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RESEARCH

Prognosis of product take-back for enhanced remanufacturing

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Abstract

Remanufacturing reduces final wastes to sinks, extraction of virgin materials and pollution from production processes by reinstating products taken back by end-users to satisfy part of overall demand. Product returns are delayed and possibly limited in periods of fast growth and excessive in the aftermath. Varying growth/demand and volatile take back by consumers and industrial end-users introduce uncertainty, regarding quantity and quality of returns. As remanufacturing expands, escalating competition for acquisition of high quality returns exacerbates uncertainty. Production planning and control for efficient remanufacturing depends on reliable prediction of quantity and quality of returns. A method is developed for prognosis of product return quantity and quality grades, as reflected by vintage flows. It is anchored on a law relating stock and end-of-life level, under random losses and arbitrary end-of-life distribution. Efficacy is tested via a model that describes stock and flows in reuse/remanufacturing, allowing for varying demand, random stock losses, random product returns with time-varying distributions and time-varying utilisation of product returns. Realisations are obtained by Marko-chain Monte-Carlo simulation. Inherently integral in nature, using scaled data and founded on rigorous balances, the method enables prognosis of returns and age-vintage flows, under realistic conditions, including unknown nonlinearities and non-stationarities. It features improved performance (mean absolute error less than one half) compared to leading methods in-use that employ black-box models with error-driven parameter adaptation (e.g. regression). Efficacy is particularly high at crucial peaks and lows (shortage or surplus periods) enabling resourceful planning of acquisition and inventory control of product returns towards sustainability.

Keywords Remanufacturing · Reuse · Product take-back · Forecasting of product returns · Closed loop supply chain · Circular economy

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Nomenclature

a_t	$= P_t + I_{net,t}$, Product inflow in period t, e.g. year t, (tons/period, e.g. t/y), a = steady state level
ARMA	Autoregressive moving average, ARIMA AR integral MA
$C_{f,t}$	Overall sales, originals+remanufactured, consumption (t/y)
$D(x)$	Polynomial of degree $2(N-\kappa) + \mu + 1$
d_k	Coefficient of order-k term in the polynomial $D(x)$
EoL	End-of-life (no further reusable product returns)
EoU	End-of-use (reusable product returns)
E_t	EoL flow (EoL product returns) or EoL exit in period t, e.g. year t, (tons/period)
$G'_{c,k}$	$= 1-g'_1-g'_2-\ldots-g'_k$, complementary cumulative distribution of the EoL exit distribution, g_i,
$g_{i,t}$, $i = 1, 2, ..., \nu$	Reusable product return distribution ($g_{i,t}$= fraction returned in period $t = t^* + j\kappa-\mu + i$, $i = 1,2,..,\nu$, $j = 1,2,\ldots,N$-1,of an originally manufactured product in time period t^*)
g_i	Expected value of the stochastic process $g_{i,t}$, $i = 1,2,..,\nu$,
h_i	Entries of vector h given by eqs. 4–6 (or coefficients of polynomial eq. A3), Appendix A
I_{net}	Flow of original net imported products = imports -exports = $I_{prod,t}-Ex_{prod,t}$ (t/period)
k_Q	Maximum age in the reusable product return sample
k_U	Maximum age in the stock sample
MAPE	Mean absolute percentage error
MRT	Mean residence time = mean lifespan, time periods, e.g. years
m_E	Minimum age in the EoL sample
m_Q	Minimum age in the reusable product return sample
N	Number of manufacturing cycles (original plus N-1 remanufacturing cycles)
P_t	Original production flow (t/period), (original items made from virgin or recycled material)
$P(x)$	Polynomial defined in eq. A4, Appendix A,
p_k	Coefficients of order-k term in the polynomial $P(x)$ found from eqs. 7
Q_t	Reusable return flow, Q = steady state value
$Q_{s,t}$	Size (mass) of the reusable product return sample at time t
$Q_{s,i,t}$	Size (mass) of vintage of age i in the reusable return sample at time t
q	Steady state product return flow rate with respect to inflow of original products = Q/a
RU_t	Actually reused/remanufactured product flow, (t/period)
s_t	Early loss ratio = $\Omega_t/(U_t + \Omega_t)$ = probability of early loss (prior to EoL exit) in period t
T	Time periods form production to centre axis of EoL exit (T = maximum lifetime for. products with non-distributed, deterministic exit in a single time period)
U_t	Product accumulation: quantity of product stock present at the end of time period t (tons)

x_t	$=1-s_t$ = retention ratio = probability of remaining in the reuse/remanufacturing cycle in period t
$y_{i,t}$	Mass fraction of vintage of age i in the reusable return flow in time period t
$y^*_{,t}$	Ratio of mass of the vintage of age i in the reusable return flow in time period t over the mass of the same vintage in the corresponding product stock

Greek

ε	$=E/a$ = steady state EoL flow ratio with respect to inflow of originals = EoL rate or yield
η_t	=Stock mean age at time period t
θ_t	=EoL flow mean age at time period t
κ	=Mean cycle duration, time periods
μ	Half spread of the take-back/EoL distribution, time periods
ν	$=2\ \mu + 1$ = spread of the take-back/EoL distribution, time periods
$\Pi\ (x)$	Polynomial in x defined in Appendix A, eqs. A5, A6
π_i	Coefficients of $\Pi(x)$, Eq. A5 given by Eq. 9 or Eq. A6
τ	$=U/a$ = mean residence time or mean product lifespan = MRT
φ	Mean take-back fraction of reusable products with respect to reusable product stock
Ω_t	=Early loss flow (t/period)

Symbols

=:	Equal by definition.
< >	Mean sample path value (MSPV)

Subscripts

$_t$	t is discrete time, $t = 1$: first time a product under consideration is launched in the market.
$_s$	Sample.

Introduction

Cyclic economy (Fig. 1a) embraces reuse/remanufacturing as a key towards more resourceful and sustainable operations [7, 36]. A new supply chain valorizes no-further usable products, cores and components returned by consumers and industrial end-users as 'raw materials' to remanufacture products in 'like new' condition [1, 47]. Compared to open loop systems, cyclic structures can attain much higher internal flows for the same input (e.g. virgin materials) and output (e.g. wastes) flows. In other words, the same consumption level can be satisfied by much lower virgin materials and wastes. In parallel to creating value for shareholders, key systemic objectives of industrial remanufacturing (IR) Fig. 1a in this vein are as follows: (*i*) to belittle, even annihilate, the flows to sinks ('zero waste') and related ecological impacts, (ii) to minimise extraction of virgin raw materials, and (iii) to reduce emissions from manufacturing processes. To attain these goals, IR requires efficient planning and control of operations [4, 30, 37, 62, 63], which include acquisition of product returns, inventory control, utilisation (sorting,

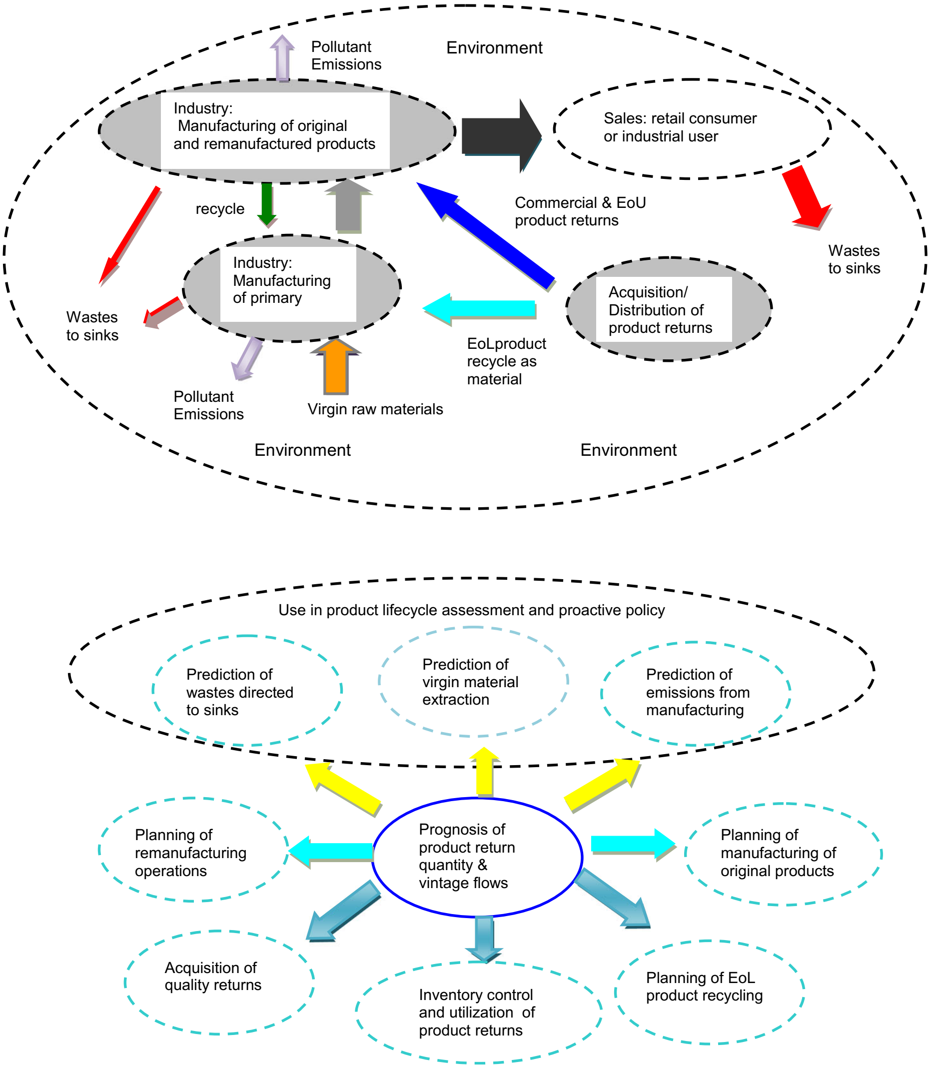

Fig. 1 **a** (Top): cyclic network (remanufacturing of commercial and end-of-use product returns and material recycling of EoL exit). **b** (Bottom): reliable prognosis of product takeback as a part of computational intelligence in remanufacturing: use for enhanced planning of operations and eco-efficiency

disassembly, refurbishing, reassembly, testing) and commercialisation. The entire chain of operations hinges on reliable prognosis of returned products, cores and components (Fig. 1b). As IR gains momentum [32, 33, 56], computational intelligence for efficient plant and remanufacturing process sizing, inventory control and production planning becomes more important [26, 69]. Besides ecological gains, tangible economic benefits from achieving higher reuse/remanufacturing include direct tax credits, or new revenue streams subsidising remanufacturing, or eco-label allowances. Instigating higher appeal to consumer and willingness to purchase, reuse/remanufacturing may increase sales' volumes [3, 38]. Safeguarding of profits intensifies competition for acquisition of high quality returns by original equipment manufacturers, independent remanufacturers, or third party collectors [16, 63].

Product/core returns include commercial or consumer returns (appearing in the early periods of use, usually within product warranty) and end-of use (EoU) returns (after use is completed by customer). Commercial and EoU returns can be reused in subsequent cycles. No-longer reusable product returns consist the end-of-life (EoL) product returns, which are recycled as material [44]. Reusable returns are already ‘loaded’ with a certain technology, which may soon become obsolete [17, 31]. As a result, the value of reusable returns declines with time, especially for high tech products and fast varying markets (e.g. electronics) and thus, a portion of acquired returns may be directed to recycling [24, 27, 65].

Although advancing, remanufacturing is still limited with volumes of remanufactured at low fractions of overall sales [12, 53, 57]. Involving reuse and minimal remanufacturing (inspection, washing and refill), refillable containers consist a notable exception [28, 40, 70]. Crucial in the design of sustainable operations, prediction of availability of ‘raw materials’ becomes more challenging in IR, due to uncertainty in quantity and quality of product returns [17, 68]. Returns can be predicted in several cases, e.g. lease-back products or service contracts [18]. Several products however, especially high-tech, feature random returns [5, 13, 23, 39, 52, 75]. Sales and returns consist of the main source of uncertainty in IR [29]. Volatility in the quantity and quality of product returns together with competition, shifting consumer priorities and ever-present uncertainty in demand, manufacturing technology and commodity prices increase uncertainty in planning remanufacturing operations [8, 64]. Factors that exert a decisive and randomising impact on product take-back include economic cycles, gross domestic product, money supply and available income, oil and metal costs, volatile consumer perception and product discard, fashion and social trends, advent of technology and innovation, environmental legislation, energy efficiency classification, diminishing operating costs, and improved eco-footprint of new products. Non-stationarities crop up, together with non-linearities, favoured by scaling and threshold effects.

Important results have appeared in IR modelling and several methods are available for predicting product returns. They include (*i*) transfer function models between sales and returns [28] of refillable containers and related models based on Gaussian distribution of the quantity of returns in terms of past sales, with known mean and variance [40] (ii) returns assumed to be a constant fraction [2] or a dynamic fraction [74] of active market demand (iii) renewal process and particularly stationary Poisson processes for the returns, with sales following a Poisson process [10, 11, 22, 41, 42, 45, 66, 67], (iv) artificial intelligence methods, following recognised patterns [34, 42, 46, 60], (v) actuary science methods [48, 49, 54], (vi) models presuming constant and known distribution of the returns [43, 51, 55, 58], (vii) smoothing, ARMA, and ARIMA type models [6] in which the forecast return flow is a projection that minimises an error metric of recent return flow data, (viii) hazard rate models and regression to estimate returns [35], (ix) prediction of product returns via tweets [13] and (x) ARIMAX forecasting [9, 67] with overall sales taken as the exogenous variable. ARIMAX estimation features lower forecasting error for commercial returns when combined with accelerated failure models for the mean retention time [59]. Frequently used in practice, ARMA, ARIMA and ARIMAX type methods presume a linear underlying model, the parameters of which are estimated via Bayesian estimation or regression methods, e.g. recursive least squares. The linearity assumption may be violated in cyclic systems, whilst the presence of non-stationarities [61] casts doubts on assumed a-priori statistics of Bayesian estimators, or on regression estimates based on linear models. Artificial intelligence methods face difficulties under shifting patterns of real markets. Well-fit for mechanical failures, accelerated failure time models may not represent consumer retention time uncertainty of product returns. Actuary science and residual life methods presuming specific exit distributions are too rigid to engulf volatility and economic cycles affecting take back. Field evidence depicts local disparities or shifting distributions with time [50, 55] not resembling any known distribution.

The endeavour of the approach herein is for enhanced prognosis of reusable product returns that does not rely on linear models or arbitrary postulates of returns and sales, or on any particular or known in advance take back distribution. Instead, a systemic procedure is sought, applicable to all IR systems, independent of take-back distribution, even adjusting to distribution variations and non-stationarities. Based on minimal monitoring, it should employ readily available, reliable data, allow for random product losses during successive reuse cycles and provide dependable prediction. The method to be developed in discrete time is based on a recent law relating the steady state stock level of the product, U and the EoL return flow E (after completion of all cycles of use as original and remanufactured product) as follows: $U - \theta E = (\eta - 1)(\alpha - E)$ [72] where a is the inflow of originals (=net demand for originally manufactured products), η is the stock mean age and θ is the EoL mean age. The mean ages are scaled variables (population averages) typically monitored in practice from samples of product stock and EoL flow. They are found in the conventional statistical way in any time period. For instance, if three vintages of ages θ_1, θ_2, θ_3 and respective quantities, E_1, E_2, E_3, $E_1 + E_2 + E_3 = E$, are present in the EoL flow, E, in time period t, then $\theta = (\theta_1 E_1 + \theta_2 E_2 + \theta_3 E_3)/(E_1 + E_2 + E_3)$. An example is a product featuring distributed EoL exit in the beginning of time periods 4, 5 and 6, after being launched in the market in the beginning of period 1, *i*.e. $\theta_1 = 3$, $\theta_2 = 4$, $\theta_3 = 5$. The product stock at the end of period t, U, includes a quantity U_1 of the current vintage (vintage U_1 of age 1) and vintages U_2, U_3, U_4 and U_5 of ages 2, 3, 4 and 5 respectively, $U_1 + U_2 + U_3 + U_4 + U_5 = U$ (all higher age vintages have exited the stock in-use). Then $\eta = (U_1 + 2U_2 + 3U_3 + 4U_4 + 5U_5)/U$ and the Eq. $U - \theta E = (\eta - 1)(\alpha - E)$ relates the potentially unknown variables U and E under any distribution of product returns and EoL exit and under any loss profile of stock. One relation between two unknown variables is better than no relations at all—if one of them is known the other can be determined. For instance, the EoL exit of passenger vehicles can be found from the mean age of the vehicle fleet and from the mean age of a representative EoL vehicle exit sample, since the size of the vehicle fleet in-use, U, is registered. Being valid under any return/EoL distribution and under arbitrary, unknown and random losses of stock, the law was used to identify the mean retention time for products with stock accumulation [73] based on measurements of η, θ and of the EoL age-vintage fractions. The quest of the present work is whether it may be used to predict quantitative and qualitative features of reusable product returns. Section 2 presents the key relations of the model used to simulate the remanufacturing loop and to assess the efficacy of the prognosis of product returns. The model is refined to include random product take-back and varying utilisation of product returns by remanufacturers. Section 3 describes the prediction method and gives a five-step implementation procedure, followed by applications in Section 4.

Modelling of stock and flows in remanufacturing systems

It is evident that a stochastic model [25] is best suited to quantitatively describe the random nature of IR product stock and flows. Table 1 presents the expressions of such a model for stock and EoL flow [71–73]. Table 2 gives the expressions for the internal (remanufacturing) cycle.

Table 1 Model for stock and EoL flow for systems with stock accumulation under random stock losses and time varying fractions of the EoL distribution [70–73]

E_t: EoL exit (EoL product returns) at the beginning of period t	$E_t = \sum_{k=1}^{\nu} g'_{\nu-k+1,t} a_{t-T+k-\mu-1} \prod_{i=1}^{T-k+\mu+1} x_{t-i}, \nu = 2\mu+1$	(T1)
(ν- step, EoL distribution $g'_{i,t}$, $i = 1, 2, ..., \nu$, ν-families of random variables, each one a stochastic process, e.g. a renewal process with mean g'_i, $i = 1, 2, ..., \nu$)	$0 \le g'_{i,t} \le 1, \sum_{i=1}^{\nu} g'_{i,t-\nu+i} = 1$	(T2)
U_t: stock of originals and remanufactured products at the end of time period t. Ω_t random losses during time period t; it includes net exports of used and exports of remanufactured.	$U_t = U_{t-1} + a_t - \Omega_t - E_t$ a_t: net demand (net inflow of originals)	(T3)
Net demand for originally manufactured products a_t and for imports of used, $I_{u,t}$	$c_t = a_t + I_{u,t}$	(T4)
Net demand or net inflow of originally manufactured products = production of originals, P_t, from virgin or recycled material + net imports of originals $I_{o,t}$.	$a_t = P_t + I_{o,t}$	(T5)
Early loss ratio:	$s_t = \Omega_t/(U_t + \Omega_t), \Omega_t \le U_t$	(T6)
Retention probability:	$x_t =: 1 - s_t = U_t/(U_t + \Omega_t)$	(T7)
Analytic expression for the stock of originals and remanufactured products	$U_t = \sum_{k=1}^{T-\mu} a_{t-k+1} \prod_{i=t-k+1}^{t} x_i + \sum_{k=T-\mu+1}^{T+\mu} a_{t-k+1} G'_{c,k-T+\mu,t} \prod_{i=t-k+1}^{t} x_i$ where	(T8)
($G'_{c,k-T+\mu,t}$ represents the part of the inflow at time $t-k+1$, a_{t-k+1}, that has not exited as EoL exit until the end of time period t).	$G'_{c,k,t} = 1 - \sum_{i=1}^{j} g'_{i,t-k+i} = 1 - g'_{1,t-k+1} - g'_{2,t-k+2} - \cdots - g'_{k-1,t-1} - g'_{k,t}$	(T9)
Key steady state relations		
Stock expressed in terms of EoL flow and retention rate	$U = \frac{x(a-E)}{1-x}$	(T10)
MRT in terms of EoL rate and retention rate	$\tau = \frac{x(1-\varepsilon)}{1-x}$	(T11)
Law of stock and EoL flow (or MRT and EoL rate)	$U - \theta E = (\eta - 1)(a - E)$ $\tau - \theta\varepsilon = (\eta - 1)(1 - \varepsilon)$	(T12) (T13)
EoL flow in terms of mean ages and retention rate	$E = a\frac{\eta x + (1-\eta)}{(\eta-\theta)x + (1-\eta+\theta)}$	(T14)
EoL rate in terms of mean ages and retention rate	$\varepsilon = \frac{\eta x + (1-\eta)}{(\eta-\theta)x + (1-\eta+\theta)}$	(T15)

Description of the model

The key equations are as follows: Eq. T1 for the EoL flow, E_t, which is distributed, stochastic and non-stationary (via the randomly varying fractions of the EoL distribution, $g'_{i,t} \ge 0$, $i = 1, 2, ..., \nu$, Eq. T2); Eq. T3 for stock evolution in which the outflow (depletion of product stock) includes random early losses, Ω_t, independent from the EoL exit, E_t; and also, Eq. T8, obtained from the lifetime product mass balance, using Eqs. T1 and T3. For each i the fraction $g'_{i,t}$ is a t-family of

Table 2 Model for reused/remanufactured products under random losses, uncertain sales and random fractions of the product take-back distribution [71]

Reusable product return flow (commercial + EoU returns) Q_t: It includes the contributions $Q_{j,t}$ from $N-1$ cycles, $j = 1,2,\ldots,N$-1. There are $(N-1)\nu$ quality grades in Q_t, differentiated by age and number of past cycles.	$Q_t = \sum_{j=1}^{N-1} Q_{j,t} = \sum_{j=1}^{N-1} \sum_{k=1}^{\nu} a_{t-j\kappa+k-\mu-1} g_{\nu-k+1,t} \prod_{i=1}^{j\kappa-k+\mu+1} x_{t-i}$ Reusable returns in time t originate from product stock at the end of the previous time period, i.e. from U_{t-1}.The fractions $g_{i,t}$ of the take-back distribution (of reusable returns) satisfy condition T17.	(T16)
(ν- step, take-back distribution with take-back fractions: $g_{i,\,t}$, $i = 1, 2, ..., \nu$, in each one of the N-1 reuse cycles).	$0 \le g_{i,t} \le 1, \sum_{i=1}^{\nu} g_{i,t-\nu+i} = \varphi_t, 0 < \varphi_t \le 1$ $g_i = \varphi g_i', i = 1, 2, ..., \nu, 0 < \varphi \le 1$	(T17)
N: number of cycles T: EoL centre axis	$T/N =: \kappa$ κ: mean cycle duration	(T18)
Consumption: products reaching the consumer (sales volume in annual mass of product) = net demand = c_t + remanufactured products, $RU_{d,\,t}$	$C_{f,\,t} = c_t + RU_{d,t}$, $a_t = C_{f,\,t} - I_{u,\,net,\,t} - RU_{d,\,t}$	(T19)
Remanufacturing: return utilisation: Remanufactured returns, $=RU_t$ is a fraction of reusable returns, Q_t: $RU_t = RU_{d,\,t} + RU_{e,\,t}$:$RU_{d,\,t}$ remain in the internal domestic cycle and $RU_{e,\,t}$ are exported.	$RU_t = RU_{d,t} + RU_{e,t} = \sum_{i=0}^{n-1} \left[\upsilon_{1,t-i,t} \;\; \upsilon_{2,t-i,t} \;\; \ldots \;\; \upsilon_{(N-1)\nu,t-i,t} \right] V_{t-i}$ where V_{t-i} is the $(N-1)\nu$- dimensional vector of the various quality grades in Q_{t-i} and $\upsilon_{1,\,t-i,\,t}$, $\upsilon_{2,\,t-i,\,t}$, ..., $\upsilon_{(N-1)\nu,\,t-i,\,t}$ are the utilization fractions of the $(N-1)\nu$ quality grades of Q_{t-i} in time period t.	(T20)
Utilised returns of period t of quality grade j, $j = 1, 2, ..., (N-1)\nu$, within the n-step utilisation horizon, i.e. within the n time intervals, t, $t+1$, $t+2,\ldots,t+n$-1.	$Q_{j,t} \sum_{i=1}^{n} \upsilon_{j,t,t+n-i}$	(T21)
Discarded fraction of returns of period t of each quality grade j	$1 - \sum_{i=1}^{n} \upsilon_{j,t,t+n-i}$	(T22)
Discarded reusable returns	$Q_t - \sum_{j=1}^{(N-1)\nu} Q_{j,t} \sum_{i=1}^{n} \upsilon_{j,t,t+n-i}$	(T23)
Product return utilisation fraction, $\upsilon_{j,t,t+i}$ =: quality grade j of product returns in period, t, utilised in period $t+i$, $i = 1, 2,\ldots$,n, by the remanufacturer	$0 \le \upsilon_{j,t,t+i} \le \sum_{i=1}^{n} \upsilon_{j,t,t+n-i} \le 1, j = 1, 2, ..., (N-1)\nu$	(T24)

random variables, independently distributed with finite mean, g'_i. It can be a discrete stochastic process at arithmetic time intervals, e.g. an arrival process of random deviates around a mean g'_i, or a renewal process, that is, for each i the t-family $g'_{i,\,t}$ is independently and identically distributed: as time progresses $g'_{i,\,t}$ takes values according to the same probability distribution. The distribution of the t-family $g'_{j,\,t}$, $j \neq i$, may be different than that of $g'_{i,\,t}$. T is the centre axis of the EoL distribution; the latter expands for μ time intervals in the past (to the left of T) and μ time intervals to the right of T. Without loss of generality it may be assumed that $\nu = 2\mu + 1$. Early loss includes all premature, i.e. non-EoL exit from stock. It entails discards of reusable products by the consumer, exports of remanufactured, exports of used, disposed-of, non-obsolete returns, which the remanufacturer decides not to use, based on marketing strategy. Early losses are represented by the dimensionless early loss ratio, s_t, eq. T6, Table 1, or its complement, the retention ratio, or retention probability in the reuse/remanufacturing cycle, x_t, $0 \le x_t \le 1$, Eq. T7, a dominant variable in stock and flow evolution. The product $\prod_{i=1}^{T-k+\mu+1} x_{t-i}$ is simply the retention probability (considering losses in each time period to be independent events) from the first entrance of the product in the market in time period, $t-T+k-\mu-1$, as originally manufactured, until and including the time period $t-1$, prior to EoL exit (beginning of period t). Figure 2 illustrates simple accumulation systems corresponding to constant and random early loss, with mean take-back fraction, $\varphi=1$ (see "Description of the model for product returns in remanufacturing systems" in regard to φ). The residual life representation [43, 48, 49, 51, 54, 55, 58] is a deterministic case, corresponding to fixed fractions $g'_{i,t}$ and no early loss, $x=1$.

Description of the model for product returns in remanufacturing systems

Cyclic systems entail accumulation of products that are returned to be remanufactured a number of cycles, say $N-1$ (original, plus $N-1$ remanufacturing $=N$ cycles) until the EoL exit, centred at the centre axis of the EoL distribution. Then, the equations in Table 1 are applicable together with those in Table 2. The key equation in Table 2 is the expression for the reusable returns (commercial and EoU), Eq. T16, which is also distributed, stochastic and non-stationary via the arbitrarily randomly varying fractions of the return distribution, $g_{i,t}$, $i=1, 2, ..., \nu$. The fractions $g_{i,t}$ sum up to φ_t, which is equal to, or less than one: $0\le g_{i,t}\le 1$, $\sum_{i=1}^{\nu} g_{i,t-\nu+i}=\varphi_t$, $0<\varphi_t\le 1$. Scaling of the reusable returned distribution by the (potentially time-varying) scale φ_t accounts for only a fraction of reusable products in-stock being returned, as it often happens in practice.

The summation $\sum_{j=1}^{N-1}$ in Eq. T16 represents reusable returns contributed from products in different cycles, *i*.e. cycles 1,2,…, $N-1$. Each cycle is centred at time interval $j\kappa$, $j=1, 2, ..., N-1$ and expands for μ time intervals in the past (to the left) and μ time intervals to the right, (Fig. 2), $\kappa\ge\mu+1$. If the mean cycle duration is denoted by κ (time periods) then $T/N=\kappa$. The first return cycle is centred at time period κ after original production. The last return cycle is centred at time period $(N-1)\kappa$ after original production and the EoL exit is centred at time period $T=N\kappa$. Cycles can be overlapping (if $N>2$ and $\kappa<\nu$) or non-overlapping (if $N=2$ or if $\kappa\ge\nu$), Fig. 2. The number of overlaps (product returns of same age but corresponding to at most two adjacent cycles) is $(N-2)(\nu-\kappa)$. For $\kappa=\nu$ the profile of returns is that of a (quantised) stationary wave with gradually fading amplitude due to early losses (as the fading riddles from a stone thrown in a lake). It becomes a standing wave, under no loss, $x=1$ and $\kappa=\nu$. There are $(N-2)\kappa+\nu$ age vintages of product returns, since $(N-1)\kappa+\mu-(\kappa-\mu)+1=(N-2)\kappa+2\mu+1=(N-2)\kappa+\nu$. Overall demand and consumption is the sum of originals and of remanufactured products, including net imports (Eq. T19) reaching the consumer in time t, irrespectively of different pricing of remanufactured products, used and originals. The model unravels the stochastic routings of returns in various realisations (e.g. Fig. 3a–f), since all variables a_t, E_t, U_t, Ω_t, x_t, $g'_{i,,t}$, Q_t, $C_{f,t}$, $g_{i,,t}$, explicitly appearing in the model, are random and the return utilisation fractions (see next section) $\upsilon_{i,t,t+j}$, and the remanufacturing level, RU_t, are time-varying. The mean duration of manufacturing cycles, κ, as well as the spread of the return distribution, μ or ν, and the centre axis, T of the EoL distribution may also be slowly time-varying, e.g. diminishing in periods of economic expansion.

Quality grades of reusable product returns and utilisation in remanufacturing

Since quality degrades with time, returns acquired in time-period t are used within a maximum time horizon, say n time-periods. Usage varies depending on management decision affected by projected sales, profitability and extent of cannibalisation of original product sales by

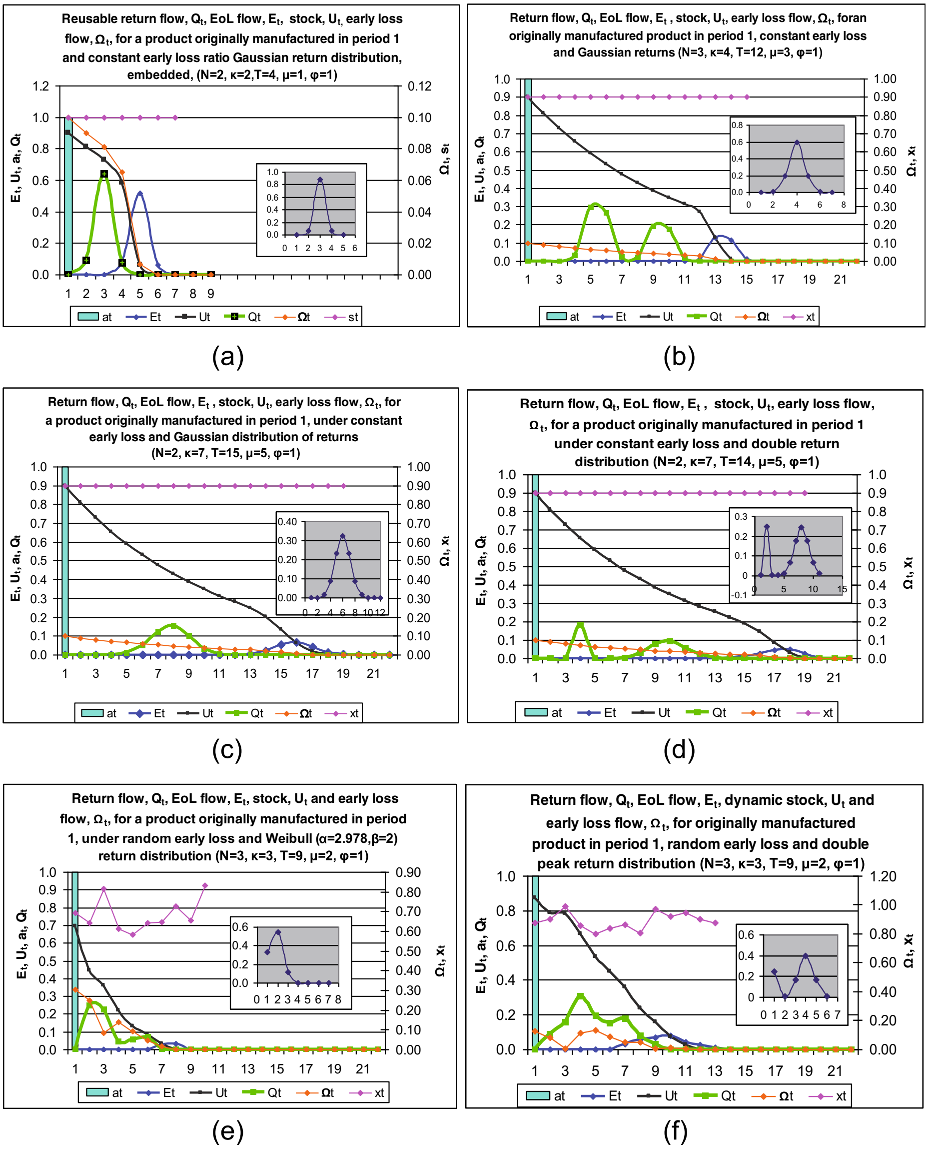

Fig. 2 Modelling of industrial remanufacturing systems. Profiles of stock, early loss, reusable product returns and EoL flow (left axis) for a product originally manufactured in a single period ($a_1 = 1$ $a_t = 0$, $t > 1$). For illustration purposes $\varphi = 1$ and the fractions of the return distributions in panels a–d are constant corresponding to the values of the mean distribution (given in the embedded graphs). Panels **a**, **b**, **c**, and **d** correspond to constant early loss (right axis, magenta). The return distribution in panel **d** includes both commercial returns (first peak of the embedded mean return distribution) and EoU returns (2nd peak). Panels **e** and **f** display two realisations of the same system under random early loss. The fractions, $g_{i,t}$, of the return distributions are randomly varying, i.e. random normal deviates around the mean distributions (embedded) obtained via the Markov-chain Monte-Carlo simulation. The stochastic nature of cyclic systems is further unfolded in Figs. 3–6 where the closed loops are driven by randomly varying demand and the level of manufacturing of originals (original inflow) in each time period is found from eq. T19 to balance overall consumption together with remanufactured products (given by eq. T20)

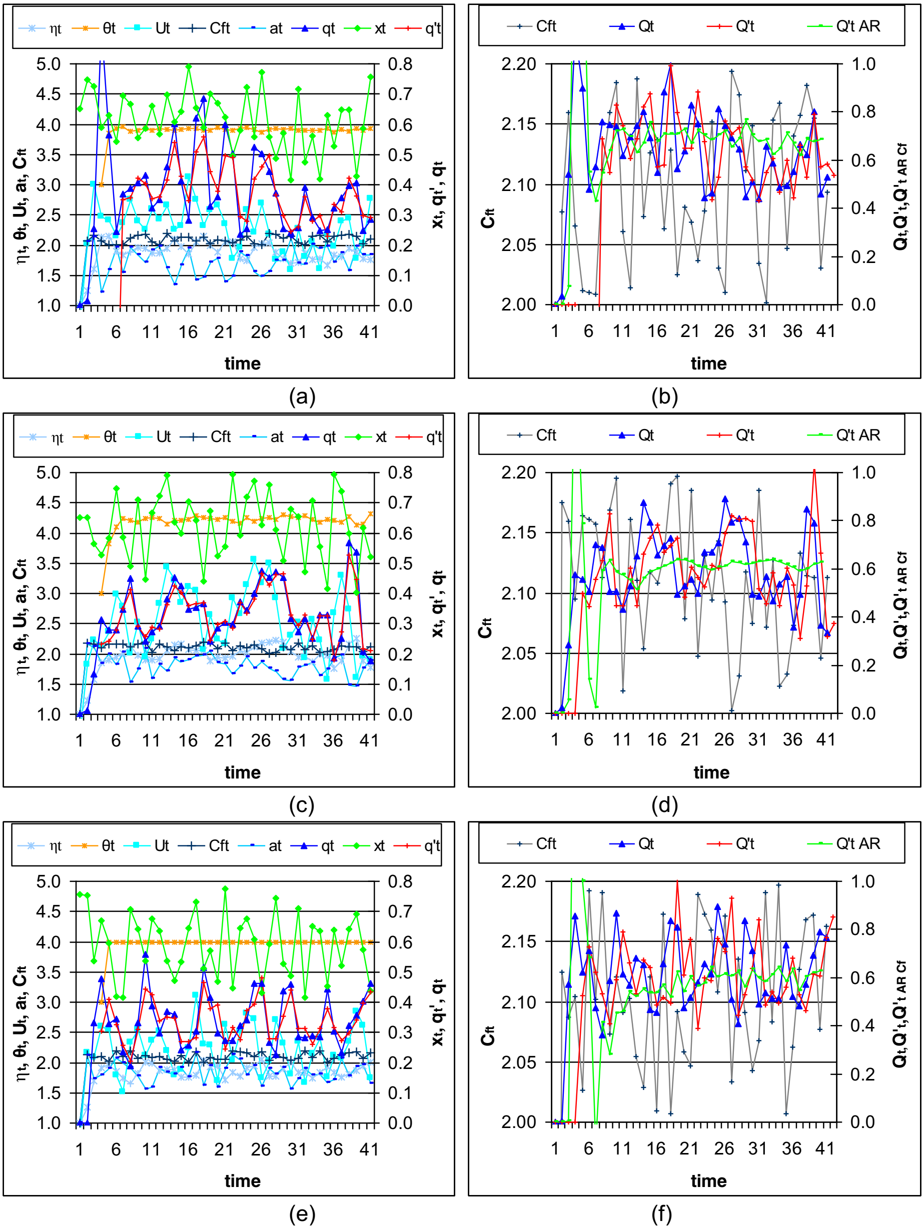

Fig. 3 Application 1. Retreaded tyres. Three different realisations via MCMC simulation, under random sales, random early loss, randomly varying return distribution and varying return utilisation policy, depicting the superior prediction efficacy of the proposed method (red line following the blue line) compared to ARIMAX (green line, right column; **b**, **d**, **f**) based on sales, $C_{f,t}$, as the exogenous variable). MAPE of the proposed method 8–12%, MAPE of the ARIMAX method 20–30%. The cyclic system is driven by consumption: Eq.T19 gives the net demand, from which the inflow of originals, a_t, is found and used in the stochastic expressions for stock, reusable returns and EoL exit, eqs. T8, T16 and T1

remanufactured and other uncertain factors. Therefore the various quality grades are utilised according to varying usage fractions. For instance, the first age vintage (youngest returns, age

κ-μ) acquired in period t, is used according to varying fractions $\upsilon_{1,t,t}$, $\upsilon_{1,t,t+1}$, ..., $\upsilon_{1,t,t+n-1}$, the second (of age κ-$\mu+1$) acquired in period t, is used according to fractions $\upsilon_{2,t,t}$, $\upsilon_{2,t,t+1}$, ..., $\upsilon_{2,t,t+n-1}$, etc. Besides age, and in the same way as used cars are differentiated both with respect to age and number of past owners, quality grades in the returns can be considered to be differentiated with respect both to age and number of past cycles. It can be shown that there are $(N-1)\nu$ such different quality grades in the return flow. Consequently, utilisation of each quality grade i to produce remanufactured products is given by the fractions $\upsilon_{i,t,t}$, $\upsilon_{i,t,t+1}$, ..., $\upsilon_{i,t,t+n-1}$, $i=1, 2, ..., (N-1)\nu$. Then if the flows (quantities) of the differing quality grades of product returns acquired by the remanufacturer in time t are stacked in a column vector V_t the utilisation of quality grades in time period t (to produce $RU_{d,t}$ products for the domestic market and $RU_{e,t}$ for exports) is given by eq. T20, $RU_t = RU_{d,t} + RU_{e,t} = \sum_{i=0}^{n-1} \left[\upsilon_{1,t-i,t} \;\; \upsilon_{2,t-i,t} \;\; \cdots \;\; \upsilon_{(N-1)\nu,t-i,t}\right] V_{t-i}$. Inventory control of product returns may then be addressed. Overall utilisation of each quality grade, as well as discard of non-utilised products within the n-time horizon (part of early losses, Ω_t) are given by Eqs. T21 and T22. The variable nature of return utilisation, as given by Eq. T20, contributes to the random variations of stock and flows.

Key asymptotic relations Table 1 gives key relations for the steady state levels of stock and EoL exit. Eqs T10 and T11 are obtained directly from Eqs. T3 and T7 at steady state. Eqs. T12 and T13 is the law of stock and EoL [72] which gives Eqs. T14 and T15 [73].

Relation between reusable product returns and EoL exit

Assumption A It is assumed that for each i, $i = 1, 2, ..., \nu$, the t-family of random variables $g'_{i,t}$ in Eqs. T1, T2 is independently and identically distributed. The t-family of random variables $g_{i,t}$ is also independently and identically distributed and the distribution of the values of the random variables $g_{i,t}$ is the same as that of the random variables $g'_{i,t}$, but it is scaled down by the factor φ_t, $0 < \varphi_t \leq 1$. It is readily shown then that as the number of samples increases, the mean sample path value of φ_t, $\langle\varphi_t\rangle$, tends to the scale of the reusable return with respect to the EoL distribution, φ (Eq. A1, Appendix A), *i*.e. φ is the mean fraction of reusable returns in stock, actually taken back by the customer. Assumption A is mild, reflecting the similarity of consumer behaviour as to product utilisation by the consumer and take-back.

Then, the following Eq. 1, resulting from eqs. T1 and T16 in Tables 1 and 2, relates the steady state levels of reusable returns, Q and EoL exit, E. It is valid under any randomly varying take-back distribution and return utilisation policy, (Appendix A). If the net demand for originals, a_t, is not vanishing, Eq. 1 takes the scaled form given in Eq. 2.

$$Q = \varphi x^{-T} E \sum_{j=1}^{N-1} x^{j\kappa} \tag{1}$$

$$q = \varphi x^{-T} \varepsilon \sum_{j=1}^{N-1} x^{j\kappa} \tag{2}$$

where q is the reusable take-back rate, $q = Q / a$, *i*.e. the steady state ratio of reusable product returns over net demand, x is the retention rate, $x = U/(U+\Omega)$, and ε is the EoL rate or EoL yield, $\varepsilon = E / a$. Being independent of net demand and overall sales, as well as of the distribution of product returns, Eq. 2 will be used to predict expected product take-back in the next section.

Prognosis of reusable product returns

The problem addressed is to predict the expected quantity and quality grades of imminent reusable product returns, with quality reflected by age vintages and past cycles. Prediction is sought under arbitrarily random and unknown early loss, without presuming any take-back distribution, *i*.e. without involving the varying fractions $g_{i,t}$, or their mean values, g_i and without any assumptions on the retention probability x_t. This will be accomplished by determining the retention rate by combining Eq. 2 with Eq. T15 (Eq. T15 results from the stock Eq. T10 or T11 and law T13). In contrast to black-box postulates, the basic relation presented in this section is based on fundamental principles (mass and product balances). It relates the long-term value of the retention probability, x_t, named retention rate, with the physical parameters N, κ, ν, via the following ensemble averages: the mean age of the stock sample at time period t, denoted by η_t and the mean age of EoL sample, denoted by θ_t (the subscript $_t$ dropped hereafter for simplicity of notation). Both are readily and reliably monitored as the mean of all (age) vintages present in a specimen at time t, even from small size or decentralised samples. The efficacy of the prognosis will be tested using the model given in the previous section.

Proposition 1 Under assumption A and for any return distribution $g_{i,\,t}$, $i = 1, 2, ..., \nu$, any sales profile and return utilisation policy, the polynomial Eq. (3), $D(x) = 0$ of degree $2(N-1)\kappa + \mu$, i.e. $\sum\limits_{j=0}^{2(N-1)\kappa+\mu} d_j x^j = 0$, relates the long-term (steady state) values of the retention probability, x, stock mean age η and EoL mean age, θ.

$$D(x) = \sum_{j=1}^{2(N-1)\kappa+\mu} d_j x^j = 0, d_j = \sum_{i=j+2}^{2(N-1)\kappa+\mu} (i-j-1) h_i \tag{3}$$

The coefficients, d_j, of the polynomial,$D(x)$ are determined in each time period from the parameters h_i; the latter are explicitly given in terms of η and θ by Eqs. 4–9: where the h_i, $i = 0$, 1, 2, ..., $2(N-1)\kappa + \mu + 1$ are found as follows in terms of the mean ages.

$$h_{2(N-1)\kappa+\mu+1} = (N-1)(\eta-\theta), h_{2(N-1)\kappa+\mu} = (N-1)(1-\eta + \theta) \tag{4}$$

$$h_j = 0, j = 2(N-2)\kappa + 2\mu + 1, 2(N-2)\kappa + 2\mu + 2, .., 2(N-1)\kappa + \mu-1 \tag{5}$$

$$h_j = \sum_{i=0}^{(N-2)\kappa+2\mu} p_{(N-2)\kappa+2\mu-i}\pi_{j-(N-2)\kappa-2\mu+i}, j = 0, 1, 2, ..., 2(N-2)\kappa + 2\mu \tag{6}$$

and the p_i are given in terms of the age vintage fractions y_i in the return sample, Q_s, by

$$p_{(N-2)\kappa+2\mu} = y_{\kappa-\mu}, p_{(N-2)\kappa+2\mu-1} = y_{\kappa-\mu+1}, ..., p_1 = y_{(N-1)\kappa+\mu-1}, p_0 = y_{(N-1)\kappa+\mu} \tag{7}$$

$$y_i = Q_{s,i}/Q_s \tag{8}$$

$$\begin{array}{l} i = \kappa-\mu, \kappa-\mu + 1, ..., \kappa + \mu, 2\kappa-\mu, 2\kappa-\mu + 1, ..., 2\kappa + \mu, ..., \\ (N-1)\kappa-\mu, (N-1)\kappa-\mu + 1, ..., (N-1)\kappa + \mu \\ \pi_{j\kappa+1} = -\eta, \pi_{j\kappa} = -(1-\eta), j = N-2, N-3, ..., 1, 0, \pi_1 = -\eta, \pi_0 = -(1-\eta) \\ \pi_{j\kappa-1} = ... = \pi_{(j-1)\kappa+2} = 0, j = 2, 3, ..., (N-1), \pi_j = 0, j < 0 \end{array} \tag{9}$$

The $(N-2)\kappa + \nu$ age-vintage fractions, y_i, in the reusable return sample, Q_s, are readily determined in each time period via Eq. 8. Neat digital mechanisation of Eq. 3 is possible using the vector notation, $d = \left[d_{2(N-1)\kappa+\mu} \quad d_{2(N-1)\kappa+\mu-1} \quad \dots \quad d_1 \quad d_0 \right]$ and corresponding matrix multiplication functions in available software: eq. 3 is written as:

$$d = Lh \tag{10}$$

L is the $(2(N-1)\kappa+\mu+1)x(2(N-1)\kappa+\mu+1)$ shift matrix $L = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 & 0 \end{bmatrix}$ with the entries h_i, $i = 0, 1, 2, \dots, 2(N-1)\kappa+\mu+1$ of the $2(N-1)\kappa+\mu+1$ dimensional vector $h^T = \left[h_{2(N-1)\kappa+\mu+1} \quad h_{2(N-1)\kappa+\mu} \quad 0 \quad 0 \quad \dots \quad 0 \quad h_{2(N-2)\kappa+2\mu+1} \quad h_{2(N-2)\kappa+2\mu} \quad \dots \quad h_1 \quad h_0 \right]$ given by Eqs. 4–9.

Examples 1) $N = 2$: (one original manufacturing cycle plus one reuse/remanufacturing cycle).

Then $2(N-1)\kappa+\mu+1 = 2\kappa+\mu+1$, Eq. 1 is $q = \varphi x^{-2\kappa}\varepsilon x^{\kappa}$ and Eq. 3 becomes $D(x) = \sum_{j=0}^{2\kappa+\mu-1} d_j x^j = 0$, with $d = L_{2\kappa+\mu+1}h$. The vector h is found from Eqs. 4–9:

$p_{2\mu} = y_{\kappa-\mu}$, $p_{2\mu-1} = y_{\kappa-\mu+1}$, ..., $p_1 = y_{\kappa+\mu-1}$, $p_0 = y_{\kappa+\mu}$ and $\pi_1 = -\eta$, $\pi_0 = -(1-\eta)$ giving $h_{2\kappa+\mu+1} = (\eta-\theta)$, $h_{2\kappa+\mu} = (1-\eta+\theta)$, $h_{2\kappa+\mu-1} = 0 = h_{2\kappa+\mu-2} = \dots h_{2\mu+2} = 0$, $h_{2\mu+1} = -\eta y_{\kappa-\mu}$, $h_{2\mu} = -\eta y_{\kappa-\mu+1} - (1-\eta)y_{\kappa-\mu}$, $h_{2\mu-1} = -\eta y_{\kappa-\mu+2} + (1-\eta)y_{\kappa-\mu+1}$, ..., $h_1 = -\eta y_{\kappa+\mu} - (1-\eta)y_{\kappa+\mu-1}$, $h_0 = -(1-\eta)y_{\kappa+\mu}$.

1a) For instance, for $\kappa = 1$ and $\mu = 0$ (fixed time of EoL returns at $T = 2$ time periods, e.g. lease-back products with EoU return at the end of period 1 and EoL exit at the end of period 2) Eq. 3, $D(x) = \sum_{j=0}^{2} d_j x^j = 0$, becomes: $(\eta_t - \theta_t)x^2 + x + (1-\eta_t) = 0$, which directly yields the nontrivial solution $x = (\eta-\theta+1)/(\theta-\eta)$, or, $x = (\eta-1)/(2-\eta)$, since in this case $\theta = 2$.

1b) In general, for any $\kappa > 0$ and for $\mu = 0$, Eq. 3 is $D(x) = \sum_{j=0}^{2\kappa} d_j x^j$, with $d_{2\kappa} = \eta - \theta$, $d_{2\kappa-1} = d_{2\kappa-2} = \dots = d_1 = 1$, $d_0 = -\eta$.

1c) Distributed product returns. For $\kappa = 2$ and $\mu = 1$ (see below what the system Eqs. T1, T8 and T16 are in this example) Eq. 3 becomes $D(x) = \sum_{j=0}^{5} d_j x^j$,

$d_5 = \eta - \theta$, $d_4 = 1$, $d_3 = 1$, $d_2 = 1 - \eta y_1$, $d_1 = 1 - \eta y_2 - y_1$, $d_0 = 1 - \eta y_3 - y_1 - y_2$.

(Eqs T1, T8 and T16: $E_t = a_{t-3}g_{1,t}\prod_{i=1}^{3} x_{t-i} + a_{t-4}g_{2,t}\prod_{i=1}^{4} x_{t-i} + a_{t-5}g_{3,t}\prod_{i=1}^{5} x_{t-i}$, $Q_t = a_{t-1}x_{t-1}g_{1,t} + a_{t-2}x_{t-1}x_{t-2}g_{2,t} + a_{t-3}g_{3,t}\prod_{i=1}^{3} x_{t-i}$, $U_t = a_t x_t + a_{t-1}x_t x_{t-1} + a_{t-2}x_t x_{t-1}x_{t-2} + a_{t-3}x_t x_{t-1}\ x_{t-2}x_{t-3}\left(1-g'_{1,t}\right) + a_{t-4}x_t x_{t-1}x_{t-2}x_{t-3}x_{t-4}\left(1-g'_{1,t-1}-g'_{2,t}\right)$

2) Two remanufacturing cycles, $N = 3$: Eq. 1 becomes, $q = \varphi x^{-3\kappa}\varepsilon(x^{\kappa} + x^{2\kappa}))$.

Then $D(x) = \sum_{j=0}^{4\kappa+\mu} d_j x^j = 0$ and from Eq. 10, $d^T = \left[d_{4\kappa+\mu} \quad d_{4\kappa+\mu-1} \quad 1 \quad 1 \quad \dots \quad 1 d_{2\kappa+2\mu} \right.$ $\left. d_{2\kappa+2\mu-1} \dots d_1 d_0 \right]$ or $d^T = \left[d_{4\kappa+\mu} = 2(\eta-\theta), \quad d_{4\kappa+\mu-1} = 2(1-\eta+\theta), 1, 1 \ \dots \ 1 d_{2\kappa+2\mu} = 1-\eta \right.$

y_1, $d_{2\kappa+2\mu-1} = \sum_{i=2\kappa+2\mu}^{4\kappa+\mu+1} h_i$,..., $d_1 = \sum_{i=2}^{4\kappa+\mu+1} h_i, d_0 = \sum_{i=1}^{4\kappa+\mu+1} h_i$] with the h_i, $i = 1,2,\ldots, 2\kappa+2\mu$ given in Appendix B.

2a) In particular, for $\kappa = 1$ and $\mu = 0$ (two reuse cycles, non-distributed product returns) Eq. 3 is $D(x) = \sum_{j=0}^{4} d_j x^j = 0$, with $d_4 = 2(\eta-\theta)$, $d_3 = 2$, $d_2 = 2-\eta y_1$, $d_1 = 2-\eta y_1-\eta y_2$, $d_0 = 2-\eta y_1-\eta y_2+(\eta-2)y_1$

2b) for $\kappa = 2$, $\mu = 1$ (distributed take-back in three time periods, see below for Eqs. T1, T8 and T16) Eq. 3 is $D(x) = \sum_{j=0}^{9} d_j x^j = 0$, with

$d_9 = 2(\eta-\theta)$, $d_8 = 2$, $d_7 = 2$, $d_6 = 2-\eta y_1$, $d_5 = 2-\eta y_2-y_1$, $d_4 = 2-\eta y_3-\eta y_1-y_2-y_1$,
$d_3 = 2-\eta y_4-\eta y_2-2y_1-y_2-y_3$, $d_2 = 2-\eta y_5-\eta y_3-2y_1-2y_2-y_3-y_4$
$d_1 = 2-\eta y_4-2y_1-2y_2-2y_3-y_4$, $d_0 = 2-\eta y_5-2y_1-2y_2-2y_3-2y_4-y_5$

(Eqs T1, T8 and T16: $E_t = a_{t-5} g_{1,t} \prod_{i=1}^{5} x_{t-i} + a_{t-6} g_{2,t} \prod_{i=1}^{6} x_{t-i} + a_{t-7} g_{3,t} \prod_{i=1}^{7} x_{t-i}$

$Q_t = a_{t-1}x_{t-1}g_{1,t} + a_{t-2}x_{t-1}x_{t-2}g_{2,t} + a_{t-3}g_{3,t}\prod_{i=1}^{3} x_{t-i} + a_{t-3}g_{1,t}\prod_{i=1}^{3} x_{t-i} + a_{t-4}g_{2,t}\prod_{i=1}^{4} x_{t-i} + a_{t-5}g_{3,t}\prod_{i=1}^{5} x_{t-i}$,

$U_t = a_t x_t + a_{t-1}x_t x_{t-1} + \ldots + a_{t-5}x_t x_{t-1}\ldots x_{t-5}\left(1-g'_{1,t}\right) + a_{t-6}x_t x_{t-1}\ldots x_{t-6}\left(1-g'_{1,t-1}-g'_{2,t}\right)$).

Eq. 3 may yield the retention rate, or retention probability, x, (x_t = probability to remain in the internal reuse/remanufacturing cycle in time period t given that it is not part of the EoL exit at time t), a key parameter which unties the cyclic knot.

Product returns and vintage flows

For practical use, the retention rate, x, is obtained via Eq. 3 in each time period, t, using the monitored values η and θ from stock and EoL samples. Then, the mean take-back rate, q, is determined from Eq. 1 as follows. First the EoL rate is found from Eq. T15. Next, the fraction of reusable products which is actually returned, φ, is found as the mean sample path value of φ_t via Eq. A1, based on assumption A and on the strong law of large numbers [25] using the ratios $y^*_{i,t-\kappa-\mu+i}$ of vintage of age i in the return sample in time period $t-\kappa-\mu+i$, over the vintage of same age i in the corresponding stock sample, (Eq. 12):

$$\varphi = <\varphi_t> = <\sum_{i=\kappa-\mu}^{\kappa+\mu} y^*_{i,t-\kappa-\mu+i}> \; if \; \nu \le \kappa \qquad (11a)$$

$$\varphi = <\varphi_t> = \sum_{i=\mu+1}^{\kappa+\mu} <y^*_{i,t-\kappa-\mu+i}> \; if \; \nu > \kappa \qquad (11b)$$

$$y^*_{i,t-\kappa-\mu+i} = Q_{s,i,t-\kappa-\mu+i}/U_{s,i,t-\kappa-\mu+i-1}, i = \kappa-\mu, \kappa-\mu+1, \ldots, \kappa, \kappa+1, \ldots, \kappa+\mu \qquad (12)$$

If the net demand, a_t, is not vanishing, the reusable return flow prediction is $Q_t^{'} = q_t^{'} a_t$. The flows of the age vintages of reusable returns can be determined based on the monitored vintage fractions of reusable returns, Eq. 8.

A 5-step implementation procedure

The following steps are applied sequentially in each time period, t.

1. Obtain a representative sample of product stock and the corresponding subsample of reusable product returns, as well as a sample of EoL returns in time t and find i) the mean age of the stock sample, η and the mean age, θ, of the EoL sample; ii) the fractions of the $(N-2)\kappa+2\mu+1$ age vintages of reusable product returns $y_{i,\ t}$, $i=\kappa-\mu,\ \kappa-\mu+1,\ \kappa-\mu+2,\ ...,\ (N-1)\kappa+\mu-1,\ (N-1)\kappa+\mu$, in the reusable product return subsample; iii) the ν ratios, $y*_{i,\ t}$, of the age vintages in the reusable product return subsample over the same age-vintages in the stock sample.
2. Determine the parameters p_j, $j=0, 1, 2, ..., (N-2)\kappa+2\mu$ from Eq. 7 and π_j, $j=0, 1, 2, ..., (N-2)\kappa+1$ from Eq. 9. Set p_j outside the range $j>(N-2)\kappa+2\mu$ and π_j, $j>(N-2)\kappa+1$ equal to zero.
3. Use Eqs. 4–6 and the current values of $y_{i,t}$, η_t, θ_t, to determine the entries h_i, $i=0, 1, 2, ..., 2(N-1)\kappa+\mu+1$, of the vector h; use Eq. 3 or Eq. 10 to determine the coefficients of Eq. 3, $D(x)=0$.
4. Solve $D(x)=0$ to obtain the retention rate, x, in time t. Use the previous value of x—solution of Eq. 3 obtained in the previous time period, $t-1$, i.e. x_{t-1}—as initial value of the numerical procedure.
5. Obtain the EoL rate, from Eq. T15 in Table 1 and φ from Eq. 11. Obtain the prediction of reusable product return rate, $q_t^{'}$, from Eq. 2 and the expected return flow using the current value of net demand, a_t; obtain the vintage flows of reusable product returns from Eq. 8.

The parameters N, κ, T, μ, can be determined, or updated, if they are slowly time-varying, using the minimum and maximum ages in the stock and return samples, $m_{E,\ t}$, $k_{E,\ t}(ork_{U,\ t})$ and $m_{Q,\ t}$ via Eqs. C1–C4 in Appendix C.

Applications

Application 1: retreaded tyres Data from Japan, Europe and the USA [14, 19, 20] suggest that the EoL exit essentially spreads in years 2–8 from production. Early losses due to wear and biodegradation amount to 2% annually. Retreading allows a second cycle of the product, yet only ~25% of returned by tyre shops are suitable for remoulding (reusable returns), the rest lost from the remanufacturing cycle. For simulation purposes the cyclic system may be represented by our model with $T=4$, $\kappa=2$, $\mu=1$ years. We allow random early loss, (x_t varies randomly with mean value 0.6). The fractions of the return and EoL distribution are randomly varying. The mean distribution is assumed Gaussian in the simulations (i.e. the fractions $g_{i,t}$ are random deviates ±50% around mean values g_i; the g_i follow a quantised Gaussian distribution) of total level $\varphi=1$. Model T1–T21, in Tables 1 and 2 is used for simulating the system. The product stock, flow and mean age values for three different realisations (among a large number of simulated realisations) are given in Fig. 3 under randomly varying sales, randomly varying early loss (around 0.4) and randomly varying return/EoL distribution. The simulations fit actual data (actually reused at about 20–25% of sales) (Fig. 3a, c, e). The return profile does not necessarily follow the sales profile, neither is there a fixed distinctive lag between sales and returns.

Expected tyre returns 5-step procedure. Using the minimum and maximum age data, Eqs. D1–D4 give $T=4$, $\kappa=2$, $\mu=1$ years. Then Eq. 2 is $N=2$, $q=\varphi x^{-2\kappa}\varepsilon x^{\kappa}$ or $q=\varphi x^{-4}\varepsilon x^2$, $h^T=[h_6 \quad h_5 \quad 0 \quad h_3 \quad -h_2 \quad h_1 \quad h_0]$
with: $h^T=[\eta-\theta \quad 1-\eta+\theta \quad 0 \quad -\eta y_1 \quad -\eta y_2-(1-\eta)y_1 \quad -\eta y_3-(1-\eta)y_2 \quad (1-\eta)y_3]$
and Eq. 3 becomes:

$$d_5=\eta-\theta, d_4=1, d_3=1, d_2=1-\eta y_1, d_1=1-\eta y_2+y_1, d_0=1-\eta y_3+y_2+y_1.$$

The mean ages and the vintage fraction values, $y_{i,\,t}$, are obtained by simulation. Solution of Eq. 3 in each time period yields the mean retention rate; x and Eq. T15 gives the EoL rate. Subsequently, the ratios of reusable return vintages over the corresponding age vintages in stock, $y'_{i,\ t+\kappa+\mu-i}$ (obtained from a stock sample and the corresponding reusable return subsample) are found,
$y*_{1,\,t-2}=a_{t-3}x_{t-3}g_{1,\,t-2}/a_{t-3}x_{t-3}=g_{1,\,t-2}$, $y*_{2,\,t-1}=a_{t-3}x_{t-2}x_{t-3}g_{2,\,t-1}/a_{t-3}x_{t-2}x_{t-3}=g_{2,\,t-1}$, $y*_{3,\ t}=a_{t-3}x_{t-1}x_{t-2}x_{t-3}g_{3,\ t}/a_{t-3}x_{t-1}x_{t-2}x_{t-3}=g_{3,\ t}$) and Eq. 11a gives the value of the mean take-back fraction: $\varphi_t=\sum_{i=1}^{3}y*_{i,t+1-i}=y*_{1,t-2}+y*_{2,t-1}+y*_{3,t}=1$.

Then Eq. 2 gives the prognosis of reusable return rate q_t' from which the reusable return flow prediction, Q_t' is obtained. It is seen (Fig. 3a, c, f) that the predicted values of reusable return rate, q_t', closely follow the MSPV of the return rate $<Q_t/a_t>$. The predicted values of the return flow, Q'_t, are also close to the actual values, Q_t, with a mean absolute percentage error (MAPE) around 10% in numerous realisations. Besides mean error metrics, the method features high accuracy at important peaks and lows of product return flow. Compared to existing methods and particularly ARIMAX, the method presented herein features about one third MAPE in numerous realisations (10.2% versus 29.3% in Fig. 3b, 8.4% versus 23.5% in Fig. 3d, 11% versus 19.3% in Fig. 3f). Predicted EoL rate, ε' and mean retention time, τ', via Eqs. T15 and T11 closely follow the time averages of E_t/a_t and U_t/a_t, as well.

Application 2: office computers. Refurbishing and upgrading of office computers [15, 21, 48, 49], extends their useful lifetime from 3 to 5 years to 8–10 years, with potentially two intermediate returns for remanufacturing ($N=3$). Model T1-T12 with $T=9$, $\kappa=3$, $\mu=2$, years and $N=3$, enables simulation of the IR (Fig. 4) under random early loss (mean at 0.89), and randomly varying fractions of the return/EoL distribution. Six different realisations are depicted in Fig. 4 with same sales profile. A return utilisation policy lasting for three periods $n=3$, i.e. Eq. T20 with $\upsilon_{i,\,t,\,t}=0.15$, $\upsilon_{i,\,t,\,t+1}=0.1$, $\upsilon_{i,\,t,\,t+2}=0.05$ $i=1,2,\ldots,10$, is assumed, with remanufactured exports assumed zero, $RU_{e,\ t}=0$. Net demand is determined from eq. T19, Table 2, to balance sales volumes. Imports of used are also zero, $I_{u,\,t}=0$. Also, $\varphi_t=0.5$ is assumed for the simulations. The mean values of the fractions of the EoL distribution follow a right skew (delayed) quantised Weibull ($\alpha=8.42$, $\beta=4.8$) distribution with deviation up to 50% (i.e. the fractions $g'_{i,t}$ are random deviates ± 50% around mean values g_i in Fig. 4a–f). Also, $g_i=0.5\,g'_i$.

Expected product returns From age data, Eqs. D1–D4 give $N=3$, $T=9$, $\kappa=3$, $\mu=2$. Then Eq. 2 becomes $q=\varphi x^{-3\kappa}\varepsilon(x^{\kappa}+x^{2\kappa})$ or, $q=\varphi x^{-9}\varepsilon(x^3+x^6)$. Following the 5-step procedure, Eqs. 4–9 give the vector h (of dimension $2(N-1)\kappa+\mu+1=2{*}2{*}3+2+1=15$, with $h_{15}=2(\eta-\theta)$,

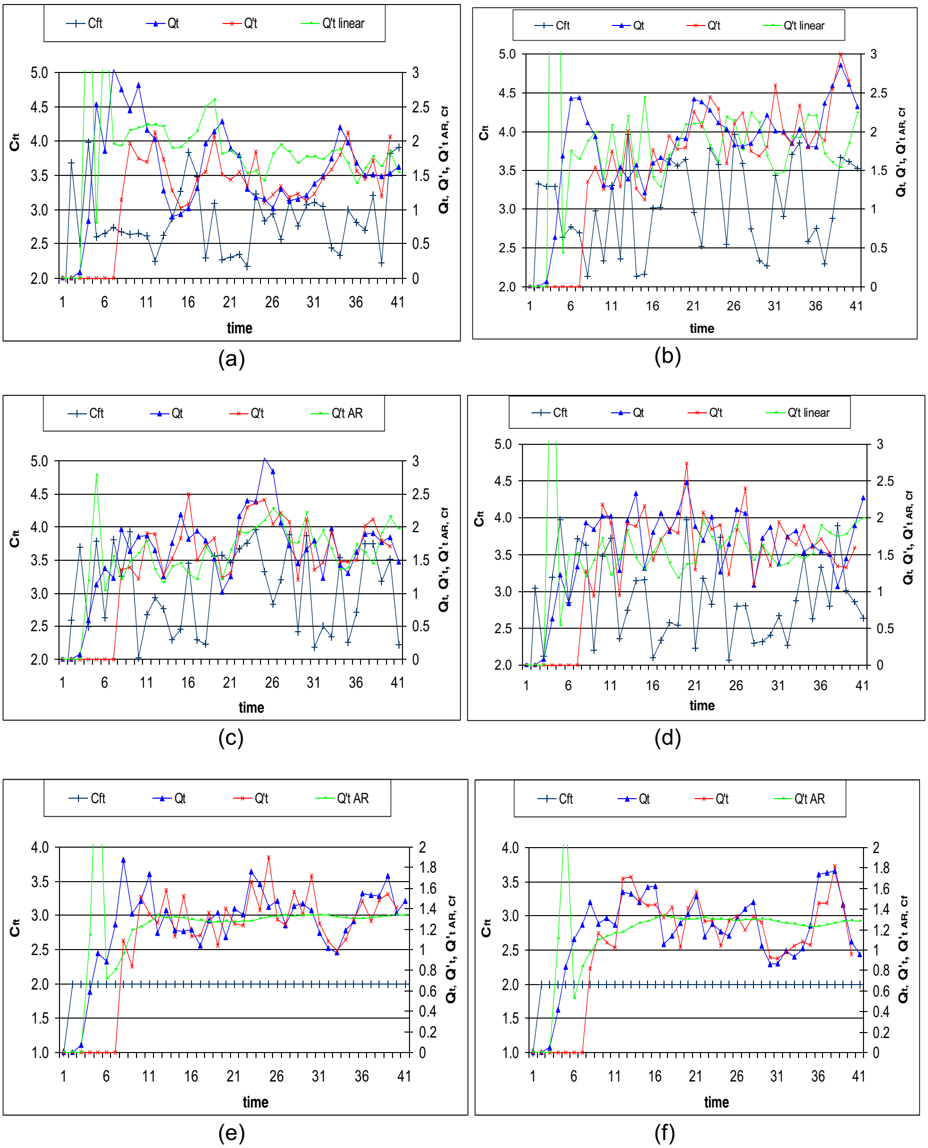

Fig. 4 Application 2. Computers: Six realisations under random early loss Ω_t, randomly varying fractions of the return distribution via MCMC simulation and various sales profiles, $C_{f,t}$. (The mean values, g_i of the random fractions $g_{i,t}$ of the take-back distribution follow the Gaussian distribution in the first four (**a**–**d**, top and middle) and the Weibull distribution with $\alpha = 8.42$, $\beta = 4.8$ in **e** and **f**, bottom). Enhanced prognosis of return flow, Q′, based on the proposed method (MAPE = 8–15%, red line following the blue line) using Eq. 3 and real-time data of stock mean age, η_t and EoL mean age, θ_t, compared to leading methods in use (MAPE at 20–30%, green line, right column, **a**–**f**)

$h_{14} = 2(1-\eta+\theta),\ h_{13} = h_{12} = 0,\ h_{11} = -\eta y_1,\ h_{10} = -\eta y_2-(1-\eta)y_1,\ h_9 = -\eta y_3-(1-\eta)y_2,$ $h_8 = -\eta y_4-(1-\eta)y_3-\eta y_1,\ h_7 = -\eta y_5-(1-\eta)y_4-\eta y_2-(1-\eta)y_1,\ h_6 = -\eta y_6-(1-\eta)y_5$ $-\eta y_3-(1-\eta)y_2,\ h_5 = -\eta y_7-(1-\eta)y_6-\eta y_4-(1-\eta)y_3,\ h_4 = -\eta y_8-(1-\eta)y_7-\eta y_5-(1$ $-\eta)y_4,\ h_3 = -(1-\eta)y_8-\eta y_6-(1-\eta)y_5,\ h_2 = -\eta y_7-(1-\eta)y_6,\ h_1 = -\eta y_8-(1-\eta)y_7,\ h_0 =$

$-(1-\eta)y_8$ and Eq. 3 or Eq. 10, $d_j = \sum_{i=j+1}^{15} h_i, j = 15, 14, 13, ..., 1, 0$. Solved in every successive time interval, Eq. 3 gives the retention rate. Then Eq. T15 yields the EoL rate and Eq. 11b ($N=3>2, \nu=5>\kappa=3$) gives $\varphi_t = \sum_{i=\mu+1}^{\kappa+\mu} y*_{i,t-\kappa-\mu+i} = \sum_{i=3}^{5} y*_{i,t-5+i} = y*_{3,t-2} + y*_{4,t-1} + y*_{5,t}$ $= a_{t-5}x_{t-5}x_{t-4}x_{t-3}g_{3,\,t-2}/a_{t-5}x_{t-5}x_{t-4}x_{t-3} + (a_{t-5}x_{t-5}x_{t-4}x_{t-3}x_{t-2}g_{4,\,t-1}/a_{t-5}x_{t-5}x_{t-4}x_{t-3}x_{t-2} + a_{t-2}x_{t-2}g_{1,\,t-1}/a_{t-2}x_{t-2}) + (a_{t-5}x_{t-5}x_{t-4}x_{t-3}x_{t-2}x_{t-1}g_{5,\,t}/a_{t-5}x_{t-5}x_{t-4}x_{t-3}x_{t-2}x_{t-1} + a_{t-2}x_{t-2}x_{t-1}g_{2,\,t}/a_{t-2}x_{t-2}x_{t-1})$ $= g_{3,\,t-2} + (g_{4,\,t-1} + g_{1,\,t-1}) + (g_{5,\,t} + g_{2,\,t})$ and $\varphi = <\varphi_t> = \sum_{i=1}^{\nu} <g_{i,t-\nu+i}> = 0.5 \sum_{i=1}^{5} <g'_{i,t-5+i}> = 0.5$, with the fractions of reusable return vintages over the corresponding vintages in stock, $y*_{i,\,t-5+i}$, i.e. $y*_{3,\,t-2}$, $y*_{4,\,t-1}$ and $y*_{5,\,t}$ obtained from the stock sample and corresponding subsample or product returns. Subsequently, the return rate forecast, $q_t^{'}$ from Eq. 2, and the return flow forecast $Q_t^{'}$ are found (Fig. 4a–f). Compared to the ARIMAX sales-based forecasting of product returns, the method features about twice lower error, e.g. in regard to the MAPE metric (9% versus 15% in Fig. 4a, 12% versus 20% in Fig. 4b, 13% versus 18% in Fig. 4c, 11% versus 22% in Fig. 4d, 8% versus 13% in Fig. 4e, 11% versus 19% in Fig. 4f). Without including any return or residual life distribution and without presuming any particular

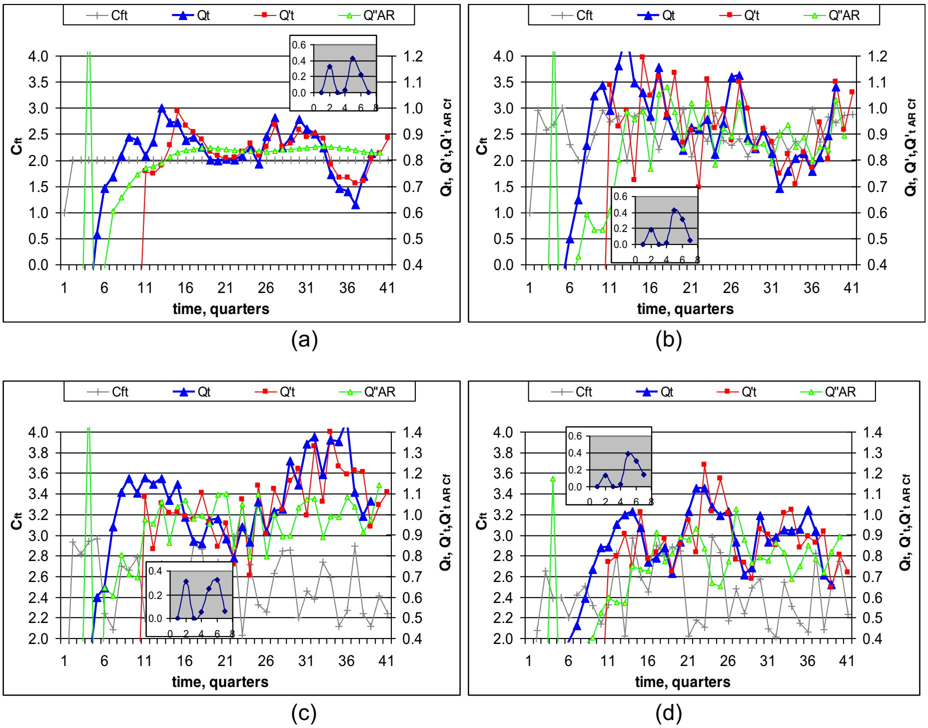

Fig. 5 Application 3 Electronic product: Four different realisations under double peak randomly varying take-back distribution, $g_{i,t}$, via MCMC simulation (embedded for time $t=10$) reflecting commercial and EoU returns, random early loss, varying sales and varying return utilisation. Prognosis based on Eq. 3 closely follows the variations of product return flow, whilst ARIMAX forecasting produces average estimates (green line, right column; **b**, **d**) especially under levelled sales, **a**. The MAPE via Eq. 3 is 4%, whilst the MAPE based on ARIMAX with sales as exogenous variable, ARIMAXCf is 20%

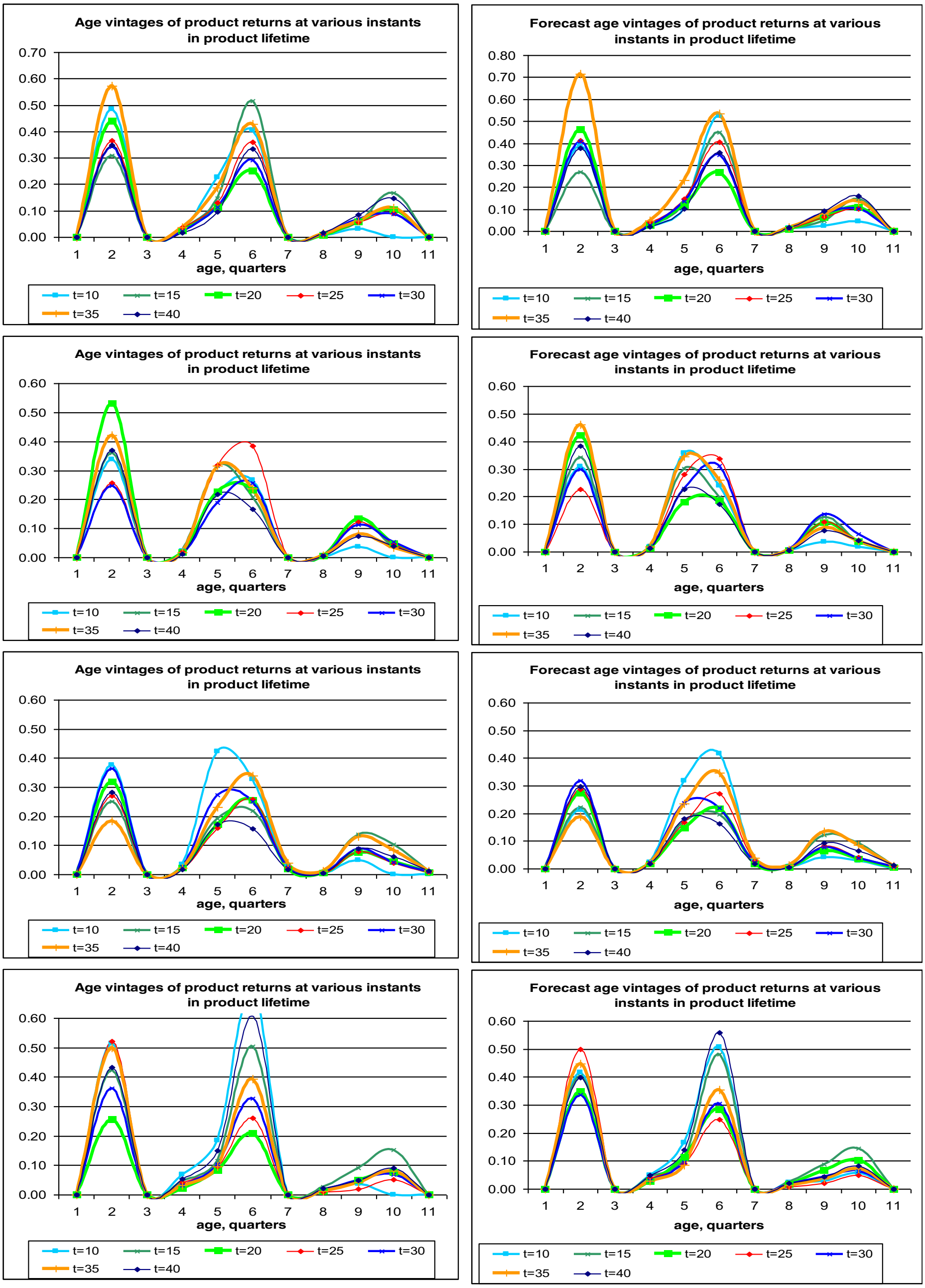

Fig. 6 Application 3. Actual (left column) and predicted via the proposed method (right column) age vintage flows corresponding to the 4 realisations in Fig. 5. The first nodule corresponds to commercial returns of the first cycle, the second represents EoU returns from the first cycle and commercial of the second cycle and the third represents EoU returns of the second cycle

dependence on sales volumes and history, the method, based on current stock mean age and EoL mean age data, shows improved performance in real time compared to the ARIMAX method.

Application 3: Electronics. Electronic products [32, 53, 57] featuring an average lifetime around 3 years may be represented by our model with $T = 12$ and a four-month discretisation interval. Products that may be remanufactured twice are assessed, i.e. $N = 3$. Then $\kappa = 4$ quarters and assuming full spread of the returns within the remanufacturing cycle gives $\mu =$ 3 quarters. As in applications 1 and 2, returns may exhibit a drastically different profile than net demand or sales, including peaks and lows (Fig. 5 presents the flows of four realisations with the fractions $g_{i,t}$ corresponding to a double-peak return distribution encompassing commercial and reusable returns, under various sales profiles and randomly varying early loss.

Expected returns Eq. 2 becomes $q = \varphi x^{-3\kappa}\varepsilon(x^{\kappa} + x^{2\kappa})$, $q = \varphi x^{-12}\varepsilon(x^4 + x^8)$. Eqs. 4–9 give the vector h (of dimension $2(N-1)\kappa + \mu + 1 = 2^*2^*4 + 3 + 1 = 20, h^T = [h_{20} \quad h_{19} \quad 0 \quad 0 \quad \ldots \quad 0 \quad h_{15} \quad h_{14} \quad \ldots \quad h_1 \quad h_0]$, $h_{20} = 2(\eta - \theta)$, $h_{19} = 2(1 - \eta + \theta)$, $h_{18} = h_{17} = h_{16} = 0$, $h_{15} = -\eta y_1$, $h_{14} = -\eta y_2 - (1-\eta)y_1$, $h_{13} = -\eta y_3 - (1-\eta)y_2$, $h_{12} = -\eta y_4 - (1-\eta)y_3$, $h_{11} = -\eta y_5 - (1-\eta)y_4 - \eta y_1$, $h_{10} = -\eta y_6 - (1-\eta)y_5 - \eta y_2 - (1-\eta)y_1$, $h_9 = -\eta y_7 - (1-\eta)y_6 - \eta y_3 - (1-\eta)y_2$, $h_8 = -\eta y_8 - (1-\eta)y_7 - \eta y_4 - (1-\eta)y_3$, $h_7 = -\eta y_9 - (1-\eta)y_8 - \eta y_5 - (1-\eta)y_4$, $h_6 = -\eta y_{10} - (1-\eta)y_9 - \eta y_6 - (1-\eta)y_5$, $h_5 = -\eta y_{11} - (1-\eta)y_{10} - \eta y_7 - (1-\eta)y_6$, $h_4 = -(1-\eta)y_{11} - \eta y_8 - (1-\eta)y_9$, $h_3 = -\eta y_9 - (1-\eta)y_8$, $h_2 = -\eta y_{10} - (1-\eta)y_9$, $h_1 = -\eta y_{11} - (1-\eta)y_{10}$, $h_0 = -(1-\eta)y_{11}$ and then Eq. 3 or Eq. 10 give Eq. 3 $(D(x) = 0)$ with $d_j = \sum_{i=j+1}^{19} h_i, j = 19, 18, 17, \ldots, 1, 0$ which, solved bumerically, gives the retention rate, x. Eq. 11b, $(N = 3 > 2, \nu = 7 > \kappa = 4)$ gives $\varphi = < \varphi_t > = < \sum_{i=\mu+1}^{\kappa+\mu} y^*_{i,t-\kappa-\mu+i} > = < \sum_{i=4}^{7} y^*_{i,t-7+i} > = < y^*_{4,t-3} + y^*_{5,t-2} + y^*_{6,t-1} + y^*_{7,t} > = \quad < g_{4,t-3} + \left(g_{5,t-2} + g_{1,t-2}\right) + \left(g_{6,t-1} + g_{2,t-1}\right) + \left(g_{7,t} + g_{3,t}\right) > = \varphi < \sum_{i=1}^{7} g'_{i,t-7+i} > = \varphi = 0.5$, (with the fractions of reusable return vintages over corresponding vintages in stock, $y*_{i,\,t-\kappa-\mu+i}$, found from corresponding return and stock samples) and Eq. T15 gives the EoL rate, ε. Then, the predicted return rate, $q_t^{'}$, is determined from Eq. 2. The flow of product returns, $Q_t^{'}$, faithfully represents the actual (simulated) value in all realisations, featuring MAPE less than 8% (4%, 9%, 10%, 8.7% in Fig. 5a, b, c, d respectively). In contrast, the ARIMAX method features MAPE about 15% (10%, 14%, 15%, 18%, respectively). The predicted vintage flows of product returns obtained from Eq. 8, using the monitored age distribution, y_i and the forecast flow $Q_t^{'}$, are in close agreement with the actual vintages, as manifested by Fig. 6 (in which the first nodule corresponds to commercial returns of the first cycle, the second represents EoU returns from the first cycle and commercial of the second cycle and the third nodule represents EoU returns of the second cycle).

Conclusions

Green and profitable industrial remanufacturing depends on the quantity and quality of product take-back. Reliable prognosis provides competitive advantages for efficient inventory control and operations' planning. Strongly influenced by random stock losses and volatile product take-back, returns are delayed, random and unobservable for several products. No fixed particular distribution can accurately represent volatile product take-back by consumer and EoL exit. The present work sought prognosis of returns via dimensionless rates that are core features in cyclic manufacturing, independent of sales and net demand. The proposed method stands on two pillars (a) a law associating the mean retention time and the EoL rate via the stock mean age and the EoL mean age (both population averages) (b) the assumption that each specific fraction of the product take-back/EoL distribution is a family of independently and identically distributed random variables. In the same line as the strong law of large numbers by which time averages are closely approximated by ensemble averages under mild conditions, we have used population averages from stock and EoL samples (the mean ages of stock and EoL which are scaled and reliably acquired data) to determine the retention rate. The method allows for random sales and returns, random early losses during lifetime and unknown/non-stationary take-back and EoL distributions. It was implemented and tested via the Markov-chain Monte-Carlo simulation in models representing car tyres, electronic products and office computers, providing improved prediction of product return quantity and age vintage flows, compared to up-to-date prognosis methods in use. The method can be used for enhanced planning of operations in remanufacturing and manufacturing of original products, prediction of life cycle impact inventory and proactive policy-making towards sustainability.

Proofs

Eq. 1: From Eqs. T2 and T17 using assumption A and since for each i the $g'_{i,\,t}$, i = 1, 2, ..., ν and the $g_{i,\,t}$, i = 1, 2, ..., ν are independently and identically distributed, with distributions scaled byφ, using the strong law of large numbers, as the number of samples becomes large the mean sample path values of the random variables $g_{i,\,t}$, $i = 1, 2, ..., \nu$ and $g'_{i,\,t}$, i = 1, 2, ..., ν are related by $g_i = \varphi g_i^{'}$, $i = 1, 2, ..., \nu$ for all possible sample paths with probability one.

$$\begin{aligned} <\varphi_t> &= <\sum_{i=1}^{\nu} g_{i,t-\nu+i}> = \sum_{i=1}^{\nu} <\left(g_{i,t-\nu+i}\right)> = \sum_{i=1}^{\nu} <\varphi g'_{t-\nu+i}> = \varphi \sum_{i=1}^{\nu} <g'_{t-\nu+i}> \\ &= \varphi^{*}1 = \varphi \end{aligned} \tag{A1}$$

Multiplying each one of the contributions of the $(N-1)$cycles in the reusable return flow, $Q_{j,\,t}$ in Eq. T16, by the retention probability from t to $t + T - j\kappa - 1$, that is by $\prod_{i=0}^{T-j\kappa-1} x_{t+i}$, gives: $\left(\sum_{k=1}^{v} a_{t-j\kappa+k-\mu-1} g_{\nu-k+1,t} \prod_{i=1}^{j\kappa-k+\mu+1} x_{t-i}\right)\left(\prod_{i=0}^{T-j\kappa-1} x_{t+i}\right) = \left(\sum_{k=1}^{v} a_{t-j\kappa+k-\mu-1} g_{\nu-k+1,t} \prod_{i=1}^{T-k+\mu} x_{t-i}\right)$, which, for fixed inflow, a, and constant retention rate, x, becomes $\left(a\sum_{k=1}^{v} g_{\nu-k+1} x^{T-k+\mu}\right)$, or $Q_j x^{T-j\kappa} = \left(a\sum_{k=1}^{v} g_{\nu-k+1} x^{T-k+\mu}\right)$. Subsequently, use of Eq. A1 gives $Q_j x^{T-j\kappa} = \left(a\sum_{k=1}^{v} \varphi_{\nu-k+1} g'_{\nu-k+1} x^{T-k+\mu}\right) = a\varphi \sum_{k=1}^{v} g'_{\nu-k+1} x^{T-k+\mu} = \varphi E$. Therefore, $Q_j x^{T-j\kappa} = \varphi E \Leftrightarrow Q_j = \varphi E x^{-T} x^{j\kappa} \Leftrightarrow \sum_{j=1}^{N-1} Q_j = \varphi x^{-T} E \sum_{j=1}^{N-1} x^{j\kappa} \Leftrightarrow Q = \varphi x^{-T} E \sum_{j=1}^{N-1} x^{j\kappa}$. ⋄

Eq. 3: q, ε and φ are substituted as follows. Denoting the age vintage flows of reusable returns by $Q_{i,\,t}$, $i = \kappa - \mu, \kappa - \mu + 1, ..., (N-1)\kappa + \mu$, the ratio q / φ is found from the product return balance

$$\left(\sum_{i=\kappa-\mu}^{(N-1)\kappa+\mu} Q_{i,t}x^{(N-1)\kappa+\mu-i}\right) = \varphi(N-1)x^{(N-1)\kappa+\mu}a, \text{or} \left(\sum_{i=\kappa-\mu}^{(N-1)\kappa+\mu} Q_{i,t}x^{(N-1)\kappa+\mu-i}\right) = \varphi(N-1)x^{(N-1)\kappa+\mu}Q/q \text{ or}$$
$$q\varphi^{-1}\left(\sum_{i=\kappa-\mu}^{(N-1)\kappa+\mu} y_{i,t}x^{(N-1)\kappa+\mu-i}\right) = (N-1)x^{(N-1)\kappa+\mu} \tag{A2}$$

Eq. A2 enables elimination of $q\varphi^{-1}$ in Eq. 1. Law T12 of stock and EoL allows to substitute ε in terms of η, θ and x (Eq. T15) in Eq. 2. Indeed, using Eq. A2 and Eq. T15, Eq. 2 becomes an algebraic equation in the retention rate, x. $x^T(N-1)x^{(N-1)\kappa+\mu} = \left(\sum_{i=\kappa-\mu}^{(N-1)\kappa+\mu} y_{i,t}x^{(N-1)\kappa+\mu-i}\right)\frac{\eta x+(1-\eta)}{(\eta-\theta)x+(1-\eta+\theta)}\sum_{j=1}^{N-1} x^{j\kappa}$. If the discretisation interval, δ, is chosen as $\delta = T/\xi$, where ξ is any common multiple of T, N, then the mean cycle time $T/N = \kappa$ is integer and Eq. 2 becomes a polynomial equation $x^T(N-1)x^{(N-1)\kappa+\mu}((\eta-\theta)x + (1-\eta+\theta)) = \left(\sum_{i=\kappa-\mu}^{(N-1)\kappa+\mu} y_{i,t}x^{(N-1)\kappa+\mu-i}\right)(\eta x + (1-\eta))\sum_{j=1}^{N-1} x^{j\kappa}$,

or, factoring out x^κ $(0 < x < 1)$,

$$H(x) = (N-1)x^{2(N-1)\kappa+\mu}[(\eta-\theta)x + (1-\eta+\theta)] - \left(\sum_{i=\kappa-\mu}^{(N-1)\kappa+\mu} y_{i,t}x^{(N-1)\kappa+\mu-i}\right)(\eta x + (1-\eta))\sum_{j=0}^{N-2} x^{j\kappa} = 0 \tag{A3}$$

The coefficients of $H(x)$ are the elements of the vector h found as follows. Define:

$$P(x) =: \sum_{i=\kappa-\mu}^{(N-1)\kappa+\mu} y_{i,t}x^{(N-1)\kappa+\mu-i} \tag{A4}$$

Then $\Pi(x) = \pi_{(N-2)\kappa+1}x^{(N-2)\kappa+1} + \pi_{(N-2)\kappa}x^{(N-2)\kappa} + \pi_{(N-3)\kappa+1}x^{(N-3)\kappa+1}$

$$+ \pi_{(N-3)\kappa}x^{(N-3)\kappa} \ldots + \pi_1 + \pi_0$$

$$\begin{gathered}\pi_{j\kappa+1} = -\eta, \pi_{j\kappa} = -(1-\eta), j = N-2, N-3, \ldots, 1, 0, \pi_1 = -\eta, \pi_0 = -(1-\eta) \\ \pi_{j\kappa-1} = \ldots = \pi_{(j-1)\kappa+2} = 0, j = 2, 3, \ldots, (N-1), \pi_j = 0, j < 0\end{gathered} \tag{A6}$$

The product of two polynomials $\alpha(x)$ of degree m and $\beta(x)$ of degree n in x is a polynomial $\gamma(x)$ of degree $(m+n)$, the coefficients of which are given by

$$\text{Coefficient of order j term}\,\gamma_j = \sum_{i=0}^{n} a_{n-i}\beta_{j-n+i} \tag{A7}$$

Applying Eq. A7 to the product $P(x)\Pi(x)$ yields the analytic formulae 4–9 for h_i, determined only in terms of the vintage fractions $y_{i,\,t}$ in the return sample and of the mean ages, η_t and θ_t. Since $H(1) = P(1) - (N-1) = 0$, the polynomial $H(x)$ is divisible by $(x-1)$, i.e. $H(x) = (x-1)D(x)$ with $D(x)$ a polynomial of degree $2(N-1)\kappa + \mu$ and then Eq. 3 becomes $D(x) = 0$.⋄.

Eq. 11. Updating of φ. From Eq. A1, φ is the MSPV of φ_t, which at steady state equals φ. If $N > 2$ then $N-1 \geq 2$ and, using the fact that $\kappa > \mu$, it follows that $(N-1)\kappa > 2\mu \Rightarrow N\kappa - \mu > \kappa + \mu \Rightarrow T - \mu > \kappa + \mu$ and the reusable age vintages in stock, which appear in Eq. 11 are limited within the first $T - \mu = N\kappa - \mu$ terms in Eq. T8, i.e. they do not include the terms featuring $G'_{c,\,k-T+\mu,\,t}$. If $\kappa \geq \nu$, the remanufacturing cycles do not overlap and all remanufacturable age vintages appear in the first (and only) cycle once. Then, since the vintage of age i in the reusable stock in time $t-\kappa-\mu+i$ is given by $a_{t-\kappa-\mu}\prod_{j=t-\kappa-\mu}^{t-\kappa-\mu+i-1}x_j$, the age vintage fractions of reusable product returns in a representative stock sample satisfy.

$$\sum_{i=\kappa-\mu}^{\kappa+\mu} y*_{i,t-\kappa-\mu+i} = \sum_{i=\kappa-\mu}^{\kappa+\mu}\left(\left(a_{t-\kappa-\mu}g_{i,t-\kappa-\mu+i}\prod_{j=t-\kappa-\mu}^{t-\kappa-\mu+i-1}x_j\right)/\left(a_{t-\kappa-\mu}\prod_{j=t-\kappa-\mu}^{t-\kappa-\mu+i-1}x_j\right)\right) = \sum_{i=1}^{\nu} g_{i,t-\nu+i} = \varphi_t.$$

If $\kappa<\nu$, adjacent cycles overlap. Since $\kappa>\mu\Rightarrow 2\kappa>2\mu$ and both are even numbers, it follows that $2\kappa>2\mu+1=\nu\Rightarrow 2\kappa-\mu>\mu+1$. The first overlap starts μ terms prior to the centre axis of the 2nd cycle, i.e. at the vintage of age $2\kappa-\mu$ and therefore the first $\mu+1$ terms do not overlap and $g_{1,t}, g_{2,t}, \ldots, g_{\mu+1,t}$ appear individually in the beginning of the first cycle. More specifically, since $\kappa<\nu$, the last ν-κ age vintages of the first cycle in the return flow corresponding to the fractions $g_{\kappa+1,t}, g_{\kappa+2,t}, \ldots, g_{\nu,t}$ of the return distribution, are of the same age (overlapping) with the first ν-κ terms of the second cycle, namely, $g_{\kappa+1,t}, g_{\kappa+2,t}, \ldots, g_{\nu,t}$, giving rise to ν-κ age vintage terms with combined fractions $(g_{\kappa+1,t}+g_{1,t})$, $(g_{\kappa+2,t}+g_{2,t})$, ..., $(g_{\nu,t}+g_{\nu-\kappa,t})$. The remaining $\nu-2(\nu-\kappa)=2\kappa-\nu$ fractions $g_{\nu-\kappa+1,t}, g_{\nu-\kappa+2,t}, \ldots, g_{\kappa,t}$ completing the return distribution, $g_{1,t}, g_{2,t}, \ldots, g_{\nu,t}$, which are not included in the combined terms, are included in terms of the first cycle as individual fractions. Since the oldest age vintage in the overlapping terms of the 1st and 2nd cycles is of age $\kappa+\mu$ and $\nu-\kappa$ age vintages have been accounted for in the combined terms and $2\kappa-\nu$ additional individual age vintages are needed, the first (most recent) age vintage needed for the completion of the return distribution is of age $\kappa+\mu-(\nu-\kappa)-(2\kappa-\nu)+1=\mu+1$. Alternatively, $\nu-2(\nu-\kappa)$ more recent vintage terms, starting from the first overlap (the first overlapping vintage of age $2\kappa-\mu$ not included) lead to the same result $2\kappa-\mu-(\nu-2(\nu-\kappa))=\mu+1$. Then, the summation in Eq. 11b collects all fractions $g_{i,t-\nu+i}$, $i=1, 2, \ldots, \nu$,once, that is, $<\sum_{i=\mu+1}^{\kappa+\mu} y*_{i,t-\kappa-\mu+i}>=<\sum_{i=1}^{\nu} g_{i,t-\nu+i}>=\sum_{i=1}^{\nu}<\left(g_{i,t-\nu+i}\right)>=\varphi.\diamond$

The vector $\underline{h}$ in examples 1 and 2

1a). Eqs. 7–9 become: $p_0=1$, $\pi_1=-\eta$, $\pi_0=-(1-\eta)$ and from Eqs. 4–6,$h_3=(\eta-\theta)$, $h_2=(1-\eta+\theta)$, $h_1=-\eta$, $h_0=-(1-\eta)$.

1b). $h_{2\kappa+1}=\eta-\theta$, $h_{2\kappa}=\eta-\theta+1$, $h_{2\kappa-1}=h_{2\kappa-2}=\ldots=h_2=0$, $h_1=-\eta$, $h_0=-(1-\eta)$.

1c). Eqs. 4–9 give $h_6=(\eta-\theta)$, $h_5=(1-\eta+\theta)$, $h_4=0$, $h_3=-\eta y_1$, $h_2=-\eta y_2-(1-\eta)y_1$, $h_1=-\eta y_3-(1-\eta)y_2$, $h_0=-(1-\eta)y_3$

2) From Eqs. 7–9 $p_{\kappa+2\mu}=y_{\kappa-\mu}$, $p_{\kappa+2\mu-1}=y_{\kappa-\mu+1}$, ..., $p_1=y_{2\kappa+\mu-1}$, $p_0=y_{2\kappa+\mu}$, $\pi_{j\kappa+1}=-\eta$, $\pi_{j\kappa}=-(1-\eta)$, $j=1, 0$, $\pi_1=-\eta$, $\pi_0=-(1-\eta)$, $\pi_{j\kappa-1}=\ldots=\pi_{(j-1)\kappa+2}=0$, $j>1$, $\pi_j=0$, $j<0$, and then from Eqs. 4–6 $h^T=[h_{4\kappa+\mu+1}\quad h_{4\kappa+\mu}\quad 0\quad 0\quad \ldots\quad 0\quad h_{2\kappa+2\mu+1}\quad h_{2\kappa+2\mu}\quad \cdots\quad h_1\quad h_0]$ $h_{4\kappa+\mu+1}=2(\eta-\theta)$, $h_{4\kappa+\mu}=2(1-\eta+\theta)$, $h_{4\kappa+\mu-1}=h_{4\kappa+\mu-2}=\ldots=h_{2\kappa+2\mu+2}=0$

$$h_j=\sum_{i=0}^{\kappa+2\mu}p_{\kappa+2\mu-i}\pi_{j-\kappa-2\mu+i}, j=0,1,2,\ldots,2\kappa+2\mu+1$$

2a). $h_5=2(\eta-\theta)$, $h_4=2(1-\eta+\theta)$, $h_3=-\eta y_1$, $h_2=-\eta y_2$, $h_1=(\eta-2)y_1$, $h_0=(\eta-2)y_2$

2b). $h_{10}=2(\eta-\theta)$, $h_9=2(1-\eta+\theta)$, $h_8=0$, $h_7=-\eta y_1$, $h_6=-(\eta y_2+(1-\eta)y_1)$, $h_5=-(\eta y_3+(1-\eta)y_2+\eta y_1)$ $h_4=-(\eta y_4+(1-\eta)y_3+\eta y_2+(1-\eta)y_1)$, $h_3=-(\eta y_5+(1-\eta)y_4+\eta y_3+(1-\eta)y_2)$, $h_2=-((1-\eta)y_5+\eta y_4+(1-\eta)y_3)$, $h_1=-(\eta y_5+(1-\eta)y_4)$, $h_0=-((1-\eta)y_5)$.

Update of parameters T, μ, κ, N

Eqs. C1–C4 enable real time update of the parameters T, N, μ and κ via the minimum and maximum ages in stock, reusable return and EoL samples ($k_{U,\,t}$, $m_{Q,\,t}$, $k_{Q,\,t}$, $m_{E,\,t}$, $k_{E,\,t}$ etc.):

$$\text{EoL distribution center axis}\, T = 0.5(m_E + k_E) = k_Q + m_Q \tag{C1}$$

$$\text{Mean cycle time}\, \kappa = m_Q + 0.5(k_U - m_E) \tag{C2}$$

$$\text{Number of remanufacturing cycles}\, (N-1)\kappa = m_E - m_Q \tag{C3}$$

$$\text{or}\, N = \kappa^{-1} T = \frac{0.5(k_U + m_E)}{m_Q + 0.5(k_U - m_E)} \tag{C4}$$

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