_2, z_3, z_4)\right| \leq \mathcal{Z}_\gamma(s)\mathcal{L}_\gamma\left(|z_1| + |z_2| + |z_3| + |z_4|\right)$$

and

$$\begin{aligned}&\left|\mathcal{S}_\gamma(s, z_1, z_2, z_3, z_4) - \mathcal{S}_\gamma(s, \tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4)\right|\\&\quad \leq \mathcal{G}_\gamma(s)\left(|z_1 - \tilde{z}_1| + |z_2 - \tilde{z}_2| + |z_3 - \tilde{z}_3| + |z_4 - \tilde{z}_4|\right)\end{aligned}$$

for all $s \in [0,1]$, $z_1, z_2, z_3, z_4, \tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4 \in \mathbb{R}$, and $\gamma = 1, 2, \dots, 18$. If

$$\Delta := \left(\mathcal{I}_4^* + \mathcal{I}_5^* + \mathcal{I}_6^*\right) \sum_{\gamma=1}^{18} \|\mathcal{G}_\gamma\| < 1,$$

then (1.4) has a solution, where $\|\mathcal{G}_\gamma\| = \sup_{s\in[0,1]} |\mathcal{G}_\gamma(s)|$, and the constants $\mathcal{I}_4^*$–$\mathcal{I}_6^*$ are given in (3.9)–(3.11), respectively.

Proof Let $\|\mathcal{Z}_\gamma\| = \sup_{s\in[0,1]} |\mathcal{Z}_\gamma(s)|$. Suppose that for suitable constants ε_γ, we have

$$\varepsilon_\gamma \geq \sum_{\gamma=1}^{18} \mathcal{L}_\gamma\left(\|z_\gamma\|_{\mathcal{U}_\gamma}\right)\|\mathcal{Z}_\gamma\|\left\{\mathcal{I}_0^* + \mathcal{I}_1^* + \mathcal{I}_2^* + \mathcal{I}_3^*\right\}, \tag{3.12}$$

where $\mathcal{I}_0^*$–$\mathcal{I}_3^*$ are given in (3.5)–(3.8). We define the set

$$\mathcal{V}_{\varepsilon_\gamma} := \left\{z = (z_1, z_2, \dots, z_{18}) \in \mathcal{U} : \|z\|_{\mathcal{U}} \leq \varepsilon_\gamma\right\},$$

where ε_γ is defined in (3.12). It is obvious that $\mathcal{V}_{\varepsilon_\gamma}$ is a nonempty, closed, bounded, and convex subset of $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_{18}$. Now we define $\mathcal{S}_1$ and $\mathcal{S}_2$ on $\mathcal{O}_{\varepsilon_\gamma}$ by

$$\begin{aligned}\mathcal{S}_1(z_1, z_2, \dots, z_{18})(s) &:= \left(\mathcal{S}_1^{(1)}(z_1, z_2, \dots, z_{18})(s), \dots, \mathcal{S}_1^{(18)}(z_1, z_2, \dots, z_{18})(s)\right),\\ \mathcal{S}_2(z_1, z_2, \dots, z_{18})(s) &:= \left(\mathcal{S}_2^{(1)}(z_1, z_2, \dots, z_{18})(s), \dots, \mathcal{S}_2^{(18)}(z_1, z_2, \dots, z_{18})(s)\right),\end{aligned}$$

where

$$\big(\mathcal{S}_1^{(\gamma)}z\big)(s) = \int_0^s \frac{(s-\xi)^{\ell-1}}{\Gamma(\ell)}\mathcal{S}_\gamma\big(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\big)\, d\xi, \tag{3.13}$$

and

$$\begin{aligned}
&\big(\mathcal{S}_2^{(\gamma)}z\big)(s)\\
&\quad= \frac{1}{V_0}\bigg(\frac{\varpi_2}{\varpi_1}+s\bigg)\bigg[\varpi_3\int_0^\theta\int_0^\xi \mathcal{S}_\gamma\big(\tau, z_\gamma(\tau), \mathfrak{D}^\nu z_\gamma(\tau), z'_\gamma(\tau), z''_\gamma(\tau)\big)\, d\tau\, d\xi\\
&\qquad+ \varpi_2\int_0^1 \mathcal{S}_\gamma\big(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\big)\, d\xi\\
&\qquad+ \varpi_1\int_0^1 \frac{(1-\xi)^{\ell-1}}{\Gamma(\ell)}\mathcal{S}_\gamma\big(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\big)\, d\xi\bigg]\\
&\qquad+ \bigg(\frac{V_0-\varpi_2-\varpi_1 s}{\varpi_1 V_0}\bigg)\\
&\qquad\times\bigg[\varpi_2\int_0^1 \frac{(1-\xi)^{\ell-2}}{\Gamma(\ell-1)}\mathcal{S}_\gamma\big(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\big)\, d\xi\\
&\qquad+ |\varpi_3|\int_0^1 \frac{(1-\xi)^{\ell-3}}{\Gamma(\ell-2)}\mathcal{S}_\gamma\big(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\big)\, d\xi\bigg]
\end{aligned} \tag{3.14}$$

for all $s \in [0,1]$ and $z = (z_1, z_2, \dots, z_{18}) \in \mathcal{V}_{\varepsilon_\gamma}$.

Let $\tilde{\mathcal{L}}_\gamma = \sup_{z_\gamma \in \mathcal{U}_\gamma} \mathcal{L}_\gamma(\|z_\gamma\|_{\mathcal{U}_\gamma})$. For all $\tilde{z} = (\tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_{18}), z = (z_1, z_2, \dots, z_{18}) \in \mathcal{V}_{\varepsilon_\gamma}$, we have

$$\begin{aligned}
&\big|\big(\mathcal{S}_1^{(\gamma)}\tilde{z} + \mathcal{S}_2^{(\gamma)}z\big)(s)\big|\\
&\quad\le \int_0^s \frac{(s-\xi)^{\ell-1}}{\Gamma(\ell)}\big|\mathcal{S}_\gamma\big(\xi, \tilde{z}_\gamma(\xi), \mathfrak{D}^\nu \tilde{z}_\gamma(\xi), \tilde{z}'_\gamma(\xi), \tilde{z}''_\gamma(\xi)\big)\big|\, d\xi\\
&\qquad+ \frac{1}{V_1}\bigg(\frac{|\varpi_2|}{|\varpi_1|}+s\bigg)\\
&\qquad\times\bigg[|\varpi_3|\int_0^\theta\int_0^\xi \big|\mathcal{S}_\gamma\big(\tau, z_\gamma(\tau), \mathfrak{D}^\nu z_\gamma(\tau), z'_\gamma(\tau), z''_\gamma(\tau)\big)\big|\, d\tau\, d\xi\\
&\qquad+ |\varpi_2|\int_0^1 \big|\mathcal{S}_\gamma\big(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\big)\big|\, d\xi\\
&\qquad+ |\varpi_1|\int_0^1 \frac{(1-\xi)^{\ell-1}}{\Gamma(\ell)}\big|\mathcal{S}_\gamma\big(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\big)\big|\, d\xi\bigg]\\
&\qquad+ \frac{|V_1-\varpi_2-\varpi_1 s|}{|\varpi_1|V_1}\\
&\qquad\times\bigg[|\varpi_2|\int_0^1 \frac{(1-\xi)^{\ell-2}}{\Gamma(\ell-1)}\big|\mathcal{S}_\gamma\big(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\big)\big|\, d\xi\\
&\qquad+ |\varpi_3|\int_0^1 \frac{(1-\xi)^{\ell-3}}{\Gamma(\ell-2)}\big|\mathcal{S}_\gamma\big(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\big)\big|\, d\xi\bigg]\\
&\quad\le \int_0^s \frac{(s-\xi)^{\ell-1}}{\Gamma(\ell)}\mathcal{Z}_\gamma(\xi)\mathcal{L}_\gamma\big(\big|\tilde{z}_\gamma(\xi)\big| + \big|\mathfrak{D}^\nu \tilde{z}_\gamma(\xi)\big| + \big|\tilde{z}'_\gamma(\xi)\big| + \big|\tilde{z}''_\gamma(\xi)\big|\big)\, d\xi
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{V_1}\left(\frac{|\varpi_2|}{|\varpi_1|} + s\right)\bigg[|\varpi_3| \int_0^\theta \int_0^\xi \mathcal{Z}_\gamma(\tau) \\
&\times \mathcal{L}_\gamma\left(\left|z_\gamma(\tau)\right| + \left|\mathfrak{D}^\nu z_\gamma(\tau)\right| + \left|z'_\gamma(\tau)\right| + \left|z''_\gamma(\tau)\right|\right) d\tau\, d\theta \\
&+ |\varpi_2| \int_0^1 \mathcal{Z}_\gamma(\xi)\mathcal{L}_\gamma\left(\left|z_\gamma(\xi)\right| + \left|\mathfrak{D}^\nu z_\gamma(\xi)\right| + \left|z'_\gamma(\xi)\right| + \left|z''_\gamma(\xi)\right|\right) d\xi \\
&+ |\varpi_1| \int_0^1 \frac{(1-\xi)^{\ell-1}}{\Gamma(\ell)} \mathcal{Z}_\gamma(\xi) \\
&\times \mathcal{L}_\gamma\left(\left|z_\gamma(\xi)\right| + \left|\mathfrak{D}^\nu z_\gamma(\xi)\right| + \left|z'_\gamma(\xi)\right| + \left|z''_\gamma(\xi)\right|\right) d\xi\bigg] \\
&+ \frac{|V_1 - \varpi_2 - \varpi_1 s|}{|\varpi_1| V_1}\bigg[|\varpi_2| \int_0^1 \frac{(1-\xi)^{\ell-2}}{\Gamma(\ell-1)} \mathcal{Z}_\gamma(\xi) \\
&\times \mathcal{L}_\gamma\left(\left|z_\gamma(\xi)\right| + \left|\mathfrak{D}^\nu z_\gamma(\xi)\right| + \left|z'_\gamma(\xi)\right| + \left|z''_\gamma(\xi)\right|\right) d\xi \\
&+ |\varpi_3| \int_0^1 \frac{(1-\xi)^{\ell-3}}{\Gamma(\ell-2)} \mathcal{Z}_\gamma(\xi) \\
&\times \mathcal{L}_\gamma\left(\left|z_\gamma(\xi)\right| + \left|\mathfrak{D}^\nu z_\gamma(\xi)\right| + \left|z'_\gamma(\xi)\right| + \left|z''_\gamma(\xi)\right|\right) d\xi\bigg] \\
\leq{}& \|\mathcal{Z}_\gamma\| \tilde{\mathcal{L}}_\tau \mathcal{I}_0^*.
\end{aligned}
$$

By using similar computations we have

$$
\begin{aligned}
&\left|\left(\mathfrak{D}^\nu \mathcal{S}_1^{(\gamma)} \tilde{z}\right)(s) + \left(\mathfrak{D}^\nu \mathcal{S}_2^{(\gamma)} z\right)(s)\right| \leq \|\mathcal{Z}_\gamma\| \tilde{\mathcal{L}}_\tau \mathcal{I}_1^*, \\
&\left|\left(\mathcal{S}_1^{(\gamma)} \tilde{z}\right)'(s) + \left(\mathcal{S}_2^{(\gamma)} z\right)'(s)\right| \leq \|\mathcal{Z}_\gamma\| \tilde{\mathcal{L}}_\tau \mathcal{I}_2^*,
\end{aligned}
$$

and

$$
\left|\left(\mathcal{S}_1^{(\gamma)} \tilde{z}\right)''(s) + \left(\mathcal{S}_2^{(\gamma)} z\right)''(s)\right| \leq \|\mathcal{Z}_\gamma\| \tilde{\mathcal{L}}_\tau \mathcal{I}_3^*.
$$

This yields that

$$
\begin{aligned}
\|\mathcal{S}_1 \tilde{z} + \mathcal{S}_2 z\|_{\mathcal{U}} &= \sum_{\gamma=1}^{18} \left\|\mathcal{S}_1^{(\gamma)} \tilde{z} + \mathcal{S}_2^{(\gamma)} z\right\|_{\mathcal{U}_\gamma} \\
&\leq \|\mathcal{Z}_\gamma\| \tilde{\mathcal{L}}_\tau \left(\mathcal{I}_0^* + \mathcal{I}_1^* + \mathcal{I}_2^* + \mathcal{I}_3^*\right) \\
&\leq \varepsilon_\gamma,
\end{aligned}
$$

and so $\mathcal{S}_1 \tilde{z} + \mathcal{S}_2 z \in \mathcal{V}_{\varepsilon_\gamma}$. Furthermore, the continuity of $\mathcal{S}_1$ is implied by the continuity of the operator $\mathcal{S}_\gamma$.

We will now demonstrate that $\mathcal{S}_1$ is uniformly bounded. For this, we have

$$
\begin{aligned}
\left|\left(\mathcal{S}_1^{(\gamma)} z\right)(s)\right| &\leq \int_0^s \frac{(s-\xi)^{\ell-1}}{\Gamma(\ell)} \left|\mathcal{S}_\gamma\left(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\right)\right| d\xi \\
&\leq \frac{1}{\Gamma(\ell+1)} \|\mathcal{Z}_\gamma\| \mathcal{L}_\gamma\left(\left|z_\gamma(\xi)\right| + \left|\mathfrak{D}^\nu z_\gamma(\xi)\right| + \left|z'_\gamma(\xi)\right| + \left|z''_\gamma(\xi)\right|\right)
\end{aligned}
$$

for all $z \in \mathcal{V}_{\varepsilon_\gamma}$. Also,

$$\begin{aligned}\left|\left(\mathfrak{D}^\nu \mathcal{S}_1^{(\gamma)} z\right)(s)\right| &\le \int_0^s \frac{(s-\theta)^{\ell-\nu-1}}{\Gamma(\ell-\nu)} \left|\mathcal{S}_\gamma\left(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\right)\right| d\xi \\ &\le \frac{1}{\Gamma(\ell-\nu+1)} \|\mathcal{Z}_\gamma\| \mathcal{L}_\gamma\left(\left|z_\gamma(\xi)\right| + \left|\mathfrak{D}^\nu z_\gamma(\xi)\right| + \left|z'_\gamma(\xi)\right| + \left|z''_\gamma(\xi)\right|\right),\end{aligned}$$

and

$$\begin{aligned}\left|\left(\mathcal{S}_1^{(\gamma)} z\right)'(s)\right| &\le \frac{1}{\Gamma(\ell)} \|\mathcal{Z}_\gamma\| \mathcal{L}_\gamma\left(\left|z_\gamma(\xi)\right| + \left|\mathfrak{D}^\nu z_\gamma(\xi)\right| + \left|z'_\gamma(\xi)\right| + \left|z''_\gamma(\xi)\right|\right), \\ \left|\left(\mathcal{S}_1^{(\gamma)} z\right)''(s)\right| &\le \frac{1}{\Gamma(\ell-1)} \|\mathcal{Z}_\gamma\| \mathcal{L}_\gamma\left(\left|z_\gamma(\xi)\right| + \left|\mathfrak{D}^\nu z_\gamma(\xi)\right| + \left|z'_\gamma(\xi)\right| + \left|z''_\gamma(\xi)\right|\right)\end{aligned}$$

for all $z \in \mathcal{V}_{\varepsilon_\gamma}$. Thus

$$\begin{aligned}\|\mathcal{S}_1 z\|_{\mathcal{U}} &= \sum_{\gamma=1}^{18} \left\|\mathcal{S}_1^{(\gamma)} z\right\|_{\mathcal{U}_\gamma} \\ &\le \left\{\frac{\ell^2}{\Gamma(\ell+1)} + \frac{1}{\Gamma(\ell-\nu+1)}\right\} \sum_{\gamma=1}^{18} \|\mathcal{Z}_\gamma\| \mathcal{L}_\gamma\left(\|z_\gamma\|_{\mathcal{U}_\gamma}\right),\end{aligned}$$

which shows that $\mathcal{S}_1$ is uniformly bounded on $\mathcal{V}_{\varepsilon_\gamma}$.

Now we will prove that $\mathcal{S}_1$ is compact on $\mathcal{V}_{\varepsilon_\gamma}$. For this, let $s_1, s_2 \in [0,1]$ with $s_1 < s_2$. Then we have

$$\begin{aligned}&\left|\left(\mathcal{S}_1^{(\gamma)} z\right)(s_2) - \left(\mathcal{S}_1^{(\gamma)} z\right)(s_1)\right| \\ &\quad \le \left|\int_0^{s_2} \frac{(s_2-\theta)^{\ell-1}}{\Gamma(\ell)} \mathcal{S}_\gamma\left(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\right) d\xi \right.\\ &\qquad \left. - \int_0^{s_1} \frac{(s_1-\theta)^{\ell-1}}{\Gamma(\ell)} \mathcal{S}_\gamma\left(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\right) d\xi\right| \\ &\quad \le \left|\int_0^{s_1} \frac{(s_2-\theta)^{\ell-1} - (s_1-\theta)^{\ell-1}}{\Gamma(\ell)} \right.\\ &\qquad \left. \times \mathcal{S}_\gamma\left(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\right) d\xi\right| \\ &\qquad + \left|\int_{s_1}^{s_2} \frac{(s_2-\theta)^{\ell-1}}{\Gamma(\ell)} \mathcal{S}_\gamma\left(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\right) d\xi\right| \\ &\quad \le \int_0^{s_1} \frac{(s_2-\theta)^{\ell-1} - (s_1-\theta)^{\ell-1}}{\Gamma(\ell)} \\ &\qquad \times \left|\mathcal{S}_\gamma\left(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\right)\right| d\xi \\ &\qquad + \int_{s_1}^{s_2} \frac{(s_2-\theta)^{\ell-1}}{\Gamma(\ell)} \left|\mathcal{S}_\gamma\left(\xi, z_\gamma(\xi), \mathfrak{D}^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi)\right)\right| d\xi \\ &\quad \le \left\{\frac{s_2^\ell - s_1^\ell - (s_2-s_1)^\ell}{\Gamma(\ell+1)} + \frac{(s_2-s_1)^\ell}{\Gamma(\ell+1)}\right\} \|\mathcal{Z}_\gamma\| \mathcal{L}_\gamma\left(\|z_\gamma\|_{\mathcal{U}_\gamma}\right).\end{aligned}$$

Hence $|(\mathcal{S}_1^{(\gamma)}z)(s_2) - (\mathcal{S}_1^{(\gamma)}z)(s_1)| \to 0$ as $s_1 \to s_2$. Also, we have

$$\lim_{s_1\to s_2}\left|\left(\mathfrak{D}^{\nu}\mathcal{S}_1^{(\gamma)}z\right)(s_2) - \left(\mathfrak{D}^{\nu}\mathcal{S}_1^{(\gamma)}z\right)(s_1)\right| = 0,$$
$$\lim_{s_1\to s_2}\left|\left(\mathcal{S}_1^{(\gamma)}z\right)'(s_2) - \left(\mathcal{S}_1^{(\gamma)}z\right)'(s_1)\right| = 0,$$
$$\lim_{s_1\to s_2}\left|\left(\mathcal{S}_1^{(\gamma)}z\right)''(s_2) - \left(\mathcal{S}_1^{(\gamma)}z\right)''(s_1)\right| = 0.$$

Hence $\|(\mathcal{S}_1z)(s_2) - (\mathcal{S}_1z)(s_1)\|_{\mathcal{U}}$ tends to zero as $s_1 \to s_2$. Thus $\mathcal{S}_1$ is equicontinuous, and therefore $\mathcal{S}_1$ is relatively compact operator on $\mathcal{V}_{\varepsilon_\gamma}$. So $\mathcal{S}_1$ is compact on $\mathcal{V}_{\varepsilon_\gamma}$ by the Arzelà–Ascoli theorem.

It remains to prove that $\mathcal{S}_2$ is a contraction. To show this, letting $\tilde{z}, z \in \mathcal{V}_{\varepsilon_\gamma}$, we obtain

$$\begin{aligned}
&\left|\left(\mathcal{S}_2^{(\gamma)}\tilde{z}\right)(s) - \left(\mathcal{S}_2^{(\gamma)}z\right)(s)\right| \\
&\quad\le \frac{1}{V_1}\left(\frac{|\varpi_2|}{|\varpi_1|} + s\right) \\
&\quad\times \biggl[|\varpi_3|\int_0^{\theta}\int_0^{\xi}\bigl|\mathcal{S}_\gamma\bigl(\tau,\tilde{z}_\gamma(\tau),\mathfrak{D}^{\nu}\tilde{z}_\gamma(\tau),\tilde{z}'_\gamma(\tau),\tilde{z}''_\gamma(\tau)\bigr) \\
&\quad- \mathcal{S}_\gamma\bigl(\tau,z_\gamma(\tau),\mathfrak{D}^{\nu}z_\gamma(\tau),z'_\gamma(\tau),z''_\gamma(\tau)\bigr)\bigr|\,d\tau\,d\xi \\
&\quad+ |\varpi_2|\int_0^1\bigl|\mathcal{S}_\gamma\bigl(\xi,\tilde{z}_\gamma(\xi),\mathfrak{D}^{\nu}\tilde{z}_\gamma(\xi),\tilde{z}'_\gamma(\xi),\tilde{z}''_\gamma(\xi)\bigr) \\
&\quad- \mathcal{S}_\gamma\bigl(\tau,z_\gamma(\tau),\mathfrak{D}^{\nu}z_\gamma(\tau),z'_\gamma(\tau),z''_\gamma(\tau)\bigr)\bigr|\,d\xi \\
&\quad+ |\varpi_1|\int_0^1\frac{(1-\xi)^{\ell-1}}{\Gamma(\ell)}\bigl|\mathcal{S}_\gamma\bigl(\xi,\tilde{z}_\gamma(\xi),\mathfrak{D}^{\nu}\tilde{z}_\gamma(\xi),\tilde{z}'_\gamma(\xi),\tilde{z}''_\gamma(\xi)\bigr) \\
&\quad- \mathcal{S}_\gamma\bigl(\tau,z_\gamma(\tau),\mathfrak{D}^{\nu}z_\gamma(\tau),z'_\gamma(\tau),z''_\gamma(\tau)\bigr)\bigr|\,d\xi\biggr] \\
&\quad+ \frac{|V_1 - \varpi_2 - \varpi_1 s|}{|\varpi_1|V_1} \\
&\quad\times \biggl[|\varpi_2|\int_0^1\frac{(1-\xi)^{\ell-2}}{\Gamma(\ell-1)}\bigl|\mathcal{S}_\gamma\bigl(\xi,\tilde{z}_\gamma(\xi),\mathfrak{D}^{\nu}\tilde{z}_\gamma(\xi),\tilde{z}'_\gamma(\xi),\tilde{z}''_\gamma(\xi)\bigr) \\
&\quad- \mathcal{S}_\gamma\bigl(\tau,z_\gamma(\tau),\mathfrak{D}^{\nu}z_\gamma(\tau),z'_\gamma(\tau),z''_\gamma(\tau)\bigr)\bigr|\,d\xi \\
&\quad+ \varpi_3\int_0^1\frac{(1-\xi)^{\ell-3}}{\Gamma(\ell-2)}\bigl|\mathcal{S}_\gamma\bigl(\xi,\tilde{z}_\gamma(\xi),\mathfrak{D}^{\nu}\tilde{z}_\gamma(\xi),\tilde{z}'_\gamma(\xi),\tilde{z}''_\gamma(\xi)\bigr) \\
&\quad- \mathcal{S}_\gamma\bigl(\tau,z_\gamma(\tau),\mathfrak{D}^{\nu}z_\gamma(\tau),z'_\gamma(\tau),z''_\gamma(\tau)\bigr)\bigr|\,d\xi\biggr] \\
&\quad\le \frac{1}{V_1}\left(\frac{|\varpi_2|}{|\varpi_1|} + s\right)\biggl[|\varpi_3|\int_0^{\theta}\int_0^{\xi}\mathcal{G}_\gamma(s)\bigl(\bigl|\tilde{z}_\gamma(\tau) - z_\gamma(\tau)\bigr| \\
&\quad+ \bigl|\mathfrak{D}^{\nu}\tilde{z}_\gamma(\tau) - \mathfrak{D}^{\nu}z_\gamma(\tau)\bigr| + \bigl|\tilde{z}'_\gamma(\tau) - z'_\gamma(\tau)\bigr| \\
&\quad+ \bigl|\tilde{z}''_\gamma(\tau) - z''_\gamma(\tau)\bigr|\bigr)\,d\tau\,d\xi + |\varpi_2|\int_0^1\mathcal{G}_\gamma(s)\bigl(\bigl|\tilde{z}_\gamma(\xi) - z_\gamma(\xi)\bigr| \\
&\quad+ \bigl|\mathfrak{D}^{\nu}\tilde{z}_\gamma(\xi) - \mathfrak{D}^{\nu}z_\gamma(\xi)\bigr| + \bigl|\tilde{z}'_\gamma(\xi) - z'_\gamma(\xi)\bigr| + \bigl|\tilde{z}''_\gamma(\xi) - z''_\gamma(\xi)\bigr|\bigr)\,d\xi
\end{aligned}$$

$$
\begin{aligned}
&+ |\varpi_1| \int_0^1 \frac{(1-\xi)^{\ell-1}}{\Gamma(\ell)} \mathcal{G}_\gamma(s)\big(\big|\tilde{z}_\gamma(\xi) - z_\gamma(\xi)\big| + \big|\mathfrak{D}^\nu \tilde{z}_\gamma(\xi) - \mathfrak{D}^\nu z_\gamma(\xi)\big| \\
&+ \big|\tilde{z}'_\gamma(\xi) - z'_\gamma(\xi)\big| + \big|\tilde{z}''_\gamma(\xi) - z''_\gamma(\xi)\big|\big)\Big]\, d\xi \\
&+ \frac{|V_1 - \varpi_2 - \varpi_1 s|}{|\varpi_1| V_1} \Big[|\varpi_2| \int_0^1 \frac{(1-\xi)^{\ell-2}}{\Gamma(\ell-1)} \mathcal{G}_\gamma(s)\big(\big|\tilde{z}_\gamma(\xi) - z_\gamma(\xi)\big| \\
&+ \big|\mathfrak{D}^\nu \tilde{z}_\gamma(\xi) - \mathfrak{D}^\nu z_\gamma(\xi)\big| + \big|\tilde{z}'_\gamma(\xi) - z'_\gamma(\xi)\big| + \big|\tilde{z}''_\gamma(\xi) - z''_\gamma(\xi)\big|\big)\, d\xi \\
&+ \varpi_3 \int_0^1 \frac{(1-\xi)^{\ell-3}}{\Gamma(\ell-2)} \mathcal{G}_\gamma(s)\big(\big|\tilde{z}_\gamma(\xi) - z_\gamma(\xi)\big| + \big|\mathfrak{D}^\nu \tilde{z}_\gamma(\xi) - \mathfrak{D}^\nu z_\gamma(\xi)\big| \\
&+ \big|\tilde{z}'_\gamma(\xi) - z'_\gamma(\xi)\big| + \big|\tilde{z}''_\gamma(\xi) - z''_\gamma(\xi)\big|\big)\Big]\, d\xi \\
\leq{}& \|\mathcal{G}_\gamma\| \mathcal{I}_4^* \|\tilde{z}_\gamma - z_\gamma\|_{\mathcal{U}_\gamma}
\end{aligned}
$$

for each $\gamma = 1, 2, \ldots, 18$, where $\mathcal{I}_4^*$ is given in (3.9). Also, by similar computations we have

$$
\begin{aligned}
&\sup_{s\in[0,1]} \big|\big(\mathfrak{D}^\nu \mathcal{S}_2^{(\gamma)}\tilde{z}\big)(s) - \big(\mathfrak{D}^\nu \mathcal{S}_2^{(\gamma)} z\big)(s)\big| \leq \|\mathcal{G}_\gamma\| \mathcal{I}_5^* \|\tilde{z}_\gamma - z_\gamma\|_{\mathcal{U}_\gamma}, \\
&\sup_{s\in[0,1]} \big|\big(\mathcal{S}_2^{(\gamma)}\tilde{z}\big)'(s) - \big(\mathcal{S}_2^{(\gamma)} z\big)'(s)\big| \leq \|\mathcal{G}_\gamma\| \mathcal{I}_6^* \|\tilde{z}_\gamma - z_\gamma\|_{\mathcal{U}_\gamma}, \\
&\sup_{s\in[0,1]} \big|\big(\mathcal{S}_2^{(\gamma)}\tilde{z}\big)''(s) - \big(\mathcal{S}_2^{(\gamma)} z\big)''(s)\big| \leq 0,
\end{aligned}
$$

where $\mathcal{I}_5^*$ and $\mathcal{I}_6^*$ are given in (3.10) and (3.11), respectively. Thus we have

$$
\begin{aligned}
\|\mathcal{S}_2\tilde{z} - \mathcal{S}_2 z\|_{\mathcal{U}} &= \sum_{\gamma=1}^{18} \big\|\mathcal{S}_2^{(\gamma)}\tilde{z} - \mathcal{S}_2^{(\gamma)} z\big\|_{\mathcal{U}_\gamma} \\
&\leq \big(\mathcal{I}_4^* + \mathcal{I}_5^* + \mathcal{I}_6^*\big) \sum_{\gamma=1}^{18} \|\mathcal{G}_\gamma\| \|\tilde{z}_\gamma - z_\gamma\|_{\mathcal{U}_\gamma},
\end{aligned}
$$

and so

$$
\|\mathcal{S}_2\tilde{z} - \mathcal{S}_2 z\|_{\mathcal{U}} \leq \Delta \|\tilde{z} - z\|_{\mathcal{U}}.
$$

Since $\Delta < 1$, $\mathcal{S}_2$ is a contraction on $\mathcal{V}_{\varepsilon_\gamma}$. As a result of Theorem 2.3, we infer that $\mathcal{S}$ contains a fixed point, which is a solution to problem (1.4). □

To illustrate the significance of our results, we provide the following example.

Example 3.3 Consider the differential equations:

$$
\begin{cases}
\mathfrak{D}^{2.01} z_1(s) = \frac{4s}{1000}|\arcsin z_1(s)| + \frac{8|\mathfrak{D}^{0.2} z_1(s)|s}{2000+2000|\mathfrak{D}^{0.2} z_1(s)|} + 0.004s|\arcsin z_1'(s)| \\
\qquad + \frac{12s|\sin z_1''(s)|}{3000(1+|\sin z_1''(s)|)}, \\
\mathfrak{D}^{2.01} z_2(s) = \frac{21e^s|\sin z_2(s)|}{3000(1+|\sin z_2(s)|)} + \frac{7e^s}{1000}|\sin(\mathfrak{D}^{0.2} z_2(s))| + \frac{14|\arctan z_2'(s)|e^s}{2000+2000|\arctan z_2'(s)|} \\
\qquad + 0.007e^s|\arcsin z_2''(s)|, \\
\mathfrak{D}^{2.01} z_3(s) = 0.011|\arctan z_3(s)|s + \frac{44s|\mathfrak{D}^{0.2} z_3(s)|}{4000(1+|\mathfrak{D}^{0.2} z_3(s)|)} + \frac{11}{1000}|\arcsin z_3'(s)|s \\
\qquad + \frac{22|\sin z_3''(s)|s}{2000+2000|\sin z_3''(s)|},
\end{cases}
\tag{3.15}
$$

associated with the boundary conditions

$$\begin{cases} \frac{9}{11}z_1(0) = \frac{3}{14}\mathfrak{D}^1 z_1(1) + \frac{6}{19}\mathfrak{D}^2 z_1(1), \\ \frac{9}{11}z_1(1) = \frac{3}{14}\mathfrak{D}^{1.01} z_1(1) + \frac{6}{19}\int_0^{0.05}\mathfrak{D}^{1.01} z_1(\xi)\,d\xi, \\ \frac{9}{11}z_2(0) = \frac{3}{14}\mathfrak{D}^1 z_2(1) + \frac{6}{19}\mathfrak{D}^2 z_2(1), \\ \frac{9}{11}z_2(1) = \frac{3}{14}\mathfrak{D}^{1.01} z_2(1) + \frac{6}{19}\int_0^{0.05}\mathfrak{D}^{1.01} z_2(\xi)\,d\xi \\ \frac{9}{11}z_3(0) = \frac{3}{14}\mathfrak{D}^1 z_3(1) + \frac{6}{19}\mathfrak{D}^2 z_3(1), \\ \frac{9}{11}z_3(1) = \frac{3}{14}\mathfrak{D}^{1.01} z_3(1) + \frac{6}{19}\int_0^{0.05}\mathfrak{D}^{1.01} z_3(\xi)\,d\xi, \end{cases} \tag{3.16}$$

where $\ell = 2.01$, $\nu = 0.2$, $\varpi_1 = \frac{9}{11}$, $\varpi_2 = \frac{3}{14}$, $\varpi_3 = \frac{6}{19}$, and $\mathfrak{D}^\ell$, $\mathfrak{D}^\nu$ represent the Caputo fractional derivatives of orders ℓ and ν, respectively. Let $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous functions given as

$$\begin{cases} \mathcal{S}_1(s, z_1(s), z_2(s), z_3(s), z_4(s)) \\ \quad = \frac{4s}{1000}|\arcsin z_1(s)| + \frac{8|\mathfrak{D}^{0.2}z_2(s)|s}{2000+2000|\mathfrak{D}^{0.2}z_2(s)|} \\ \qquad + 0.004s|\arcsin z_3'(s)| + \frac{12s|\sin z_4''(s)|}{3000(1+|\sin z_4''(s)|)}, \\ \mathcal{S}_2(s, z_1(s), z_2(s), z_3(s), z_4(s)) \\ \quad = \frac{21e^s|\sin z_1(s)|}{3000(1+|\sin z_1(s)|)} + \frac{7e^s}{1000}|\sin(\mathfrak{D}^{0.2}z_2(s))| \\ \qquad + \frac{14|\arctan z_3'(s)|e^s}{2000+2000|\arctan z_3'(s)|} + 0.007e^s|\arcsin z_4''(s)|, \\ \mathcal{S}_3(s, z_1(s), z_2(s), z_3(s), z_4(s)) \\ \quad = 0.011|\arctan z_1(s)|s + \frac{44s|\mathfrak{D}^{0.2}z_2(s)|}{4000(1+|\mathfrak{D}^{0.2}z_2(s)|)} \\ \qquad + \frac{11}{1000}|\arcsin z_3'(s)|s + \frac{22|\sin z_4''(s)|s}{2000+2000|\sin z_4''(s)|}. \end{cases}$$

Let $z_1, z_2, z_3, z_4, \tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4 \in \mathbb{R}$. Then we have

$$\begin{aligned} &\bigl|\mathcal{S}_1\bigl(s, z_1(s), z_2(s), z_3(s), z_4(s)\bigr) - \mathcal{S}_1\bigl(s, \tilde{z}_1(s), \tilde{z}_2(s), \tilde{z}_3(s), \tilde{z}_4(s)\bigr)\bigr| \\ &\quad \le \frac{4s}{1000}\bigl(\bigl|\arcsin z_1(s) - \arcsin \tilde{z}_1(s)\bigr| + \bigl|z_2(s) - \tilde{z}_2(s)\bigr| \\ &\qquad + \bigl|\sin z_3(s) - \sin \tilde{z}_3(s)\bigr| + \bigl|\sin z_4(s) - \sin \tilde{z}_4(s)\bigr|\bigr), \\ &\bigl|\mathcal{S}_2\bigl(s, z_1(s), z_2(s), z_3(s), z_4(s)\bigr) - \mathcal{S}_2\bigl(s, \tilde{z}_1(s), \tilde{z}_2(s), \tilde{z}_3(s), \tilde{z}_4(s)\bigr)\bigr| \\ &\quad \le \frac{7e^s}{1000}\bigl(\bigl|\sin z_1(s) - \sin \tilde{z}_1(s)\bigr| + \bigl|\sin z_2(s) - \sin \tilde{z}_2(s)\bigr| \\ &\qquad + \bigl|\arctan z_3(s) - \arctan \tilde{z}_3(s)\bigr| + \bigl|\arcsin z_4(s) - \arcsin \tilde{z}_4(s)\bigr|\bigr), \\ &\bigl|\mathcal{S}_3\bigl(s, z_1(s), z_2(s), z_3(s), z_4(s)\bigr) - \mathcal{S}_3\bigl(s, \tilde{z}_1(s), \tilde{z}_2(s), \tilde{z}_3(s), \tilde{z}_4(s)\bigr)\bigr| \\ &\quad \le \frac{11s}{1000}\bigl(\bigl|\arctan z_1(s) - \arctan \tilde{z}_1(s)\bigr| + \bigl|z_2(s) - \tilde{z}_2(s)\bigr| \\ &\qquad + \bigl|\arcsin z_3(s) - \arcsin \tilde{z}_3(s)\bigr| + \bigl|\sin z_4(s) - \sin \tilde{z}_4(s)\bigr|\bigr). \end{aligned}$$

Here $\mathcal{G}_1(s) = \frac{4s}{1000}, \mathcal{G}_2(s) = \frac{7e^s}{1000}, \mathcal{G}_3(s) = \frac{11s}{1000}$, where $\|\mathcal{G}_1\| = \frac{4}{1000}, \|\mathcal{G}_2\| = \frac{7}{1000}, \|\mathcal{G}_3\| = \frac{11}{1000}$. Let $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 : [0,\infty) \to \mathbb{R}$ be the identity functions. Then we obtain

$$\begin{aligned}\left|\mathcal{S}_1\left(s, z(s), \mathfrak{D}^{0.2}z(s), z'(s), z''(s)\right)\right| &\leq \frac{4s}{1000}\left(|\arcsin z| + |\mathfrak{D}z| + \left|\sin z'\right| + \left|\sin z''\right|\right)\\ &\leq \frac{4s}{1000}\left(|z| + |\mathfrak{D}z| + \left|z'\right| + \left|z''\right|\right).\end{aligned}$$

Also,

$$\begin{aligned}\left|\mathcal{S}_2\left(s, z(s), \mathfrak{D}^{0.2}z(s), z'(s), z''(s)\right)\right| &\leq \frac{7e^s}{1000}\left(|\sin z| + \left|\sin(\mathfrak{D}z)\right| + \left|\arctan z'\right| + \left|\arcsin z''\right|\right)\\ &\leq \frac{7e^s}{1000}\left(|z| + |\mathfrak{D}z| + \left|z'\right| + \left|z''\right|\right)\end{aligned}$$

and

$$\begin{aligned}\left|\mathcal{S}_3\left(s, z(s), \mathfrak{D}^{0.2}z(s), z'(s), z''(s)\right)\right| &\leq \frac{11s}{1000}\left(|\arctan z| + |\mathfrak{D}z| + \left|\arcsin z'\right| + \left|\sin z''\right|\right)\\ &\leq \frac{11s}{1000}\left(|z| + |\mathfrak{D}z| + \left|z'\right| + \left|z''\right|\right),\end{aligned}$$

where the continuous functions $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3 : [0,1] \to \mathbb{R}$ are defined by

$$\mathcal{Z}_1(s) = \frac{4s}{1000}, \qquad \mathcal{Z}_2(s) = \frac{7e^s}{1000}, \qquad \mathcal{Z}_3(s) = \frac{11s}{1000}.$$

Also,

$$\mathcal{I}_4^* \simeq 0.9227, \qquad \mathcal{I}_5^* \simeq 1.2360 \quad \text{and} \quad \mathcal{I}_6^* \simeq 1.1512,$$

and so

$$\Delta := \left(\mathcal{I}_4^* + \mathcal{I}_5^* + \mathcal{I}_6^*\right)\left(\|\mathcal{G}_1\| + \|\mathcal{G}_2\| + \|\mathcal{G}_3\|\right) \simeq 0.0728 < 1.$$

Hence by Theorem 3.2 problem (3.15)–(3.16) has a solution.

4 Conclusion

Chemical graph theory is a broad area of research, which uses theoretical and practical techniques to analyze the molecular structure of a chemical substance on graphs while considering particular mathematical challenges. As a result of the fast growth of this area over the last few decades, many new concepts and techniques for conducting such research have emerged. In this paper, we used the graph of a hexasilinane compound and defined the Caputo fractional boundary value problem on each of its edges. We utilized the fixed point theorems of Krasnoselskii and Schaefer to prove the existence of solutions to the proposed boundary value problem. Our approach is easy to implement and can be used in a wide range of graphs, particularly digraphs, which are often used in medical technology for protein networks.

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References

1. Lumer, G.: Connecting of local operators and evolution equations on a network. Lect. Notes Math. **787**, 219–234 (1985). https://doi.org/10.1007/BFb0086338
2. Zavgorodnii, M.G., Pokornyi, Y.V.: On the spectrum of second-order boundary value problems on spatial networks. Usp. Mat. Nauk **44**, 220–221 (1989)
3. Gordeziani, D.G., Kupreishvli, M., Meladze, H.V., Davitashvili, T.D.: On the solution of boundary value problem for differential equations given in graphs. Appl. Math. Lett. **13**, 80–91 (2008)
4. Graef, J.R., Kong, L.J., Wang, M.: Existence and uniqueness of solutions for a fractional boundary value problem on a graph. Fract. Calc. Appl. Anal. **17**, 499–510 (2014). https://doi.org/10.2478/s13540-014-0182-4
5. Mehandiratta, V., Mehra, M., Leugering, G.: Existence and uniqueness results for a nonlinear Caputo fractional boundary value problem on a star graph. J. Math. Anal. Appl. **477**(2), 1243–1264 (2019). https://doi.org/10.1016/j.jmaa.2019.05.011
6. Mophou, G., Leugering, G., Fotsing, P.S.: Optimal control of a fractional Sturm–Liouville problem on a star graph. Optimization **70**(3), 659–687 (2021). https://doi.org/10.1080/02331934.2020.1730371
7. Etemad, S., Rezapour, S.: On the existence of solutions for fractional boundary value problems on the ethane graph. Adv. Differ. Equ. **2020**, 276 (2020). https://doi.org/10.1186/s13662-020-02736-4
8. Baleanu, D., Etemad, S., Mohammadi, H., Rezapour, S.: A novel modeling of boundary value problems on the glucose graph. Commun. Nonlinear Sci. Numer. Simul. **100**, Article ID 105844 (2021). https://doi.org/10.1016/j.cnsns.2021.105844
9. Turab, A., Sintunavarat, W.: The novel existence results of solutions for a nonlinear fractional boundary value problem on the ethane graph. Alex. Eng. J. **60**(6), 5365–5374 (2021). https://doi.org/10.1016/j.aej.2021.04.020
10. Turab, A., Sintunavarat, W.: A unique solution of the iterative boundary value problem for a second-order differential equation approached by fixed point results. Alex. Eng. J. **60**(6), 5797–5802 (2021). https://doi.org/10.1016/j.aej.2021.04.031
11. Lazreg, J.E., Abbas, S., Benchohra, M., Karapınar, E.: Impulsive Caputo–Fabrizio fractional differential equations in b-metric spaces. Open Math. **19**(1), 363–372 (2021). https://doi.org/10.1515/math-2021-0040
12. Karapınar, E., Fulga, A., Rashid, M., Shahid, L., Aydi, H.: Large contractions on quasi-metric spaces with an application to nonlinear fractional differential equations. Mathematics **7**(5), Article ID 444 (2019). https://doi.org/10.3390/math7050444
13. Alqahtani, B., Aydi, H., Karapınar, E., Rakočević, V.A.: Solution for Volterra fractional integral equations by hybrid contractions. Mathematics **7**, 694 (2019). https://doi.org/10.3390/math7080694
14. Abdeljawad, T., Agarwal, R.P., Karapınar, E., Kumari, P.S.: Solutions of the nonlinear integral equation and fractional differential equation using the technique of a fixed point with a numerical experiment in extended b-metric space. Symmetry **11**, 686 (2019). https://doi.org/10.3390/sym11050686
15. Afshari, H., Atapour, M., Karapinar, E.: A discussion on a generalized Geraghty multi-valued mappings and applications. Adv. Differ. Equ. **2020**, Article ID 356 (2020). https://doi.org/10.1186/s13662-020-02819-2
16. Afshari, H.: Solution of fractional differential equations in quasi-b metric and b-metric-like spaces. Adv. Differ. Equ. **2018**, Article ID 285 (2018). https://doi.org/10.1186/s13662-019-2227-9
17. Afshari, H., Kalantari, S., Karapinar, E.: Solution of fractional differential equations via coupled fixed point. Electron. J. Differ. Equ. **2015**(286), 1 (2015)

18. Adiguzel, R.S., Aksoy, U., Karapinar, E., Erhan, I.M.: On the solutions of fractional differential equations via Geraghty type hybrid contractions. Appl. Comput. Math. **20**(2), 313–333 (2021)
19. Adiguzel, R.S., Aksoy, U., Karapinar, E., Erhan, I.M.: On the solution of a boundary value problem associated with a fractional differential equation. Math. Methods Appl. Sci. (2020). https://doi.org/10.1002/mma.665
20. Afshari, H., Karapınar, E.: A discussion on the existence of positive solutions of the boundary value problems via ψ-Hilfer fractional derivative on b-metric spaces. Adv. Differ. Equ. **2020**, Article ID 616 (2020). https://doi.org/10.1186/s13662-020-03076-z
21. Sevinik-Adıgüzel, R., Aksoy, Ü., Karapinar, E., Erhan, I.M.: Uniqueness of solution for higher-order nonlinear fractional differential equations with multi-point and integral boundary conditions. RACSAM **115**, 155 (2021). https://doi.org/10.1007/s13398-021-01095-3
22. Karapinar, E., Binh, H.D., Luc, N.H., Can, N.H.: On continuity of the fractional derivative of the time-fractional semilinear pseudo-parabolic systems. Adv. Differ. Equ. **2021**, Article ID 70 (2021). https://doi.org/10.1186/s13662-021-03232-z
23. Afshari, H., Shojaat, H., Siahkali Moradi, M.: Existence of the positive solutions for a tripled system of fractional differential equations via integral boundary conditions. Res. Nonlinear Anal. **4**(3), 186–199 (2021). https://doi.org/10.53006/rna.938851
24. Jangid, N., Joshi, S., Purohit, S.D., Suthar, D.L.: Fractional derivatives and expansion formulae of incomplete H and $\bar{H}$-functions. Adv. Theory Nonlinear Anal. Appl. **5**(2), 193–202 (2021). https://doi.org/10.31197/atnaa.755309
25. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equation. Wiley, New York (1993)
26. Smart, D.R.: Fixed Point Theorems. Cambridge University Press, Cambridge (1990)