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# The weak solutions of a nonlinear parabolic equation from two-phase problem

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## Abstract

A nonlinear parabolic equation from a two-phase problem is considered in this paper. The existence of weak solutions is proved by the standard parabolically regularized method. Different from the related papers, one of diffusion coefficients in the equation,  $b(x)$ , is degenerate on the boundary. Then the Dirichlet boundary value condition may be overdetermined. In order to study the stability of weak solution, how to find a suitable partial boundary value condition is the foremost work. Once such a partial boundary value condition is found, the stability of weak solutions will naturally follow.

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**Keywords:** Nonlinear parabolic equation; Two-phase problem; Partial boundary value condition; Stability

## 1 Introduction

In this paper, we study the following initial-boundary value problem:

$$u_t = \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u + b(x)|\nabla u|^{q(x)-2} \nabla u), \quad (x, t) \in Q_T = \Omega \times (0, T), \quad (1.1)$$

$$u|_{t=0} = u_0(x), \quad x \in \Omega, \quad (1.2)$$

$$u|_{\Gamma_T} = 0, \quad (x, t) \in \Gamma_T = \partial\Omega \times (0, T), \quad (1.3)$$

where  $1 < p(x), q(x) \in C(\overline{\Omega})$ ,  $b(x) \in C^1(\overline{\Omega})$  and satisfies

$$b(x) > 0, \quad x \in \Omega, \quad b(x) = 0, \quad x \in \partial\Omega. \quad (1.4)$$

For any  $h(x) \in C(\overline{\Omega})$ , we denote

$$h^+ = \max_{\overline{\Omega}} h(x), \quad h^- = \min_{\overline{\Omega}} h(x),$$

as usual.

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Let us give a brief review of the related works. We first noticed that the initial-boundary value problem of the equation

$$u_t = \operatorname{div}((|u|^{\sigma(x,t)} + d_0)|\nabla u|^{p(x,t)-2}\nabla u) + c(x,t) - b_0u(x,t), \quad (x,t) \in Q_T, \tag{1.5}$$

has been considered in [14, 19, 23], where  $\sigma(x,t) > 1, d_0 > 0, c(x,t) \geq 0,$  and  $b_0 > 0, \Omega \subset \mathbb{R}^N$  is a bounded domain with smooth boundary  $\partial\Omega$ . This model may describe some properties of image restoration in space and time,  $u(x,t)$  represents a recovering image,  $p(x,t)$  reflects the corresponding observed noisy image. The authors of [14] obtained the existence and uniqueness of weak solutions with the assumption that the exponent  $\sigma(x,t) \equiv 0, 1 < p^- < p^+ < 2$ . If  $\sigma(x,t) \equiv 0$  and  $b_0 = 0$ , the existence of weak solutions was proved in [23] by Galerkin’s method. Next, in [19], they proved the existence and uniqueness of weak solution when  $\sigma(x,t) \in (2, \frac{2p^+}{p^+-1})$  or  $\sigma(x,t) \in (1, 2), 1 < p^- < p^+ \leq 1 + \sqrt{2}$ . Moreover, they applied energy estimates and Gronwall’s inequality to obtain the extinction of solutions when the exponents  $p^-$  and  $p^+$  belong to different intervals.

Secondly, the nonlinear parabolic equation from the double phase problems

$$u_t = \operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u + b(x)|\nabla u|^{q-2}\nabla u) + f(x,t), \quad (x,t) \in Q_T, \tag{1.6}$$

has been studied in [6–10] and [16, 17, 26] in recent years, where the diffusion coefficients  $a(x)$  and  $b(x)$  satisfy

$$a(x) + b(x) > 0, \quad x \in \overline{\Omega}. \tag{1.7}$$

If  $f(x,t) = 0$ , the author of [6] studied the existence of weak solutions to equation (1.6) by the energy functional method. If  $f(x,t) \in L^r(0, T; L^s(\Omega))$  with some given positive constants  $r$  and  $s$ , by defining the local parabolic potential, the author of [12] obtained the local boundedness of weak solutions. In addition, there are many papers that worked on the double phase elliptic equations studied in the framework of the Musielak–Orlicz spaces, see [5, 13, 15, 20, 21, 27, 30].

In this paper, we use the parabolically regularized method to prove the existence of the weak solution to equation (1.1). If  $p(x) \geq q(x)$ , it is not difficult to show that the weak solution  $u$  is in  $L^\infty(0, T; W_0^{1,p(x)}(\Omega))$ . Then, based on the usual Dirichlet boundary value condition (1.3), the stability of weak solutions can be obtained in a simple way. So, in this paper, we assume that  $p(x) \leq q(x)$ . Since  $b(x)$  satisfies (1.4), in general,  $u \in L^\infty(0, T; W_0^{1,q(x)}(\Omega))$  is impossible. The greatest contribution of this paper lies in that, instead of using the usual boundary value condition (1.3), it proves the stability of weak solutions only under a partial boundary value condition

$$u|_{\Gamma_{1T}} = 0, \quad (x,t) \in \Gamma_{1T} = \Sigma_1 \times (0, T), \tag{1.8}$$

and the uniqueness follows naturally. Here,  $\Sigma_1 \subset \partial\Omega$  is a relative open subset and will be specified below.

The method used in this paper may be generalized to study the well-posedness problem of the following double phase equation with the variable exponents:

$$u_t = \operatorname{div}(a(x)|\nabla u|^{p(x)-2}\nabla u + b(x)|\nabla u|^{q(x)-2}\nabla u), \quad (x,t) \in Q_T. \tag{1.9}$$

We are ready to study this problem in the future. For a general degenerate parabolic equation, the well-posedness of weak solutions based on a partial boundary value condition has been studied for a long time, relevant literature can be referred to [4, 22, 28, 29, 31–37].

## 2 The definitions of weak solution and the main results

We assume that  $r(x) \in C(\overline{\Omega})$ ,

$$1 < r^- \leq r(x), \quad \forall x \in \Omega,$$

and quote some function spaces with variable exponents.

1.  $L^{r(x)}(\Omega)$  space

$$L^{r(x)}(\Omega) = \left\{ u : u \text{ is a measurable real-valued function, } \int_{\Omega} |u(x)|^{r(x)} dx < \infty \right\}$$

is equipped with the Luxemburg norm

$$\|u\|_{L^{r(x)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{r(x)} dx \leq 1 \right\},$$

which is a separable, uniformly convex Banach space.

2.  $W^{1,r(x)}(\Omega)$  space

$$W^{1,r(x)}(\Omega) = \{ u \in L^{r(x)}(\Omega) : |\nabla u| \in L^{r(x)}(\Omega) \}$$

is endowed with the norm

$$\|u\|_{W^{1,r(x)}} = \|u\|_{L^{r(x)}(\Omega)} + \|\nabla u\|_{L^{r(x)}(\Omega)}, \quad \forall u \in W^{1,r(x)}(\Omega).$$

3.  $W_0^{1,r(x)}(\Omega)$  is the closure of  $C_0^\infty(\Omega)$  in  $W^{1,r(x)}(\Omega)$ .

Let us recall some properties of the function spaces  $W^{1,r(x)}(\Omega)$  according to [18, 24].

**Lemma 2.1** (i) *The spaces  $(L^{r(x)}(\Omega), \|\cdot\|_{L^{r(x)}(\Omega)})$ ,  $(W^{1,r(x)}(\Omega), \|\cdot\|_{W^{1,r(x)}(\Omega)})$ , and  $W_0^{1,r(x)}(\Omega)$  are reflexive Banach spaces.*

(ii)  *$r(x)$ -Hölder’s inequality. Let  $r_1(x)$  and  $r_2(x)$  be real functions with  $\frac{1}{r_1(x)} + \frac{1}{r_2(x)} = 1$  and  $r_1(x) > 1$ . Then the conjugate space of  $L^{r_1(x)}(\Omega)$  is  $L^{r_2(x)}(\Omega)$ . For any  $u \in L^{r_1(x)}(\Omega)$  and  $v \in L^{r_2(x)}(\Omega)$ , there is*

$$\left| \int_{\Omega} uv dx \right| \leq 2 \|u\|_{L^{r_1(x)}(\Omega)} \|v\|_{L^{r_2(x)}(\Omega)}.$$

(iii)

$$\text{If } \|u\|_{L^{r(x)}(\Omega)} = 1, \text{ then } \int_{\Omega} |u|^{r(x)} dx = 1,$$

$$\text{If } \|u\|_{L^{r(x)}(\Omega)} > 1, \text{ then } \|u\|_{L^{r(x)}(\Omega)}^{r^-} \leq \int_{\Omega} |u|^{r(x)} dx \leq \|u\|_{L^{r(x)}(\Omega)}^{r^+},$$

$$\text{If } \|u\|_{L^{r(x)}(\Omega)} < 1, \text{ then } \|u\|_{L^{r(x)}(\Omega)}^{r^+} \leq \int_{\Omega} |u|^{r(x)} dx \leq \|u\|_{L^{r(x)}(\Omega)}^{r^-}.$$

(iv) If  $r_1(x) \leq r_2(x)$ , then

$$L^{r_1(x)}(\Omega) \supset L^{r_2(x)}(\Omega).$$

(v) If  $r_1(x) \leq r_2(x)$ , then

$$W^{1,r_1(x)}(\Omega) \hookrightarrow W^{1,r_2(x)}(\Omega).$$

Besides this trivial embedding, it would be useful to know finer estimates of the type of Sobolev inequality.

(vi)  $r(x)$ -Poincaré inequality. If  $r(x) \in C(\overline{\Omega})$ , then there is a constant  $C > 0$  such that

$$\|u\|_{L^{r(x)}(\Omega)} \leq C \|\nabla u\|_{L^{r(x)}(\Omega)}, \quad \forall u \in W_0^{1,r(x)}(\Omega).$$

This implies that  $\|\nabla u\|_{L^{r(x)}(\Omega)}$  and  $\|u\|_{W^{1,r(x)}(\Omega)}$  are equivalent norms of  $W_0^{1,r(x)}(\Omega)$ .

But Zhikov [38] pointed out that

$$W_0^{1,r(x)}(\Omega) \neq \{v \in W_0^{1,r(x)}(\Omega) \mid v|_{\partial\Omega} = 0\} = \mathring{W}^{1,p(x)}(\Omega),$$

unless  $r(x) \in C_{\log}(\Omega)$ . Here,  $r(x) \in C_{\log}(\Omega)$  means that  $r(x)$  is a logarithmic Hölder continuity function, i.e., it satisfies

$$|r(x) - r(y)| \leq \omega(|x - y|), \quad \forall x, y \in Q_T, |x - y| < \frac{1}{2},$$

where  $\omega(s)$  is with the property

$$\overline{\lim}_{s \rightarrow 0^+} \omega(s) \ln\left(\frac{1}{s}\right) = C < \infty.$$

Let  $\rho(x)$  be the Friedrichs mollifying kernel

$$\rho(x) = k \begin{cases} \exp\left(\frac{-1}{1-|x|^2}\right), & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

where  $k$  is a constant such that  $\int_{\mathbb{R}^N} \rho(x) dx = 1$ . Denote that  $\rho_\varepsilon(x) = \varepsilon^{-N} \rho\left(\frac{x}{\varepsilon}\right)$ . For  $f \in W_0^{1,p(x)}(\Omega)$ , denote that

$$f_\varepsilon(x) = f * \rho_\varepsilon = \int_{\mathbb{R}^N} f(y) \rho_\varepsilon(y - x) dy.$$

**Lemma 2.2** Let  $\Omega' \subset\subset \Omega$ . If  $r(x) \in C_{\log}(\Omega)$ , then for every  $f \in L^{r(x)}(\Omega)$ ,

$$\|f_\varepsilon\|_{L^{r(x)}(\Omega')} \leq c(\|f\|_{L^{r(x)}(\Omega)} + \|f\|_{L^1(\Omega)}),$$

$$\|f - f_\varepsilon\|_{L^{r(x)}(\Omega')} \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

**Lemma 2.3** If  $r(x) \in C_{\log}(\Omega)$ , then the set  $C_0^\infty(\Omega)$  is dense in  $\mathring{W}^{1,r(x)}(\Omega)$ .

These two lemmas can be found in [2]. Certainly, for a constant  $p \geq 1$ , it is well known that, if  $u \in L^p(\Omega)$ , then  $\rho_\varepsilon * u \in L^p(\Omega)$  and

$$\|u - \rho_\varepsilon * u\|_{L^p(\Omega)} \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

Let us give the definition of weak solution.

**Definition 2.4** *If  $0 \leq u(x, t) \in L^\infty(Q_T)$  satisfies*

$$u_t \in L^2(Q_T), \quad b(x)|\nabla u|^{q(x)} \in L^1(Q_T), \quad u \in L^1(0, T; W_0^{1,p(x)}(\Omega)),$$

*and for any function  $\varphi \in L^\infty(0, T; W_0^{1,p(x)}(\Omega)) \cap L^\infty(0, T; W_{loc}^{1,q(x)}(\Omega))$ ,*

$$\int_\tau^s \int_\Omega [u_t \varphi + |\nabla u|^{p(x)-2} \nabla u \nabla \varphi + b(x)|\nabla u|^{q(x)-2} \nabla u \nabla \varphi] dx dt = 0, \quad \forall \tau, s \in [0, T], \quad (2.1)$$

*then  $u(x, t)$  is said to be a weak solution of equation (1.1) with the initial value (1.2), provided that*

$$\lim_{t \rightarrow 0} \int_\Omega u(x, t) \phi(x) dx = \int_\Omega u_0(x) \phi(x) dx, \quad \forall \phi(x) \in C_0^\infty(\Omega). \quad (2.2)$$

Throughout this paper, we assume that  $q(x), p(x)$  both are logarithmic Hölder continuous functions and satisfy

$$q(x) \geq q^- \geq p^+ \geq p(x) \geq p^- > 1. \quad (2.3)$$

The main results are the following theorems.

**Theorem 2.5** *If  $p(x)$  and  $q(x)$  are  $C^1(\overline{\Omega})$  functions,  $q(x) \geq q^- \geq 2$ ,*

$$\frac{2N}{N+2} \leq p^- \leq p(x) \leq q(x) < p(x) + \frac{4p^-}{2N + p^-(N+2)} \quad (2.4)$$

*and  $0 \leq u_0(x) \in L^\infty(\Omega)$  satisfies*

$$b(x)|\nabla u_0|^{q(x)} \in L^1(\Omega), \quad u_0(x) \in W_0^{1,q^+}(\Omega), \quad (2.5)$$

*then equation (1.1) with the initial boundary values (1.2)–(1.3) has a solution  $u(x, t)$ .*

**Theorem 2.6** *If  $u(x, t)$  and  $v(x, t)$  are two weak solutions with the same homogeneous boundary value (1.3) and with different initial values  $u_0(x), v_0(x)$  respectively, then there holds*

$$\int_\Omega |u(x, t) - v(x, t)| dx \leq c \int_\Omega |u_0(x) - v_0(x)| dx, \quad \forall t \in [0, T]. \quad (2.6)$$

The unusual thing is that, since  $b(x)$  satisfies (1.4), the stability of weak solutions can be proved under a partial boundary value condition (1.8) in which  $\Sigma_1$  has the form

$$\Sigma_1 = \left\{ x \in \partial\Omega : b(x) \left| \frac{\nabla b}{b(x)} \right|^{p(x)} \neq 0 \right\}. \tag{2.7}$$

For example,  $d(x) = \text{dist}(x, \partial\Omega)$ ,  $b(x) = d^\alpha$ ,

$$b(x) \left| \frac{\nabla b}{b(x)} \right|^{p(x)} = |\alpha \nabla d|^{p(x)} d^{\alpha-p(x)} = |\alpha|^{p(x)} d^{\alpha-p(x)}, \quad x \in \partial\Omega,$$

thus, when  $\alpha \geq p^+$ ,  $\Sigma_1 = \emptyset$ ; when  $\alpha < p^-$ ,  $\Sigma_1 = \partial\Omega$  is the entire boundary. Moreover, if there is a subset  $\Sigma_{11} \subset \partial\Omega$  such that  $p(x) = \alpha, x \in \Sigma_{11}$ , and  $p(x) < \alpha, x \in \partial\Omega \setminus \Sigma_{11}$ , then the partial boundary appearing in (1.8)

$$\Sigma_1 = \Sigma_{11} = \{x \in \partial\Omega : p(x) = \alpha\}$$

is just a part of  $\partial\Omega$ .

We denote that

$$\Omega_\eta = \{x \in \Omega : b(x) > \eta\}.$$

**Theorem 2.7** *Let  $u(x, t)$  and  $v(x, t)$  be two solutions of equation (1.1) with the initial values  $u_0(x)$  and  $v_0(x)$  respectively, with the same partial boundary value condition (1.8) and  $\Sigma_1$  given by (2.7). If*

$$\frac{1}{\eta} \left( \int_{\Omega \setminus \Omega_\eta} b(x) |\nabla b(x)|^{p(x)} dx \right)^{\frac{1}{q^+}} \leq c, \tag{2.8}$$

then the stability of (2.6) is true.

### 3 The proof of Theorem 2.5

Let  $q(x) \geq q^- \geq p^+ \geq p(x)$ . Consider the following regularized problem:

$$u_{\varepsilon t} = \text{div}((b(x) + \varepsilon)(|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{q(x)-2}{2}} \nabla u_\varepsilon + |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon), \quad (x, t) \in Q_T, \tag{3.1}$$

$$u_\varepsilon(x, t) = 0, \quad (x, t) \in \Gamma_T, \tag{3.2}$$

$$u_\varepsilon(x, 0) = u_{0\varepsilon}(x), \quad x \in \Omega, \tag{3.3}$$

where  $u_{0\varepsilon} \in C_0^\infty(\Omega)$  and  $(b(x) + \varepsilon)|\nabla u_{0\varepsilon}|^{q(x)} \in L^1(\Omega)$  are uniformly bounded, and  $u_{0\varepsilon}$  converges to  $u_0$  in  $W_0^{1,q^+}(\Omega)$  and  $\|u_{0\varepsilon}(x)\|_{L^\infty} \leq \|u_0(x)\|_{L^\infty}$ .

If  $p(x)$  and  $q(x)$  are with logarithmic Hölder continuous property, similar to [1, 3, 19, 23], by constructing suitable function spaces and applying Galerkin’s method, we can prove that there is a weak solution to problem (3.1)–(3.3),  $u_\varepsilon \in L^{p^-}(0, T; W_0^{1,p(x)}(\Omega)) \cap L^{q^-}(0, T; W_0^{1,q(x)}(\Omega))$ , which satisfies

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \leq c_1. \tag{3.4}$$

In what follows, we shall show that the constant  $c_1$  in (3.4) is independent of  $\varepsilon$ .

**Lemma 3.1** *Assume that  $a, b, \lambda$  are positive constants, where  $\lambda > \frac{1}{2} + \frac{b}{a}$ . Define*

$$\varphi(s) = \begin{cases} e^{\lambda s-1} & s \geq 0, \\ -e^{-\lambda s} + 1 & s \leq 0. \end{cases} \tag{3.5}$$

*Then the following properties hold:*

1. *For any  $s \in \mathbb{R}$ , we have*

$$|\varphi(s)| \geq \lambda|s|, \quad a\varphi'(s) - b|\varphi(s)| \geq \frac{a}{2}e^{\lambda|s|}. \tag{3.6}$$

2. *For any  $s \geq d$ , there hold constants  $d \geq 0, M > 1$ , we have*

$$\varphi'(s) \leq \lambda M \left[ \varphi\left(\frac{s}{p^-}\right) \right]^{p^-}, \quad \varphi(s) \leq M \left[ \varphi\left(\frac{s}{p^-}\right) \right]^{p^-}. \tag{3.7}$$

3. *Let  $\Phi(s) = \int_0^s \varphi(\sigma) d\sigma$ . For any  $s \geq 0$ , if  $p^- > 2$ , there holds constant  $c^* > 0$ , we have*

$$\Phi(s) \geq c^* \left[ \varphi\left(\frac{s}{p^-}\right) \right]^{p^-}. \tag{3.8}$$

*If  $1 < p^- < 2$ , then there exist  $d \geq 0$  and  $c^* = c^*(p^-, d)$  such that*

$$\begin{cases} \Phi(s) \geq c^* [\varphi(\frac{s}{p^-})]^{p^-}, & \forall s \geq d, \\ \Phi(s) \geq c^* [\varphi(\frac{s}{p^-})]^2, & \forall 0 \leq s \leq d. \end{cases} \tag{3.9}$$

We introduce a function space

$$V = \{v \in L^{p^-}(0, T; W_0^{1,p(x)}(\Omega)) : |\nabla v| \in L^{p(x)}(Q_T)\},$$

endowed with the norm  $\|u\|_V = |\nabla u|_{L^{p(x)}(Q_T)}$ , or equivalent norm  $\|u\|_V = |u|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))} + |\nabla u|_{L^{p(x)}(Q_T)}$ , and the equivalence follows from the  $p(x)$ -Poincare inequality. Then  $V$  is a separable and reflexive Banach space. We denote by  $V^*$  its dual space.

**Lemma 3.2** *Assume that  $\pi : \mathbb{R} \rightarrow \mathbb{R}$  is a piecewise function in  $C^1$  satisfying  $\pi(0) = 0$ , and out of a bicomact set  $\pi' = 0$ . Let  $\Pi(s) = \int_0^s \pi(\sigma) d\sigma$ . If  $u \in V$  and  $u_t \in V^* + L^1(Q_T)$ , we have*

$$\begin{aligned} \int_0^T \langle u_t, \pi(u) \rangle dt &:= \langle u_t, \pi(u) \rangle_{V^*+L^1(Q_T), V \cap L^\infty(Q_T)} \\ &= \int_\Omega \Pi(u(T)) dx - \int_\Omega \Pi(u(0)) dx. \end{aligned} \tag{3.10}$$

Lemmas 3.1 and 3.2 can be found in [25].

**Lemma 3.3** *Assume that  $u_\varepsilon \in V \cap L^\infty(Q_T)$  is a weak solution of (3.1), then there is a constant  $c$  (independent of  $\varepsilon$ ) that depends on  $p^-, N, T, \Omega$ , let*

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \leq \|u_0\|_{L^\infty(\Omega)} + c.$$

*Proof* In the proof, we simply denote that  $u_\varepsilon = u$ . If  $k$  is a real number and  $\|u_0\|_{L^\infty(\Omega)} \leq k$ , function (3.5) is defined in  $\varphi$ . Define

$$G_k(u) = \begin{cases} u - k, & u > k, \\ u + k, & u < -k, \\ 0, & |u| \leq k. \end{cases}$$

We can see  $u \in V \cap L^\infty(Q_T)$ , so  $\varphi(G_k(u)) \in V \cap L^\infty(Q_T)$ . So, for any  $\tau \in [0, T]$ , we can choose  $v = \varphi(G_k(u))\chi_{[0,\tau]}$  as a test function (where  $\chi_A$  is an eigenfunction on the set  $A$ ). At the same time, we know that  $v_{x_i} = \chi_{[0,\tau]}\chi\{|u| > k\}\varphi'(G_k(u))u_{x_i}$ , and  $\nabla v = \chi_{[0,\tau]}\chi\{|u| > k\}\varphi'(G_k(u))\nabla u$ , so we have

$$\begin{aligned} & \int_0^\tau \langle u_t, \varphi(G_k(u)) \rangle dt \\ & + \int_0^\tau \int_\Omega [(b(x) + \varepsilon)(|\nabla u|^2 + \varepsilon)^{\frac{q(x)-2}{2}} |\nabla u|^2 \\ & + |\nabla u|^{p(x)}] \varphi'(G_k(u)) \chi\{|u| > k\} dx dt \\ & = 0. \end{aligned} \tag{3.11}$$

Let  $A_k(t) = \{x \in \Omega : |u(x, t)| > k\}$  depend on  $k$ , we have

$$\begin{aligned} \int_0^\tau \langle u_t, \varphi(G_k(u)) \rangle dt &= \int_\Omega \Phi(G_k(u))(\tau) dx - \int_\Omega \Phi(G_k(u_0)) dx \\ &= \int_{A_k(\tau)} \Phi(G_k(u))(\tau) dx - \int_{A_k(0)} \Phi(G_k(u_0)) dx \\ &= \int_{A_k(\tau)} \Phi(G_k(u))(\tau) dx. \end{aligned} \tag{3.12}$$

Substituting (3.12) into (3.11), we can deduce that

$$\begin{aligned} & \int_{A_k(\tau)} \Phi(G_k(u))(\tau) dx + \int_0^\tau \int_{A_k(t)} |\nabla u|^{p(x)} \varphi' dx dt \\ & \leq \int_{A_k(\tau)} \Phi(G_k(u))(\tau) dx \\ & + \int_0^\tau \int_{A_k(t)} [(b(x) + \varepsilon)(|\nabla u|^2 + \varepsilon)^{\frac{q(x)-2}{2}} |\nabla u|^2 + |\nabla u|^{p(x)}] \varphi' dx dt \\ & = 0, \end{aligned} \tag{3.13}$$

which implies

$$\int_{A_k(\tau)} \Phi(G_k(u))(\tau) dx = 0, \quad \forall \tau \in [0, T],$$

so the measure  $\mu(A_k(\tau)) = 0$ , and the conclusion follows naturally. □

Lemma 3.3 implies that one can choose a subsequence of  $u_\varepsilon$  (we still denote it as  $u_\varepsilon$ ) such that

$$u_\varepsilon \rightharpoonup *u, \text{ weakly star in } L^\infty(Q_T), \tag{3.14}$$

where  $u(x, t) \in L^\infty(Q_T)$ . Now, we can show that  $u(x, t)$  is a weak solution of equation (1.1) with the initial value (1.2) in the sense of Definition 2.4.

*Proof of Theorem 2.5* First, for any  $t \in [0, T)$ , we multiply (3.1) by  $u_\varepsilon$  to obtain

$$\begin{aligned} & \frac{1}{2} \int_\Omega u_\varepsilon^2 dx + \iint_{Q_T} [|\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon \nabla u_\varepsilon \\ & \quad + (b(x) + \varepsilon)(|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{q(x)-2}{2}} |\nabla u_\varepsilon|^2] dx dt \\ & = \frac{1}{2} \int_\Omega u_{0\varepsilon}^2 dx, \end{aligned} \tag{3.15}$$

and so we have

$$\iint_{Q_T} |\nabla u_\varepsilon|^{p(x)} dx dt \leq c, \tag{3.16}$$

$$\begin{aligned} & \iint_{Q_T} (b(x) + \varepsilon) |\nabla u_\varepsilon|^{q(x)} \\ & \leq \iint_{Q_T} (b(x) + \varepsilon) (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{q(x)-2}{2}} |\nabla u_\varepsilon|^2 dx dt \leq c. \end{aligned} \tag{3.17}$$

Secondly, by condition (2.4),

$$\frac{2N}{N+2} \leq p^- \leq p(x) \leq q(x) < p(x) + \frac{4p^-}{2N + p^-(N+2)}.$$

Bögelein, Duzaar, and Marcellini [9–11] proved

$$\|u_{\varepsilon t}\|_{L^2(Q_T)} \leq c, \tag{3.18}$$

where the constant  $c$  is independent of  $\varepsilon$ .

By (3.4), (3.14), (3.16), (3.17), and (3.18), there exist a function  $u$  and two  $n$ -dimensional vectors  $\vec{\zeta} = (\zeta_1, \dots, \zeta_N)$  and  $\vec{\xi} = (\xi_1, \dots, \xi_N)$  which satisfy that

$$u \in L^\infty(Q_T), \quad |\zeta_i| \in L^1(0, T; L^{\frac{p(x)}{p(x)-1}}(\Omega)), \quad |\xi_i| \in L^1(0, T; L^{\frac{q(x)}{q(x)-1}}(\Omega)),$$

$u_\varepsilon \rightarrow u$  a.e. in  $Q_T$ , and

$$\begin{aligned} & u_{\varepsilon t} \rightharpoonup u_t, \quad \text{in } L^2(Q_T), \\ & |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon \rightharpoonup \vec{\zeta}, \quad \text{in } L^1(0, T; L^{\frac{p(x)}{p(x)-1}}(\Omega)), \\ & b(x) |\nabla u_\varepsilon|^{q(x)-2} \nabla u_\varepsilon \rightharpoonup \vec{\xi}, \quad \text{in } L^1(0, T; L^{\frac{q(x)}{q(x)-1}}(\Omega)). \end{aligned}$$

In order to prove that  $u$  satisfies equation (2.1), we have to show that

$$\begin{aligned} & \iint_{Q_T} |\nabla u|^{p(x)-2} \nabla u \nabla \varphi_1 \, dx \, dt + \iint_{Q_T} b(x) |\nabla u|^{q(x)-2} \nabla u \nabla \varphi_1 \, dx \, dt \\ &= \iint_{Q_T} (\vec{\zeta} + \vec{\xi}) \cdot \nabla \varphi_1 \, dx \, dt \end{aligned} \tag{3.19}$$

for any  $\varphi_1 \in C_0^1(Q_T)$ .

In the first place, for any  $\varphi \in C_0^1(Q_T)$ , we have

$$\begin{aligned} & \iint_{Q_T} [u_{\varepsilon t} \varphi + (b(x) + \varepsilon)(|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{q(x)-2}{2}} \nabla u_\varepsilon \nabla \varphi \\ &+ |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon \nabla \varphi] \, dx \, dt = 0. \end{aligned} \tag{3.20}$$

Letting  $\varepsilon \rightarrow 0$  in (3.20) yields

$$\iint_{Q_T} \left[ \frac{\partial u}{\partial t} \varphi + \sum_{i=1}^N (\zeta_i + \xi_i) \varphi_{x_i} \right] \, dx \, dt = 0. \tag{3.21}$$

In the second place, let  $0 \leq \psi \in C_0^\infty(Q_T)$  and  $\psi = 1$  in  $\text{supp} \varphi$ ,  $v \in L^\infty(Q_T)$ ,  $b(x) |\nabla v|^{q(x)} \in L^1(Q_T)$ ,  $|\nabla v|^{p(x)} \in L^1(Q_T)$ .

If we choose  $\psi u_\varepsilon$  as the test function of equation (3.1), then

$$\begin{aligned} & \frac{1}{2} \iint_{Q_T} \psi_t |u_\varepsilon|^2 \, dx \, dt - \iint_{Q_T} (b(x) + \varepsilon) u_\varepsilon (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{q(x)-2}{2}} \nabla u_\varepsilon \nabla \psi \, dx \, dt \\ & - \iint_{Q_T} u_\varepsilon |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon \nabla \psi \, dx \, dt \\ &= \iint_{Q_T} \psi (b(x) + \varepsilon) (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{q(x)-2}{2}} |\nabla u_\varepsilon|^2 \, dx \, dt + \iint_{Q_T} \psi |\nabla u_\varepsilon|^{p(x)} \, dx \, dt \\ &\geq \iint_{Q_T} \psi (b(x) + \varepsilon) |\nabla u_\varepsilon|^{q(x)} \, dx \, dt + \iint_{Q_T} \psi |\nabla u_\varepsilon|^{p(x)} \, dx \, dt. \end{aligned} \tag{3.22}$$

By the facts

$$\iint_{Q_T} \psi (b(x) + \varepsilon) [|\nabla u_\varepsilon|^{q(x)-2} \nabla u_\varepsilon - |\nabla v|^{q(x)-2} \nabla v] (\nabla u_\varepsilon - \nabla v) \, dx \, dt \geq 0$$

and

$$\iint_{Q_T} \psi [|\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon - |\nabla v|^{p(x)-2} \nabla v] (\nabla u_\varepsilon - \nabla v) \, dx \, dt \geq 0,$$

from (3.22) we can deduce that

$$\begin{aligned}
 & \frac{1}{2} \iint_{Q_T} \psi_t |u_\varepsilon|^2 \, dx \, dt - \iint_{Q_T} (b(x) + \varepsilon) u_\varepsilon (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{q(x)-2}{2}} \nabla u_\varepsilon \nabla \psi \, dx \, dt \\
 & \quad - \iint_{Q_T} u_\varepsilon |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon \nabla \psi \, dx \, dt \\
 & \quad - \iint_{Q_T} \psi (b(x) + \varepsilon) |\nabla v|^{q(x)-2} \nabla v \nabla (u_\varepsilon - v) \, dx \, dt \\
 & \quad - \iint_{Q_T} \psi (b(x) + \varepsilon) |\nabla u_\varepsilon|^{q(x)-2} \nabla u_\varepsilon \nabla v \, dx \, dt \\
 & \quad - \iint_{Q_T} \psi |\nabla v|^{p(x)-2} \nabla v \nabla (u_\varepsilon - v) \, dx \, dt \\
 & \quad - \iint_{Q_T} \psi |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon \nabla v \, dx \, dt \\
 & \geq 0.
 \end{aligned} \tag{3.23}$$

Now, since

$$(|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{q(x)-2}{2}} \nabla u_\varepsilon = |\nabla u_\varepsilon|^{q(x)-2} \nabla u_\varepsilon + \frac{q(x)-2}{2} \varepsilon \int_0^1 (|\nabla u_\varepsilon|^2 + \varepsilon s)^{\frac{q(x)-4}{2}} \, ds \nabla u_\varepsilon,$$

we have

$$\lim_{\varepsilon \rightarrow 0} \iint_{Q_T} \frac{q(x)-2}{2} \varepsilon \int_0^1 (|\nabla u_\varepsilon|^2 + \varepsilon s)^{\frac{q(x)-4}{2}} \, ds \nabla u_\varepsilon \nabla \psi \, dx \, dt = 0. \tag{3.24}$$

Let  $\varepsilon \rightarrow 0$  in (3.23). By (3.24) and using the Hölder inequality, we can deduce that

$$\begin{aligned}
 & \frac{1}{2} \iint_{Q_T} \psi_t |u|^2 \, dx \, dt - \iint_{Q_T} u \overrightarrow{\xi} \nabla \psi \, dx \, dt \\
 & \quad - \iint_{Q_T} u \overrightarrow{\zeta} \nabla \psi \, dx \, dt \\
 & \quad - \iint_{Q_T} \psi b(x) |\nabla v|^{q(x)-2} \nabla v \nabla (u - v) \, dx \, dt \\
 & \quad - \iint_{Q_T} \psi \overrightarrow{\xi} \nabla v \, dx \, dt \\
 & \quad - \iint_{Q_T} \psi |\nabla v|^{p(x)-2} \nabla v \nabla (u - v) \, dx \, dt \\
 & \quad - \iint_{Q_T} \psi \overrightarrow{\zeta} \nabla v \, dx \, dt \\
 & \geq 0.
 \end{aligned} \tag{3.25}$$

In the third place, let  $\varphi = \psi u$  in (3.21). We get

$$\iint_{Q_T} \left[ -\frac{1}{2} u^2 \varphi_t + (\overrightarrow{\zeta} + \overrightarrow{\xi})(u \nabla \psi + \nabla u \psi) \right] \, dx \, dt = 0. \tag{3.26}$$

Combining (3.25) with (3.26), we have

$$\begin{aligned} & \iint_{Q_T} \psi \left[ \vec{\zeta} (\nabla u - \nabla v) + \vec{\xi} (\nabla u - \nabla v) \right] dx dt \\ & - \iint_{Q_T} \psi (b(x) |\nabla v|^{q(x)-2} \nabla v + |\nabla v|^{p(x)-2} \nabla v) (\nabla u - \nabla v) dx dt \\ & \geq 0. \end{aligned} \tag{3.27}$$

At last, when we choose  $v = u - \lambda \varphi_1, \lambda > 0$ , we have

$$\begin{aligned} & \iint_{Q_T} \psi \left( \vec{\zeta} + \vec{\xi} - |\nabla(u - \lambda \varphi_1)|^{p(x)-2} \nabla(u - \lambda \varphi_1) \right. \\ & \quad \left. - b(x) |\nabla(u - \lambda \varphi_1)|^{q(x)-2} \nabla \varphi_1 \right) \cdot \nabla \varphi_1 dx dt \\ & = 0. \end{aligned} \tag{3.28}$$

If  $\lambda \rightarrow 0$ , then

$$\iint_{Q_T} \psi \left( \vec{\zeta} + \vec{\xi} - |\nabla u|^{p(x)-2} \nabla u - b(x) |\nabla u|^{q(x)-2} \nabla u \right) \cdot \nabla \varphi_1 dx dt \geq 0.$$

Simultaneously, if we choose  $v = u - \lambda \varphi_1, \lambda < 0$ , then  $\lambda \rightarrow 0$  similarly yields

$$\iint_{Q_T} \psi \left( \vec{\zeta} + \vec{\xi} - |\nabla u|^{p(x)-2} \nabla u - b(x) |\nabla u|^{q(x)-2} \nabla u \right) \cdot \nabla \varphi_1 dx dt \leq 0.$$

Thus

$$\iint_{Q_T} \psi \left( \vec{\zeta} + \vec{\xi} - |\nabla u|^{p(x)-2} \nabla u - b(x) |\nabla u|^{q(x)-2} \nabla u \right) \cdot \nabla \varphi_1 dx dt = 0. \tag{3.29}$$

Since  $\psi = 1$  on  $\text{supp } \varphi_1$ , namely we know that (3.19) is true, for any  $\varphi_1 \in C_0^1(Q_T)$ , we have

$$\begin{aligned} & \iint_{Q_T} |\nabla u|^{p(x)-2} \nabla u \nabla \varphi_1 dx dt + \iint_{Q_T} b(x) |\nabla u|^{q(x)-2} \nabla u \nabla \varphi_1 dx dt \\ & = \iint_{Q_T} (\vec{\zeta} + \vec{\xi}) \cdot \nabla \varphi_1 dx dt. \end{aligned} \tag{3.30}$$

Now, if

$$\varphi_2 \in L^\infty(0, T; W_0^{1,p(x)}(\Omega)) \cap L^\infty(0, T; W_{\text{loc}}^{1,q(x)}(\Omega)),$$

then

$$\varphi_2 \in L^r(0, T; W_0^{1,p(x)}(\Omega)) \cap L^r(0, T; W_{\text{loc}}^{1,q(x)}(\Omega)) \tag{3.31}$$

is true for any given  $r > 1$ . For any given  $t \in (0, T)$ , if we denote by  $\Omega_t$  the compact support set of  $\varphi_2(x, t)$ , for any  $\Omega_{1t}$  satisfying  $\Omega_t \subset\subset \Omega_{1t} \subset\subset \Omega$ , by (3.31), we get

$$\varphi_2 \in L^r(0, T; W_0^{1,p(x)}(\Omega)) \cap L^r(0, T; W_0^{1,q(x)}(\Omega_{1t})). \tag{3.32}$$

Here, we have used the assumption that  $q(x)$  satisfies the logarithmic Hölder continuity condition, then

$$\mathring{W}^{1,q(x)}(\Omega_{1t}) = W_0^{1,q(x)}(\Omega_{1t}).$$

Thus, there is a sequence  $\varphi_{n2}(x, t) \in C_0^\infty(Q_T)$  such that

$$\varphi_{n2}(x, t) \rightarrow \varphi_2(x, t), \quad \text{in } L^r(0, T; W_0^{1,q(x)}(\Omega_{1t})). \tag{3.33}$$

Since  $q(x) \geq p(x)$ , using the  $p(x)$ -Hölder inequality, we know

$$\|\varphi_{n2}(x, t)\|_{L^r(0,T;W_0^{1,p(x)}(\Omega_{1t}))} \leq c.$$

By choosing a subsequence of  $\varphi_{n2}(x, t)$  (we still denote it as  $\varphi_{n2}(x, t)$ ), we may think that  $\varphi_{n2}(x, t)$  satisfies

$$\varphi_{n2}(x, t) \rightarrow \varphi_2(x, t), \quad \text{in } L^r(0, T; W_0^{1,p(x)}(\Omega)). \tag{3.34}$$

Then, by (3.30), we have

$$\begin{aligned} & \iint_{Q_T} |\nabla u|^{p(x)-2} \nabla u \nabla \varphi_{n2} \, dx \, dt + \iint_{Q_T} b(x) |\nabla u|^{q(x)-2} \nabla u \nabla \varphi_{n2} \, dx \, dt \\ &= \iint_{Q_T} (\vec{\zeta} + \vec{\xi}) \cdot \nabla \varphi_{n2} \, dx \, dt. \end{aligned}$$

Letting  $n \rightarrow \infty$ , we get

$$\begin{aligned} & \iint_{Q_T} |\nabla u|^{p(x)-2} \nabla u \nabla \varphi_2 \, dx \, dt + \iint_{Q_T} b(x) |\nabla u|^{q(x)-2} \nabla u \nabla \varphi_2 \, dx \, dt \\ &= \iint_{Q_T} (\vec{\zeta} + \vec{\xi}) \cdot \nabla \varphi_2 \, dx \, dt \end{aligned} \tag{3.35}$$

for any  $\varphi_2 \in L^r(0, T; W_0^{1,p(x)}(\Omega)) \cap L^r(0, T; W_{loc}^{1,q(x)}(\Omega))$ .

As for the initial value, (2.3) can be showed as in [1], the proof of Theorem 2.5 ends.  $\square$

#### 4 The stability of weak solutions

For small  $\eta > 0$ , we define

$$S_\eta(s) = \int_0^s h_\eta(\tau) \, d\tau,$$

where  $h_\eta(s) = \frac{2}{\eta}(1 - \frac{|s|}{\eta})_+$ , and it is clear that

$$\lim_{\eta \rightarrow 0^+} sS'_\eta(s) = \lim_{\eta \rightarrow 0} s h_\eta(s) = 0,$$

$$\lim_{\eta \rightarrow 0^+} S_\eta(s) = \text{sgn}(s),$$

where  $\text{sgn}(s)$  is the sign function.

*Proof of Theorem 2.6* By Definition 2.4, for any

$$\varphi \in L^\infty(0, T; W_0^{1,p(x)}(\Omega)) \cap L^\infty(0, T; W_{loc}^{1,q(x)}(\Omega)),$$

there holds

$$\iint_{Q_t} [u_t \varphi + |\nabla u|^{p(x)-2} \nabla u \nabla \varphi + b(x) |\nabla u|^{q(x)-2} \nabla u \nabla \varphi] dx dt = 0, \tag{4.1}$$

where  $Q_t = \Omega \times (0, t)$ .

Thus, if we choose  $S_\eta(u - v)$  as the test function, then we have

$$\begin{aligned} & \iint_{Q_t} S_\eta(u - v) \frac{\partial(u - v)}{\partial t} dx dt \\ &= - \iint_{Q_t} [|\nabla u|^{p(x)-2} \nabla u - |\nabla v|^{p(x)-2} \nabla v] \nabla(u - v) h_\eta(u - v) dx dt \\ & \quad - \iint_{Q_t} b(x) [|\nabla u|^{q(x)-2} \nabla u - |\nabla v|^{q(x)-2} \nabla v] \nabla(u - v) h_\eta(u - v) dx dt \\ & \leq 0. \end{aligned} \tag{4.2}$$

Since  $u_t, v_t \in L^2(Q_T)$ , we have

$$\lim_{\eta \rightarrow 0^+} \int_\Omega S_\eta(a(u) - a(v)) \frac{\partial(u - v)}{\partial t} dx = \frac{d}{dt} \|u - v\|_{L^1(\Omega)}. \tag{4.3}$$

Let  $\eta \rightarrow 0^+$  in (4.2). Then, by (4.3), we have

$$\int_\Omega |u(x, t) - v(x, t)| dx - \int_\Omega |u_0(x) - v_0(x)| dx = \int_0^t \frac{d}{dt} \|u - v\|_{L^1(\Omega)} dt \leq 0,$$

so

$$\int_\Omega |u(x, t) - v(x, t)| dx \leq c \int_\Omega |u_0(x) - v_0(x)| dx, \quad \forall t \in [0, T].$$

Theorem 2.6 is proved. □

### 5 The partial boundary value condition

*Proof of Theorem 2.7* If  $u(x, t)$  and  $v(x, t)$  are two weak solutions of equation (1.1) with the partial homogeneous boundary value condition

$$u(x, t) = v(x, t) = 0, \quad (x, t) \in \Sigma_1 \times [0, T], \Sigma_1 = \left\{ x \in \partial\Omega : b(x) \left| \frac{\nabla b}{b(x)} \right|^{p(x)} \neq 0 \right\} \tag{5.1}$$

and with the different initial values  $u(x, 0)$  and  $v(x, 0)$  respectively.

For small  $\eta > 0$ , let

$$\Omega_\eta = \{x \in \Omega : b(x) > \eta\}$$

and

$$\phi_\eta(x) = \begin{cases} 1 & \text{if } x \in \Omega_\eta, \\ \frac{b(x)}{\eta} & \text{if } x \in \Omega \setminus \Omega_\eta. \end{cases} \tag{5.2}$$

Then  $\nabla \phi_\eta = \frac{\nabla b(x)}{\eta}$  when  $x \in \Omega \setminus \Omega_\eta$ , and in the other place, it is identically zero.

Choosing  $\phi_\eta S_\eta(u - v)$  as the test function, we have

$$\begin{aligned} & \int_\tau^s \int_\Omega \phi_\eta S_\eta(u - v) \frac{\partial(u - v)}{\partial t} dx dt \\ & + \int_\tau^s \int_\Omega [|\nabla u|^{p(x)-2} \nabla u - |\nabla v|^{p(x)-2} \nabla v] \nabla(u - v) h_\eta(u - v) \phi_\eta(x) dx dt \\ & + \int_\tau^s \int_\Omega b(x) [|\nabla u|^{q(x)-2} \nabla u - |\nabla v|^{q(x)-2} \nabla v] \nabla(u - v) h_\eta(u - v) \phi_\eta(x) dx dt \\ & + \int_\tau^s \int_\Omega [|\nabla u|^{p(x)-2} \nabla u - |\nabla v|^{p(x)-2} \nabla v] \nabla \phi_\eta(x) S_\eta(u - v) dx dt \\ & + \int_\tau^s \int_\Omega b(x) [|\nabla u|^{q(x)-2} \nabla u - |\nabla v|^{q(x)-2} \nabla v] \nabla \phi_\eta(x) S_\eta(u - v) dx dt \\ & = 0. \end{aligned} \tag{5.3}$$

In the first place, following [9, Lemma 3.1] we find

$$\begin{aligned} & \lim_{\eta \rightarrow 0} \int_\tau^s \int_\Omega \phi_\eta(x) S_\eta(u - v) \frac{\partial(u - v)}{\partial t} dx dt \\ & = \int_\Omega |u - v|(x, s) dx - \int_\Omega |u - v|(x, \tau) dx. \end{aligned} \tag{5.4}$$

In the second place, it is easy to see that

$$\int_\tau^s \int_\Omega [|\nabla u|^{p(x)-2} \nabla u - |\nabla v|^{p(x)-2} \nabla v] \nabla(u - v) h_\eta(u - v) \phi_\eta(x) dx dt \geq 0 \tag{5.5}$$

and

$$\int_\tau^s \int_\Omega b(x) [|\nabla u|^{q(x)-2} \nabla u - |\nabla v|^{q(x)-2} \nabla v] \nabla(u - v) h_\eta(u - v) \phi_\eta(x) dx dt \geq 0. \tag{5.6}$$

In the third place, to evaluate the third term on the left-hand side of (5.3), in consideration of (5.2), by a straightforward calculation we obtain

$$\begin{aligned} & \left| \int_\tau^s \int_\Omega [|\nabla u|^{p(x)-2} \nabla u - |\nabla v|^{p(x)-2} \nabla v] \nabla \phi_\eta(x) S_\eta(u - v) dx dt \right| \\ & = \left| \int_\tau^s \int_{\Omega \setminus \Omega_\eta} [|\nabla u|^{p(x)-2} \nabla u - |\nabla v|^{p(x)-2} \nabla v] \nabla \phi_\eta(x) S_\eta(u - v) dx dt \right| \\ & \leq \int_\tau^s \int_{\Omega \setminus \Omega_\eta} (|\nabla u|^{p(x)-1} + |\nabla v|^{p(x)-1}) |S_\eta(u - v) \nabla \phi_\eta| dx dt \end{aligned}$$

$$\begin{aligned}
 &\leq \int_{\tau}^s \int_{\Omega \setminus \Omega_{\eta}} \left[ \frac{b(x)}{\eta} \right]^{\frac{p(x)-1}{p(x)}} (|\nabla u|^{p(x)-1} + |\nabla v|^{p(x)-1}) \left| S_{\eta}(u-v) \frac{(\frac{1}{\eta})^{\frac{1}{p(x)}} \nabla b}{[b(x)]^{\frac{p(x)-1}{p(x)}}} \right| dx dt \\
 &\leq \left( \int_{\tau}^s \int_{\Omega \setminus \Omega_{\eta}} \frac{1}{\eta} b(x) (|\nabla u|^{p(x)} + |\nabla v|^{p(x)}) dx dt \right)^{\frac{1}{q_1}} \\
 &\quad \times \left( \int_{\tau}^s \int_{\Omega \setminus \Omega_{\eta}} \frac{1}{\eta} |S_{\eta}(u-v)|^{p(x)} \frac{|\nabla b(x)|^{p(x)}}{[b(x)]^{p(x)-1}} dx dt \right)^{\frac{1}{p_1}} \\
 &\leq c \left( \int_{\tau}^s \int_{\Omega \setminus \Omega_{\eta}} (|\nabla u|^{p(x)} + |\nabla v|^{p(x)}) dx dt \right)^{\frac{1}{q_1}} \\
 &\quad \times \left( \int_{\tau}^s \frac{1}{\eta} \int_{\Omega \setminus \Omega_{\eta}} |S_{\eta}(u-v)|^{p(x)} \frac{|\nabla b|^{p(x)}}{[b(x)]^{p(x)-1}} dx \right)^{\frac{1}{p_1}},
 \end{aligned} \tag{5.7}$$

where  $p_1 = p^+$  or  $p^-$  according to (iii) of Lemma 2.2,  $q(x) = \frac{p(x)}{p(x)-1}$  and  $q_1 = q^+$  or  $q^-$ .

If we denote  $\Sigma_2 = \partial\Omega \setminus \Sigma_1$  and define

$$\begin{aligned}
 \Omega_{\eta 1} &= \{x \in \Omega \setminus \Omega_{\eta} : \text{dist}(x, \Sigma_2) > \text{dist}(x, \Sigma_1)\}, \\
 \Omega_{\eta 2} &= \{x \in \Omega \setminus \Omega_{\eta} : \text{dist}(x, \Sigma_2) \leq \text{dist}(x, \Sigma_1)\},
 \end{aligned}$$

then

$$\begin{aligned}
 &\frac{1}{\eta} \int_{\Omega \setminus \Omega_{\eta}} |S_{\eta}(u-v)|^{p(x)} \frac{|\nabla b|^{p(x)}}{[b(x)]^{p(x)-1}} dx \\
 &\leq \frac{1}{\eta} \int_{\Omega_{\eta 1}} |S_{\eta}(u-v)|^{p(x)} \frac{|\nabla b|^{p(x)}}{[b(x)]^{p(x)-1}} dx \\
 &\quad + \frac{1}{\eta} \int_{\Omega_{\eta 2}} |S_{\eta}(u-v)|^{p(x)} \frac{|\nabla b|^{p(x)}}{[b(x)]^{p(x)-1}} dx.
 \end{aligned} \tag{5.8}$$

Since

$$u(x, t) = v(x, t) = 0, \quad (x, t) \in \Sigma_1 \times (0, T),$$

we have

$$\begin{aligned}
 &\lim_{\eta \rightarrow 0} \frac{1}{\eta} \int_{\Omega_{\eta 1}} |S_{\eta}(u-v)|^{p(x)} \frac{|\nabla b|^{p(x)}}{[b(x)]^{p(x)-1}} dx \\
 &= \int_{\Sigma_1} |\text{sgn}(u-v)|^{p(x)} \frac{|\nabla b|^{p(x)}}{[b(x)]^{p(x)-1}} d\Sigma \\
 &= 0.
 \end{aligned} \tag{5.9}$$

Moreover, by using the identity

$$\frac{|\nabla b|^{p(x)}}{[b(x)]^{p(x)-1}} = 0, \quad x \in \Sigma_2,$$

we derive that

$$\begin{aligned}
 & \lim_{\eta \rightarrow \infty} \frac{1}{\eta} \int_{\Omega_{\eta^2}} |S_{\eta}(u - v)|^{p(x)} \frac{|\nabla b|^{p(x)}}{[b(x)]^{p(x)-1}} dx \\
 & \leq \lim_{\eta \rightarrow \infty} \frac{1}{\eta} \int_{\Omega_{\eta^2}} \frac{|\nabla b|^{p(x)}}{[b(x)]^{p(x)-1}} dx \\
 & = \int_{\Sigma_2} \frac{|\nabla b|^{p(x)}}{[b(x)]^{p(x)-1}} d\Sigma \\
 & = 0.
 \end{aligned} \tag{5.10}$$

From (5.7)–(5.10), we obtain

$$\lim_{\eta \rightarrow 0} \left| \int_{\tau}^s \int_{\Omega} [|\nabla u|^{p(x)-2} \nabla u - |\nabla v|^{p(x)-2} \nabla v] \nabla \phi_{\eta}(x) S_{\eta}(u - v) dx dt \right| = 0. \tag{5.11}$$

In the fourth place, to evaluate the fourth term on the left-hand side of (5.3), by a direct calculation, we have

$$\begin{aligned}
 & \left| \int_{\tau}^s \int_{\Omega} b(x) [|\nabla u|^{q(x)-2} \nabla u - |\nabla v|^{q(x)-2} \nabla v] \nabla \phi_{\eta}(x) S_{\eta}(u - v) dx dt \right| \\
 & = \left| \int_{\tau}^s \int_{\Omega \setminus \Omega_{\eta}} b(x) [|\nabla u|^{q(x)-2} \nabla u - |\nabla v|^{q(x)-2} \nabla v] \nabla \phi_{\eta}(x) S_{\eta}(u - v) dx dt \right| \\
 & \leq \int_{\tau}^s \frac{1}{\eta} \int_{\Omega \setminus \Omega_{\eta}} b(x) (|\nabla u|^{q(x)-1} + |\nabla v|^{q(x)-1}) |\nabla b S_{\eta}(u - v)| dx dt \\
 & \leq c \int_{\tau}^s \left( \int_{\Omega \setminus \Omega_{\eta}} b(x) (|\nabla u|^{q(x)} + |\nabla v|^{q(x)}) dx \right)^{\frac{1}{q^+}} \\
 & \quad \times \frac{1}{\eta} \left( \int_{\Omega \setminus \Omega_{\eta}} b(x) |\nabla b|^{p(x)} dx \right)^{\frac{1}{p^+}} dt.
 \end{aligned}$$

By (2.8), we have

$$\lim_{\eta \rightarrow 0} \left| \int_{\tau}^s \int_{\Omega \setminus \Omega_{\eta}} b(x) [|\nabla u|^{q(x)-2} \nabla u - |\nabla v|^{q(x)-2} \nabla v] \nabla \phi_{\eta}(x) S_{\eta}(u - v) dx dt \right| = 0. \tag{5.12}$$

By the above discussion, letting  $\eta \rightarrow 0$  in (5.3), we find there is a constant  $l < 1$  such that

$$\int_{\Omega} |u(x, s) - v(x, s)| dx \leq \int_{\Omega} |u(x, \tau) - v(x, \tau)| dx + c \left( \int_{\tau}^s \int_{\Omega} |u - v| dx dt \right)^l. \tag{5.13}$$

Using a generalization of the Gronwall inequality [34], we easily extrapolate that

$$\int_{\Omega} |u(x, s) - v(x, s)| dx \leq c \int_{\Omega} |u(x, \tau) - v(x, \tau)| dx.$$

and by the arbitrariness of  $\tau$ , we have

$$\int_{\Omega} |u(x, s) - v(x, s)| dx \leq c \int_{\Omega} |u_0(x) - v_0(x)| dx. \quad \square$$

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**Authors' contributions**

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