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Research article

A delayed synthetic drug transmission model with two stages of addiction and Holling Type-II functional response

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Abstract: This paper gropes the stability and Hopf bifurcation of a delayed synthetic drug transmission model with two stages of addiction and Holling Type-II functional response. The critical point at which a Hopf bifurcation occurs can be figured out by using the escalating time delay of psychologically addicts as a bifurcation parameter. Directly afterwards, properties of the Hopf bifurcation are explored with aid of the central manifold theorem and normal form theory. Specially, global stability of the model is proved by constructing a suitable Lyapunov function. To underline effectiveness of the obtained results and analyze influence of some influential parameters on dynamics of the model, some numerical simulations are ultimately addressed.

Keywords: delay; Hopf bifurcation; synthetic drugs model; stability; periodic solution
Mathematics Subject Classification: 34C23

1. Introduction

In recent years, as a kind of new infectious disease, sucking drugs has attracted attention of scholars all over the world. According to the World Drug Situation Report (2019), in 2017, about 271 million people abused drugs, accounting for 5.5% of the global population aged 15-64 [1]. It also showed that 0.585 million people died of drug abuse in 2017. In addition, there are also 1.4 million drug users infected with HIV and 5.6 million drug users suffering from hepatitis C. Compared with the traditional drugs such as heroin and cocaine, synthetic drugs which are mainly composed of chemicals, are relatively cheap and easy to obtain. Therefore, synthetic drug abusers are increasing rapidly. According to the report of China Drug Situation Report (2017), among 2.553 million drug abusers, 1.538 million people abused synthetic drugs, accounting for 60.2% [2]. In 2018, the current number of drug addicts in China declined for the first time, accounting for 0.18% of the total population [3]. From the statistical data, it indicates that China has made some achievements in drug abuse control. However, the abuse of synthetic drugs is still spreading, and types and structures of drug abuse have changed. Due to these facts, it is urgent to take extraordinary effective measures to curb and eliminate the spread of synthetic drugs.

In the last decades, social problems such as binge drinking, heavy smoking and drug abuse have been referred to in terms of epidemics [4,5]. There have been some mathematical models developed to explore the dynamics of drinking [6–9], smoking [10–15] and drug abuse [16–22]. Recently, Saha and Samanta proposed synthetic drug transmission model with two stages of addiction and Holling Type-II functional response [23]:

$$\begin{cases} \frac{dS(t)}{dt} &= \Lambda - \delta S(t) - \frac{\beta S(t)(P_1(t)+bP_2(t))}{A+S(t)}, \\ \frac{dP_1(t)}{dt} &= \frac{\beta S(t)(P_1(t)+bP_2(t))}{A+S(t)} - \alpha P_1(t) - (\delta+\gamma)P_1(t), \\ \frac{dP_2(t)}{dt} &= \alpha P_1(t) + \eta T(t) - \rho P_2(t) - \delta P_2(t), \\ \frac{dT(t)}{dt} &= \gamma P_1(t) + \rho P_2(t) - \eta T(t) - \delta T(t), \end{cases} \tag{1.1}$$

where $S(t)$ is standing for the number of susceptible individuals at time t; $P_1(t)$ is denoting the number of psychologically addicts at time t; $P_2(t)$ is indicating the number of physiologically addicts at time t; $T(t)$ is representing the number of addicts under treatment at time t. Λ is the recruitment rate of susceptible individuals; δ is the natural death rate of all the populations; β is the transmission rate of psychologically addicts; $b\beta$ is the transmission rate of psychologically addicts; A is the average number of contacts with others per unit time; α, γ, η and ρ are state transition rates. Obviously, Saha and Samanta [23] assumed that the contact rate of physiologically addicts is multiple of the contact rate of psychologically addicts.

Generally speaking, a susceptible individual is more likely to initiate drug abuse when he contacts with physiological addicts compared to psychologically addicts [22]. In this sense, there should be a delay before psychologically addicts can escalate into physiologically addicts. Meanwhile, it is worthy to note that time delay plays extremely important role on population dynamics. Numerous studies have revealed that time delay can vary the amplitude and cause the occurrence of Hopf bifurcation. Guo et al. [24] considered the effect of the conversion time delay on a predator-prey model with food subsidies and derived the sufficient conditions for local Hopf bifurcations at the positive equilibrium, and analyzed the direction of Hopf bifurcations and stability of the bifurcating periodic solutions. Kundu et al. [25,26] analyzed the existence for different form of the predator-prey system with stage structure by regarding the corresponding time delay as the bifurcation parameter. Huang et al. [27] studied Hopf bifurcation of a delayed fractional predator-prey model. Zheng et al. [28] analyzed Hopf bifurcation for a fractional order delayed paddy ecosystem in the fallow season. Miao and Kang [29] investigated Hopf bifurcation for an HIV infection model with Beddington-DeAngelis incidence by using different combinations of two delays as the bifurcation parameters. Zhang et al. [30] studied Hopf bifurcation for an SVEIR epidemic model with vaccination with multiple delays. Zhang et al. [31] proposed a delayed e-epidemic model for computer virus with graded infection rates and performed

Hopf bifurcation analysis and nonlinear stability analysis. Zhang and Zhao [32] formulated an e-SEIARS model with three delays for point-to-group worm propagation and analyzed Hopf bifurcation by taking different combinations of the three delays as bifurcation parameter. Motivated by the works above, we introduce the escalating time delay of psychologically addicts into system (1.1) and consider the following delayed synthetic drug transmission model with two stages of addiction and Holling Type-II functional response:

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \delta S(t) - \frac{\beta S(t)(P_1(t)+bP_2(t))}{A+S(t)}, \\ \frac{dP_1(t)}{dt} = \frac{\beta S(t)(P_1(t)+bP_2(t))}{A+S(t)} - \alpha P_1(t-\tau) - (\delta+\gamma)P_1(t), \\ \frac{dP_2(t)}{dt} = \alpha P_1(t-\tau) + \eta T(t) - \rho P_2(t) - \delta P_2(t), \\ \frac{dT(t)}{dt} = \gamma P_1(t) + \rho P_2(t) - \eta T(t) - \delta T(t), \end{cases} \tag{1.2}$$

where τ is the escalating time delay of psychologically addicts. The foundation of the current paper is arranged as follows. In Section 2, existence of Hopf bifurcation is analyzed. In Section 3, properties of the Hopf bifurcation are determined. In Section 4, global stability of the model is proved by constructing a suitable Lyapunov function. The effectiveness of our findings is certified by performing some numerical simulations in Section 5. Finally, we end our paper with a conclusion.

2. Existence of Hopf bifurcation

If $\mathbb{R}_0 = \frac{\beta\Lambda(\rho+\delta)+b\alpha\beta\Lambda}{(\rho+\delta)(A\delta+\Lambda)(\alpha+\delta+\gamma)} > 1$ drug addiction equilibrium $E^*(S^*, P_1^*, P_2^*, T^*)$, where

$$\begin{aligned} S^* &= \frac{\Lambda - (\alpha+\delta+\gamma)P_1^*}{\delta}, \\ P_1^* &= \frac{\beta\Lambda\delta(\eta+\delta+\rho) + b\beta\Lambda(\alpha\eta+\alpha\delta+\eta\gamma) - \delta(\alpha+\delta+\gamma)(\eta+\delta+\rho)(A\delta+\Lambda)}{(\alpha+\delta+\gamma)[\beta\delta(\eta+\delta+\rho) + b\beta(\alpha\eta+\alpha\delta+\eta\gamma) - \delta(\alpha+\delta+\gamma)(\eta+\delta+\rho)]}, \\ P_2^* &= \frac{(\alpha\eta+\alpha\delta+\eta\gamma)P_1^*}{\delta(\eta+\delta+\rho)}, \\ T^* &= \frac{\gamma P_1^* + \rho P_2^*}{\eta+\delta}. \end{aligned}$$

The linear section of system (1.2) at $E^*(S^*, P_1^*, P_2^*, T^*)$ is

$$\begin{cases} \frac{dS(t)}{dt} = S_{11}S(t) + S_{12}P_1(t) + S_{13}P_2(t), \\ \frac{dP_1(t)}{dt} = S_{21}S(t) + S_{22}P_1(t) + S_{23}P_2(t) + P_{22}P_1(t-\tau), \\ \frac{dP_2(t)}{dt} = S_{33}P_2(t) + S_{34}T(t) + P_{32}P_1(t-\tau), \\ \frac{dT(t)}{dt} = S_{42}P_1(t) + S_{43}P_2(t) + S_{44}T(t), \end{cases} \tag{2.1}$$

with

$$\begin{aligned} S_{11} &= -\Big(\delta + \frac{\beta A(P_1^*+bP_2^*)}{(A+S^*)^2}\Big), S_{12} = -\frac{\beta S^*}{A+S^*}, S_{13} = -\frac{b\beta S^*}{A+S^*}, \\ S_{21} &= \frac{\beta A(P_1^*+bP_2^*)}{(A+S^*)^2}, S_{22} = \frac{\beta S^*}{A+S^*} - (\delta+\gamma), S_{23} = \frac{b\beta S^*}{A+S^*}, P_{22} = -\alpha, \end{aligned}$$

$$S_{33} = -(\rho+\delta), S_{34} = \eta, P_{32} = \alpha, S_{42} = \gamma, S_{43} = \rho, S_{44} = -(\eta+\delta).$$

The corresponding characteristic equation is

$$\lambda^4 + S_3\lambda^3 + S_2\lambda^2 + S_1\lambda + S_0 + (P_3\lambda^3 + P_2\lambda^2 + P_1\lambda + P_0)e^{-\lambda\tau} = 0, \tag{2.2}$$

where

$$\begin{aligned}
S_0 &= S_{11}S_{34}(S_{23}S_{42} - S_{22}S_{43}) + S_{33}S_{44}(S_{11}S_{22} - S_{12}S_{21}) \\
&\quad + S_{21}S_{34}(S_{12}S_{43} - S_{13}S_{42}), \\
S_1 &= S_{34}S_{43}(S_{11} + S_{22}) + S_{12}S_{21}(S_{33} + S_{44}) - S_{23}S_{34}S_{42} \\
&\quad - S_{11}S_{44}(S_{22} + S_{33}) - S_{22}S_{33}(S_{11} + S_{44}), \\
S_2 &= S_{11}S_{44} + S_{22}S_{33} - S_{12}S_{21} - S_{34}S_{43} + (S_{11} + S_{44})(S_{22} + S_{33}), \\
S_3 &= -(S_{11} + S_{44} + S_{22} + S_{33}), \\
P_0 &= S_{11}S_{44}S_{33}P_{22} + S_{13}S_{21}S_{44}P_{32} - S_{11}S_{44}S_{23}P_{32} - S_{11}S_{34}S_{43}P_{22}, \\
P_1 &= S_{23}P_{32}(S_{11} + S_{44}) - P_{22}(S_{11}S_{44} + S_{11}S_{33} + S_{33}S_{44}) \\
&\quad - S_{13}S_{21}P_{32} + S_{34}S_{43}P_{22}, \\
P_2 &= P_{22}(S_{11} + S_{44} + S_{33}) - S_{23}P_{32}, P_3 = -P_{22}.
\end{aligned}$$

For $\tau = 0$, Eq (2.2) becomes

$$\lambda^4 + S_{03}\lambda^3 + S_{02}\lambda^2 + S_{01}\lambda + S_{00} = 0, \tag{2.3}$$

with

$$S_{00} = S_0 + P_0, S_{01} = S_1 + P_1, S_{02} = S_2 + P_2, S_{03} = S_3 + P_3.$$

Thus, it is straightforward to know that if (T_1): $S_{00} > 0$, $S_{03} > 0$, $S_{02}S_{03} > S_{01}$, $S_{01}S_{02}S_{03} > S_{00}S_{03}^2 + S_{01}^2$ is satisfied, then system (1.2) is locally asymptotically stable.

For $\tau > 0$, $\lambda = i\varpi$ ($\varpi > 0$) is the root of Eq (2.2) if and only if

$$\begin{cases} (P_1\varpi - P_3\varpi^3)\sin\tau\varpi + (P_0 - P_2\varpi^2)\cos\tau\varpi = S_2\varpi^2 - \varpi^4 - S_0, \\ (P_1\varpi - P_3\varpi^3)\cos\tau\varpi - (P_0 - P_2\varpi^2)\sin\tau\varpi = S_3\varpi^3 - S_1\varpi. \end{cases} \tag{2.4}$$

As a result,

$$\varpi^8 + \tilde{S}_3\varpi^6 + \tilde{S}_2\varpi^4 + \tilde{S}_1\varpi^2 + \tilde{S}_0 = 0, \tag{2.5}$$

where

$$\begin{aligned}
\tilde{S}_0 &= S_0^2 - P_0^2, \\
\tilde{S}_1 &= S_1^2 - 2S_0S_2 + 2P_0P_2 - P_1^2, \\
\tilde{S}_2 &= S_2^2 + 2S_0 - 2S_1S_3 - P_2^2 + 2P_1P_3, \\
\tilde{S}_3 &= S_3^2 - 2S_2 - P_3^2.
\end{aligned}$$

Let $\omega = \varpi^2$. Then Eq (2.4) becomes

$$\omega^4 + \tilde{S}_3\omega^3 + \tilde{S}_2\omega^2 + \tilde{S}_1\omega + \tilde{S}_0 = 0. \tag{2.6}$$

Based on the discussion about the distribution of roots of Eq (2.6) in [33], we have the following lemma.

Lemma 1. If (a) $\tilde{S}_0 < 0$; or (b) $\tilde{S}_0 \geq 0$, $\Delta \geq 0$, $\omega_1 > 0$ and $F(\omega_1) < 0$; or (c) if $\tilde{S}_0 \geq 0$, $\Delta < 0$, and there exists $\omega^* \in \{\omega_1, \omega_2, \omega_3\}$, such that $\omega^* > 0$ and $F(\omega^*) \leq 0$, then Eq (2.6) has at leat one positive root, where

$$\begin{aligned}
x_1 &= \frac{\tilde{S}_2}{2} - \frac{3}{16}\tilde{S}_3^2, x_2 = \frac{\tilde{S}_3^3}{32} - \frac{\tilde{S}_2\tilde{S}_3}{8} + \tilde{S}_1, \\
\Delta &= \left(\frac{x_2}{2}\right)^2 + \left(\frac{x_1}{3}\right)^2, \Gamma = \frac{-1 + \sqrt{3}i}{2}, \\
z_1 &= \sqrt[3]{-\frac{x_2}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{x_2}{2} - \sqrt{\Delta}}, \\
z_2 &= \sqrt[3]{-\frac{x_2}{2} + \sqrt{\Delta}\Gamma} + \sqrt[3]{-\frac{x_2}{2} - \sqrt{\Delta}\Gamma^2}, \\
z_3 &= \sqrt[3]{-\frac{x_2}{2} + \sqrt{\Delta}\Gamma^2} + \sqrt[3]{-\frac{x_2}{2} - \sqrt{\Delta}\Gamma}, \\
\omega_i &= z_i - \frac{3\tilde{S}_3}{4}, z = \omega + \frac{3\tilde{S}_3}{4}, \\
F(\omega) &= \omega^4 + \tilde{S}_3\omega^3 + \tilde{S}_2\omega^2 + \tilde{S}_1\omega + \tilde{S}_0.
\end{aligned}$$

In what follows, we suppose that (T_2): Eq (2.6) has four positive roots, which are denoted by ω_1, ω_2, ω_3 and ω_4, respectively. Then, the roots of Eq (2.5) can be denoted as $\varpi_i = \sqrt{\omega_i}$, $i = 1, 2, \cdots, 4$. For ϖ_i, by the aid of Eq (2.4), we have

$$\tau_i^j = \frac{1}{\varpi_i} \times \arccos\left\{\frac{\tilde{P}_1(\varpi_i)}{\tilde{P}_2(\varpi_i)}\right\} + 2j\pi, i = 1, 2, \cdots, 4, j = 0, 1, 2, \cdots. \tag{2.7}$$

where

$$\begin{aligned}
\tilde{P}_1(\varpi_i) &= (P_2 - P_3S_3)\varpi_i^6 + (P_3S_1 - P_0 - P_2S_2)\varpi_i^4 \\
&\quad + (P_0S_2 - P_1S_1 + P_2S_0)\varpi_i^2 - P_0S_0, \\
\tilde{P}_2(\varpi_i) &= P_3^2\varpi_i^6 + (P_2^2 - 2P_1P_3)\varpi_i^4 + (P_1^2 - 2P_0P_2)\varpi_i^2 + P_0^2.
\end{aligned}$$

Define

$$\tau_0 = \min\{\tau_i^0 | i = 1, 2, 3, 4.\} \tag{2.8}$$

Then, when $\tau = \tau_0$, Eq (2.2) has a pair of purely imaginary roots $\pm i\varpi_0$. Differentiating Eq (2.2) with regard to τ, it can be calculated as

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = -\frac{4\lambda^3 + 3S_3\lambda^2 + 2S_2\lambda + S_1}{\lambda(\lambda^4 + S_3\lambda^3 + S_2\lambda^2 + S_1\lambda + S_0)}$$

$$+\frac{3P_3^2+2P_2\lambda+P_1}{\lambda(P_3\lambda^3+P_2\lambda^2+P_1\lambda+P_0)}-\frac{\tau}{\lambda}. \tag{2.9}$$

Replacing $\lambda = i\varpi_0$, then

$$\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}=\frac{F'(\omega_0)}{\tilde{P}_2(\varpi_0)}, \tag{2.10}$$

where $F(\omega)=\omega^4+\tilde{S}_3\omega^3+\tilde{S}_2\omega^2+\tilde{S}_1\omega+\tilde{S}_0$ and $\omega_0=\varpi_0^2$.

To further cast about for the conditions for the occurrence of Hopf bifurcation, the hypothesis (T_3): $F'(\omega_0)\neq 0$ is necessary. In conclusion, the following theorem can be concluded.

Theorem 2.1. *If the conditions (T_1)-(T_3) are satisfied, then system (1.2) is locally asymptotically stable when $\tau_\in[0,\tau_0)$; system (1.2) undergoes a Hopf bifurcation near $\tau=\tau_0$ and a family of periodic solutions bifurcate from the drug addiction equilibrium $E^*(S^*,P_1^*,P_2^*,T_*)$.*

3. Properties of Hopf bifurcation

Let $\tau=\tau_0+\mu$, $\mu\in R$, $u_1(t)=S(\tau t)$, $u_2(t)=P_1(\tau t)$, $u_3(t)=P_2(\tau t)$ and $u_4(t)=T(\tau t)$. Then, system (1.2) becomes

$$\dot{u}(t)=L_\mu(u_t)+F(\mu,u_t), \tag{3.1}$$

where

$$L_\mu\phi=(\tau_0+\mu)\Big(S_{\mathrm{trix}}\phi(0)+P_{\mathrm{trix}}\phi(-1)\Big), \tag{3.2}$$

$$F(\mu,\phi)=(F_1,F_2,0,0)^T, \tag{3.3}$$

$$S_{\mathrm{trix}}=\begin{pmatrix} S_{11} & S_{12} & S_{13} & 0\\ S_{21} & S_{22} & S_{23} & 0\\ 0 & 0 & S_{33} & S_{34}\\ 0 & S_{42} & S_{43} & S_{44}\end{pmatrix}, P_{\mathrm{trix}}=\begin{pmatrix} 0&0&0&0\\ 0&P_{22}&0&0\\ 0&P_{32}&0&0\\ 0&0&0&0\end{pmatrix},$$

and

$$\begin{aligned}
F_1 &= S_{14}\phi_1^2(0)+S_{15}\phi_1(0)\phi_2(0)+S_{16}\phi_1(0)\phi_3(0)+S_{17}\phi_1^3(0)\\
&\quad +S_{18}\phi_1^2(0)\phi_2(0)+S_{19}\phi_1^2(0)\phi_3(0)+\cdots,\\
F_2 &= S_{24}\phi_1^2(0)+S_{25}\phi_1(0)\phi_2(0)+S_{26}\phi_1(0)\phi_3(0)+S_{27}\phi_1^3(0)\\
&\quad +S_{28}\phi_1^2(0)\phi_2(0)+S_{29}\phi_1^2(0)\phi_3(0)+\cdots,
\end{aligned}$$

with

$$\begin{aligned}
S_{14} &= \frac{A\beta(P_1^*+bP_2^*)}{(A+S^*)^3}, S_{15}=\frac{A\beta}{(A+S^*)^2}, S_{16}=\frac{bA\beta}{(A+S^*)^2},\\
S_{17} &= -\frac{A\beta(P_1^*+bP_2^*)}{(A+S^*)^4}, S_{18}=\frac{A\beta}{(A+S^*)^3}, S_{19}=\frac{bA\beta}{(A+S^*)^3},\\
S_{24} &= -\frac{A\beta(P_1^*+bP_2^*)}{(A+S^*)^3}, S_{25}=-\frac{A\beta}{(A+S^*)^2}, S_{26}=-\frac{bA\beta}{(A+S^*)^2},
\end{aligned}$$

$$S_{27} = \frac{A\beta(P_1^* + bP_2^*)}{(A+S^*)^4}, S_{28} = -\frac{A\beta}{(A+S^*)^3}, S_{29} = -\frac{bA\beta}{(A+S^*)^3}.$$

According to Riesz representation theorem, there exists a 4×4 matrix function $\eta(\theta, \mu)$ $(-1 \le \theta \le 0)$, satisfying For $\phi \in C[-1, 0]$, let

$$L_\mu \phi = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta). \tag{3.4}$$

Further

$$\eta(\theta, \mu) = (\tau_0 + \mu)\Big(S_{\text{trix}}\delta(\theta) + P_{\text{trix}}\delta(\theta + 1)\Big). \tag{3.5}$$

Here, $\delta(\theta)$ is the Dirac delta function.
Define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0, \\ \int_{-1}^{0} d\eta(\theta, \varrho)\phi(\theta), & \theta = 0, \end{cases} \tag{3.6}$$

and

$$R(\mu)\phi = \begin{cases} 0, & -1 \le \theta < 0, \\ F(\mu, \phi), & \theta = 0, \end{cases} \tag{3.7}$$

where $\phi \in C([-1, 0], R^4)$. Then system (3.1) is equivalent to

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t. \tag{3.8}$$

For $\varphi \in C^1([0, 1], (R^4)^*)$, define

$$A^*(\varphi) = \begin{cases} -\frac{d\varphi(s)}{ds}, & 0 < s \le 1, \\ \int_{-1}^{0} d\eta^T(s, 0)\varphi(-s), & s = 0, \end{cases} \tag{3.9}$$

and a inner product form

$$\langle \varphi(s), \phi(\theta) \rangle = \bar{\varphi}(0)\phi(0) - \int_{\theta=-1}^{0} \int_{\xi=0}^{\theta} \bar{\varphi}(\xi - \theta) d\eta(\theta)\phi(\xi) d\xi, \tag{3.10}$$

with $\eta(\theta) = \eta(\theta, 0)$. Let $y(\theta) = (1, y_2, y_3, y_4)^T e^{i\tau_0\varpi_0\theta}$ be the eigenvector of $A(0)$ associated with $+i\tau_0\varpi_0$ and $y^*(\theta) = D(1, y_2^*, y_3^*, y_4^*)^T e^{i\tau_0\varpi_0 s}$ be the eigenvector of $A^*(0)$ associated with $-i\tau_0\varpi_0$. Then, through some calculations, we can obtain

$$\begin{aligned} y_2 &= \frac{S_{13}S_{21} - S_{23}(i\varpi_0 - S_{11})}{S_{13}(i\varpi_0 - S_{22} - P_{22}e^{-i\tau_0\varpi_0}) - S_{12}S_{23}}, \\ y_3 &= \frac{i\varpi_0 - S_{11}}{S_{13}} - \frac{S_{12}y_2}{S_{13}}, \\ y_4 &= \frac{(i\varpi_0 - S_{33})y_3 - P_{32}e^{-i\tau_0\varpi_0}y_2}{S_{34}}, \\ y_2^* &= -\frac{i\varpi_0 + S_{11}}{S_{21}}, \end{aligned}$$

$$
\begin{aligned}
y_3^* &= \frac{i\varpi_0 + S_{22} + P_{22}e^{i\tau_0\varpi_0}y_2^* + S_{12}}{S_{34}S_{42} - (i\varpi_0 + S_{44})P_{32}e^{i\tau_0\varpi_0}},\\
y_4^* &= -\frac{S_{34}y_3^*}{i\varpi_0 + S_{44}}.
\end{aligned}
$$

In terms of Eq (3.10), we obtain

$$
\bar{D} = [1 + y_2\bar{y}_2^* + y_3\bar{y}_3^* + y_4\bar{y}_4^* + \tau_0\bar{y}_2^* e^{-i\tau_0\omega_0}(P_{22}y_2 + P_{32}y_3)]^{-1}. \tag{3.11}
$$

Let

$$
z(t) = \langle y^*, u_t\rangle, W(t,\theta) = u_t(\theta) - 2\mathrm{Re}\{z(t)y(\theta)\}, \tag{3.12}
$$

be on the center manifold C_0, and then

$$
W(t,\theta) = W(z(t), \bar{z}(t), \theta), \tag{3.13}
$$

where

$$
W(z(t), \bar{z}(t), \theta) = W(z, \bar{z}) = W_{20}\frac{z^2}{2} + W_{11}z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + \cdots . \tag{3.14}
$$

Then,

$$
\begin{aligned}
\dot{z}(t) &= \langle y^*, \dot{u}_t\rangle = \langle y^*, A(0)u_t\rangle + \langle y^*, R(0)u_t\rangle\\
&= \langle A^*(0)y^*, u_t\rangle + \langle y^*, R(0)u_t\rangle\\
&= \langle A^*(0)y^*, u_t\rangle + \langle y^* + \bar{y}^*(0)R(0)u_t - \int_{-1}^{0}\int_{0}^{\theta}\bar{y}^*(\xi-\theta)d\eta(\theta)A(0)R(0)u_t(\xi)d\xi\\
&= i\varpi_0\tau_0 z(t) + \bar{q}_0^* f(0, u_t(\theta))\\
&:= i\varpi_0\tau_0 z(t) + \bar{q}_0^* f_0(z(t), \bar{z}(t)).
\end{aligned} \tag{3.15}
$$

$$
\dot{z}(t) = i\varpi_0\tau_0 z(t) + g(z, \bar{z}), \tag{3.16}
$$

where

$$
g(z, \bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots . \tag{3.17}
$$

Thus,

$$
g(z, \bar{z}) = \bar{q}^*(0)f_0(z, \bar{z}) = \bar{D}(1, \bar{y}_2^*, \bar{y}_3^*, \bar{y}_4^*)(f_1(0, u_t), f_2(0, u_t), 0, 0)^T, \tag{3.18}
$$

with

$$
\begin{aligned}
f_1(0, u_t) &= \tau_0[S_{14}\phi_1^2(0) + S_{15}\phi_1(0)\phi_2(0) + S_{16}\phi_1(0)\phi_3(0) + S_{17}\phi_1^3(0)\\
&\quad + S_{18}\phi_1^2(0)\phi_2(0) + S_{19}\phi_1^2(0)\phi_3(0) + \cdots],\\
f_2(0, u_t) &= \tau_0[S_{24}\phi_1^2(0) + S_{25}\phi_1(0)\phi_2(0) + S_{26}\phi_1(0)\phi_3(0) + S_{27}\phi_1^3(0)\\
&\quad + S_{28}\phi_1^2(0)\phi_2(0) + S_{29}\phi_1^2(0)\phi_3(0) + \cdots],
\end{aligned}
$$

According to the above discussion, we know that

$$
\begin{aligned}
u_t &= u(t+\theta) = W(z, \bar{z}, \theta) + zy(\theta) + \bar{z}y(\theta),\\
y(\theta) &= (1, y_2, y_3, y_4)^T e^{i\varpi_0\tau_0\theta},
\end{aligned}
$$

we can obtain

$$u_t = \begin{bmatrix} u_1(t+\theta) \\ u_2(t+\theta) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} W^{(1)}(t+\theta) \\ W^{(2)}(t+\theta) \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} e^{i\varpi_0\tau_0\theta} + \bar{z} \begin{bmatrix} 1 \\ \bar{y}_2 \\ \bar{y}_3 \\ \bar{y}_4 \end{bmatrix} e^{-i\varpi_0\tau_0\theta}, \tag{3.19}$$

and

$$\begin{aligned} \phi_1(0) &= z + \bar{z} + W_{20}^{(1)}\frac{z^2}{2} + W_{11}^{(1)} z\bar{z} + W_{02}^{(1)}\frac{\bar{z}^2}{2} + \cdots, \\ \phi_2(0) &= zq_2 + \bar{z}\bar{y}_2 + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)} z\bar{z} + W_{02}^{(2)}\frac{\bar{z}}{2} + \cdots, \\ \phi_3(0) &= zq_3 + \bar{z}\bar{y}_3 + W_{20}^{(3)}(0)\frac{z^2}{2} + W_{11}^{(3)} z\bar{z} + W_{02}^{(3)}\frac{\bar{z}}{2} + \cdots, \end{aligned}$$

From Eq (3.17) and Eq (3.18), we have

$$g(z,\bar{z}) = \bar{D}(1, \bar{y}_2^*, \bar{y}_3^*, \bar{y}_4^*) \begin{bmatrix} N_{11}z^2 + N_{12}z\bar{z} + N_{13}\bar{z}^2 + N_{14}z^2\bar{z} \\ N_{21}z^2 + N_{22}z\bar{z} + N_{23}\bar{z}^2 + N_{24}z^2\bar{z} \\ 0 \\ 0 \end{bmatrix} + \cdots, \tag{3.20}$$

where

$$\begin{aligned} N_{11} &= \tau_0(S_{14} + S_{15}y_2 + S_{16}y_3), \\ N_{21} &= \tau_0(S_{24} + S_{25}y_2 + S_{26}y_3), \\ N_{12} &= \tau_0[2S_{14} + S_{15}(y_2 + \bar{y}_2) + S_{16}(y_3 + \bar{y}_3)], \\ N_{22} &= \tau_0[2S_{24} + S_{25}(y_2 + \bar{y}_2) + S_{26}(y_3 + \bar{y}_3)], \\ N_{13} &= \tau_0(S_{14} + S_{15}\bar{y}_2 + S_{16}\bar{y}_3), \\ N_{23} &= \tau_0(S_{24} + S_{25}\bar{y}_2 + S_{26}\bar{y}_3), \end{aligned}$$

$$\begin{aligned} N_{14} &= \tau_0[S_{14}(2W_{11}^{(1)}(0) + w_{20}^{(1)}(0)) \\ &\quad + S_{15}\Big(W_{11}^{(1)}(0)y_2 + \frac{1}{2}W_{11}^{(1)}(0)\bar{y}_2 + W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0)\Big) \\ &\quad + S_{16}\Big(W_{11}^{(1)}(0)y_3 + \frac{1}{2}W_{11}^{(1)}(0)\bar{y}_3 + W_{11}^{(3)}(0) + \frac{1}{2}W_{20}^{(3)}(0)\Big) \\ &\quad + 3S_{17} + S_{18}(\bar{y}_2 + 2y_2) + S_{19}(\bar{y}_3 + 2y_3)], \\ N_{24} &= \tau_0\Big(S_{24}(2W_{11}^{(1)}(0) + w_{20}^{(1)}(0)) \\ &\quad + S_{25}\Big(W_{11}^{(1)}(0)y_2 + \frac{1}{2}W_{11}^{(1)}(0)\bar{y}_2 + W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0)\Big) \\ &\quad + S_{26}\Big(W_{11}^{(1)}(0)y_3 + \frac{1}{2}W_{11}^{(1)}(0)\bar{y}_3 + W_{11}^{(3)}(0) + \frac{1}{2}W_{20}^{(3)}(0)\Big) \\ &\quad + 3S_{27} + S_{28}(\bar{y}_2 + 2y_2) + S_{29}(\bar{y}_3 + 2y_3)\Big). \end{aligned}$$

Comparing the coefficients in Eq (3.20) with those in Eq (3.17), one can obtain

$$\begin{aligned}
g_{20} &= 2\bar{D}\tau_0[S_{14} + S_{15}y_2 + S_{16}y_3 + \bar{y}_2^*(S_{24} + S_{25}y_2 + S_{26}y_3)],\\
g_{11} &= \bar{D}\tau_0[2S_{14} + S_{15}(y_2 + \bar{y}_2) + S_{16}(y_3 + \bar{y}_3)\\
&\quad +\bar{y}_2^*(2S_{24} + S_{25}(y_2 + \bar{y}_2) + S_{26}(y_3 + \bar{y}_3))],\\
g_{02} &= 2\bar{D}\tau_0[S_{14} + S_{15}\bar{y}_2 + S_{16}\bar{y}_3 + \bar{y}_2^*(S_{24} + S_{25}\bar{y}_2 + S_{26}\bar{y}_3)],\\
g_{21} &= 2\bar{D}\tau_0\Big[S_{14}(2W_{11}^{(1)}(0) + w_{20}^{(1)}(0))\\
&\quad +S_{15}\Big(W_{11}^{(1)}(0)y_2 + \frac{1}{2}W_{11}^{(1)}(0)\bar{y}_2 + W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0)\Big)\\
&\quad +S_{16}\Big(W_{11}^{(1)}(0)y_3 + \frac{1}{2}W_{11}^{(1)}(0)\bar{y}_3 + W_{11}^{(3)}(0) + \frac{1}{2}W_{20}^{(3)}(0)\Big)\\
&\quad +3S_{17} + S_{18}(\bar{y}_2 + 2y_2) + S_{19}(\bar{y}_3 + 2y_3)\\
&\quad +\bar{y}_2^*\Big(S_{24}(2W_{11}^{(1)}(0) + w_{20}^{(1)}(0))\\
&\quad +S_{25}\Big(W_{11}^{(1)}(0)y_2 + \frac{1}{2}W_{11}^{(1)}(0)\bar{y}_2 + W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0)\Big)\\
&\quad +S_{26}\Big(W_{11}^{(1)}(0)y_3 + \frac{1}{2}W_{11}^{(1)}(0)\bar{y}_3 + W_{11}^{(3)}(0) + \frac{1}{2}W_{20}^{(3)}(0)\Big)\\
&\quad +3S_{27} + S_{28}(\bar{y}_2 + 2y_2) + S_{29}(\bar{y}_3 + 2y_3)\Big)\Big],
\end{aligned}$$

with

$$W_{20}(\theta) = \frac{ig_{20}y(0)}{\tau_0\omega_0}e^{i\tau_0\varpi_0\theta} + \frac{i\bar{g}_{02}\bar{g}(0)}{3\tau_0\varpi_0}e^{-i\tau_0\varpi_0\theta} + E_1e^{2i\tau_0\varpi_0\theta}, \tag{3.21}$$

$$W_{11}(\theta) = -\frac{ig_{11}g(0)}{\tau_0\omega_0}e^{i\tau_0\varpi_0\theta} + \frac{i\bar{g}_{11}\bar{g}(0)}{\tau_0\varpi_0}e^{-i\tau_0\varpi_0\theta} + E_2. \tag{3.22}$$

where

$$E_1 = 2\begin{pmatrix} 2i\varpi_0 - S_{11} & -S_{12} & -S_{13} & 0\\ -S_{21} & 2i\varpi_0 - S_{22} - e^{-2i\tau_0\varpi_0} & -S_{23} & 0\\ 0 & -S_{32}e^{-2i\tau_0\varpi_0} & 2i\varpi_0 - S_{33} & -S_{34}\\ 0 & -S_{42} & -S_{43} & 2i\varpi_0 - S_{44}\end{pmatrix}^{-1} \times \begin{pmatrix} E_1^{(1)}\\ E_1^{(2)}\\ 0\\ 0\end{pmatrix},$$

$$E_2 = \begin{pmatrix} S_{11} & S_{12} & S_{13} & 0\\ S_{21} & S_{22} + P_{22} & S_{23} & 0\\ 0 & P_{32} & S_{33} & S_{34}\\ 0 & S_{42} & S_{43} & S_{44}\end{pmatrix}^{-1} \times \begin{pmatrix} E_2^{(1)}\\ E_2^{(2)}\\ 0\\ 0\end{pmatrix},$$

with

$$\begin{aligned}
E_1^{(1)} &= S_{14} + S_{15}y_2 + S_{16}y_3,\\
E_1^{(2)} &= S_{24} + S_{25}y_2 + S_{26}y_3,\\
E_2^{(1)} &= 2S_{14} + S_{15}(y_2 + \bar{y}_2) + S_{16}(y_3 + \bar{y}_3),
\end{aligned}$$

$$E_2^{(2)} = 2S_{24} + S_{25}(y_2 + \bar{y}_2) + S_{26}(y_3 + \bar{y}_3).$$

Thus,

$$\begin{aligned}
&C_1(0) = \tfrac{i}{2\tau_0\omega_0}\Big(g_{11}g_{20} - 2|g_{11}|^2 - \tfrac{|g_{02}|^2}{3}\Big) + \tfrac{g_{21}}{2}\\
&\mu_2 = -\tfrac{\mathrm{Re}\{C_1(0)\}}{\mathrm{Re}\{\lambda'(\tau_0)\}},\\
&\beta_2 = 2\mathrm{Re}\{C_1(0)\},\\
&T_2 = -\tfrac{\mathrm{Im}\{C_1(0)\}+\mu_2\mathrm{Im}\{\lambda'(\tau_0)\}}{\tau_0\omega_0},
\end{aligned} \tag{3.23}$$

According to the results about the direction and stability of Hopf bifurcation for system (1.2) in [34], we have the following theorem.

Theorem 3.1. *The Hopf bifurcation is supercritical (subcritical) when $\mu_2 > 0$ ($\mu_2 < 0$); The bifurcating periodic solutions are stable (unstable) when $\beta_2 < 0$ ($\beta_2 > 0$); The period of the bifurcating periodic solutions increase (decrease) when $T_2 > 0$ ($T_2 < 0$).*

4. Global stability criteria

Theorem 4.1. *If $-h = max\{h_1, h_2, h_3, h_4\} < 0$, with*

$$\begin{aligned}
h_1 &= \Big[\frac{1}{n_1}S_{11} + \frac{1}{n_2}S_{21} + \frac{P_{32}S_{21}N_2\tau}{n_2n_3}\Big] < 0,\\
h_2 &= \Big[\frac{1}{n_1}S_{12} + \frac{1}{n_3}P_{32} + \frac{1}{n_4}S_{42} + \frac{\big(S_{22}+P_{22}\big)}{n_2} + \frac{P_{32}\big(P_{22}+S_{22}\big)N_2\tau}{n_2n_3}\Big] < 0,\\
h_3 &= \Big[\frac{1}{n_1}S_{13} + \frac{1}{n_2}S_{23} + \frac{1}{n_3}S_{33} + \frac{1}{n_4}S_{43} + \frac{S_{32}P_{32}N_2\tau}{n_2n_3}\Big] < 0,\\
h_4 &= \Big[\frac{1}{n_3}S_{34} + \frac{1}{n_4}S_{44}\Big] < 0,
\end{aligned}$$

where $n_1 < S(t) < N_1$, $n_2 < P_1(t) < N_2$, $n_3 < P_2(t) < N_3$ and $n_4 < T(t) < N_4$ for $t > 0$, the equilibrium E^* of linear system (2.1) is global asymptotically stable.

Proof. Let $S(t) - S^* = S^*\big(e^{q(t)} - 1\big)$, $P_1(t) - P_1^* = P_1^*\big(e^{u(t)} - 1\big)$, $P_2(t) - P_2^* = P_2^*\big(e^{v(t)} - 1\big)$ and $T(t) - T^* = T^*\big(e^{w(t)} - 1\big)$. Then, $E^*(S^*, P_1^*, P_2^*, T^*)$ becomes the trivial equilibrium for $q(t) = u(t) = v(t) = w(t) = 0$ for all $t > 0$, and system (2.1) can be reduced in the following form:

$$\frac{dq(t)}{dt} = \frac{1}{S}S_{11}S^*\big(e^{q(t)} - 1\big) + \frac{1}{S}S_{12}P_1^*\big(e^{u(t)} - 1\big) + \frac{1}{S}S_{13}P_2^*\big(e^{v(t)} - 1\big), \tag{4.1}$$

$$\frac{du(t)}{dt} = \frac{1}{P_1}S_{21}S^*\big(e^{q(t)} - 1\big) + \frac{1}{P_1}S_{22}P_1^*\big(e^{u(t)} - 1\big) + \frac{1}{P_1}S_{23}P_2^*\big(e^{v(t)} - 1\big) \tag{4.2}$$

$$+\frac{1}{P_1}P_{22}P_1^*\big(e^{u(t-\tau)} - 1\big), \tag{4.3}$$

$$\frac{dv(t)}{dt} = \frac{1}{P_2}S_{33}P_2^*\big(e^{v(t)} - 1\big) + \frac{1}{P_2}S_{34}T^*\big(e^{w(t)} - 1\big) + \frac{1}{P_2}P_{32}P_1^*\big(e^{u(t-\tau)} - 1\big), \tag{4.4}$$

$$\frac{dw(t)}{dt} = \frac{1}{T}S_{42}P_1^*\big(e^{u(t)} - 1\big) + \frac{1}{T}S_{43}P_2^*\big(e^{v(t)} - 1\big) + \frac{1}{T}S_{44}T^*\big(e^{w(t)} - 1\big). \tag{4.5}$$

Let $V_1(t) = |q(t)|$, then

$$D^+V_1(t) \le \frac{1}{n_1}S_{11}S^*\Big|e^{q(t)}-1\Big| + \frac{1}{n_1}S_{12}P_1^*\Big|e^{u(t)}-1\Big| + \frac{1}{n_1}S_{13}P_2^*\Big|e^{v(t)}-1\Big|. \tag{4.6}$$

Let $\tilde{V}_2(t) = |u(t)|$, then

$$\begin{aligned} D^+\tilde{V}_2(t) &\le \frac{1}{n_2}S_{21}S^*\Big|e^{q(t)}-1\Big| + \frac{1}{n_2}S_{22}P_1^*\Big|e^{u(t)}-1\Big| + \frac{1}{n_2}S_{23}P_2^*\Big|e^{v(t)}-1\Big| && (4.7)\\ &\quad +\frac{1}{n_3}P_{22}P_1^*\Big|e^{u(t-\tau)}-1\Big|. && (4.8)\end{aligned}$$

Again, due to the form of (4.8), we consider the following functional

$$V_2(t) \le \tilde{V}_2(t) + \frac{P_{22}P_1^*}{n_2}\int_{t-\tau}^{t}|e^{u(m)}-1|\,\mathrm{d}\,m$$

whose time derivative along trajectories of system (4.3) is given by

$$\begin{aligned} D^+V_2(t) &\le D^+\tilde{V}_2(t) + \frac{P_{22}P_1^*}{n_2}\Big[\Big|e^{u(t)}-1\Big| - \Big|e^{u(t-\tau)}-1\Big|\Big] && (4.9)\\ &\le \frac{1}{n_2}S_{21}S^*\Big|e^{q(t)}-1\Big| + \frac{1}{n_2}\Big(S_{22}+P_{22}\Big)P_1^*\Big|e^{u(t)}-1\Big| + \frac{1}{n_2}S_{23}P_2^*\Big|e^{v(t)}-1\Big|. && (4.10)\end{aligned}$$

Now, Eq (4.4) can be rewritten as

$$\begin{aligned} \frac{dv(t)}{dt} &= \frac{1}{P_2}S_{33}P_2^*\Big(e^{v(t)}-1\Big) + \frac{1}{P_2}S_{34}T^*\Big(e^{w(t)}-1\Big) + \frac{1}{P_2}P_{32}P_1^*\Big(e^{u(t)}-1-\int_{t-\tau}^{t}e^{u(m)}\frac{du}{dm}dm\Big) && (4.11)\\ &= \frac{1}{P_2}S_{33}P_2^*\Big(e^{v(t)}-1\Big) + \frac{1}{P_2}S_{34}T^*\Big(e^{w(t)}-1\Big) + \frac{1}{P_2}P_{32}P_1^*\Big(e^{u(t)}-1\Big) && (4.12)\\ &\quad -\frac{1}{P_2}P_{32}P_1^*\int_{t-\tau}^{t}e^{u(m)}\Big[\frac{1}{P_1}S_{21}S^*\Big(e^{q(m)}-1\Big) + \frac{1}{P_1}S_{22}P_1^*\Big(e^{u(m)}-1\Big) && (4.13)\\ &\quad +\frac{1}{P_1}S_{23}P_2^*\Big(e^{v(m)}-1\Big) + \frac{1}{P_1}P_{22}P_1^*\Big(e^{u(m-\tau)}-1\Big)\Big]dm. && (4.14)\end{aligned}$$

Let $\tilde{V}_3(t) = |v(t)|$, then

$$\begin{aligned} D^+\tilde{V}_3(t) &\le \frac{1}{n_3}S_{33}P_2^*\Big|e^{v(t)}-1\Big| + \frac{1}{n_3}S_{34}T^*\Big|e^{w(t)}-1\Big| + \frac{1}{n_3}P_{32}P_1^*\Big|e^{u(t)}-1\Big|\\ &\quad +\frac{1}{n_3}P_{32}P_1^*\int_{t-\tau}^{t}e^{u(m)}\Big[\frac{1}{n_2}S_{21}S^*\Big|e^{q(m)}-1\Big| + \frac{1}{n_2}S_{22}P_1^*\Big|e^{u(m)}-1\Big|\\ &\quad +\frac{1}{n_2}S_{23}P_2^*\Big|e^{v(m)}-1\Big| + \frac{1}{n_2}P_{22}P_1^*\Big|e^{u(m-\tau)}-1\Big|\Big]dm\end{aligned}$$

We find that there exists $t_1 > 0$ such that $P_1^*e^{u(t)} < N_2$ for $t > t_1 + \tau$, we have

$$D^+\tilde{V}_3(t) \le \frac{1}{n_3}S_{33}P_2^*\Big|e^{v(t)}-1\Big| + \frac{1}{n_3}S_{34}T^*\Big|e^{w(t)}-1\Big| + \frac{1}{n_3}P_{32}P_1^*\Big|e^{u(t)}-1\Big|$$

$$+\frac{1}{n_3}P_{32}N_2\int_{t-\tau}^{t}\Big[\frac{1}{n_2}S_{21}S^*\Big|e^{q(m)}-1\Big|+\frac{1}{n_2}S_{22}P_1^*\Big|e^{u(m)}-1\Big|$$
$$+\frac{1}{n_2}S_{23}P_2^*\Big|e^{v(m)}-1\Big|+\frac{1}{n_2}P_{22}P_1^*\Big|e^{u(m-\tau)}-1\Big|\Big]dm. \tag{4.15}$$

Again, due to the form of Eq (4.15), we consider the following functional

$$\begin{aligned}V_3(t) \ \leq\ & \tilde{V}_3(t)+\frac{1}{n_3}P_{32}N_2\int_{t-\tau}^{t}\int_{z}^{t}\Big[\frac{1}{n_2}S_{21}S^*\Big|e^{q(m)}-1\Big|+\frac{1}{n_2}S_{22}P_1^*\Big|e^{u(m)}-1\Big| \\ & +\frac{1}{n_2}S_{23}P_2^*\Big|e^{v(m)}-1\Big|+\frac{1}{n_2}P_{22}P_1^*\Big|e^{u(m-\tau)}-1\Big|\Big]dmdz \\ & +\frac{1}{n_2n_3}P_{22}P_{32}N_2P_1^*\tau\int_{t-\tau}^{t}\Big|e^{u(m)}-1\Big|dm\end{aligned}$$

whose time derivative along trajectories of system (4.4) is given by

$$\begin{aligned}D^+V_3(t) \ \leq\ & D^+\tilde{V}_3(t)+\frac{1}{n_3}P_{32}N_2\tau\Big[\frac{1}{n_2}S_{21}S^*\Big|e^{q(t)}-1\Big|+\frac{1}{n_2}S_{22}P_1^*\Big|e^{u(t)}-1\Big| \\ & +\frac{1}{n_2}S_{23}P_2^*\Big|e^{v(t)}-1\Big|\Big]+\frac{1}{n_2n_3}P_{22}P_{32}N_2P_1^*\tau\Big|e^{u(t)}-1\Big| \\ & -\frac{1}{n_3}P_{32}N_2\int_{t-\tau}^{t}\Big[\frac{1}{n_2}S_{21}S^*\Big|e^{q(m)}-1\Big|+\frac{1}{n_2}S_{22}P_1^*\Big|e^{u(m)}-1\Big| \\ & +\frac{1}{n_2}S_{23}P_2^*\Big|e^{v(m)}-1\Big|+\frac{1}{n_2}P_{22}P_1^*\Big|e^{u(m-\tau)}-1\Big|\Big]dm \\ \leq\ & \frac{1}{n_2n_3}P_{32}S_{21}N_2\tau S^*\Big|e^{q(t)}-1\Big|+\Big[\frac{1}{n_3}P_{32}+\frac{1}{n_2n_3}P_{32}N_2\Big(P_{22}+S_{22}\Big)\tau\Big]P_1^*\Big|e^{u(t)}-1\Big| \\ & +\Big[\frac{1}{n_3}S_{33}+\frac{1}{n_2n_3}P_{32}S_{23}N_2\tau\Big]P_2^*\Big|e^{v(t)}-1\Big|+\frac{1}{n_3}S_{34}T^*\Big|e^{w(t)}-1\Big|.\end{aligned}$$

Let $V_4(t)=|w(t)|$, then

$$D^+V_4(t) \ \leq\ \frac{1}{n_4}S_{42}P_1^*\Big|e^{u(t)}-1\Big|+\frac{1}{n_4}S_{43}P_2^*\Big|e^{v(t)}-1\Big|+\frac{1}{n_4}S_{44}T^*\Big|e^{w(t)}-1\Big|.$$

Let us define functional $V(t)=V_1(t)+V_2(t)+V_3(t)+V_4(t)$. Then

$$\begin{aligned}D^+V(t) \ =\ & D^+V_1(t)+D^+V_2(t)+D^+V_3(t)+D^+V_4(t) \\ \leq\ & h_1S^*|e^{q(t)}-1|+h_2P_1^*|e^{u(t)}-1|+h_3P_2^*|e^{v(t)}-1|+h_4T^*|e^{w(t)}\end{aligned}$$

where $h_1,\ h_2,\ h_3,h_4,h_5$ are already defined in hypothesis of Theorem 3.

Let $-h=\max\{h_1,h_2,h_3,h_4\}<0$, then (2.1) becomes

$$D^+V(t) \ \leq\ -h\Big[S^*|e^{q(t)}-1|+P_1^*|e^{u(t)}-1|+P_2^*|e^{v(t)}-1|+T^*|e^{w(t)}-1|\Big]$$

Since, the model (2.1) is positive invariant for all $t>t_1^*$, we have

$$S(t) \ =\ S^*e^{q(t)}>\underline{S},$$

$$\begin{aligned} P_1(t) &= P_1^* e^{u(t)} > \underline{P}_1, \\ P_2(t) &= P_2^* e^{v(t)} > \underline{P}_2, \\ T(t) &= T^* e^{w(t)} > \underline{T}. \end{aligned}$$

According to mean value theorem, we have $S^*|e^{q(t)} - 1| = S^* e^{\omega_1(t)}|q(t)| > n_1|q(t)|$, $P_1^*|e^{u(t)} - 1| = P_1^* e^{\omega_2(t)}|u(t)| > n_2|u(t)|$, $P_2^*|e^{v(t)} - 1| = P_2^* e^{\omega_3(t)}|v(t)| > n_3|v(t)|$ and $T^*|e^{w(t)} - 1| = T^* e^{\omega_4(t)}|w(t)| > n_4|w(t)|$, where $S^* e^{\omega_1(t)}$ lies between S^* and $S(t)$, $P_1^* e^{\omega_2(t)}$ lies between P_1^* and $P_1(t)$, $P_2^* e^{\omega_3(t)}$ lies between P_2^* and $P_2(t)$, $T^* e^{\omega_4(t)}$ lies between T^* and $T(t)$. Therefore,

$$D^+ V(t) \leq -\tilde{h}\Big(|q(t)| + |u(t)| + |v(t)| + |w(t)|\Big)$$

where $\tilde{h} = \min\{h\underline{S},\ h\underline{P}_1,\ h\underline{P}_2,\ h\underline{T}\}$. Hence, by using Lyapunov stability theory, the equilibrium E^* of linear system (2.1) is globally asymptotically stable. Hence, the proof is completed.

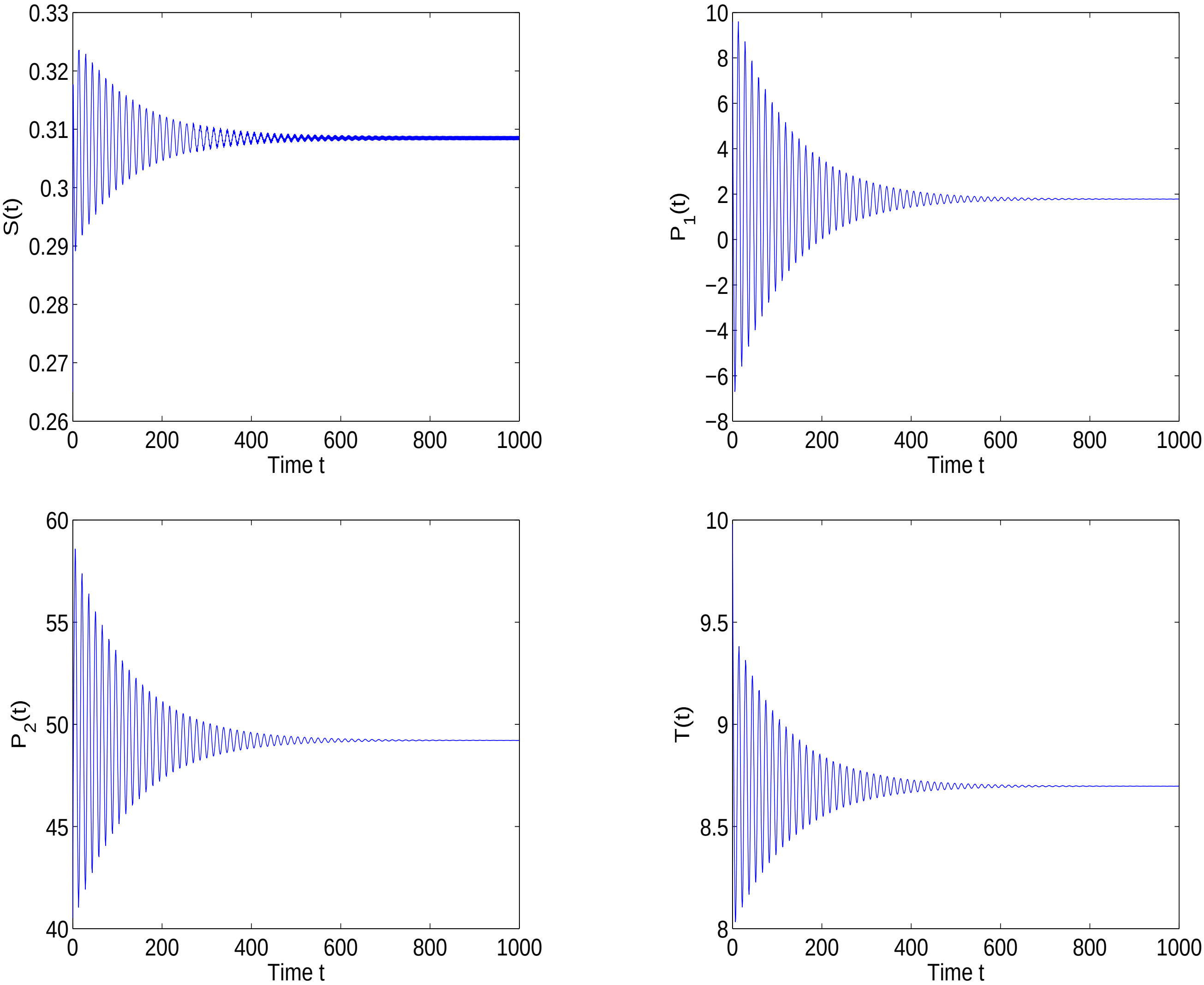

Figure 1. Waveform plots of system (5.1) with $\tau = 4.9064 \in [0, \tau_0)$.

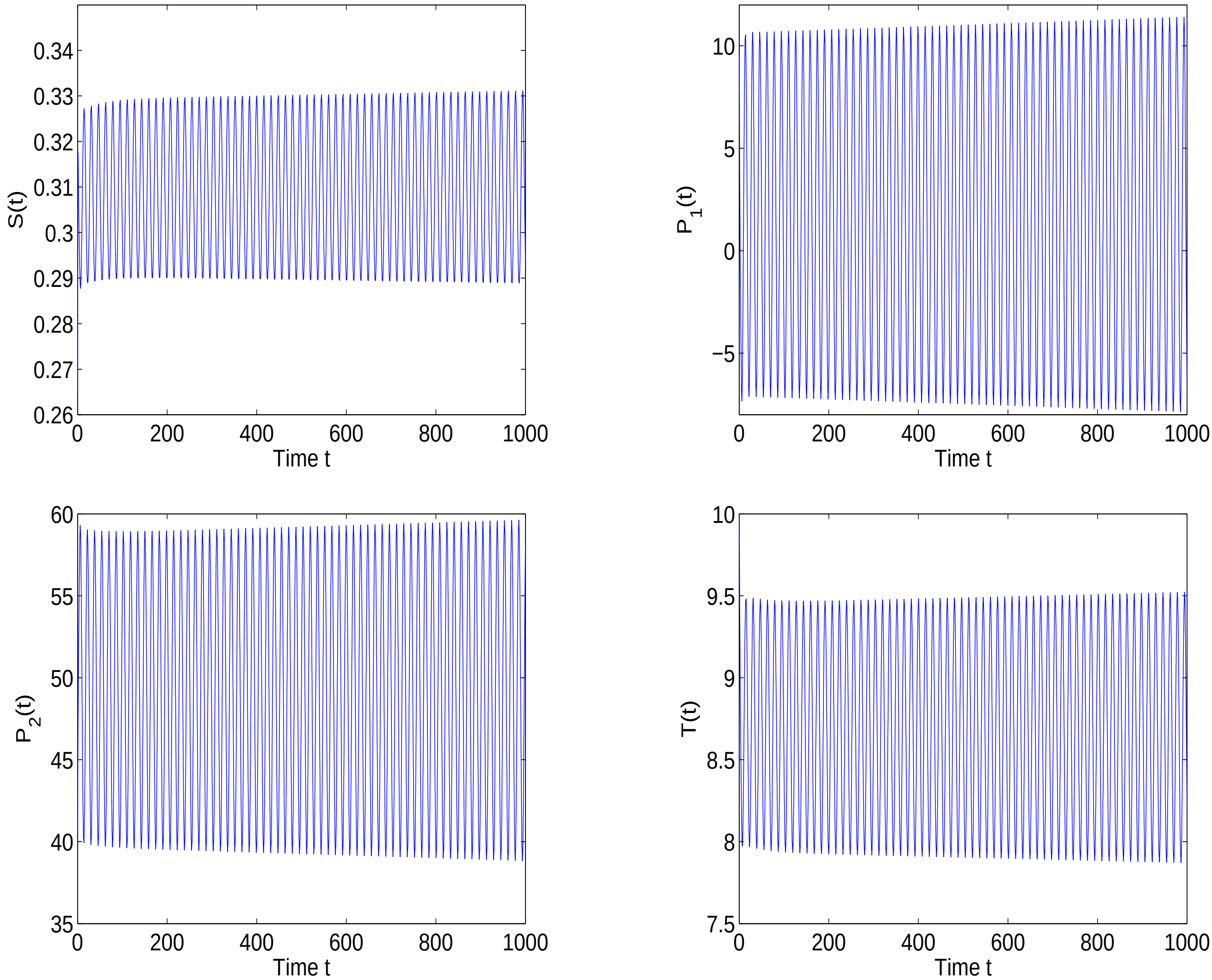

Figure 2. Waveform plots of system (5.1) with $\tau = 5.3202 > \tau_0$.

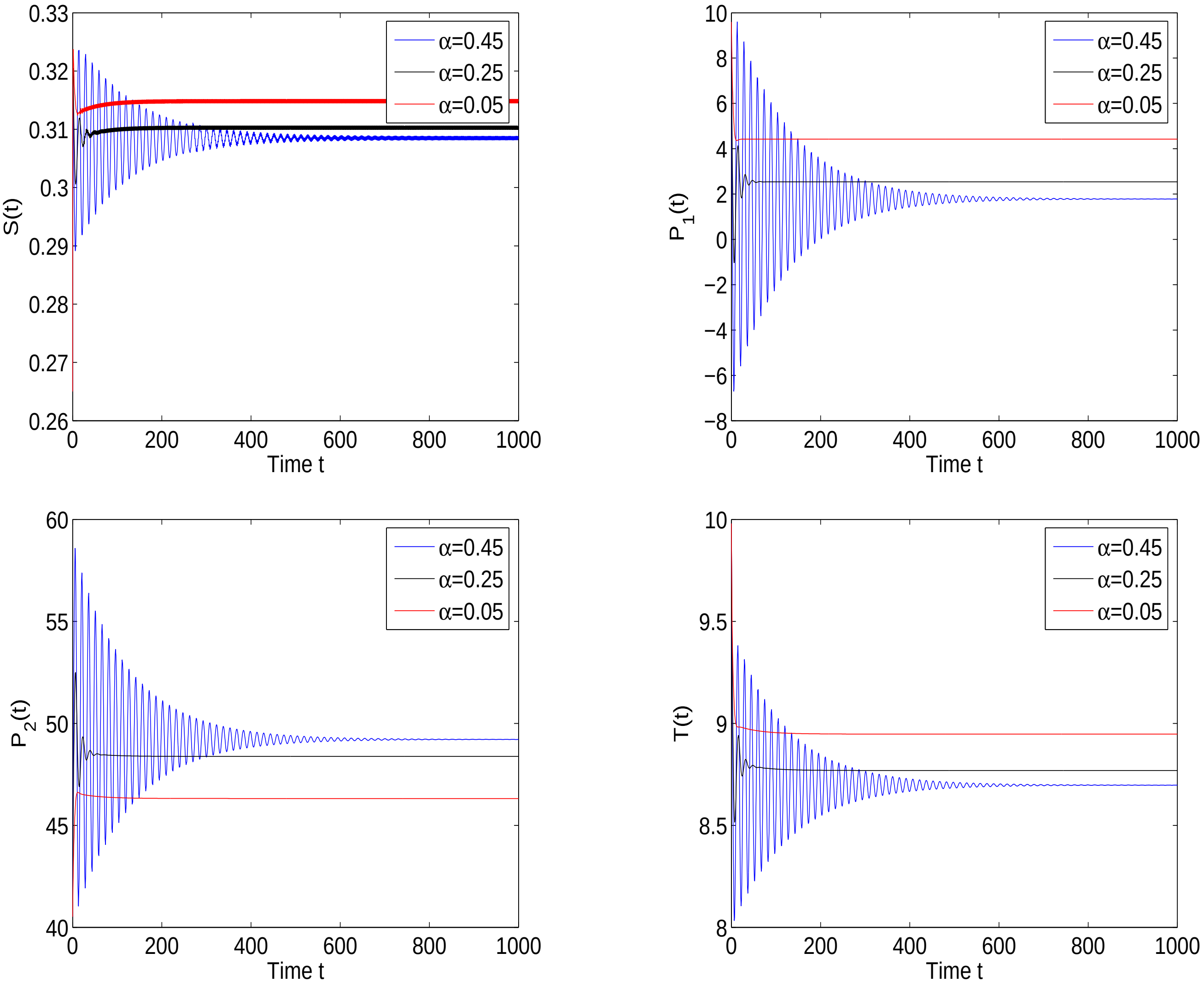

Figure 3. Waveform plots of system (5.1) for different α at $\tau = 4.9064$. Rest of the parameters are taken as given in the text.

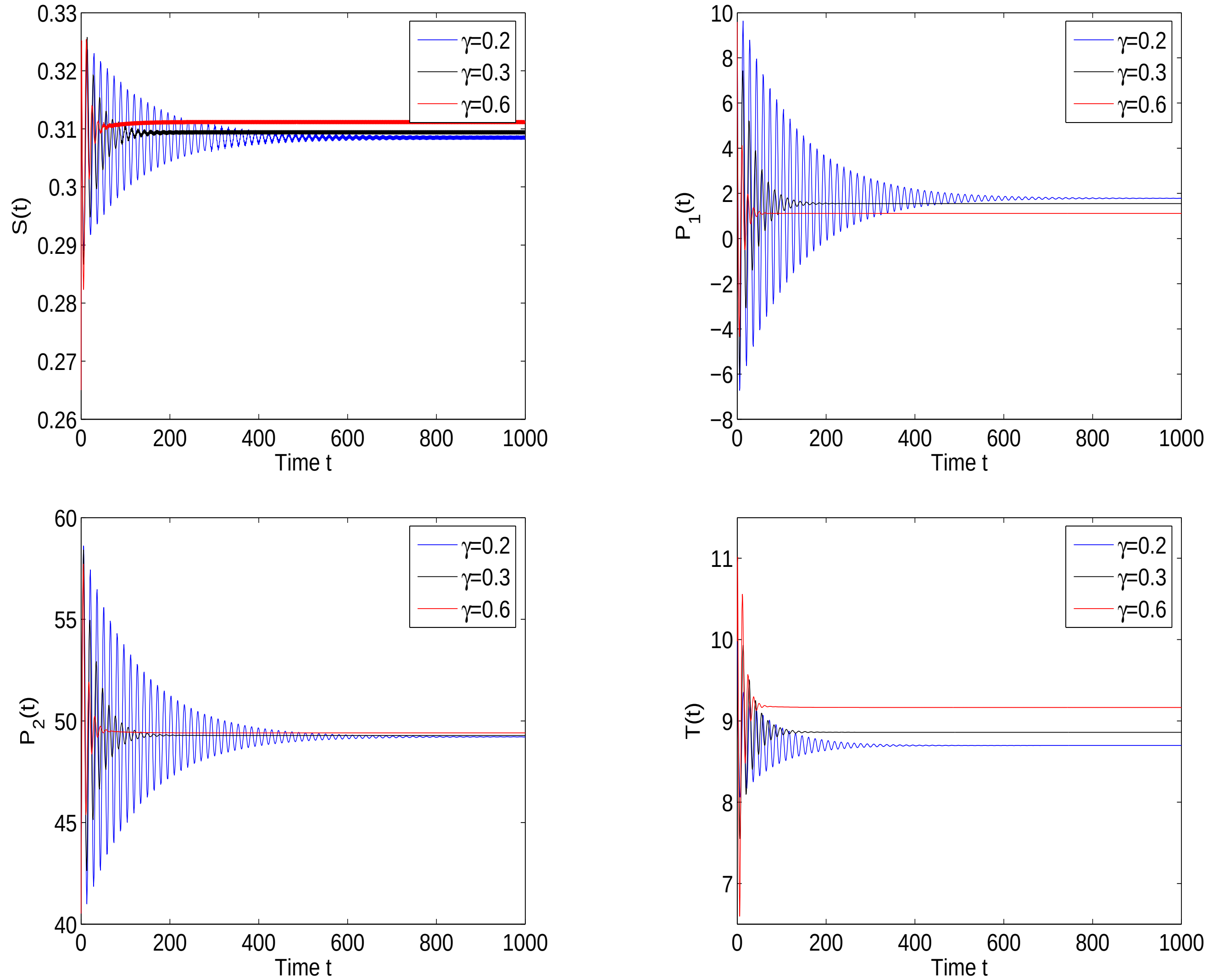

Figure 4. Waveform plots of system (5.1) for different γ at $\tau = 4.9064$. Rest of the parameters are taken as given in the text.

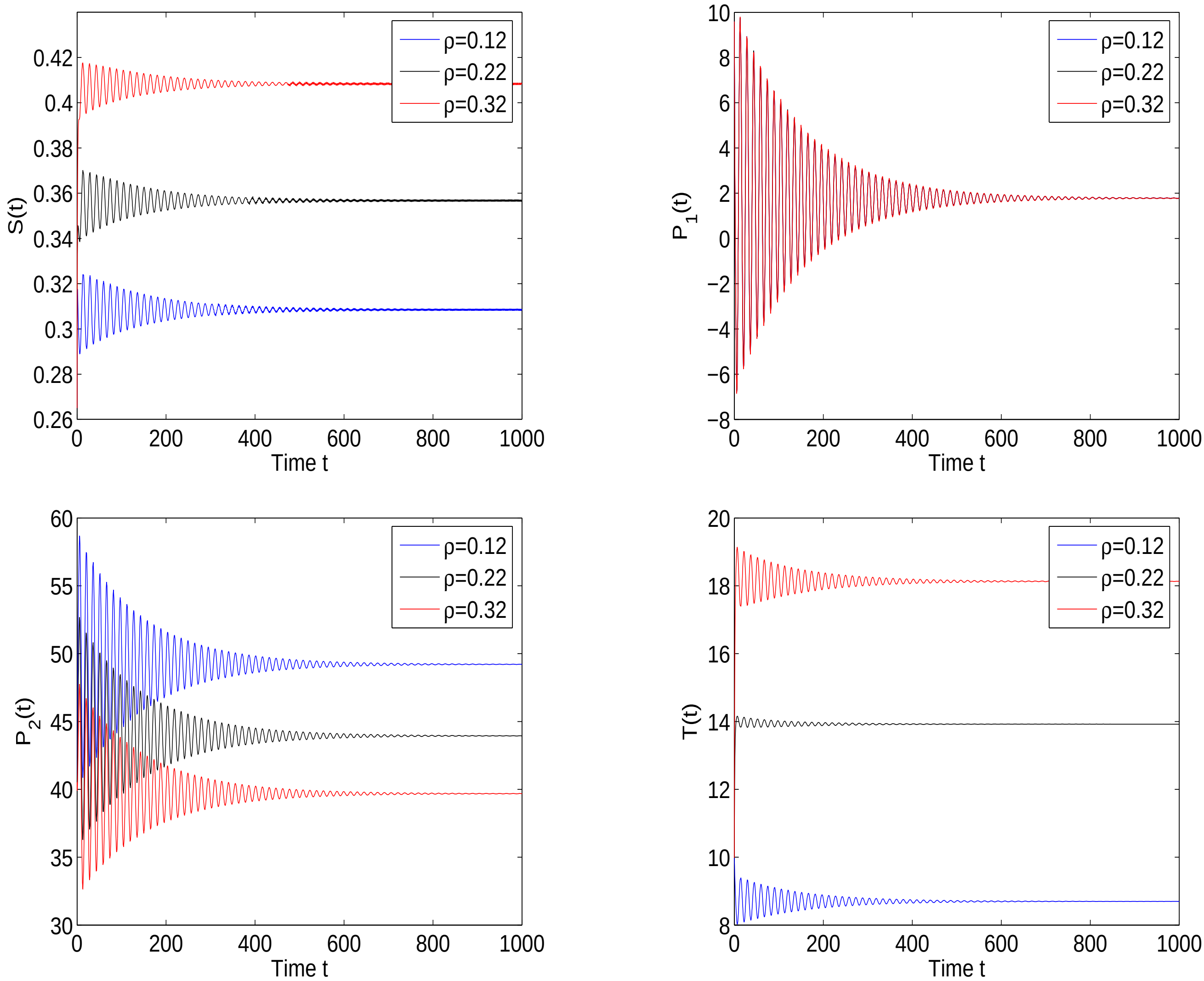

Figure 5. Waveform plots of system (5.1) for different ρ at $\tau = 4.9064$. Rest of the parameters are taken as given in the text.

5. Numerical simulations

Choosing $\Lambda = 1.2, \delta = 0.2, \beta = 0.08, b = 1.25, A = 1, \alpha = 0.45, \gamma = 0.2, \eta = 0.7, \rho = 0.12$.

$$\begin{cases} \frac{dS(t)}{dt} &= 1.2 - 0.02S(t) - \frac{0.08S(t)(P_1(t)+1.25P_2(t))}{1+S(t)}, \\ \frac{dP_1(t)}{dt} &= \frac{0.08S(t)(P_1(t)+1.25P_2(t))}{1+S(t)} - 0.45P_1(t-\tau) - 0.22P_1(t), \\ \frac{dP_2(t)}{dt} &= 0.45P_1(t-\tau) + 0.7T(t) - 0.14P_2(t), \\ \frac{dT(t)}{dt} &= 0.2P_1(t) + 0.12P_2(t) - 0.72T(t), \end{cases} \tag{5.1}$$

The unique drug addiction equilibrium $E^*(0.3097, 1.7818, 49.2127, 8.6971)$. Then, we can verify that $S_{00} = 0.781529$, $S_{01} = 4.061691$, $S_{02}S_{03} = 40.212361$, $S_{03} = 5.203203$. It is obviously that $S_{00} > 0, S_{03} > 0, S_{02}S_{03} > S_{01}$ and $S_{01}S_{02}S_{03} > S_{00}S_{03}^2 + S_{01}^2$ are satisfied. Now, Eq (2.6) becomes

$$\omega^4 + 10.847296\omega^3 + 10.223340\omega^2 + 0.894157\omega - 0.210990 = 0, \tag{5.2}$$

which has a unique positive root $\omega_0 = 0.102541$, leading to the unique positive root $\varpi_0 = 0.320220$, and $\tau_0 = 5.105862$, $F'(\omega_0) = 1.123348e-006 > 0$. Thus, we can conclude that the conditions for the occurrence of Hopf bifurcation of system (5.1) are satisfied.

From Theorem 1, system (5.1) is locally asymptotically stable when $\tau \in [0, \tau_0 = 5.105862)$. Letting $\tau = 4.9064 \in [0, \tau_0)$, it is shown in Figure 1 that system (5.1) is locally asymptotically stable. If choosing $\tau = 5.3202 > \tau_0$, then (5.1) loses its stability and a Hopf bifurcation occurs, which can be illustrated in Figure 2. And then we obtain $\mu_2 = 2.452198e-006 > 0, \beta_2 = -0.099276-006 < 0$ and $T_2 = 2.004967-007 > 0$. From Theorem 2, we know that the Hopf bifurcation is supercritical and stable, and period of the bifurcating periodic solutions increases.

In Figure 3, we can see that number of the physiologically addicts in system (5.1) decreases whereas numbers of the susceptible individuals, the psychologically addicts and the addicts under treatment increase, when the value of α decreases. As the value of γ increases, number of the psychologically addicts decreases and number of the susceptible individuals, the physiologically addicts and the addicts under treatment increase. This can be seen from Figure 4. In addition, we can observe that when the value of ρ increases, number of the physiologically addicts decreases and numbers of the susceptible individuals and the addicts under treatment increase, which can be illustrated in Figure 5. However, it can be also seen from Figure 5 that ρ does not affect the number of the physiologically addicts.

6. Conclusions

The production and abuse of synthetic drugs has been a major problem all over the world, with immense health and social consequences. In 2017 alone, about 35 million people around the world suffered from drug abuse barriers and needed treatment services [1]. In this paper, we proposed a delayed delayed synthetic drug transmission model with two stages of addiction and Holling Type-II functional response by introducing the escalating time delay of psychologically addicts into the model formulated in [23].

Existence of Hopf bifurcation is analyzed by using the escalating time delay of psychologically addicts as a bifurcation parameter. It is found that when the escalating time delay is suitable small (below τ_0), system (1.2) is locally asymptotically stable, then the transmission of synthetic drug can be predicted and controlled. Once the escalating time delay exceeds the critical value τ_0, system (1.2) will lose stability and a Hopf bifurcation occurs, then the transmission of synthetic drug will be out of control. Thus, we should control and postpone the occurrence of the Hopf bifurcation. In this respect, it is strongly suggested that someone who think that they will not become an addict after taking once should never take the first slip. Moreover, global stability of the model is proved by constructing a suitable Lyapunov function.

On the other hand, from the influence of α on numbers of the four populations in system (1.2), we appeal that anyone should never be contaminated with drugs. Since number of the psychologically addicts decreases and number of the susceptible individuals, the physiologically addicts and the addicts under treatment increase when the value of γ increases, it is indicated that psychotherapy for drug abusers is not enough. What is more important is that they need scientific and compulsory abandonment of drug abuse, which can be also seen from the influence of ρ on numbers of the four populations in system (1.2).

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Competing interests

The authors declare that there is no conflict of interests.

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