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Cauchy type means for some generalized convex functions

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Abstract

In this paper, we establish Jensen's inequality for s-convex functions in the first sense. By using Jensen's inequalities, we obtain some Cauchy type means for p-convex and s-convex functions in the first sense. Also, by using Hermite–Hadamard inequalities for the respective generalized convex functions, we find new generalized Cauchy type means.

Keywords: Cauchy mean value theorem; Jensen's inequality; Hermite–Hadamard inequality; p-convex function; s-convex function in the first sense

1 Introduction

Cauchy mean value theorem is of huge importance in mathematical analysis. Mercer [18] and Pečarić [21] made connection between Cauchy type means and Jensen's inequality. These are given both in discrete and in integral form and have many applications. A meaningful advancement in theory of Cauchy type means is given in [1–5, 18–21]. Also see [8–11, 15–17] for more information about means. The following result is given in [19], which involves Jensen's inequality both in numerator and denominator.

Theorem 1.1 ([19]) *Let $G \subseteq \mathbb{R}$ be an interval and $r_i > 0$ for all $1 \leq i \leq n$ such that $\Sigma_{i=1}^n r_i = S_n$ and $c_1, \ldots, c_n \in G$ not all the same. Consider the twice differentiable functions $\zeta_1, \zeta_2 : G \to \mathbb{R}$ such that*

$$0 \leq l \leq \zeta_1''(c) \leq L \quad \textit{and} \quad 0 \leq m \leq \zeta_2''(x) \leq M \quad \textit{for all } c \in G.$$

Then

$$\frac{l}{M} \leq \frac{\frac{1}{S_n}\Sigma_{i=1}^n r_i \zeta_1(c_i) - \zeta_1(\frac{1}{S_n}\Sigma_{i=1}^n r_i c_i)}{\frac{1}{S_n}\Sigma_{i=1}^n r_i \zeta_2(c_i) - \zeta_2(\frac{1}{S_n}\Sigma_{i=1}^n r_i c_i)} \leq \frac{L}{m}. \tag{1}$$

Here our aim is to find some Cauchy type means for p-convex and s-convex functions in the first sense using Jensen's and Hermite–Hadamard inequalities, respectively.

Let M, N be two bivariable means defined in a real interval G, and let $J \subseteq G$ be a subinterval of G. According to Aumann [6], a function $\zeta : J \to G$ is convex with respect to the pair of means (M,N) if

$$\zeta\big(M(j_1,j_2)\big) \leq N\big(\zeta(j_1),\zeta(j_2)\big), \quad j_1,j_2 \in J;$$

and ζ is convex with respect to M if

$$\zeta\big(M(j_1,j_2)\big) \leq M\big(\zeta(j_1),\zeta(j_2)\big), \quad j_1,j_2 \in J.$$

These notions generalize the classical notions of convexity. Moreover, taking for M the weighted power mean, i.e.,

$$M(j_1,j_2) = \big[rj_1^p + (1-r)j_2^p\big]^{\frac{1}{p}},$$

and for N the weighted arithmetic mean

$$N(j_1,j_2) = [rj_1 + (1-r)j_2,$$

one gets the following definition.

Definition 1.1 ([13, 14]) Let $G \subset (0,\infty)$ be a real interval and $p \in \mathbb{R}\backslash\{0\}$. A function $\zeta : G \to \mathbb{R}$ is said to be a p-convex function if

$$\zeta\big[\big[rg_1^p + (1-r)g_2^p\big]^{\frac{1}{p}}\big] \leq r\zeta(g_1) + (1-r)\zeta(g_2) \tag{2}$$

for all $g_1,g_2 \in G$ and $r \in [0,1]$. If inequality (2) is reversed, then ζ is called p-concave function.

Definition 1.2 ([12]) Let $s \in (0,1]$. A function $\zeta : [0,\infty) \to \mathbb{R}$ is called an s-convex function (in the first sense) or $\zeta \in K_s^1$ if

$$\zeta(r_1g_1 + r_2g_2) \leq r_1^s\zeta(g_1) + r_2^s\zeta(g_2) \tag{3}$$

for all $g_1,g_2 \in \mathbb{R}^+ = [0,\infty)$ and $r_1,r_2 \geq 0$ with $r_1^s + r_2^s = 1$.

2 Cauchy type means for p-convex functions in Jensen's sense

Toplu et al. [22] proved Jensen's inequality for p-convex functions as follows.

Theorem 2.1 ([22]) *Let $p \in \mathbb{R}\backslash\{0\}$ and $\zeta : G \subset (0,\infty) \to \mathbb{R}$ be a p-convex function. Let $g_i \in G$ and $r_i \in [0,1]$, $0 \leq i \leq n$, then the following inequality holds*:

$$\zeta\left(\left(\sum_1^n r_i g_i^p\right)^{\frac{1}{p}}\right) \leq \sum_1^n r_i\zeta(g_i), \tag{4}$$

where $\sum_1^n r_i = 1$.

Now, by using Theorem 2.1, we state and prove the following theorem, which gives the Cauchy type mean for p-convex function.

Theorem 2.2 *Let $G \subset (0,\infty)$ be an interval, $p \in \mathbb{R}\backslash\{0\}$, and $r_i \in [0,1]$. Let $\zeta_1, \zeta_2 \in C^2(G)$ be p-convex functions. Then there exist some $\chi \in G$ such that the following equality holds*:

$$\frac{\sum_1^n r_i\zeta_1(g_i) - \zeta_1((\sum_1^n r_i g_i^p)^{\frac{1}{p}})}{\sum_1^n r_i\zeta_2(g_i) - \zeta_2((\sum_1^n r_i g_i^p)^{\frac{1}{p}})} = \frac{\zeta_1''(\chi)}{\zeta_2''(\chi)}, \tag{5}$$

with each $r_i \in [0,1]$ such that $\sum_1^n r_i = 1$ and provided that the denominators are non-zero.

Proof Let us define

$$H := \left(\sum_1^n r_i g_i^p\right)^{\frac{1}{p}}$$

and

$$(T\zeta_1)(\lambda) := \sum_1^n r_i\zeta_1\bigl(\lambda g_i + (1-\lambda)H\bigr) - \zeta_1(H),$$

where $\lambda \in [0,1]$. Similarly, we define $(T\zeta_2)(\lambda)$.

Note that

$$(T\zeta_1)'(\lambda) := \sum_1^n r_i(g_i - H)\zeta_1'\bigl(\lambda g_i + (1-\lambda)H\bigr)$$

and

$$(T\zeta_1)''(\lambda) := \sum_1^n r_i(g_i - H)^2\zeta_1''\bigl(\lambda g_i + (1-\lambda)H\bigr).$$

Now consider a function $Q(\lambda)$ defined as follows:

$$Q(\lambda) = (T\zeta_2)(1)(T\zeta_1)(\lambda) - (T\zeta_1)(1)(T\zeta_2)(\lambda),$$

such that we have

$$Q(0) = Q(1) = Q'(0) = 0.$$

Then from two applications of mean value theorem, we have $\upsilon \in G$ so that

$$Q''(\upsilon) = 0.$$

It implies that

$$\sum_{i=1}^n r_i(g_i - H)^2\bigl[(T\zeta_2)(1)\zeta_1''\bigl(\upsilon g_i + (1-\upsilon)H\bigr) - (T\zeta_1)(1)\zeta_2''\bigl(\upsilon g_i + (1-\upsilon)H\bigr)\bigr] = 0. \tag{6}$$

For some fixed υ, the expression in the square brackets in (6) is a continuous function of g_i, so it vanishes. Corresponding to that value of g_i, we can have a number

$$\chi = \upsilon g_i + (1-\upsilon)H$$

such that

$$(T\zeta_2)(1).\zeta_1''(\chi) - (T\zeta_1)(1).\zeta_2''(\chi) = 0.$$

This gives equality (5). □

Corollary 2.3 *Let $G \subset (0,\infty)$ be an interval, $p \in \mathbb{R}\backslash\{0\}$, and $r_i \in [0,1]$. Let $\zeta_1, \zeta_2 \in C^2(G)$ be p-convex functions such that $\frac{\zeta_1''}{\zeta_2''}$ is invertible. Then there exist some $\chi \in G$ such that the following equality holds*:

$$\chi = \left(\frac{\zeta_1''}{\zeta_2''}\right)^{-1}\left(\frac{\sum_1^n r_i\zeta_1(g_i) - \zeta_1((\sum_1^n r_i g_i^p)^{\frac{1}{p}})}{\sum_1^n r_i\zeta_2(g_i) - \zeta_2((\sum_1^n r_i g_i^p)^{\frac{1}{p}})}\right), \tag{7}$$

with each $r_i \in [0,1]$ such that $\sum_1^n r_i = 1$ and provided that the denominators are non-zero.

Corollary 2.4 *Let $G \subset (0,\infty)$ be an interval, $p \in \mathbb{R}\backslash\{0\}$, and $r_i \in [0,1]$. Let $\zeta \in C^2(G)$ be a p-convex function. Then there exist some $\chi \in G$ such that the following equality holds*:

$$\sum_1^n r_i\zeta(g_i) - \zeta\left(\left(\sum_1^n r_i g_i^p\right)^{\frac{1}{p}}\right) = \frac{\zeta''(\chi)}{2}\left(\sum_1^n r_i g_i^2 - \left(\left(\sum_1^n r_i g_i^p\right)^{\frac{1}{p}}\right)^2\right) \tag{8}$$

with each $r_i \in [0,1]$ such that $\Sigma_1^n r_i = 1$.

Proof By letting $\zeta_1 = \zeta$ and $\zeta_2(w) = w^2$, where $w \in (0,\infty)$, in Theorem 2.2, we achieve equality (8). □

3 Cauchy type means for *p*-convex functions in the Hermite–Hadamard sense

Let $\zeta : G \subset (0,\infty) \to \mathbb{R}$ be a p-convex function, $p \in \mathbb{R}\backslash\{0\}$, and $g_1, g_2 \in G$ with $g_1 < g_2$. If $\zeta \in L_1[g_1, g_2]$, then we have (e.g., see [13])

$$\zeta\left(\left(\frac{g_1^p + g_2^p}{2}\right)^{\frac{1}{p}}\right) \le \frac{p}{g_2^p - g_1^p}\int_{g_1}^{g_2} \frac{\zeta(w)}{w^{1-p}}\,dw \le \frac{\zeta(g_1)+\zeta(g_2)}{2}. \tag{9}$$

By using the right half of inequality (9), we have following result.

Theorem 3.1 *Let $G \subset (0,\infty)$ be an interval, $p \in \mathbb{R}\backslash\{0\}$, and $g_1, g_2 \in G$ with $g_1 < g_2$. Let $\zeta_1, \zeta_2 \in C^2(G)$ be p-convex functions. Then there exists some $\chi \in G$ such that the following equality holds*:

$$\frac{\frac{p}{g_2^p - g_1^p}\int_{g_1}^{g_2} \frac{\zeta_1(w)}{w^{1-p}}\,dw - \zeta_1((\frac{g_1^p+g_2^p}{2})^{\frac{1}{p}})}{\frac{p}{g_2^p - g_1^p}\int_{g_1}^{g_2} \frac{\zeta_2(w)}{w^{1-p}}\,dw - \zeta_2((\frac{g_1^p+g_2^p}{2})^{\frac{1}{p}})} = \frac{\zeta_1''(\chi)}{\zeta_2''(\chi)}, \tag{10}$$

provided that the denominators are non-zero.

Proof Let

$$H := \left(\frac{g_1^p + g_2^p}{2}\right)^{\frac{1}{p}}$$

and

$$(T\zeta_1)(\lambda) := \frac{p}{g_2^p - g_1^p} \int_{g_1}^{g_2} \frac{\zeta_1(\lambda w + (1-\lambda)H)}{w^{1-p}}\,dw - \zeta_1(H),$$

where $\lambda \in [0,1]$. Similarly, we can define $(T\zeta_2)(\lambda)$.

Observe that

$$(T\zeta_1)'(\lambda) := \frac{p}{g_2^p - g_1^p} \int_{g_1}^{g_2} (w-H)\frac{\zeta_1'(\lambda w + (1-\lambda)H)}{w^{1-p}}\,dw$$

and

$$(T\zeta_1)''(\lambda) := \frac{p}{g_2^p - g_1^p} \int_{g_1}^{g_2} (w-H)^2\frac{\zeta_1''(\lambda w + (1-\lambda)H)}{w^{1-p}}\,dw.$$

Now consider the function $Q(\lambda)$ defined by

$$Q(\lambda) = (T\zeta_2)(1)(T\zeta_1)(\lambda) - (T\zeta_1)(1)(T\zeta_2)(\lambda)$$

such that we have

$$Q(0) = Q(1) = Q'(0) = 0.$$

Then, from two applications of mean value theorem, we find $\upsilon \in G$ such that

$$Q''(\upsilon) = 0.$$

It implies

$$\begin{aligned}\frac{p}{g_2^p - g_1^p} \int_{[g_1,g_2]} (w-H)^2\big[&(T\zeta_2)(1)\zeta_1''\big(w\upsilon - (1-\upsilon)H\big)\\ &- (T\zeta_1)(1)\zeta_2''\big(w\upsilon - (1-\upsilon)H\big)\big] = 0.\end{aligned} \tag{11}$$

For any fixed υ, the expression in the square brackets in (11) is a continuous function of w, so it vanishes. Corresponding to that value of w, we get a number

$$\chi = w\upsilon + (1-\upsilon)H$$

such that

$$(T\zeta_2)(1).\zeta_1''(\chi) - (T\zeta_1)(1).\zeta_2''(\chi) = 0.$$

This gives equality (10). □

Corollary 3.2 *If $\frac{\zeta_1''}{\zeta_2''}$ is invertible, then we have*

$$\chi = \left(\frac{\zeta_1''(\chi)}{\zeta_2''(\chi)}\right)^{-1}\left(\frac{\frac{p}{g_2^p-g_1^p}\int_{g_1}^{g_2}\frac{\zeta_1(w)}{w^{1-p}}\,dw - \zeta_1((\frac{g_1^p+g_2^p}{2})^{\frac{1}{p}})}{\frac{p}{g_2^p-g_1^p}\int_{g_1}^{g_2}\frac{\zeta_2(w)}{w^{1-p}}\,dw - \zeta_2((\frac{g_1^p+g_2^p}{2})^{\frac{1}{p}})}\right). \tag{12}$$

Corollary 3.3 *By taking $\zeta_2(w) = w^2$ and $\zeta_1 = \zeta$ in Theorem 3.1, we have*

$$\begin{aligned}&\frac{p}{g_2^p - g_1^p}\int_{g_1}^{g_2}\frac{\zeta(w)}{w^{1-p}}\,dw - \zeta\left(\left(\frac{g_1^p + g_2^p}{2}\right)^{\frac{1}{p}}\right)\\ &\quad = \frac{\zeta''(\chi)}{2}\left[\frac{p}{g_2^p - g_1^p}\left(\frac{g_2^{p+2} - g_1^{p+2}}{p+2}\right) - \left(\frac{g_1^p + g_2^p}{2}\right)^{\frac{2}{p}}\right].\end{aligned} \tag{13}$$

4 Cauchy type means for s-convex functions in Jensen's sense

Here first we prove Jensen's inequality for s-convex function.

Lemma 4.1 *Let $s \in (0,1]$ and $\zeta : G \subset \mathbb{R}^+ \to \mathbb{R}$ be an s-convex function. Let $\sum_1^n r_i g_i$ be convex combinations of points $g_i \in G$ with coefficients $r_i \in [0,1]$. Then each s-convex function (in the first sense) satisfies the inequality*

$$\zeta\left(\sum_1^n r_i g_i\right) \le \sum_1^n r_i^s \zeta(g_i), \tag{14}$$

where $\sum_1^n r_i^s = 1$.

Proof We apply induction on the number of points in convex combination.

Basis step: for $n = 1$, equality (14) is true since

$$\zeta(r_1 g_1) \le r_1^s \zeta(g_1),$$

where $r_1^s = 1$ since $r_1 = 1$.

Induction step: suppose that (14) holds for all convex combinations of points containing less than or equal to $n-1$ points. Let $r_n \neq 1$ and

$$w = \sum_1^{n-1}\frac{r_i}{1-r_n}g_i,$$

where the sum $\sum_1^{n-1}(\frac{r_i}{1-r_n})g_i \in G$. Then, by induction hypothesis, we have

$$\zeta(w) \le \sum_1^{n-1}\left(\frac{r_i}{1-r_n}\right)^s \zeta(g_i). \tag{15}$$

By using (3) and (15), we get

$$\begin{aligned}\zeta\left(\sum_1^n r_i g_i\right) &= \zeta\big((1-r_n)w + r_n g_n\big)\\ &\leq (1-r_n)^s \zeta(w) + r_n^s \zeta(g_n)\\ &\leq (1-r_n)^s \sum_1^{n-1}\left(\frac{r_i}{1-r_n}\right)^s \zeta(g_i) + r_n^s\zeta(g_n)\\ &= \sum_1^n r_i^s \zeta(g_i). \end{aligned} \tag{16}$$

Thus we get (14). □

Remark 4.1 By taking $s = 1$ in Lemma 4.1 we can get Jensen's inequality for convex function.

Now, by using the above lemma, we state and prove the following theorem, which gives the Cauchy type means for s-convex function.

Theorem 4.1 *Let $s \in (0,1]$ and $r_i \in [0,1]$. Let $\zeta_1, \zeta_2 \in C^2(G \subset [0,\infty))$ be s-convex functions (in the first sense). Then there exist some $\chi \in G$ such that the following equality holds*:

$$\frac{\sum_1^n r_i^s \zeta_1(g_i) - \zeta_1(\sum_1^n r_i g_i)}{\sum_1^n r_i^s \zeta_2(g_i) - \zeta_2(\sum_1^n r_i g_i)} = \frac{\zeta_1''(\chi)}{\zeta_2''(\chi)} \tag{17}$$

with each $r_i \in [0,1]$ such that $\sum_1^n r_i^s = 1$ and provided that the denominators are non-zero.

Proof Define

$$H := \sum_1^n r_i g_i$$

and

$$(T\zeta_1)(\lambda) := \sum_1^n r_i^s \zeta_1\big(\lambda g_i + (1-\lambda)H\big) - \zeta_1(H),$$

where $\lambda \in [0,1]$. Accordingly, we can define $(T\zeta_2)(\lambda)$.

Note that

$$(T\zeta_1)'(\lambda) := \sum_1^n r_i^s (g_i - H)\zeta_1'\big(\lambda g_i + (1-\lambda)H\big)$$

and

$$(T\zeta_1)''(\lambda) := \sum_1^n r_i^s (g_i - H)^2 \zeta_1''\big(\lambda g_i + (1-\lambda)H\big).$$

Now consider the function $Q(\lambda)$ defined by

$$Q(\lambda) = (T\zeta_2)(1)(T\zeta_1)(\lambda) - (T\zeta_1)(1)(T\zeta_2)(\lambda)$$

such that we have

$$Q(0) = Q(1) = Q'(0) = 0.$$

Then, from two applications of mean value theorem, we find $\upsilon \in G$ such that

$$Q''(\upsilon) = 0.$$

It follows that

$$\sum_{i=1}^{n} r_i^s (g_i - H)^2 \bigl[(T\zeta_2)(1).\zeta_1''\bigl(\upsilon g_i + (1-\upsilon)H\bigr) - (T\zeta_1)(1).\zeta_2''\bigl(\upsilon g_i + (1-\upsilon)H\bigr)\bigr] = 0. \tag{18}$$

For any fixed υ, the expression in the square brackets in (18) is a continuous function of g_i, so it vanishes. Corresponding to that value of g_i, we get a number

$$\chi = \upsilon + (1-\upsilon)H,$$

so that

$$(T\zeta_2)(1).\zeta_1''(\chi) - (T\zeta_1)(1).\zeta_2''(\chi) = 0.$$

This gives equality (17). □

Corollary 4.2 *Let $s \in (0,1]$. Let $\zeta_1, \zeta_2 \in C^2(G \subset [0,\infty))$ be s-convex functions (in the first sense) such that $\frac{\zeta_1''}{\zeta_2''}$ is invertible. Then there exist some $\chi \in G$ such that the following equality holds*:

$$\chi = \left(\frac{\zeta_1''}{\zeta_2''}\right)^{-1} \left(\frac{\sum_1^n r_i^s \zeta_1(g_i) - \zeta_1(\sum_1^n r_i g_i)}{\sum_1^n r_i^s \zeta_2(g_i) - \zeta_2(\sum_1^n r_i g_i)}\right), \tag{19}$$

with each $r_i \in [0,1]$ such that $\sum_1^n r_i^s = 1$ and provided that the denominators are non-zero.

Corollary 4.3 *Let $s_1, s_2 \in (0,1)$. Let $\zeta_1, \zeta_2 \in C^2((0,\infty))$ be an s_1-convex function and an s_2-convex function (in the first sense), respectively, defined as $\zeta_1(w) = w^{s_1}$ and $\zeta_2(w) = w^{s_2}$. Then, from Theorem 4.1, we get*

$$\frac{\sum_1^n r_i^{s_1} (g_i)^{s_1} - (\sum_1^n r_i g_i)^{s_1}}{\sum_1^n r_i^{s_2} (g_i)^{s_2} - (\sum_1^n r_i g_i)^{s_2}} = \frac{s_1(s_1-1)}{s_2(s_2-1)} (\chi)^{s_1-s_2}. \tag{20}$$

5 Cauchy type means for *s*-convex functions in the Hermite–Hadamard sense

Drgomir and Fitzpatrick [7] gave the following result.

Theorem 5.1 *Suppose that $\zeta : [0,\infty) \to \mathbb{R}$ is an s-convex function in the first sense, where $s \in (0,1)$, and let $g_1, g_2 \in [0,\infty)$, $g_1 \leq g_2$. Then the following inequality holds*:

$$\zeta\left(\frac{g_1 + g_2}{2}\right) \leq \frac{1}{g_2 - g_1} \int_{g_1}^{g_2} \zeta(w)\,dw \leq \frac{\zeta(g_1) + s\zeta(g_2)}{s+1}. \tag{21}$$

The above inequalities are sharp.

From inequality (21) we give the following result.

Theorem 5.2 *Suppose that $\zeta_1, \zeta_2 : [0,\infty) \to \mathbb{R}$ is an s-convex function in the first sense, where $s \in (0,1)$, and let $g_1, g_2 \in [0,\infty)$, $g_1 \leq g_2$. Let $\zeta_1, \zeta_2 \in C^2([g_1, g_2])$. Then there exist some $\chi \in [g_1, g_2]$ such that the following equality holds*:

$$\frac{\frac{1}{g_2-g_1}\int_{g_1}^{g_2} \zeta_1(w)\,dw - \zeta_1(\frac{g_1+g_2}{2})}{\frac{1}{g_2-g_1}\int_{g_1}^{g_2} \zeta_2(w)\,dw - \zeta_2(\frac{g_1+g_2}{2})} = \frac{\zeta_1''(\chi)}{\zeta_2''(\chi)}, \tag{22}$$

provided that the denominators are non-zero.

Proof Let

$$H := \frac{g_1 + g_2}{2}$$

and

$$(T\zeta)(\lambda) := \frac{1}{g_2 - g_1} \int_{g_1}^{g_2} \zeta_1\big(\lambda w + (1-\lambda)H\big)\,dw - \zeta_1(H),$$

where $\lambda \in [0,1]$. Accordingly, we can define $(T\zeta_2)(\lambda)$.

We can have

$$(T\zeta_1)'(\lambda) := \frac{1}{g_2 - g_1} \int_{g_1}^{g_2} (w - H)\zeta_1'\big(\lambda w + (1-\lambda)H\big)\,dw$$

and

$$(T\zeta_1)''(\lambda) := \frac{1}{g_2 - g_1} \int_{g_1}^{g_2} (w - H)^2\zeta_1''\big(\lambda w + (1-\lambda)H\big)\,dw.$$

Now consider the function $Q(\lambda)$ defined by

$$Q(\lambda) = (T\zeta_2)(1)(T\zeta_1)(\lambda) - (T\zeta_1)(1)(T\zeta_2)(\lambda)$$

such that we have

$$Q(0) = Q(1) = Q'(0) = 0.$$

Then, from two applications of mean value theorem, we find $\upsilon \in [g_1, g_2]$ such that

$$Q''(\upsilon) = 0.$$

It implies

$$\begin{aligned}&\frac{1}{g_2-g_1}\int_{[g_1,g_2]}(w-H)^2\big[(T\zeta_2)(1).\zeta_1''\big(w\upsilon-(1-\upsilon)H\big)\\&\quad-(T\zeta_1)(1).\zeta_2''\big(w\upsilon-(1-\upsilon)H\big)\big]=0.\end{aligned}\tag{23}$$

For some fixed υ, the expression in the square brackets in (23) is a continuous function of w, so it vanishes. Corresponding to that value of w, we get a number

$$\chi = w\upsilon + (1-\upsilon)H$$

such that

$$(T\zeta_2)(1).\zeta_1''(\chi) - (T\zeta_1)(1).\zeta_2''(\chi) = 0.$$

Thus we get (22). □

Corollary 5.3 *If $\frac{\zeta_1''}{\zeta_2''}$ is invertible, then we have*

$$\chi = \left(\frac{\zeta_1''(\chi)}{\zeta_2''(\chi)}\right)^{-1}\left(\frac{\frac{1}{g_2-g_1}\int_{g_1}^{g_2}\Psi_1(w)\,dw - \zeta_1(\frac{g_1+g_2}{2})}{\frac{1}{g_2-g_1}\int_{g_1}^{g_2}\zeta_2(w)\,dw - \zeta_2(\frac{g_1+g_2}{2})}\right).\tag{24}$$

Corollary 5.4 *Let $s_1, s_2 \in (0,1)$. By taking $\zeta_1(w) = w^{s_1}$ and $\zeta_2(w) = w^{s_2}$, where $w \in (0,\infty)$, in Theorem 5.2 we have*

$$\frac{\frac{g_2^{s_1+1}-g_1^{s_1+1}}{(s_1+1)(g_2-g_1)} - (\frac{g_1+g_2}{2})^{s_1}}{\frac{g_2^{s_2+1}-g_1^{s_2+1}}{(s_2+1)(g_2-g_1)} - (\frac{g_1+g_2}{2})^{s_2}} = \frac{s_1(s_1-1)}{s_2(s_2-1)}(\chi)^{s_1-s_2}.\tag{25}$$

Now we define the following definition.

Definition 5.1 Let $s \in (0,1)$ and $g_1, g_2 \in [0,\infty)$, $g_1 \le g_2$. Then quasi-arithmetic mean for the strictly monotonic function Φ defined on $[g_1, g_2]$ is as follows:

$$\widehat{M}_{\Phi}(g_1,g_2) = \Phi^{-1}\left(\frac{1}{g_2-g_1}\int_{g_1}^{g_2}\Phi(w)\,dw - \Phi\left(\frac{g_1+g_2}{2}\right)\right).\tag{26}$$

Theorem 5.5 *Let $s \in (0,1)$ and $g_1, g_2 \in [0,\infty)$, $g_1 \le g_2$. Let $\Phi_1, \Phi_2, \Phi_3 \in C^2([g_1,g_2])$ be strictly monotonic real-valued functions. Then*

$$\frac{\Phi_1(\widehat{M}_{\Phi_1}(g_1,g_2)) - \Phi_1(\widehat{M}_{\Phi_3}(g_1,g_2))}{\Phi_2(\widehat{M}_{\Phi_2}(g_1,g_2)) - \Phi_2(\widetilde{M}_{\Phi_3}(g_1,g_2))} = \frac{\Phi_1''(\upsilon)\Phi_3'(\upsilon) - \Phi_1'(\upsilon)\Phi_3''(\upsilon)}{\Phi_2''(\upsilon)\Phi_3'(\upsilon) - \Phi_2'(\eta)\Phi_3''(\upsilon)}\tag{27}$$

for some υ, provided that the denominators are non-zero.

Proof Let us choose functions $\zeta_1 = \Phi_1 \circ \Phi_3^{-1}$, $\zeta_2 = \Phi_2 \circ \Phi_3^{-1}$, $w = \Phi_3(w)$, and $\frac{g_1+g_2}{2} = \frac{1}{g_2-g_1}\int_{g_1}^{g_2} \Phi_3(w)\,dw$ in Theorem 5.2, we observe that there exists some $\upsilon \in [g_1, g_2]$ such that

$$\begin{aligned}&\frac{\Phi_1(\widehat{M}_{\Phi_1}(g_1,g_2)) - \Phi_1(\widehat{M}_{\Phi_3}(g_1,g_2))}{\Phi_2(\widehat{M}_{\Phi_2}(g_1,g_2)) - \Phi_2(\widehat{M}_{\Phi_3}(g_1,g_2))} \\ &\quad = \frac{\Phi_1''(\Phi_3^{-1}(\chi))\Phi_3'(\Phi_3^{-1}(\chi)) - \Phi_1'(\Phi_3^{-1}(\chi))\Phi_3''(\Phi_3^{-1}(\chi))}{\Phi_2''(\Phi_3^{-1}(\chi))\Phi_3'(\Phi_3^{-1}(\chi)) - \Phi_2'(\Phi_3^{-1}(\chi))\Phi_3''(\Phi_3^{-1}(\chi))}.\end{aligned} \tag{28}$$

Then, by letting $\Phi_3^{-1}(\chi) = \upsilon$, we notice that we have $\upsilon \in [g_1, g_2]$ such that

$$\frac{\Phi_1(\widehat{M}_{\Phi_1}(g_1,g_2)) - \Phi_1(\widehat{M}_{\Phi_3}(g_1,g_2))}{\Phi_2(\widehat{M}_{\Phi_2}(g_1,g_2)) - \Phi_2(\widehat{M}_{\Phi_3}(g_1,g_2))} = \frac{\Phi_1''(\upsilon)\Phi_3'(\upsilon) - \Phi_1'(\upsilon)\Phi_3''(\upsilon)}{\Phi_2''(\upsilon)\Phi_3'(\upsilon) - \Phi_2'(\upsilon)\Phi_3''(\upsilon)}. \tag{29}$$

□

Again from inequality (21) we have following result.

Theorem 5.6 *Suppose that $\zeta_1, \zeta_2 : [0,\infty) \to \mathbb{R}$ is an s-convex function in the first sense, where $s \in (0,1)$, and let $g_1, g_2 \in [0,\infty)$, $g_1 \le g_2$. Let $\zeta_1, \zeta_2 \in C^2([g_1, g_2])$. Then there exist some $\chi \in [g_1, g_2]$ such that the following equality holds*:

$$\frac{\frac{\zeta_1(g_1)+s\zeta_1(g_2)}{s+1} - \frac{1}{g_2-g_1}\int_{g_1}^{g_2}\zeta_1(w)\,dw}{\frac{\zeta_2(g_1)+s\zeta_2(g_2)}{s+1} - \frac{1}{g_2-g_1}\int_{g_1}^{g_2}\zeta_2(w)\,dw} = \frac{\zeta_1''(\chi)}{\zeta_2''(\chi)}, \tag{30}$$

provided that the denominators are non-zero.

Proof Consider the function

$$(T\zeta_1)(w) = \frac{s\zeta_1(w) + \zeta_1(g_1)}{s+1}(w - g_1) - \int_{g_1}^{w} \zeta_1(x)\,dx. \tag{31}$$

Similarly, we can define $T\zeta_2(w)$.

Note that

$$(T\zeta_1)'(w) = \frac{s\zeta_1'(w)}{s+1}(w-g_1) - \frac{\zeta_1(w) - \zeta_1(g_1)}{s+1} \tag{32}$$

and

$$(T\zeta_1)''(w) = \frac{s\zeta_1''(w)}{s+1}(w-g_1). \tag{33}$$

We observe that

$$(T\zeta_1)(g_1) = (T\zeta_1)'(g_1) = (T\zeta_1)''(g_1) = 0.$$

Now we define $D(w)$ as follows:

$$D(w) = (T\zeta_2)(g_2)(T\zeta_1)(w) - (T\zeta_1)(g_2)(T\zeta_2)(w). \tag{34}$$

Then note that

$$D(g_1) = D'(g_2) = D''(g_1) = D(g_2) = 0.$$

Thus, by application of the mean-value theorem, we get

$$D''(\chi) = 0$$

for some $\chi \in [g_1, g_2]$. Consequently, this completes the proof of the theorem. □

Corollary 5.7 *If $\frac{\zeta_1''}{\zeta_2''}$ is invertible, then we have*

$$\chi = \left(\frac{\zeta_1''(\chi)}{\zeta_2''(\chi)}\right)^{-1}\left(\frac{\frac{\zeta_1(g_1)+s\zeta_1(g_2)}{s+1} - \frac{1}{g_2-g_1}\int_{g_1}^{g_2}\zeta_1(w)\,dw}{\frac{\zeta_2(g_1)+s\zeta_2(g_2)}{s+1} - \frac{1}{g_2-g_1}\int_{g_1}^{g_2}\zeta_2(w)\,dw}\right). \tag{35}$$

Corollary 5.8 *Let $s_1, s_2 \in (0,1)$. By taking $\zeta_1(w) = w^{s_1}$ and $\zeta_2(w) = w^{s_2}$, where $w \in (0,\infty)$, in Theorem 5.6, we have*

$$\frac{(g_1^{s_1} + s_1 g_2^{s_1}) - (\frac{g_2^{s_1+1} - g_1^{s_1+1}}{g_2-g_1})}{(g_1^{s_2} + s_2 g_2^{s_2}) - (\frac{g_2^{s_2+1} - g_1^{s_2+1}}{g_2-g_1})} = \frac{s_1(s_1-1)(s_2+1)}{s_2(s_2-1)(s_1+1)}(\chi)^{s_1-s_2}. \tag{36}$$

6 Conclusion

In Sect. 2, we proved Cauchy type mean for p-convex functions. In Sect. 3, Cauchy type theorem in the Hermite–Hadamard sense was obtained for p-convex functions. In Sect. 4, we proved Jensen's inequality for s-convex functions in the first sense, and then a Cauchy type theorem was obtained. In Sect. 5, a Cauchy type theorem in the Hermite–Hadamard sense was obtained for s-convex functions in the first sense.

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