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Existence of solutions for a class of Kirchhoff-type equations with indefinite potential

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Abstract
In this paper, we consider the existence of solutions of the following Kirchhoff-type problem:

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,dx)\Delta u+V(x)u=f(x,u), & \text{in } \mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3), \end{cases}$$

where $a,b>0$ are constants, and the potential $V(x)$ is indefinite in sign. Under some suitable assumptions on f, the existence of solutions is obtained by Morse theory.

Keywords: Kirchhoff-type equation; Variational methods; Palais–Smale condition; Local linking; Morse theory

1 Introduction and main result

This paper is concerned with the following Kirchhoff-type problem:

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,dx)\Delta u+V(x)u=f(x,u) & \text{in } \mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3), \end{cases} \tag{1.1}$$

where $a,b>0$ are constants, and the potential $V(x)$ is indefinite in sign, f satisfies some conditions which will be stated later.

In recent years, more and more attention has been devoted to study the following Kirchhoff-type problem:

$$\begin{cases} -(a+b\int_{\mathbb{R}^N}|\nabla u|^2\,dx)\Delta u+V(x)u=f(x,u) & \text{in } \mathbb{R}^N,\\ u\in H^1(\mathbb{R}^N), \end{cases} \tag{1.2}$$

where $V:\mathbb{R}^N\to\mathbb{R}$ and $a,b>0$ are constants. (1.2) is a nonlocal problem as the appearance of the term $\int_{\mathbb{R}^N}|\nabla u|^2\,dx\Delta u$, which implies that (1.2) is not a pointwise identity. This causes some mathematical difficulties which make the study of (1.2) particularly interesting. Problem (1.2) appears in an interesting physical context. Indeed, if we consider the case $V(x) = 0$ and replace $\mathbb{R}^N$ with a bounded domain $\Omega \subset \mathbb{R}^N$ in (1.2), then we get the following Dirichlet problem of Kirchhoff type:

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2\,dx)\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \Omega, \end{cases} \tag{1.3}$$

which is a nonlocal problem due to the presence of the nonlocal term $b\int_\Omega |\nabla u|^2\,dx\Delta u$ and is related to the stationary analogue of the equation

$$u_{tt} - \left(a + b\int_\Omega |\nabla u|^2\,dx\right)\Delta u = f(x,u). \tag{1.4}$$

(1.4) was first proposed by Kirchhoff in [12] as a generalization of the classical D'Alembert wave equations, particularly taking into account the subsequent change in string length caused by oscillations. The readers can learn some early classical research of Kirchhoff equations from [4, 22]. For the results concerning the existence of sign-changing solutions for (1.3), we refer the reader to papers [20, 24, 34], which depend heavily on the nonlinear term with 4-superlinear growth at infinity in the sense that

$$\lim_{|t|\to\infty} \frac{F(x,t)}{t^4} = +\infty, \quad x \in \Omega,$$

where $F(x,t) = \int_0^t f(x,s)\,ds$. And [19, 32] deal with the fact that the nonlinearity $f(x,u)$ may not be 4-superlinear at infinity.

Motivated by the strong physical background, equations (1.2) and (1.3) have been extensively studied in recent years under variant assumptions on V and f. There are many papers involving the existence of nontrivial solutions of equation (1.2). In [21], Perera and Zhang obtained a nontrivial solution of (1.2) via Yang index and critical group. By using the local minimum methods and the fountain theorems, He and Zou [9] obtained infinitely many solutions. Later, Jin and Wu [11] proved the existence of infinitely many radial solutions by applying a fountain theorem. Using the Nehari manifold and fibering map methods, equation (1.2) was studied with concave and convex nonlinearities, the existence of multiple positive solutions was obtained by Chen et al. [6]. Moreover, the existence of infinitely many solutions to equation (1.2) has been derived by a variant version of fountain theorem in [18]. Subsequently, in [13] Li and Ye, using a monotone method and a global compactness lemma, showed the existence of a positive ground state solution for the corresponding limiting problem of equation (1.2). After that, Guo [8] generalized the result in [13] to a general nonlinearity. In [26] Tang and Cheng proposed a new approach to recover compactness for the (PS)-sequence, and they proved that equation (1.3) possesses one ground state sign-changing solution, and its energy is strictly larger than twice that of the ground state solutions of Nehari type. In [25] Tang and Chen proved that equation (1.2) admits a ground state solution of Nehari–Pohozaev type and a least energy solution under some mild assumptions on V and f by using a new approach to recover compactness for the minimizing sequence.

Recently, Xiang et al. [31] considered the existence and multiplicity of solutions for a class of Schrödinger–Kirchhoff type problems involving the fractional p-Laplacian and critical exponent. By using the concentration compactness principle in fractional Sobolev spaces, they obtained m pairs of solutions, by using Krasnoselskii's genus theory, the existence of infinitely many solutions were obtained. Later, Xiang et al. [30] developed the fractional Trudinger–Moser inequality in the singular case and used it to study the existence and multiplicity of solutions for a class of perturbed fractional Kirchhoff-type problems with singular exponential nonlinearity. For further important and interesting results, one can refer to [3, 10, 14, 27, 29] and the references therein.

In all the above-mentioned studies, we notice that the potential $V(x)$ was assumed to be equipped with some "compact" condition or positive definite. But in this paper the potential $V(x)$ is indefinite in sign, the methods and arguments for the cases $V(x) \geq 0$ are not applicable to the indefinite cases. So, this article is a complement to the indefinite Kirchhoff problems in the literature. Our main aim is to study the existence of nontrivial solutions for problem (1.1) by means of Morse theory and local linking, which are different from the literature mentioned above. Before stating our main results, we need to describe the eigenvalue of Schrödinger operator $-a\Delta + V$. Consider the following increasing sequence $\lambda_1 \leq \lambda_2 \leq \cdots$ of minimax values defined by

$$\lambda_n := \inf_{S\in\Gamma_n} \sup_{u\in S\setminus\{0\}} \frac{\int_{\mathbb{R}^3}(a|\nabla u|^2 + V(x)u^2)\,dx}{\int_{\mathbb{R}^3} u^2\,dx},$$

where Γ_n denotes the family of n-dimensional subspaces of $C_0^\infty(\mathbb{R}^3)$, remember $a \neq 0$. Let

$$\lambda_\infty := \lim_{n\to\infty} \lambda_n,$$

then λ_∞ is the bottom of the essential spectrum of $-a\Delta + V$ if it is finite, and for every $n \in \mathbb{N}$, the inequality $\lambda_n < \lambda_\infty$ implies that λ_n is an eigenvalue of $-a\Delta + V$ of finite multiplicity (see [23, Chapt. XIII] for details). Note that if V is bounded from below, then λ_n is well defined and is finite.

Set $F(x,u) := \int_0^u f(x,t)\,dt$. We assume that V and f satisfy the following conditions:

(V) $V(x) \in C(\mathbb{R}^3,\mathbb{R})$ bounded and there exists an integer $k \geq 1$ such that $\lambda_k < 0 < \lambda_{k+1}$.

(f_1) $f \in C(\mathbb{R}^3 \times \mathbb{R},\mathbb{R})$, and there exist $C > 0$ and $p \in (2,6)$ such that

$$\bigl|f(x,u)\bigr| \leq C\bigl(1 + |u|^{p-1}\bigr) \quad \text{for all } (x,u) \in \mathbb{R}^3 \times \mathbb{R}.$$

(f_2) $f(x,u) = o(u)$ as $u \to 0$ uniformly in $x \in \mathbb{R}^3$.

(f_3) There exists $\mu > 4$ such that $0 < \mu F(x,u) \leq f(x,u)u$ for all $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$ and $u \neq 0$.

(f_4) For any $r > 0$, we have

$$\lim_{|x|\to\infty} \sup_{0<|u|\leq r} \left|\frac{f(x,u)}{u}\right| = 0.$$

It is easy to see that condition (f_3) implies that

$$\lim_{|u|\to\infty} \frac{F(x,u)}{u^4} = +\infty.$$

Concerning the existence of solutions for problem (1.1), we have the following result.

Theorem 1.1 *Suppose that* (V) *and* (f_1)–(f_4) *hold. Then problem* (1.1) *has at least one nontrivial solution.*

Remark 1.2 We should also mention two recent papers [15, 33] related to problem (1.1). In these two papers, the variational functional is coercive and bounded from below. When the nonlinearity is odd, infinitely many nontrivial solutions of (1.1) were obtained in both [15] and [33] by using critical point theory of even functional; while if the nonlinearity is not odd, two nontrivial solutions were obtained in [15] via Morse theory. In the current paper, the variational functional is neither bounded from above nor bounded from below, this is quite different from the situation in [15, 33].

Remark 1.3 To deal with problem (1.1), one encounters various difficulties. On the one hand, the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$ is not compact. To overcome this, one can restrict the energy functional Φ to a subspace of $H^1(\mathbb{R}^3)$, which embeds compactly into $L^2(\mathbb{R}^3)$ with certain qualifications or consists of radially symmetric functions. In [7], Chen and Liu considered the standing waves of (1.1) with the nonlinearity f is 4-superlinear and the potential V satisfying assumption (V) and

$$\mu\bigl(V^{-1}(-\infty, M]\bigr) < \infty \tag{1.5}$$

for all $M > 0$, where μ denotes the Lebesgue measure in $\mathbb{R}^N$, then the working space

$$X := \left\{ u \in H^1\bigl(\mathbb{R}^N\bigr) \,\middle|\, \int_{\mathbb{R}^N} V(x) u^2 \, dx < \infty \right\}$$

embeds compactly into $L^2(\mathbb{R}^N)$, which is crucial in verifying the Palais–Smale condition. However, our assumptions on V are much weaker.

On the other hand, under our assumptions the potential $V(x)$ is indefinite in sign, then the quadratic part of the functional Φ (defined in (2.2)) possesses a nontrivial negative space, and the functional Φ does not satisfy the linking geometry any more, so that the linking theorem is not applicable. We will use the idea of local linking to overcome the difficulty. To our knowledge, there are a few results on this case.

The remainder of this paper is organized as follows. In Sect. 2, we give the variational framework for problem (1.1) and prove that Φ satisfies the (PS) condition. In Sect. 3, we recall some concepts and results in infinite-dimensional Morse theory [5] and give the proof of Theorem 1.1.

2 Variational setting and Palais–Smale condition

In this section, we give the variational setting for problem (1.1) and establish the compactness conditions. By $|\cdot|_s$ as follows, we denote the usual L^s-norm for $s \geq 2$, and C, C_i stand for different positive constants.

Let

$$H^1\bigl(\mathbb{R}^3\bigr) := \bigl\{ u \in L^2\bigl(\mathbb{R}^3\bigr) | \nabla u \in L^2\bigl(\mathbb{R}^3\bigr) \bigr\},$$

with the usual norm

$$\|u\|_H = \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2 \right) dx \right)^{\frac{1}{2}}$$

and

$$E := \left\{ u \in H^1(\mathbb{R}^3) \,\middle|\, \int_{\mathbb{R}^3} V(x) u^2 \, dx < \infty \right\},$$

be a linear subspace of $H^1(\mathbb{R}^3)$. Let E^- be the space spanned by the eigenfunctions with respect to $\lambda_1, \dots, \lambda_k$ and $E^+ = (E^-)^\perp$. From (V), we deduce that $E = E^- \oplus E^+$, where E^-, E^+ are the negative eigenspace and the positive eigenspace of the operator $-a\Delta + V$. Moreover, $k \leq \dim E^- < \infty$.

For any $u, v \in E$, we define

$$(u, v) := \int_{\mathbb{R}^3} \left(a \nabla u^+ \nabla v^+ + V(x) u^+ v^+ \right) dx - \int_{\mathbb{R}^3} \left(a \nabla u^- \nabla v^- + V(x) u^- v^- \right) dx,$$

where $u = u^- + u^+$, $v = v^- + v^+$, $u^+, v^+ \in E^+$, $u^-, v^- \in E^-$. Then $(\cdot, \cdot)$ is an inner product in E. Hence, E is a Hilbert space with the norm $\|u\| = (u, u)^{\frac{1}{2}}$, which is an equivalent norm on $H^1(\mathbb{R}^3)$. It is easy to see that

$$\int_{\mathbb{R}^3} \left(a |\nabla u|^2 + V(x) u^2 \right) dx = \left\| u^+ \right\|^2 - \left\| u^- \right\|^2.$$

For any $s \in [2, 6]$, since the embedding $E \hookrightarrow L^s(\mathbb{R}^3)$ is continuous, then there exists a constant $d_s > 0$ such that

$$|u|_s \leq d_s \|u\| \quad \text{for all } u \in E. \tag{2.1}$$

It is easy to see that, from (f_1) and (f_2), the functional $\Phi : E \to \mathbb{R}$,

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^3} a |\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} V(x) u^2 \, dx + \frac{b}{4} \left(\int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)^2 - \int_{\mathbb{R}^3} F(x, u) \, dx, \tag{2.2}$$

is of class C^1 with derivative

$$\begin{aligned} \langle \Phi'(u), v \rangle = {} & \int_{\mathbb{R}^3} \left(a \nabla u \cdot \nabla v + V(x) u v \right) dx + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \int_{\mathbb{R}^3} \nabla u \cdot \nabla v \, dx \\ & - \int_{\mathbb{R}^3} f(x, u) v \, dx \end{aligned} \tag{2.3}$$

for all $u, v \in E$. It can be proved that a weak solution of problem (1.1) is a critical point of the functional Φ.

We say that a functional $I \in C^1(E, \mathbb{R})$ satisfies the (PS) condition if any sequence $\{u_n\} \subset E$ such that

$$\sup_n \left| I(u_n) \right| < \infty \quad \text{and} \quad I'(u_n) \to 0 \quad \text{in } E^{-1}$$

(called a (PS) sequence) has a convergent subsequence.

Lemma 2.1 *Under assumptions (V), (f_3), every (PS) sequence of functional Φ is bounded in E.*

Proof Let $\{u_n\}$ be a (PS) sequence of functional Φ, that is,

$$\sup_n \bigl|\Phi(u_n)\bigr| < \infty \quad \text{and} \quad \Phi'(u_n) \to 0 \quad \text{in } E^{-1}.$$

If the conclusion is not true, we may assume $\|u_n\| \to \infty$, as $n \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$ up to a subsequence

$$v_n = v_n^+ + v_n^- \rightharpoonup v = v^+ + v^- \in E, \quad v_n^\pm, v^\pm \in E^\pm.$$

If $v = 0$, then $v_n^- \to v^- = 0$ due to $\dim E^- < \infty$. Since

$$\|v_n\|^2 = \bigl\|v_n^-\bigr\|^2 + \bigl\|v_n^+\bigr\|^2 = 1,$$

for n large enough, we obtain

$$\bigl\|v_n^+\bigr\|^2 - \bigl\|v_n^-\bigr\|^2 \geq \frac{1}{2}.$$

Now, using (f_3), for n large enough, we deduce

$$\begin{aligned}
&1 + \sup_n \bigl|\Phi(u_n)\bigr| + \|u_n\| \\
&\quad \geq \Phi(u_n) - \frac{1}{\mu}\bigl\langle \Phi'(u_n), u_n \bigr\rangle \\
&\quad = \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_{\mathbb{R}^3} \bigl(a|\nabla u_n|^2 + V(x)u_n^2\bigr)\,dx + \left(\frac{1}{4} - \frac{1}{\mu}\right) b \left(\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx\right)^2 \\
&\qquad + \int_{\mathbb{R}^3} \left(\frac{1}{\mu} f(x,u_n)u_n - F(x,u_n)\right) dx \\
&\quad \geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_{\mathbb{R}^3} \bigl(a|\nabla u_n|^2 + V(x)u_n^2\bigr)\,dx \\
&\quad = \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_n\|^2 \bigl(\bigl\|v_n^+\bigr\|^2 - \bigl\|v_n^-\bigr\|^2\bigr) \\
&\quad \geq \left(\frac{1}{4} - \frac{1}{2\mu}\right) \|u_n\|^2,
\end{aligned} \tag{2.4}$$

contradicting $\|u_n\| \to \infty$, thus $v \neq 0$.

It follows from (2.4) that

$$0 \leftarrow \frac{1 + \sup_n |\Phi(u_n)| + \|u_n\|}{\|u_n\|^4} \geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\left(\frac{\|u_n^+\|^2}{\|u_n\|^4} - \frac{\|u_n^-\|^2}{\|u_n\|^4}\right) + \left(\frac{1}{4} - \frac{1}{\mu}\right) b \frac{|\nabla u_n|_2^4}{\|u_n\|^4},$$

which implies that

$$\lim_{n\to\infty} \frac{|\nabla u_n|_2^4}{\|u_n\|^4} = 0.$$

Define $\Upsilon(u) := |\nabla u|_2$, it is easy to see that Υ is continuous and convex, hence it is weak lower semi-continuous. Consequently,

$$\liminf_{n\to\infty} |\nabla v_n|_2 \geq |\nabla v|_2,$$

and then

$$0 = \liminf_{n\to\infty} \frac{|\nabla u_n|_2^4}{\|u_n\|^4} = \liminf_{n\to\infty} |\nabla v_n|_2^4 \geq |\nabla v|_2^4 > 0.$$

This is a contradiction, thus the proof is completed. □

Lemma 2.2 *Under the assumptions of Theorem 1.1, Φ satisfies the (PS) condition.*

Proof Assume that $\{u_n\}$ is a (PS) sequence of Φ. It follows from Lemma 2.1 that $\{u_n\}$ is bounded in E, then up to a subsequence

$$u_n \rightharpoonup u, \quad \text{in } E, \qquad u_n \to u \quad \text{in } L^s_{\text{loc}}(\mathbb{R}^3), 2 \leq s < 6, \qquad u_n(x) \to u(x) \quad \text{a.e. in } \mathbb{R}^3.$$

We have

$$\int_{\mathbb{R}^3} \big(a\nabla u_n \cdot \nabla u + V(x)u_n u\big)\,dx \to \int_{\mathbb{R}^3} \big(a|\nabla u_n|^2 + V(x)u^2\big)\,dx = \|u^+\|^2 - \|u^-\|^2.$$

Consequently,

$$\begin{aligned}
o(1) &= \langle \Phi'(u_n), u_n - u\rangle \\
&= \int_{\mathbb{R}^3} \big(a\nabla u_n \cdot \nabla(u_n - u) + V(x)u_n(u_n - u)\big)\,dx \\
&\quad + b\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx \int_{\mathbb{R}^3} \nabla u_n \cdot \nabla(u_n - u)\,dx - \int_{\mathbb{R}^3} f(x,u_n)(u_n - u)\,dx \\
&= \int_{\mathbb{R}^3} \big(a|\nabla u_n|^2 + V(x)u_n^2\big)\,dx - \int_{\mathbb{R}^3} \big(a\nabla u_n \cdot \nabla u + V(x)u_n u\big)\,dx \\
&\quad + b\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx \int_{\mathbb{R}^3} \nabla u_n \cdot \nabla(u_n - u)\,dx - \int_{\mathbb{R}^3} f(x,u_n)(u_n - u)\,dx \qquad (2.5)\\
&= \big(\|u_n^+\|^2 - \|u_n^-\|^2\big) - \big(\|u^+\|^2 - \|u^-\|^2\big) \\
&\quad + b\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx \int_{\mathbb{R}^3} \nabla u_n \cdot \nabla(u_n - u)\,dx - \int_{\mathbb{R}^3} f(x,u_n)(u_n - u)\,dx + o(1) \\
&= \big(\|u_n^+\|^2 - \|u^+\|^2\big) - \big(\|u_n^-\|^2 - \|u^-\|^2\big) \\
&\quad + b\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx \int_{\mathbb{R}^3} \nabla u_n \cdot \nabla(u_n - u)\,dx - \int_{\mathbb{R}^3} f(x,u_n)(u_n - u)\,dx + o(1).
\end{aligned}$$

Since $\dim E^- < \infty$, we have $u_n^- \to u^-$, thus $\|u_n^-\| \to \|u^-\|$. Collecting all infinitesimal terms, we obtain

$$\|u_n^+\|^2 - \|u^+\|^2 = \int_{\mathbb{R}^3} f(x,u_n)(u_n - u)\,dx$$

$$- b\int_{\mathbb{R}^3}|\nabla u_n|^2\,dx\int_{\mathbb{R}^3}\nabla u_n\cdot\nabla(u_n-u)\,dx+o(1). \tag{2.6}$$

With the assumption (f_4), it has been shown in [2, p.29] that

$$\overline{\lim}\int_{\mathbb{R}^3}f(x,u_n)(u_n-u)\,dx\le 0.$$

Because $\Upsilon(u)=|\nabla u|_2$ is weak lower semi-continuous, we have

$$\begin{aligned}\underline{\lim}\int_{\mathbb{R}^3}\nabla u_n\cdot\nabla(u_n-u)\,dx&=\underline{\lim}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2\,dx-\int_{\mathbb{R}^3}\nabla u_n\cdot\nabla u\,dx\right)\\&\ge\int_{\mathbb{R}^3}|\nabla u|^2\,dx-\int_{\mathbb{R}^3}|\nabla u|^2\,dx\\&=0.\end{aligned}$$

Now, from (2.6), we have

$$\begin{aligned}&\overline{\lim}\left(\|u_n^+\|^2-\|u^+\|^2\right)\\&\quad=\overline{\lim}\left(\int_{\mathbb{R}^3}f(x,u_n)(u_n-u)\,dx-b\int_{\mathbb{R}^3}|\nabla u_n|^2\,dx\int_{\mathbb{R}^3}\nabla u_n\cdot\nabla(u_n-u)\,dx\right)\\&\quad\le\overline{\lim}\int_{\mathbb{R}^3}f(x,u_n)(u_n-u)\,dx-\underline{\lim}b\int_{\mathbb{R}^3}|\nabla u_n|^2\,dx\int_{\mathbb{R}^3}\nabla u_n\cdot\nabla(u_n-u)\,dx\\&\quad\le\overline{\lim}\int_{\mathbb{R}^3}f(x,u_n)(u_n-u)\,dx\\&\quad\le 0,\end{aligned}$$

which implies $\overline{\lim}\|u_n^+\|^2\le\|u^+\|^2$. From the weak lower semi-continuity of the norm, we have

$$\|u^+\|^2\le\underline{\lim}\|u_n^+\|^2\le\overline{\lim}\|u_n^+\|^2\le\|u^+\|^2,$$

that is, $\|u_n^+\|^2\to\|u^+\|^2$, combining $\|u_n^-\|^2\to\|u^-\|^2$, we get $\|u_n\|\to\|u\|$. Thus $u_n\to u$ in E. The proof is completed. □

3 Critical groups and the proof of Theorem 1.1

In Sect. 2, we have established the (PS) condition for Φ. Now, we recall some concepts and results in infinite-dimensional Morse theory [5], then analyze the critical groups of Φ at infinity, and give the proof of Theorem 1.1.

Let X be a Banach space, $\varphi: X\to\mathbb{R}$ be a C^1-functional, u be an isolated critical point of φ and $\varphi(u)=c$. Then

$$C_q(\varphi,u):=H_q\left(\varphi_c,\varphi_c\backslash\{u\}\right),\quad q=0,1,2,\ldots,$$

is called the qth critical group of φ at u, where $\varphi_c:=\varphi^{-1}(-\infty,c]$ and $H_q(\cdot,\cdot)$ stands for the qth singular relative homology group with integer coefficients.

If φ satisfies the (PS) condition and the critical values of φ are bounded from below by α, then following Bartsch and Li [1], we call

$$C_q(\varphi,\infty) := H_q(X,\varphi_\alpha), \quad q = 0,1,2,\ldots,$$

the qth critical group of φ at ∞. It is well known that the homology on the right does not depend on the choice of α.

Proposition 3.1 ([1]) *If $\varphi \in C^1(X,\mathbb{R})$ satisfies the (PS) condition and $C_l(\varphi,0) \neq C_l(\varphi,\infty)$ for some $l \in \mathbb{N}$, then φ has a nonzero critical point.*

Proposition 3.2 ([16]) *Suppose $\varphi \in C^1(X,\mathbb{R})$ has a local linking at 0, that is, $X = Y \oplus Z$ and*

$$\begin{aligned}
&\varphi(u) \le 0 \quad \text{for } u \in Y \cap B_\rho,\\
&\varphi(u) > 0 \quad \text{for } u \in \big(Z\backslash\{0\}\big) \cap B_\rho,
\end{aligned}$$

for some $\rho > 0$, where $B_\rho := \{u \in X \mid \|u\| \le \rho\}$. If $l = \dim Y < \infty$, then $C_l(\varphi,0) \neq 0$.

To prove Theorem 1.1 with Proposition 3.1, we need the following lemma to investigate the critical groups of Φ at infinity.

Lemma 3.3 *Under the assumptions of Theorem 1.1, there exists $A > 0$ such that, if $\Phi(u) \le -A$, then*

$$\langle \Phi'(u), u\rangle = \frac{d}{dt}\bigg|_{t=1} \Phi(tu) < 0.$$

Proof We argue by contradiction. Assume that there exists a sequence $\{u_n\} \subset E$ such that $\Phi(u_n) \le -n$, but

$$\langle \Phi'(u_n), u_n\rangle = \frac{d}{dt}\bigg|_{t=1} \Phi(tu_n) \ge 0. \tag{3.1}$$

By (f_3), we have

$$\begin{aligned}
\|u_n^+\|^2 - \|u_n^-\|^2 &\le \big(\|u_n^+\|^2 - \|u_n^-\|^2\big) + \int_{\mathbb{R}} \big[f(x,u_n)u_n - 4F(x,u_n)\big]\,dx\\
&= 4\Phi(u_n) - \langle \Phi'(u_n), u_n\rangle\\
&\le -4n.
\end{aligned} \tag{3.2}$$

This implies $\|u_n\| \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$, and $v_n^\pm$ be the orthogonal projection of v_n on $E^\pm$. Then $v_n^- \to v^-$ for some $v^- \in E^-$ since $\dim E^- < \infty$.

If $v^- \neq 0$, then for some $v \in E\backslash\{0\}$, we have $v_n \rightharpoonup v$ in E, and the set $\Theta = \{x \in \mathbb{R}^3 : v \neq 0\}$ has positive Lebesgue measure. For $x \in \Theta$, we have $|u_n(x)| \to \infty$, from (f_3), that

$$\frac{F(x,u_n(x))}{u_n^4(x)} v_n^4(x) \to +\infty.$$

Then, by Fatou's lemma and (f_3), we have

$$\begin{aligned}\int_{\mathbb{R}^3}\frac{f(x,u_n)u_n}{\|u_n\|^4}\,dx &\geq \int_{\mathbb{R}^3}\frac{\mu F(x,u_n)}{\|u_n\|^4}\,dx\\ &\geq \mu\int_{v\neq 0}\frac{F(x,u_n)}{u_n^4(x)}v_n^4(x)\\ &\to +\infty.\end{aligned}$$

Hence, using (3.1), we have

$$\begin{aligned}0 &\leq \frac{\langle\Phi'(u_n),u_n\rangle}{\|u_n\|^4}\\ &= \frac{\|u_n^+\|^2-\|u_n^-\|^2}{\|u_n\|^4}+b\frac{|\nabla u_n|_2^4}{\|u_n\|^4}-\int_{\mathbb{R}^3}\frac{f(x,u_n)u_n}{\|u_n\|^4}\,dx\\ &\leq \frac{1}{\|u_n\|^2}+C_1b-\int_{\mathbb{R}^3}\frac{f(x,u_n)u_n}{\|u_n\|^4}\,dx\\ &\leq 1+C_1b-\int_{\mathbb{R}^3}\frac{f(x,u_n)u_n}{\|u_n\|^4}\,dx\\ &\to -\infty,\end{aligned}$$

a contradiction. Therefore $v^- = 0$, from

$$\|v_n\|^2 = \left\|v_n^+\right\|^2+\left\|v_n^-\right\|^2 = 1,$$

we see that $\|v_n^+\|\to 1$. Consequently, for n large enough, we have

$$\left\|u_n^+\right\| = \|u_n\|\left\|v_n^+\right\| \geq \|u_n\|\left\|v_n^-\right\| = \left\|u_n^-\right\|,$$

a contradiction to (3.2). Thus the conclusion of the lemma is true. □

Lemma 3.4 *Under the assumptions of Theorem 1.1, $C_i(\Phi,\infty)\cong 0$ for all $i\in\mathbb{N}$.*

Proof Let $B=\{v\in E \mid \|v\|\leq 1\}$ be the unit ball in E, $S=\partial B$ be the unit sphere. Let $A>0$ be the number given in Lemma 3.3. Without loss of generality, we may assume that

$$\inf_{\|u\|\leq 2}\Phi(u) > -A.$$

Then, for $v\in S$ and (f_3), we deduce that

$$\begin{aligned}\Phi(sv) &= \frac{s^2}{2}\left(\left\|v^+\right\|^2-\left\|v^-\right\|^2\right)+\frac{bs^4}{4}\left(\int_{\mathbb{R}^3}|\nabla v|^2\,dx\right)^2-\int_{\mathbb{R}^3}F(x,sv)\,dx\\ &= s^4\left\{\frac{\|v^+\|^2-\|v^-\|^2}{2s^2}+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla v|^2\,dx\right)^2-\int_{\mathbb{R}^3}\frac{F(x,sv)}{s^4}\,dx\right\}\\ &\to -\infty,\quad \text{as } s\to+\infty.\end{aligned}$$

So there is $s_v>0$ such that $\Phi(s_v v) = -A$.

Set $w = s_v v$, a direct computation and Lemma 3.3 give

$$\left\langle \Phi'(s_v v), v \right\rangle = \frac{d}{ds}\bigg|_{s=s_v} \Phi(sv) = \frac{1}{s_v}\frac{d}{dt}\bigg|_{t=1} \Phi(tw) < 0.$$

By the implicit function theorem, $T : v \to s_v$ is a continuous function on S. Using the function T and a standard argument (see, e.g., [17, 28]), we can construct a deformation from $X\backslash B$ to $\Phi_{-A} = \Phi^{-1}(-\infty, -A]$, and deduce via the homotopic invariance of singular homology

$$C_i(\Phi, \infty) = H_i(X, \Phi_{-A}) \cong H_i(X, X\backslash B) = 0, \quad \text{for all } i \in \mathbb{N}.$$

The proof is completed. □

Lemma 3.5 *Under assumptions* (V), (f_1), *and* (f_2), *the functional* Φ *has a local linking at* 0 *with respect to* $E = E^- \oplus E^+$.

Proof By (f_1) and (f_2), there exists a constant $C > 0$ such that

$$\left|F(x,u)\right| \le \varepsilon u^2 + Cu^p.$$

If $u \in E^-$, by (2.1) and the equivalence of norms on finite dimensional space E^-, we have

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\int_{\mathbb{R}^3} \left(a|\nabla u|^2 + V(x)u^2\right) dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\, dx\right)^2 - \int_{\mathbb{R}^3} F(x,u)\, dx \\
&= -\frac{1}{2}\|u\|^2 + \frac{b}{4}|\nabla u|_2^4 - \int_{\mathbb{R}^3} F(x,u)\, dx \\
&\le -\frac{1}{2}\|u\|^2 + C_2\frac{b}{4}\|u\|^4 + \varepsilon|u|_2^2 + C|u|_p^p \\
&\le \left(-\frac{1}{2} + \varepsilon d_2^2\right)\|u\|^2 + C_2\frac{b}{4}\|u\|^4 + Cd_p^p\|u\|^p.
\end{aligned}$$

If $u \in E^+$, we have

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\int_{\mathbb{R}^3} \left(a|\nabla u|^2 + V(x)u^2\right) dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\, dx\right)^2 - \int_{\mathbb{R}^3} F(x,u)\, dx \\
&= \frac{1}{2}\|u\|^2 + \frac{b}{4}|\nabla u|_2^4 - \int_{\mathbb{R}^3} F(x,u)\, dx \\
&\ge \frac{1}{2}\|u\|^2 - \varepsilon|u|_2^2 - C|u|_p^p \\
&\ge \left(\frac{1}{2} - \varepsilon d_2^2\right)\|u\|^2 - Cd_p^p\|u\|^p.
\end{aligned}$$

So, there exists $0 < \rho < 1$ small enough such that

$$\begin{aligned}
&\Phi(u) \le 0 \quad \text{for } u \in E^- \text{ with } \|u\| \le \rho, \\
&\Phi(u) > 0 \quad \text{for } u \in E^+ \text{ with } \|u\| \le \rho.
\end{aligned}$$

The proof is completed. □

We are now ready to prove Theorem 1.1.

Proof It follows from Lemma 3.5 that Φ has a local linking at 0 with respect to $E = E^- \oplus E^+$. Therefore Proposition 3.2 yields

$$C_k(\Phi, 0) \neq 0,$$

where $k = \dim E^- < \infty$. By Lemma 3.4, we have

$$C_k(\Phi, \infty) = 0.$$

Applying Proposition 3.1, we see that Φ has a nontrivial critical point. The proof of Theorem 1.1 is completed. □

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