

TRANSFER EQUATION IN GENERAL CURVILINEAR COORDINATES

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Abstract. The differential operator of the monochromatic polarized radiative transfer equation is studied in case of statistically homogeneous turbid medium in Euclidean three-dimensional space, with arbitrary curvilinear coordinate system defined in it. An apparent rotation of the polarization plane along the light ray with respect to the chosen polarization reference plane generally takes place, due to purely geometric reasons. Using methods of tensor analysis, analytic expressions for the differential operator of the transfer equation depending on the components of the metric tensor and their derivatives are found. Considerable simplifications take place if the coordinate system is orthogonal. As an example, the differential operator of the vector radiative transfer equation in both elliptical conical coordinate system and oblate spheroidal coordinate system is written down. Nonstandard parameterization of standard elliptical conical coordinate system is proposed.

Key words: radiative transfer – polarization

1. INTRODUCTION

At the 200th anniversary of Tartu Observatory it is a pleasure to remark the continuous development and many achievements of Tartu scientists in the theory of radiative transfer (A. Sapaar, T. Viik, R. Rõõm, A. Heinlo, I. Vurm and others). Topics covered include Ambartsumian's/Chandrasekhar's and Hopf's functions, a modified version of X- and Y-functions (Viik 2010), and Sobolev's resolvent functions (Rõõm et al. 1981; Viik & Rõõm 2008); novel algorithms for the calculation of Voigt and Holtsmark functions; multilayer atmospheres, and Viik's principle of invariance (Viik 1981; Viik 1982b); Rayleigh and molecular scattering of polarized radiation; scattering within spectral lines with redistribution of radiation over frequencies (Heinlo 1973; Sapaar & Sapaar 1999); the connection between radiative transfer equation, classical and quantum electrodynamics (Sapaar 1968); NLTE phenomena in stellar atmospheres; theory of Compton scattering (Vurm & Poutanen 2009; Poutanen & Vurm 2010), etc. And now it seems that the long term durability of this research in Tartu is ensured.

Stationary integro-differential radiative transfer equation (RTE) in a polydis-

perse medium consists of the following basic terms:

- the differential operator, sometimes called “the streaming operator”,
- the extinction term,
- the integral term describing scattering, and
- the term describing the primary sources of radiation.

The first of those will be the object of study in this contribution, because this item depends on the chosen coordinate system. The other terms describe the physical interactions between the matter and the radiation field, and they are essentially the same in all coordinate systems.

As it is shown, e.g., in Rytov (1938), Landau & Lifshitz (1982), as well as Apresyan & Kravtsov (1996), sometimes the linear polarization plane of electromagnetic radiation physically rotates around the direction of propagation in an inhomogeneous and / or anisotropic medium. In particular, this happens if, due to refraction within the inhomogeneous medium, the light ray moves along a curve with nonzero torsion.

Polarized RTE in curved spacetime where general relativity is essential, e.g. in accretion disks around neutron stars and black holes, was considered by Fanton et al. (1997), Gammie et al. (2003), Broderick and Blandford (2003, 2004), Pitrou (2009) and others. As was shown by Broderick and Blandford (2004), due to general relativistic rotation of the polarization basis vectors $\mathbf{e}_{\parallel}^{\mu}$ and \mathbf{e}_{\perp}^{μ} along the path of propagation, the linear polarization angle apparently rotates in vacuum.

But there are many astronomical objects in approximately flat spacetime with nontrivial (both nonplanar and non-spherical) symmetry, e.g. protoplanetary nebulae, circumstellar dust envelopes, reflection nebulae, some of them being very strongly polarized (Sahai et al. 1998; Su et al. 2003; Ueta et al. 2007; Ortiz et al. 2010). In pictures given in those papers one sees conical, toroidal and maybe ellipsoidal structures. The most accurate results of numerical solution of the problems of mathematical physics (in this case, the polarized RTE with appropriate boundary conditions) are generally obtained if the symmetry of coordinate system reflects the symmetry of the physical problem. The symmetry of the radiation field is often lower than the symmetry of the distribution of matter (e.g. in conical dust cloud).

Until recently, the differential operator of polarized RTE was widely known only in Cartesian coordinate system, as well as in the case of plane-parallel, spherical and cylindrical symmetry (in the last case – not always completely). This contribution is largely based on Freimanis (2011), where general expressions for this differential operator were found; they contain the components of spatial metric tensor and their first-order derivatives. It is shown that in general case the linear polarization plane apparently rotates around the direction of propagation if the spatial coordinate system is curvilinear, because, as a rule, the polarization reference vectors are tied to local spatial basis vectors. The derivation of a differential operator of RTE in an arbitrary coordinate system has been made a standard procedure, similarly as it is if one needs to calculate divergence, curl or Laplace operator of some vector. Finally, two examples are explicitly written down: differential operator of vector RTE in elliptical conical and oblate spheroidal coordinate system.

2. THE BASIC ASSUMPTIONS

Let us assume that:

1. The 4-dimensional spacetime is a pseudo-Euclidean (i.e., Minkowski) one. Consequently, the 3-dimensional space is Euclidean. It is equipped with the coordinate system x^i , $i = 1, 2, 3$, and the corresponding metric tensor γ_{ik} :

$$ds^2 = \gamma_{ik} dx^i dx^k, \quad \Gamma = \det \gamma_{ik} > 0, \quad (1)$$

with Riemann curvature tensor $R_{klm}^i \equiv 0$ (Golab 1974).

2. The host medium is homogeneous and isotropic, and it is piecewise homogeneously filled with polydisperse scatterers. Consequently, plane electromagnetic waves propagate along straight lines within each homogeneous part of the space. The real part of the effective refractive index is almost isotropic, and there is no significant birefringence in the medium.

Let us note that the coordinate system is not assumed to be orthogonal. Orthogonal coordinate systems are treated as a special (but very convenient) case.

3. SPATIAL BASIS VECTORS AND THE DIRECTIONAL ANGLES

According to Korn and Korn (1968) one can define the covariant local spatial basis vectors \mathbf{e}_i , $i = 1, 2, 3$,

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial x^i}, \quad (2)$$

\mathbf{r} being the radius vector, and the contravariant basis vectors \mathbf{e}^i , with the biorthogonality property

$$\mathbf{e}^i \mathbf{e}_k = \delta_k^i, \quad (3)$$

where δ_k^i is Kronecker symbol. The direction of propagation of radiation is characterized by unit vector $\mathbf{\Omega}$ with contravariant components $\Omega^i = dx^i/ds$, i.e. the derivatives of coordinates by path of propagation. But it is desirable to describe $\mathbf{\Omega}$ by the angles (ϑ, φ) of the standard spherical coordinate system. This can be done using one of the following options:

Option A. The polar angle ϑ is the angle between the vector $\mathbf{\Omega}$ and the contravariant spatial basis vector \mathbf{e}^c , where c is a fixed integer, $1 \leq c \leq 3$. The azimuth φ is the angle between two half-planes: i) the half-plane delimited by the polar axis (i.e. the basis vector \mathbf{e}^c) and containing the covariant spatial basis vector \mathbf{e}_a , where a is another fixed integer, $1 \leq a \leq 3$, $a \neq c$, and ii) the half-plane delimited by the polar axis and containing the vector $\mathbf{\Omega}$. It is assumed that $\varphi > 0$ if the rotation from the half-plane $(\mathbf{e}^c, \mathbf{e}_a)$ to the half-plane $(\mathbf{e}^c, \mathbf{\Omega})$ is counter-clockwise when looking from the positive direction of vector \mathbf{e}^c . Denoting with b the remaining integer $1 \leq b \leq 3$, $a \neq b \neq c$, one can mention that the angle φ is measured in plane $(\mathbf{e}_a, \mathbf{e}_b)$ being orthogonal to \mathbf{e}^c .

Option B. The covariant basis vector \mathbf{e}_c is taken as the polar axis, and the plane $(\mathbf{e}_c, \mathbf{e}^a)$ – as zero azimuth plane. Azimuth φ is defined in right-handed version again; it is measured in the plane $(\mathbf{e}^a, \mathbf{e}^b)$ being orthogonal to \mathbf{e}_c .

Orthogonal coordinate systems. In this case the respective covariant and contravariant basis vectors are parallel, the definitions of angles (ϑ, φ) according to Option A and Option B coincide, and all the formulae are greatly simplified.

Further in this paper it is assumed that values of the tensor indices a, b and c are fixed as stated above, and no summation is done over them if they are in one and the same tensor expression at both covariant and contravariant position (e.g., $E^a T_{abc}^c E^b$). For all other values of tensor indices (i, j, k, l, \dots) , usual summation rule is applied (Golab 1974).

Explicit expressions for the components of $\mathbf{\Omega}$ through (ϑ, φ) and the derivatives by path $d\vartheta/ds, d\varphi/ds$ for all the above mentioned options were obtained in (Freimanis 2011). For nonorthogonal coordinate systems Christoffel symbols of the second kind (Golab 1974, Henderson 1998) appear. For orthogonal coordinate systems the simplifications lead to obvious formulae

$$\Omega^a = \frac{\sin \vartheta \cos \varphi}{H_a}, \quad \Omega^b = \frac{S \varepsilon_{abc}}{H_b} \sin \vartheta \sin \varphi, \quad \Omega^c = \frac{\cos \vartheta}{H_c}, \quad (4)$$

where H_i is Lamé coefficient corresponding to the i th coordinate (Morse and Feshbach 1953), ε_{ijk} is Ricci symbol (Golab 1974), and

$$S = \begin{cases} 1, & \text{if } (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \text{ is the right-handed trihedron,} \\ -1, & \text{if } (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \text{ is the left-handed trihedron,} \end{cases} \quad (5)$$

as well as moderately long expressions for $d\vartheta/ds$ and $d\varphi/ds$ containing Lamé coefficients and their first order derivatives.

4. POLARIZATION REFERENCE BASIS VECTORS

Either for Option A, Option B or orthogonal spatial coordinate system, it is usual that Stokes parameters are determined using two mutually orthogonal polarization reference basis unit vectors $(\mathbf{e}_{\parallel}, \mathbf{e}_{\perp})$ defined so that $(\mathbf{e}_{\parallel}, \mathbf{e}_{\perp}, \mathbf{\Omega})$ is a right-handed trihedron of mutually orthogonal unit vectors (e.g., Mishchenko et al. 2006).

Let \mathbf{u}_3 be the (unnormalized) spatial basis vector used as local polar axis $\vartheta = 0$, i.e. $\mathbf{u}_3 = \mathbf{e}^c$ for Option A, $\mathbf{u}_3 = \mathbf{e}_c$ for Option B, and any of the two vectors $(\mathbf{e}^c, \mathbf{e}_c)$ for orthogonal coordinate systems. The traditional way how to define $(\mathbf{e}_{\parallel}, \mathbf{e}_{\perp})$ is as follows:

$$\mathbf{e}_{\perp} = S \frac{\mathbf{u}_3 \times \mathbf{\Omega}}{|\mathbf{u}_3 \times \mathbf{\Omega}|}, \quad \mathbf{e}_{\parallel} = S \mathbf{e}_{\perp} \times \mathbf{\Omega}, \quad (6)$$

where $|\dots|$ denotes Euclidean vector norm. From the physical conditions specified in Section 2 it follows that $\mathbf{\Omega} = \text{const}$ along the path of propagation of radiation, but the basis vector \mathbf{u}_3 is generally changing in curvilinear coordinate system. Due to this, the vectors $(\mathbf{e}_{\parallel}, \mathbf{e}_{\perp})$ generally rotate around $\mathbf{\Omega}$, and the speed of rotation per unit path is

$$\frac{d\psi}{ds} = \mathbf{e}_{\perp} \frac{d\mathbf{e}_{\parallel}}{ds} = e_{\perp}^i \frac{D e_{\parallel i}}{Ds} = -\mathbf{e}_{\parallel} \frac{d\mathbf{e}_{\perp}}{ds} = -e_{\parallel}^i \frac{D e_{\perp i}}{Ds}, \quad (7)$$

where D/Ds is the absolute derivative along the path of propagation (Golab 1974). Again, explicit expressions for $d\psi/ds$ were derived in (Freimanis 2011); if the coordinate system is orthogonal, then

$$\begin{aligned} \frac{d\psi}{ds} &= \frac{\sin 2\varphi}{2H_c} \frac{\partial}{\partial x^c} \ln \frac{H_a}{H_b} \\ &+ \cot \vartheta \left(\frac{S\varepsilon_{abc} \cos \varphi}{H_b} \frac{\partial \ln H_c}{\partial x^b} - \frac{\sin \varphi}{H_a} \frac{\partial \ln H_c}{\partial x^a} \right). \end{aligned} \quad (8)$$

5. RADIATIVE TRANSFER EQUATION: THE GENERAL CASE

With the above definitions and expressions, radiative transfer equation for Stokes vector $\mathbf{I} = (I, Q, U, V)^T$ (Mishchenko et al. 2006) is

$$\begin{aligned} \Omega^j \frac{\partial \mathbf{I}(\mathbf{r}, \vartheta, \varphi)}{\partial x^j} &+ \frac{d\vartheta}{ds} \frac{\partial \mathbf{I}(\mathbf{r}, \vartheta, \varphi)}{\partial \vartheta} + \frac{d\varphi}{ds} \frac{\partial \mathbf{I}(\mathbf{r}, \vartheta, \varphi)}{\partial \varphi} - \frac{d\psi}{ds} \mathbf{U}_1 \mathbf{I}(\mathbf{r}, \vartheta, \varphi) \\ &= -k'' \mathbf{I}(\mathbf{r}, \vartheta, \varphi) - n_0 \mathbf{K}(\vartheta, \varphi) \mathbf{I}(\mathbf{r}, \vartheta, \varphi) \\ &+ n_0 \int_{4\pi} \mathbf{Z}(\boldsymbol{\Omega}, \boldsymbol{\Omega}') \mathbf{I}(\mathbf{r}, \boldsymbol{\Omega}') d\boldsymbol{\Omega}' + \mathbf{B}_{(0)}(\mathbf{r}, \vartheta, \varphi), \end{aligned} \quad (9)$$

where k'' is the imaginary part of wavenumber in host medium, $\mathbf{K}(\vartheta, \varphi)$ and $\mathbf{Z}(\boldsymbol{\Omega}, \boldsymbol{\Omega}')$ are Stokes extinction matrix and Stokes phase matrix, respectively (averaged over the compositions, forms, physical states and orientations of the polydisperse scatterers), n_0 is the volume concentration of the scatterers, and $\mathbf{B}_{(0)}(\mathbf{r}, \vartheta, \varphi)$ is the vector of primary sources. Matrix \mathbf{U}_1 characterizes the change of the components of Stokes vector created by infinitesimal rotation of polarization reference basis vectors, i.e. the seeming rotation of linear polarization plane in vacuum due to purely geometric reasons:

$$\mathbf{U}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (10)$$

Formulae equivalent to Equation (10) were given by Broderick and Blandford (2004) but the authors attributed all that only to specifically general relativistic effects.

Due to general approach used in this study, all the terms in the left-hand side of Equation (9) can be explicitly written down using well-defined standard procedures, and no tricky calculations of differential geometry are needed. The application of these routine methods will be shown below, as a couple of examples.

6. EXAMPLE: ELLIPTICAL CONICAL COORDINATE SYSTEM

The standard definition of elliptical conical coordinate system (Morse and Feshbach 1953; Korn and Korn 1968) uses three coordinates (u, v, w) defining mutually orthogonal families of spheres, elliptic cones around z axis, and elliptic cones around x axis. The coordinates (u, v, w) are 8-fold degenerated, and they have several other disadvantages.

Alternative parameterization of essentially the same coordinate system can be offered as follows. Let us express the Descartes coordinates (x, y, z) through elliptical conical coordinates (r, Θ, Φ) by formulae

$$x = \frac{r \sin \Theta \cos \Phi}{\sqrt{1 - \beta^2 \sin^2 \Phi}}, \quad y = r \sin \Phi \sqrt{\frac{\sin^2 \Theta - \beta^2}{1 - \beta^2 \sin^2 \Phi}}, \quad z = \frac{r \cos \Theta}{\sqrt{1 - \beta^2 \sin^2 \Phi}}, \quad (11)$$

where the arithmetic values of all square roots are taken. The ellipticity parameter β and the coordinates (r, Θ, Φ) are bounded by the conditions $0 \leq \beta < 1$, $r \geq 0$, $\arcsin \beta \leq \Theta \leq \pi - \arcsin \beta$, $0 \leq \Phi < 2\pi$.

This is an orthogonal coordinate system, with the basis vectors $(\mathbf{e}_r, \mathbf{e}_\Theta, \mathbf{e}_\Phi)$ constituting the right-handed trihedron almost everywhere. Let us assume the "conical" viewpoint, setting the basis vector \mathbf{e}_Θ to be the polar axis. Now the differential operator of radiative transfer equation, i.e. the left-hand side of Equation (9), is

$$\begin{aligned} \hat{\mathbf{L}}\mathbf{I}(\mathbf{r}, \vartheta, \varphi) &= \sin \vartheta \cos \varphi \frac{\partial \mathbf{I}(r, \Theta, \Phi; \vartheta, \varphi)}{\partial r} \\ &+ \frac{\cos \vartheta}{r} \sqrt{\frac{(\sin^2 \Theta - \beta^2)(1 - \beta^2 \sin^2 \Phi)}{\sin^2 \Theta - \beta^2(1 - \cos^2 \Theta \sin^2 \Phi)}} \frac{\partial \mathbf{I}(r, \Theta, \Phi; \vartheta, \varphi)}{\partial \Theta} \\ &- \frac{(1 - \beta^2 \sin^2 \Phi) \sin \vartheta \sin \varphi}{r \sqrt{\sin^2 \Theta - \beta^2(1 - \cos^2 \Theta \sin^2 \Phi)}} \frac{\partial \mathbf{I}(r, \Theta, \Phi; \vartheta, \varphi)}{\partial \Phi} \\ &+ \frac{\cos \vartheta \cos \varphi - \chi_0(\Theta, \Phi, \vartheta, \varphi) \sin \vartheta \sin \varphi}{r} \frac{\partial \mathbf{I}(r, \Theta, \Phi; \vartheta, \varphi)}{\partial \vartheta} \\ &- \frac{1}{r} \left[\frac{\sin \varphi}{\sin \vartheta} + \chi_0(\Theta, \Phi, \vartheta, \varphi) \cos \vartheta \cos \varphi \right] \frac{\partial \mathbf{I}(r, \Theta, \Phi; \vartheta, \varphi)}{\partial \varphi} \\ &+ \frac{\cot \vartheta \sin \varphi + \chi_0(\Theta, \Phi, \vartheta, \varphi) \cos \varphi}{r} \mathbf{U}_1 \mathbf{I}(r, \Theta, \Phi; \vartheta, \varphi), \end{aligned} \quad (12)$$

where

$$\begin{aligned} \chi_0(\Theta, \Phi, \vartheta, \varphi) &= \frac{1}{2 [\sin^2 \Theta - \beta^2 (1 - \cos^2 \Theta \sin^2 \Phi)]^{\frac{3}{2}}} \\ &\times \left[\sin 2\Theta \sqrt{\sin^2 \Theta - \beta^2 (1 - \beta^2 \sin^2 \Phi)}^{\frac{3}{2}} \sin \varphi \right. \\ &\quad \left. + \beta^2 (1 - \beta^2) \sin 2\Phi \cot \vartheta \right]. \end{aligned} \quad (13)$$

7. EXAMPLE: OBLATE SPHEROIDAL COORDINATE SYSTEM

Here the standard definition of oblate spheroidal coordinate system (Korn and Korn 1968) is used:

$$x = a \cosh \xi \sin \Theta \cos \Phi, \quad y = a \cosh \xi \sin \Theta \sin \Phi, \quad z = a \sinh \xi \cos \Theta, \quad (14)$$

with the conditions $a > 0$, $\xi \geq 0$, $0 \leq \Theta \leq \pi$, $0 \leq \Phi < 2\pi$. The differential operator of polarized radiative transfer equation in this case can be obtained both directly using the definition Equation (14) as well as a special case of that for (triaxial) ellipsoidal coordinate system. It is as follows:

$$\begin{aligned}
\hat{\mathbf{L}}\mathbf{I}(\mathbf{r}, \vartheta, \varphi) = & \frac{\cos \vartheta}{a\sqrt{\sinh^2 \xi + \cos^2 \Theta}} \frac{\partial \mathbf{I}(\xi, \Theta, \Phi; \vartheta, \varphi)}{\partial \xi} \\
& + \frac{\sin \vartheta \cos \varphi}{a\sqrt{\sinh^2 \xi + \cos^2 \Theta}} \frac{\partial \mathbf{I}(\xi, \Theta, \Phi; \vartheta, \varphi)}{\partial \Theta} \\
& + \frac{\sin \vartheta \sin \varphi}{a \cosh \xi \sin \Theta} \frac{\partial \mathbf{I}(\xi, \Theta, \Phi; \vartheta, \varphi)}{\partial \Phi} \\
& + \frac{(2 \tanh \xi \sin^2 \Theta \sin^2 \varphi - \sinh 2\xi) \sin \vartheta - \sin 2\Theta \cos \vartheta \cos \varphi}{2a (\sinh^2 \xi + \cos^2 \Theta)^{\frac{3}{2}}} \\
& \quad \times \frac{\partial \mathbf{I}(\xi, \Theta, \Phi; \vartheta, \varphi)}{\partial \vartheta} \\
& + \left(\tanh \xi \sin^2 \Theta \cos \vartheta \cos \varphi - \cosh^2 \xi \cot \Theta \sin \vartheta + \frac{\sin 2\Theta}{2 \sin \vartheta} \right) \\
& \quad \times \frac{\sin \varphi}{a (\sinh^2 \xi + \cos^2 \Theta)^{\frac{3}{2}}} \frac{\partial \mathbf{I}(\xi, \Theta, \Phi; \vartheta, \varphi)}{\partial \varphi} \\
& - \frac{\sin \Theta \sin \varphi (\tanh \xi \sin \Theta \cos \varphi + \cos \Theta \cot \vartheta)}{a (\sinh^2 \xi + \cos^2 \Theta)^{\frac{3}{2}}} \mathbf{U}_1 \mathbf{I}(\xi, \Theta, \Phi; \vartheta, \varphi).
\end{aligned} \tag{15}$$

8. CONCLUSIONS

Using the methods of tensor analysis, general expressions for the differential operator of vector radiative transfer equation in arbitrary curvilinear coordinate system have been obtained. It has been shown that the plane of linear polarization can seemingly rotate in the vacuum if the polarization reference frame is tied to the basis vectors of the spatial coordinate system. Feasibility of the general method has been demonstrated by derivation of polarized RTE in elliptical conical and oblate spheroidal coordinate systems.

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