Journal of Control, Automation and Electrical Systems (2021) 32:378–389
https://doi.org/10.1007/s40313-020-00680-y

Multivariable Binary MRAC with Guaranteed Transient Performance Using Non-homogeneous Robust Exact Differentiators

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Received: 2 September 2019 / Revised: 28 September 2020 / Accepted: 5 December 2020 / Published online: 8 February 2021

Abstract
In this paper, we propose an extension of the binary model reference adaptive control (BMRAC) for uncertain multivariable linear plants with non-uniform arbitrary relative degree. The BMRAC is a robust adaptive strategy which has good transient properties and robustness of sliding mode control with the important advantage of having a continuous control signal free of chattering. The relative degree obstacle is circumvented using a multivariable version of a hybrid estimation scheme, named global robust exact differentiator (GRED). The hybrid estimator switches between robust exact differentiators (RED) based on higher-order sliding modes and lead filters in a way that the exact derivatives are globally obtained in finite time. To improve the robustness and the transient performance of the GRED, we propose a modification of the switching scheme replacing the conventional RED with a non-homogeneous one. Global exact output tracking is obtained with robustness and guaranteed transient performance without requiring stringent symmetry assumptions on the plant high-frequency-gain matrix.

Keywords Multivariable adaptive control · Global exact tracking · Higher-order sliding modes · Arbitrary relative degree · Output feedback · Uncertain systems · Binary adaptive control

1 Introduction

Recent progress regarding the relaxation of the symmetry assumption have brought renewed attention to the multivariable model reference adaptive control (MIMO MRAC) problem. While conventional direct[1] MIMO MRAC based on bilinear parametric model, referred here as bilinear MRAC, techniques require the knowledge of a multiplier S_p for the plant high-frequency-gain (HFG) matrix K_p, such that $S_p K_p$ is symmetric positive definite (SPD) (Tao 2003; Ioannou and Sun 1996), newly proposed techniques exempt such requirement. This is an important result, since symmetry conditions are not generic and can be destroyed by arbitrarily small parametric perturbations.

Some quite general solutions use matrix factorization to exempt the need of a symmetrizing matrix S_p to deal with uncertain and possibly non-symmetric HFG such as Tao (2003), Costa et al. (2003), Xie and Zhang (2005) and Xie (2008). However, these approaches lead to controller overparametrization, which may be an undesirable drawback. The necessity of overparametrization not only increases the number of adapted parameters in a square system of M inputs and M outputs by a scale of $M(M-1)/2$ as it can lead to loss of robustness, since the matching parameters form a linear, and therefore unbounded, manifold (Tao and Ioannou 1990; Hsu et al. 2015).

New designs were recently proposed to plants of relative degree one (Gerasimov et al. 2018), plants with HFG with nonzero leading minors of unknown sign (Wang et al. 2020) and process with actuator faults (Arici and Kara 2020).

[1] In direct adaptive control, the controller parameters are directly updated from an adaptive law.

This work was partly supported by FAPERJ, CAPES and CNPq.

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Some new possibilities appeared recently based on the generalized passivity concept named WSPR (Fradkov 2003). The WASPR concept was also introduced in Barkana et al. (2006) and consists in requiring the plant to become WSPR through some static output feedback. In Hsu et al. (2015), it is shown that the condition for minimum phase plants with uniform relative degree one to be WASPR is that the HFG matrix has a positive real diagonal Jordan form (referred as PDJ condition). Inspired by these results, a new bilinear MRAC solution for plants with non-uniform arbitrary relative degree was proposed in Hsu et al. (2014). In contrast to the conventional solutions (Tao 2003; Ioannou and Sun 1996) that require stringent assumption of symmetry or symmetrization of the plant HFG, the result established in Hsu et al. (2014) requires only the PDJ condition and avoids controller overparametrization.

However, it is known that pure gradient adaptive control may suffer from lack of robustness and poor adaptation transient. This has motivated the proposal of the BMRAC (binary model reference adaptive control) (Hsu and Costa 1994) which combines desirable features of sliding mode control with those of parameter adaptation algorithms. The BMRAC consists basically of a conventional MRAC modified by parameter projection combined with high adaptation gain. The BMRAC tends to behave as a sliding mode controller as the adaptation gain is increased. However, since the BMRAC has a continuous control signal such gain can be tuned up to a sufficient value while avoiding chattering achieving robustness with predictable transient behavior. An indirect adaptive approach to BMRAC was recently proposed in Teixeira et al. (2015) and, however, restricted to SISO plants of relative degree one.

Considering the generalized passivity concepts, a MIMO extension of the BMRAC was developed in Yanque et al. (2012) successfully mitigating the symmetry assumption related to the plant HFG matrix. As in Hsu et al. (2014), the method proposed in Yanque et al. (2012) relies only on the PDJ condition and does not lead to controller overparametrization. However, the development was restricted to plants of uniform relative degree one.

In this context, this paper seeks a MIMO MRAC that (i) circumvents symmetry requirements; (ii) deals with plants with non-uniform arbitrary relative degree; (iii) achieves robustness and global exact tracking with predictable transient behavior; and (iv) does not present overparametrization.

This can be obtained through a further extension to MIMO BMRAC that uses a hybrid estimation scheme recently generalized to a MIMO framework in Nunes et al. (2014). Such estimator, named global robust exact differentiator (GRED), switches between a standard MIMO lead filter and a nonlinear one which uses robust exact differentiators (RED) (Levant 2003) based on higher-order sliding modes. The conventional RED, proposed in Levant (1998, 2003), is based on the homogeneity principle and is derived from the super-twisting algorithm (STA) introduced in Levant (1993). Such differentiator is not only able to obtain exact derivatives, but also presents asymptotic optimal performance in the presence of small noise (Levant 2003). However, its convergence can be slow when the initial errors are large, which is a consequence of the homogeneity.

More recently, the seminal work (Moreno and Osorio 2008) has proposed a simple strong Lyapunov function for the STA, which has allowed further developments and improvements for the STA and STA-based observers (Shtessel et al. 2010; Gonzalez et al. 2012; Nagesh and Edwards 2014; Oliveira et al. 2017). A modification to the conventional RED was proposed in Levant (2009) by introducing non-homogeneous higher-order linear terms, which allow faster convergence with large initial errors while preserving finite time convergence. Moreover, as shown in Moreno and Osorio (2008) the inclusion of linear terms into the STA not only enhances the performance, but also improves the robustness to perturbations.

In this paper, we introduce a modification to the hybrid estimator of Nunes et al. (2014), replacing the conventional RED with a non-homogeneous RED in the switching scheme to take advantage of its improved robustness and fast convergence. Thus, the derivatives of the output provided by the modified hybrid estimator can be used to render a system with uniform relative degree one to which the MIMO BMRAC can be applied. We show that global exact output tracking is obtained with the new controller with predictable transient performance.

Note that the main added complexity is the inclusion of MIMO GRED, since the stabilizing multiplier is of little complexity. This is a solution to allow the use of the BMRAC approach which requires the error equation to be of relative degree one with respect to the input u. The overall trade-off is worth since overparameterization is avoided while maintaining the good performance and predictable transient of the BMRAC.

2 Problem Description

Consider an uncertain square MIMO LTI plant described by

$$\dot{x}_p = A_p x_p + B_p u \,, \qquad y = H_p x_p \,, \tag{1}$$

where $x_p \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^M$ is the input, $y \in \mathbb{R}^M$ is the output and A_p, B_p and H_p are constant uncertain matrices. All the uncertain parameters belong to some compact set Υ, such that the necessary uncertainty bounds to be defined later are available for design. The plant input–output model is given by

$$y = G(s)u, \qquad G(s) = H_p(sI - A_p)^{-1}B_p\,.$$

The variable s is used to denote both the Laplace variable and the differential operator, according to the context. The following assumptions are made as usual in MIMO adaptive control literature:

(A1) $G(s)$ is minimum phase and has full rank.
(A2) The plant is controllable and observable.
(A3) The observability index ν of $G(s)$, or an upper bound of ν, is known.
(A4) There exists a known diagonal polynomial matrix $\iota_m(s)$, defined as the modified left interactor (MLI) matrix of $G(s)$ of the form $\iota_m(s) = \text{diag}\,\{d_1(s),\, d_2(s),\, \ldots,\, d_M(s)\}$ where $d_i(s)$ are monic stable polynomials of degrees $\rho_i > 0$.

Despite the fact that Assumption (A4) may seem restrictive, it can be argued that a diagonal interactor can be achieved by means of an appropriate precompensator. According to Lemma 2.6 in Tao and Ioannou (1988), a precompensator $W_p(s)$ exists so that $G(s)W_p(s)$ has diagonal interactor matrix. Moreover, $W_p(s)$ does not depend on the plant parameters. Once the interactor is known to be diagonal and if the relative degree of each element of G(s) (or $G(s)W_p(s)$) is known, then $\xi(s)$ can be determined without any prior knowledge on the transfer function parameters.

(A5) The high-frequency-gain matrix of $G(s)$ defined as $K_p = \lim_{s\to\infty} \xi_m(s)G(s)$ is finite and non-singular, with positive eigenvalues and diagonal Jordan form (PDJ condition).

Hence, from Assumption (A4) the vector relative degree $[\rho_1, \rho_2, \ldots, \rho_M]^T$ is arbitrary and known.

Let the reference signal y_m be generated by the following reference model[2]

$$y_m = W_m(s)\,r\,; \qquad r, y_m \in \mathbb{R}^M \tag{2}$$

$$W_m(s) = \text{diag}\left\{(s+a)^{-1}, \ldots, (s+a)^{-1}\right\} L^{-1}(s), \tag{3}$$

where $a > 0$ and L(s) is given by

$$L(s) = \text{diag}\,\{L_1(s), L_2(s), \ldots, L_M(s)\}\,, \tag{4}$$

where $L_i(s), i = 1, \ldots, M$ are Hurwitz polynomials

$$L_i(s) = s^{(\rho_i-1)} + l^{[i]}_{\rho_i-2}s^{(\rho_i-2)} + \cdots + l^{[i]}_1 s + l^{[i]}_0. \tag{5}$$

[2] The tracking of more general reference models could be obtained by simply preshaping the reference signal r through a precompensator at the input of the above model.

The choice of the reference model follows the idea of reducing an arbitrary relative degree problem to one with uniform relative degree one, which is achieved through differentiation of output signals. The transfer matrix $W_m(s)$ has the same vector relative degree as $G(s)$ and its HFG is the identity matrix. The tracking error is then given by

$$e = y - y_m. \tag{6}$$

When the plant is known, matching between the closed-loop transfer function matrix and $W_m(s)$ is achieved by the control law

$$u^* = {\theta^*}^T \omega, \tag{7}$$

where the parameter matrix is written as

$$\theta^* = \left[{\theta_1^*}^T \; {\theta_2^*}^T \; {\theta_3^*}^T \; {\theta_4^*}^T\right]^T, \tag{8}$$

with $\theta_1^*, \theta_2^* \in \mathbb{R}^{M(\nu-1)\times M}$, $\theta_3^*, \theta_4^* \in \mathbb{R}^{M\times M}$ and the regressor vector

$$\omega = [\omega_u^T \; \omega_y^T \; y^T \; r^T]^T, \quad w_u, w_y \in \mathbb{R}^{M(\nu-1)} \tag{9}$$

is obtained from I/O state variable filters given by:

$$\omega_u = A(s)\Lambda^{-1}(s)u\,, \quad \omega_y = A(s)\Lambda^{-1}(s)y\,, \tag{10}$$

where $A(s) = [Is^{\nu-2} \;\; Is^{\nu-3} \;\cdots\; Is \;\; I]^T$, $\Lambda(s) = \lambda(s)I$ with $\lambda(s)$ being a monic stable polynomial of degree $\nu - 1$. The matching conditions require that $\theta_4^{*T} = K_p^{-1}$.

The plant transfer matrix can be expresses as a product $G(s) = Z_0(s)P_0^{-1}(s)$. With the matching control of Eq. (7), the parameter matrix (8) and the regressor (9), the matching equation

$$\begin{aligned}&\theta_1^{*T}A(s)P_0(s) + \left(\theta_2^{*T}A(s) + \theta_3^{*T}\Lambda(s)\right)Z_0(s)\\&\quad = \Lambda(s)\left(P_0(s) - \theta_4^{*T}\xi_m(s)\right)\end{aligned} \tag{11}$$

defines the matching parameters θ_1^{*T}, $\theta_2^{*T}\theta_3^{*T}$, and θ_4^{*T} (Tao 2003). However, since the plant is unknown, the desired parameters matrix θ^* is also unknown. In this case, the following control law can be used

$$u(t) = \theta^T(t)\omega(t). \tag{12}$$

It is important to note that in the MIMO case, solution of matching equation is possibly non-unique, such that Eq. (11) may admit a set of solutions to θ^*. This is worthy of attention since in this case it is not possible to identify the plant even with a rich signal (Mathelin and Bodson 1994). In this work, we are interested in boundedness of the closed-loop signals and tracking of the reference model. The problem of parameter convergence is discussed in detail in Mathelin and Bodson (1994) and Willner et al. (1992).

An error equation can be developed extending the usual approach for SISO MRAC to the multivariable case (Tao 2003). Defining the state vector $X = \left[x_p^T,\ \omega_u^T,\ \omega_y^T\right]^T$

$$\dot{X} = A_0 X + B_0 u, \qquad y = H_0 X. \tag{13}$$

Then, adding and subtracting $B_0 u^*$ and noting that there are matrices Ω_1 and Ω_2 such that $\omega = \Omega_1 X + \Omega_2 r$, one has

$$\dot{X} = A_c X + B_c K_p \left[u - u^*\right] + B_c r, \qquad y = H_0 X \tag{14}$$

with $A_c = A_0 + B_0 \theta^{*T} \Omega_1$, $B_c = B_0 \theta^{*T} \Omega_2 = B_0 \theta_4^{*T} = B_0 K_p^{-1}$. The reference model can be described by

$$\dot{X}_m = A_c X_m + B_c r, \qquad y_m = H_0 X_m. \tag{15}$$

The error state $x_e := X - X_m$ dynamics is given by

$$\dot{x}_e = A_c\, x_e + B_c K_p [u - \theta^{*T} \omega], \qquad e = H_0\, x_e, \tag{16}$$

where $\{A_c, B_c, H_0\}$ is a non-minimal realization of $W_m(s)$, so that the error equation can be written in input–output form:

$$e = W_m(s) K_p \left[u - \theta^{*T} \omega\right]. \tag{17}$$

In Yanque et al. (2012), an extension of the BMRAC to MIMO systems with uniform vector relative degree one was proposed. A simple way to extend the result to the non-uniform arbitrary relative degree case would be to use the derivatives of y such that a system with relative degree one is rendered. Thus, instead of using the tracking error $e = y - y_m$, one could use a modified error of uniform relative degree one

$$\bar{e} = \xi_y - \xi_m = L(s) W_m(s) K_p \left[u - \theta^{*T} \omega\right], \tag{18}$$

where the modified outputs are defined as

$$\xi_y = L(s) y, \qquad \xi_m = L(s) y_m = L(s) W_m(s) r = \frac{1}{s+a} I r$$

$$\xi_y = L(s) y = \begin{bmatrix} y_1^{(\rho_1 - 1)} + \cdots + l_1^{[1]} \dot{y}_1 + l_0^{[1]}\, y_1 \\ \vdots \\ y_M^{(\rho_M - 1)} + \cdots + l_1^{[M]} \dot{y}_M + l_0^{[M]} y_M \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{j=0}^{\rho_1 - 1} l_j^{[1]} h_1^T A_c^{(j)} X \\ \vdots \\ \sum_{j=0}^{\rho_M - 1} l_j^{[M]} h_M^T A_c^{(j)} X \end{bmatrix} = \bar{H} X, \tag{19}$$

where $h_i^T \in \mathbb{R}^{n+2M(\nu-1)}$ is the ith row of matrix H_0 and the second equality comes from Assumption (A4) and (14).

Thus, to overcome the relative degree restriction, the idea would be to employ the operator defined in Eq. (4) such that $L(s)G(s)$ and $L(s)W_m(s)$ have uniform vector relative degree one. Note that since $W_m(s)$ is chosen by design, ξ_m is easily generated without the need of calculating derivatives. A further requirement to guarantee stability is a passivity property of the error system. The generalized WSPR passivity concept will be used to cope with uncertain and not necessarily symmetric K_p. Main results and further discussion on WSPR and WASPR can be found in Barkana et al. (2006), Hsu et al. (2011). If K_p is PDJ, it is possible to conclude that the error system of Eq. (18) is WSPR since $L(s) W_m(s) = \frac{1}{s+a} I$. However, if the PDJ condition is not satisfied on K_p, it is possible to use a stabilizing multiplier $\bar{L}$ such that $\bar{L} K_p$ is PDJ. The modified tracking error is, in this case,

$$\begin{aligned} \bar{e}_L &= \bar{L}\left(\xi_y - \xi_m\right) \\ \dot{x}_e &= A_c x_e + B_c K_p \left[u - u^*\right], \quad \bar{e}_L = \bar{L} \bar{H} x_e \end{aligned} \tag{20}$$

which can also be rewritten in input–output form as follows, since $L(s) W_m(s)$ commutes with $\bar{L}$

$$\bar{e}_L(t) = L(s) W_m(s) \bar{L} K_p \left[u - \theta^{*T} \omega\right]. \tag{21}$$

However, we still have to address the problem of obtaining ξ_y since it is not directly available as the operator (4) is not implementable. To solve this problem, we use a global robust exact differentiator (GRED) that switches between a lead filter and a non-homogenous RED.

3 BMRAC Using a MIMO Lead Filter

Note that ξ_m is directly available for implementation, while the signal ξ_y needed to overcome the relative degree obstacle is not. A possible way to solve this problem is to estimate ξ_y by means of a lead filter.

$$\hat{\xi}_l = L_a(s) y, \ \ L_a(s) = L(s) F^{-1}(\tau s), \tag{22}$$

where $F(\tau s) = \mathrm{diag}\{(\tau s+1)^{\rho_1 - 1}, \ldots, (\tau s+1)^{\rho_m - 1}\}$. One can note that as $\tau > 0$ tends to zero, $\hat{\xi}_l$ approximates ξ_y. Defining the lead filter estimation error as the difference between the estimate of ξ_y obtained by the lead filter and its actual value

$$\varepsilon_l = \hat{\xi}_l - \xi_y, \tag{23}$$

its dynamics can be described by:

$$\dot{x}_\varepsilon = \frac{1}{\tau} A_\varepsilon x_\varepsilon + B_\varepsilon \dot{\xi}_y, \quad \varepsilon_l = H_\varepsilon x_\varepsilon\,, \tag{24}$$

where $\dot{\xi}_y = \bar{H} A_c X + \bar{H} B_c K_p \tilde{\vartheta}^T \Omega + \bar{H} B_c r$ (see (14) and (19), $A_\varepsilon = \text{block diag}\{A_\varepsilon^{[1]}, \ldots, A_\varepsilon^{[M]}\}$,

$B_\varepsilon = \text{block diag}\{B_\varepsilon^{[1]}, \ldots, B_\varepsilon^{[M]}\}$, $H_\varepsilon = \text{block diag}\{H_\varepsilon^{[1]}, \ldots, H_\varepsilon^{[M]}\}$, with $A_\varepsilon^{[i]} \in \mathbb{R}^{\rho_i - 1 \times \rho_i - 1}$, $B_\varepsilon^{[i]} \in \mathbb{R}^{\rho_i - 1 \times 1}$, $H_\varepsilon^{[i]} \in \mathbb{R}^{1 \times \rho_i - 1}$,

$$A_\varepsilon^{[i]} = \begin{bmatrix} -a_{\rho_i-2}^{[i]} & 1 & 0 & \ldots & 0 \\ -a_{\rho_i-3}^{[i]} & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1^{[i]} & 0 & 0 & \ldots & 1 \\ -a_0^{[i]} & 0 & 0 & \ldots & 0 \end{bmatrix}, \qquad B_\varepsilon^{[i]} = \begin{bmatrix} -b_{\rho_i-2}^{[i]} \\ -b_{\rho_i-3}^{[i]} \\ \vdots \\ -b_1^{[i]} \\ -b_0^{[i]} \end{bmatrix},$$

$$H_\varepsilon^{[i]} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \end{bmatrix},$$

$a_j^{[i]} = \mathcal{C}_{\rho_i-1-j}^{\rho_i-1}$, $b_j^{[i]} = \mathcal{C}_{j+1}^{\rho_i-1}$, and $\mathcal{C}_l^n = n!/(k!(n-k)!)$.

In the stability analysis of the closed-loop error system, with state $z^T = \left[x_e^T \; x_\varepsilon^T\right]$, we will consider the presence of a uniformly bounded output disturbance $\beta_\alpha(t)$ of order τ. Further, the following parametrization is adopted in order to extend the BMRAC to the MIMO case in a more natural way

$$\vartheta = \text{vec}(\theta) = \begin{bmatrix} \theta^{[1]} \\ \theta^{[2]} \\ \vdots \\ \theta^{[N]} \end{bmatrix}, \qquad \Omega = I_M \otimes \omega = \begin{bmatrix} \omega & & \\ & \ddots & \\ & & \omega \end{bmatrix} \tag{25}$$

with $\Omega \in \mathbb{R}^{NM \times M}$, $\vartheta \in \mathbb{R}^{NM}$, where N is the number of elements of the regressor vector ω, $\theta^{[i]}$ is the ith column of the parameter matrix θ and $\otimes$ is the Kronecker product. Taking into account the presence of $\beta_\alpha(t)$, the adaptation law of the MIMO BMRAC using a lead filter is given by

$$\dot{\vartheta} = -\vartheta\sigma - \gamma \Omega \bar{L}(\hat{\xi}_l - \xi_m + \beta_\alpha) \tag{26}$$

with σ given by a projection

$$\sigma = \begin{cases} 0, & \text{if} \quad \|\vartheta\| < M_\vartheta \quad \text{or} \quad \sigma_{\text{eq}} < 0 \\ \sigma_{\text{eq}}, & \text{if} \quad \|\vartheta\| \geq M_\vartheta \quad \text{and} \quad \sigma_{\text{eq}} \geq 0 \end{cases} \tag{27}$$

$$\sigma_{\text{eq}} = \frac{-\gamma \vartheta^T \Omega \bar{L}(\hat{\xi}_l - \xi_m + \beta_\alpha)}{\|\vartheta\|^2} \tag{28}$$

where $M_\vartheta > \|\vartheta^*\|$. The control law can be rewritten as

$$u(t) = \theta^T(t)\omega(t) = \Omega^T(t)\vartheta(t). \tag{29}$$

At this point, the following theorem can be stated.

Theorem 1 *Consider the plant (1) and the reference model (2)–(4) with control signal (29) and adaptation law (26)–(28). Suppose that assumptions (A1) to (A4) hold and $\vartheta(0) \leq M_\vartheta$. If the disturbance $\beta_\alpha(t)$ is uniformly bounded by $\|\beta_\alpha(t)\| \leq \tau K_R$, where $K_R > 0$ is a constant, then for sufficiently small $\tau > 0$ and sufficiently large $\gamma > 0$, the closed-loop error system (20), (29), (19), (24), (26)–(28) with state $z^T = \left[x_e^T \; x_\varepsilon^T\right]$, is uniformly globally exponentially practically stable (GEpS) with respect to a residual set, i.e., there exist constants $c_z, a > 0$ such that $\|z(t)\| \leq c_z e^{-a(t-t_0)}\|z(t_0)\| + \mathcal{O}(\sqrt{\tau}) + \mathcal{O}(\sqrt{1/\gamma})$ holds $\forall z(t_0)$, $\forall t \geq t_0 > 0$.*

Proof See Battistel et al. (2014). □

Corollary 1 *For all $R > 0$, there exists $\tau > 0$ sufficiently small and γ sufficiently large to an invariant compact set $D_R := \{z : \|z\| \leq R\}$.*

Corollary 2 *Signals $y_j^{(i)}(t), i = 0, \ldots, \rho_j,\ j = 1, \ldots, M$ are uniformly bounded, i.e., $\exists K_i^{[j]} > 0$ such that $|y_j^{(i)}(t)| \leq K_i^{[j]}, \forall\, t \geq t_0 \geq 0,\ i = 0, \ldots, \rho_j,\ j = 1, \ldots, M$. Moreover, if $\|x_e(t)\| \leq R, \forall t > T$, then, $\exists C_{\rho_j}^{[j]} > 0$ such that $\left\| y_{j[T,t]}^{(\rho_j)} \right\|_\infty \leq C_{\rho_j}^{[j]}$, $j = 1, \ldots, M$.* □

Proof The proof follows the same steps of the proof of Corollary 2 in Nunes et al. (2014). □

4 MIMO Robust Exact Differentiators

The control scheme based on BMRAC and an estimate of ξ_y obtained by a MIMO lead filter presented in the previous section cannot guarantee exact tracking, though it can ensure global stability properties.

Exact tracking can be achieved following similar steps to Nunes et al. (2014), where the conventional homogeneous REDs (Levant 2003) were used in a hybrid estimation scheme. In this work, we employ a modified scheme proposed in Levant (2009) where higher-order terms were added to improve robustness and provide faster convergence while preserving the optimal asymptotic features of the conventional RED.

Let the input signal $f(t)$ be a function defined on $[0, \infty)$ with the nth derivative having a known Lipschitz constant, which means that $f^{(n+1)}$ is uniformly bounded. The nonhomogeneous RED can be described by Levant (2009):

$$\begin{aligned} \dot{\zeta}_i &= v_i, \\ v_i &= -\lambda_i \left|\zeta_i - v_{i-1}\right|^{\frac{n-i}{n-i+1}} \text{sgn}\,(\zeta_i - v_{i-1}) + \\ &\quad - \mu_i(\zeta_i - v_{i-1}) + \zeta_{i+1}, \\ &\;\;\vdots \end{aligned}$$

$$\dot{\zeta}_n = -\lambda_n \operatorname{sgn}(\zeta_n - v_{n-1}) + \mu_n(\zeta_n - v_{n-1}) \tag{30}$$
$$v_{-1} = f. \tag{31}$$

If the positive sequences $\{\lambda_i\}$ and $\{\mu_i\}$ are properly recursively chosen, then the equalities

$$\zeta_0 = f(t); \quad \zeta_i = f^{(i)}(t), \quad i = 1, \ldots, n$$

are established in finite time [see Levant (2009)].

The sequences $\{\lambda_i\}$ and $\{\mu_i\}$ are to be chosen recursively in such a way that $\{\lambda_1, \ldots, \lambda_n\}$ and $\{\mu_1, \ldots, \mu_n\}$ provide for the convergence of the $(n-1)$th-order differentiator with the same Lipschitz constant, and λ_0, μ_0 be sufficiently large [see Levant (2009)]. A possible sequences choice for the second-order differentiator is $\{\lambda_i\} = \{3C_3^{1/3},\ 1.5C_3^{1/2},\ 1.1C_3\}$ and $\{\mu_i\} = \{8,\ 6,\ 3\}$, where C_3 is such that the inequality $\left|f^{(3)}(t)\right| \le C_3$ holds.

In this paper, the idea is to modify the hybrid estimator such that non-homogeneous REDs are employed. The idea is to use a non-homogeneous RED of order $p_j = \rho_j - 1$ for each output $y_j \in \mathbb{R}$, $j = 1, \ldots, M$.

$$\begin{aligned}
v_{-1}^{[j]} &= y_j(t)\\
\dot{\zeta}_i^{[j]} &= v_i^{[j]}, \\
v_i^{[j]} &= -\lambda_i^{[j]} \left|\zeta_i^{[j]} - v_{i-1}^{[j]}\right|^{\frac{p_j - i}{p_j - i + 1}} \operatorname{sgn}(\zeta_i^{[j]} - v_{i-1}^{[j]}) +\\
&\quad - \mu_i^{[j]}(\zeta_i - v_{i-1}) + \zeta_{i+1}^{[j]}\\
&\ \ \vdots
\end{aligned} \tag{32}$$
$$\dot{\zeta}_{p_j}^{[j]} = -\lambda_{p_j}^{[j]} \operatorname{sgn}(\zeta_{p_j}^{[j]} - v_{p_j-1}^{[j]}) - \mu_{p_j}(\zeta_{p_j}^{[j]} - v_{p_j-1}^{[j]}), \tag{33}$$

where $i = 0, \ldots, p_j$ and $v_{-1}^{[j]} = y_j(t)$. A similar approach was considered in Fridman et al. (2008) and Nunes et al. (2014).

Under the foregoing conditions, the above differentiator can provide the exact $y_j(t)$ derivatives. According to Levant (2003, 2009), the RED's performance is asymptotically optimal in the presence of small Lebesgue-measurable input noise. Moreover, it is important to stress that the variables of each individual RED, described in (32), cannot escape in finite time if each y_j is bounded together with its ρ_j first derivatives. As in Nunes et al. (2014), this result is essential in the development of the hybrid estimator scheme and allows the use of the non-homogeneous RED not only in the BMRAC arbitrary relative degree extension, but in any other GRED application. It is formalized in the following lemma.

Lemma 1 *Consider system (30), with state* $\zeta = [\zeta_0 \ \ldots \ \zeta_n]^T$. *If* $\left|f^{(i)}(t)\right| \le K_i,\ i = 0, \ldots, n+1,\ \forall t$ *(finite), for some positive constants* $K_i, i = 0, \ldots, n+1$, *then* $\zeta(t)$ *cannot diverge in finite time.*

Proof See "Appendix A.1". □

Using a MIMO RED, composed by M REDs of order $\rho_j - 1$ for each output y_j, the following estimate for ξ_y can be obtained:

$$\hat{\xi}_r = \begin{bmatrix} \zeta_{\rho_1-1}^{[1]} + \cdots + l_1^{[1]}\zeta_1^{[1]} + l_0^{[1]}\zeta_0^{[1]} \\ \vdots \\ \zeta_{\rho_M-1}^{[M]} + \cdots + l_1^{[M]}\zeta_1^{[M]} + l_0^{[M]}\zeta_0^{[M]} \end{bmatrix}. \tag{34}$$

Using the non-homogeneous MIMO RED, the adaptation law is given by

$$\dot{\vartheta} = -\vartheta\sigma - \gamma\Omega\bar{L}(\hat{\xi}_r - \xi_m) \tag{35}$$

with σ given by a projection

$$\sigma = \begin{cases} 0, & \text{if} \quad \|\vartheta\| < M_\vartheta \quad \text{or} \quad \sigma_{\text{eq}} < 0 \\ \sigma_{\text{eq}}, & \text{if} \quad \|\vartheta\| \ge M_\vartheta \quad \text{and} \quad \sigma_{\text{eq}} \ge 0 \end{cases} \tag{36}$$
$$\sigma_{\text{eq}} = \frac{-\gamma\vartheta^T\Omega\bar{L}(\hat{\xi}_r - \xi_m)}{\|\vartheta\|^2} \tag{37}$$

where $M_\vartheta > \|\vartheta^*\|$.

However, only local/semi-global stability properties can be guaranteed if the control (29) is combined with the adaptation law (35)–(37), since the signals $y_j^{(\rho_j)}(t)$, $j = 1, \ldots, M$ should be uniformly bounded to ensure the non-homogeneous MIMO RED convergence.

5 Global RED-Based BMRAC

The global RED is a hybrid compensator which consists of a (time-varying) convex combination of a lead filter estimate (22) and a RED estimate (34) according to:

$$\hat{\xi}_g = \alpha(\tilde{v}_{rl})\,\hat{\xi}_l + \left[1 - \alpha(\tilde{v}_{rl})\right]\,\hat{\xi}_r\,, \tag{38}$$

where $\tilde{v}_{rl} = \hat{\xi}_r - \hat{\xi}_l$ is the difference between both estimates. The switching function $\alpha(\tilde{v}_{rl})$ is a continuous, state-dependent modulation which assumes values in the interval $[0, 1]$ and allows the controller to smoothly change from one estimator to the other.

Specifically, $\alpha(\cdot)$ is designed such that $\left\|\hat{\xi}_g - \hat{\xi}_l\right\| \le \tau K_R$:

$$\alpha(\tilde{v}_{rl}) \begin{cases} 0, & \|\tilde{v}_{rl}\| < \varepsilon_M - \Delta \\ (\|\tilde{v}_{rl}\| - \varepsilon_M + \Delta)/\Delta, & \varepsilon_M - \Delta \le \|\tilde{v}_{rl}\| < \varepsilon_M \\ 1, & \|\tilde{v}_{rl}\| \ge \varepsilon_M \end{cases} \tag{39}$$

where $0 < \Delta < \varepsilon_M$ is a boundary layer used to smoothen the switching function, and $\varepsilon_M := \tau K_R$ with K_R being an

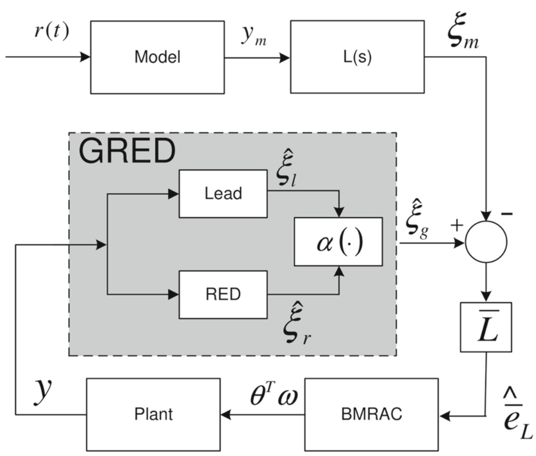

Fig. 1 Block diagram of GRED-BMRAC

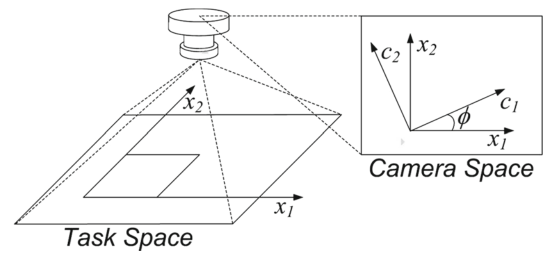

Fig. 2 Camera–robot system representation

appropriate positive design parameter that is selected such that $\varepsilon_M - \Delta > \bar{\varepsilon}_l$. This implies that after some finite time only the MIMO RED is active ($\alpha = 0$), providing exact estimation of the output derivatives ξ_y, as desired. Some insight into how to tune MIMO GRED parameters is given below [for further reference, see Nunes et al. (2014)].

In order to guarantee global exponential stability with respect to a small residual set and to achieve global convergence of the error state to zero, we show that the BMRAC using a MIMO lead filter presented in Sect. 3 can be combined with the MIMO RED (Sect. 4).

It should be noted that global stability to an invariant compact set D_R is guaranteed independently of switching between both estimators since it is possible to show that the resulting system is equivalent to a BMRAC using a MIMO lead filter with a uniformly bounded output disturbance of order τ. Thus, global practical stability and convergence to the compact set D_R are guaranteed, according to Theorem 1. The switching function is properly chosen to ensure that after some finite time only the estimate provided by the MIMO RED is used.

Using the GRED to estimate ξ_y, the adaptive law is

$$\dot{\vartheta} = -\vartheta\sigma - \gamma\Omega\hat{\bar{e}}_L, \tag{40}$$

where $\hat{\bar{e}}_L = \bar{L}(\hat{\xi}_g - \xi_m)$ and σ is given by a projection

$$\sigma = \begin{cases} 0, & \text{if} \quad \|\vartheta\| < M_\vartheta \quad \text{or} \quad \sigma_{\text{eq}} < 0 \\ \sigma_{\text{eq}}, & \text{if} \quad \|\vartheta\| \geq M_\vartheta \quad \text{and} \quad \sigma_{\text{eq}} \geq 0 \end{cases} \tag{41}$$

$$\sigma_{\text{eq}} = \frac{-\gamma\vartheta^T\Omega\hat{\bar{e}}_L}{\|\vartheta\|^2}, \tag{42}$$

where $M_\vartheta > \|\vartheta^*\|$. The stability and convergence results of the proposed control scheme are stated in the following theorem. A block diagram of such scheme is shown in Fig. 1.

Theorem 2 *Consider the plant (1) and the reference model (2)–(4) with control law given by (29) and adaptation law (40)–(42). The switching function $\alpha(\cdot)$ is defined in (39). Suppose that assumptions (A1) to (A5) hold and $\vartheta(0) \leq M_\vartheta$. Then, for sufficiently small $\tau > 0$ and sufficiently large $\gamma > 0$, the closed-loop error system, with state $z^T = [x_e^T \; x_\varepsilon^T]$, described by (29), (20), (19), (24), (40)–(42) is uniformly globally exponentially practically stable (GEpS) with respect to a residual set and the non-homogeneous MIMO RED estimation and all closed-loop signals are uniformly bounded. Moreover, for $\lambda_i^{[j]}$, $j = 1, \ldots, M$, $i = 0, \ldots, \rho_j - 1$, and K_R properly chosen, the estimation of the output derivatives ξ_y becomes exact, being made exclusively by the non-homogeneous MIMO RED ($\alpha(\cdot) = 0$) after some finite time and then, the output tracking error e, converge exponentially to zero.*

Proof See Appendix A.2. □

6 Application to Visual Servoing and Simulation Results

Consider the problem of direct adaptive visual tracking for planar manipulators using a fixed camera (plant) with optical axis orthogonal to the robot workspace. The camera orientation is uncertain with respect to the robot workspace coordinates (Hsu and Lizarralde 2000).

The objective is to control the robot such that the image of the effectuator tracks the desired trajectory in the image plane (Fig. 2). The motivation to choose this example is that the HFG is essentially a rotation matrix which, except for the trivial cases, is neither symmetric nor PDJ since its eigenvalues are complex.

As in Hsu et al. (2007), the following linear uncertain 2×2 plant inspired from the visual servoing problem is considered

$$G(s) = \frac{1}{s^2}K_p, \qquad K_p = \begin{bmatrix} h_1\cos(\alpha) & h_1\sin(\alpha) \\ -h_2\sin(\alpha) & h_2\cos(\alpha) \end{bmatrix}$$

where the camera misalignment angle $\alpha \in (-\pi/2, \pi/2)$, and the scaling factors h_1 and h_2 are unknown parameters.

The reference model is given by

$$W_m(s) = \frac{\lambda_c^2}{(s+\lambda_c)^2} I, \qquad \lambda_c > 0$$

with $L(s) = (s+\lambda_c)I$.

Considering the fact that K_p is the only unknown parameter of the plant, it is possible to reduce the number of parameters to be adjusted. Let $\bar{u} = K_p u$, then $y = (1/s^2)\bar{u}$ and the control law $\bar{u}^*$ that ensures the matching between the closed-loop transfer function matrix and the reference model is given by

$$\begin{aligned} \bar{u}^* &= -2\lambda_c \tfrac{s(s+\lambda_0)}{(s+\lambda_0)} y - \lambda_c^2 y + \lambda_c^2 r \\ &= -2\lambda_c \left\{ \tfrac{1}{(s+\lambda_0)}\bar{u} + \lambda_0 y - \tfrac{\lambda_0^2}{(s+\lambda_0)} y \right\} - \lambda_c^2 y + \lambda_c^2 r, \end{aligned}$$

where $\lambda_0 > 0$. However, $\bar{u}$ is not measurable, since K_p is unknown. Therefore, since $u = K_p^{-1}\bar{u}$, the model matching control law u^* is given by

$$u^* = \theta^{*T}\omega - \frac{2\lambda_c}{(s+\lambda_0)} u,$$

with

$$\omega = \frac{2\lambda_c\lambda_0^2}{(s+\lambda_0)} y - (\lambda_c^2 + 2\lambda_c\lambda_0) y + \lambda_c^2 r$$

and $\theta^* = K_p^{-T}$. Note that θ^* is the only unknown parameter in the model matching control law.

In order to achieve the model following objective, the following GRED-BMRAC-based control law can be used

$$u = \theta^T\omega - \frac{2\lambda_c}{(s+\lambda_0)} u$$

where θ is given by the BMRAC adaptation law (40).

If the PDJ condition is not satisfied on K_p, it is possible to use a stabilizing multiplier $\bar{L}$ such that $\bar{L}K_p$ is PDJ (for details see Yanque et al. (2012)).

In this simulation, we chose the same settings considered in Hsu et al. (2007) to allow a comparison. Thus, the following parameters were used: plant: $\alpha = 1$ rad, $h_1 = 1$, $h_2 = 0.5$, $y(0) = \dot{y}(0) = \left[0,\ 0\right]^T$; model: $\lambda_c = 10$, $y_m(0) = \left[-0.1,\ -0.14\right]^T$; I/O filters $\lambda_0 = 60$; lead filter: $\tau = 0.01$; RED: $C_2 = 10$, $\lambda_0 = 1.1C_2$, $\lambda_1 = 1.5C_2$, $\mu_0 = 3$, $\mu_1 = 6$; GRED: $\varepsilon_M = 0.5$, $\Delta = 0.2$; $\bar{L}$ multiplier: $\hat{\alpha} = 1 - \frac{\pi}{6}$ rad, $\hat{h}_1 = 0.5$, $\hat{h}_2 = 1$, $D_0 = \operatorname{diag}\{1,\ 4\}$; BMRAC: $\gamma = 60$, $M_\vartheta = 2.5$, $\theta(0) = 0_{2\times 2}$; and step size: $h = 5\,.\,10^{-5}$. Reference signal is

$$r = r_1 = \begin{bmatrix} 0.1 + 0.02\,(\sin(0.5t) + \sin(0.75t)) \\ 0.1 + 0.02\,(\sin(0.7t + 1.6) + \sin(t + 1.6)) \end{bmatrix}.$$

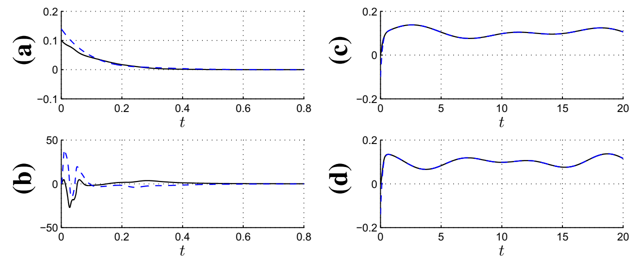

Fig. 3 GRED-BMRAC performance for reference signal $r = r_1$: **a** output tracking errors (—) e_1, (- -) e_2; **b** control signals (—) u_1, (- -) u_2; **c** (—) plant output y_1, (- -) model output y_{m1}; **d** (—) plant output y_2, (- -) model output y_{m2}

In this particular case, the GRED switches instantly to non-homogeneous MIMO RED estimate and $\alpha = 0$ during the entire simulation time. As shown in Fig. 3, the tracking errors converge to zero with a smooth control signal. The GRED-BMRAC obtains faster tracking errors convergence in comparison with the MRAC Lyapunov-based scheme of Hsu et al. (2007) for the same plant and model.

To illustrate the advantage of using the hybrid switching scheme, we now consider a different reference signal with a larger amplitude. In this case, the reference is chosen as

$$r = r_2 = \begin{bmatrix} 5 + (\sin(0.5t) + \sin(0.75t)) \\ 5 + (\sin(0.7t + 1.6) + \sin(t + 1.6)) \end{bmatrix}.$$

As expected, Fig. 4a shows that exact tracking is obtained with the GRED-BMRAC. The switching function α is presented in Fig. 4b. As desired, after a short finite transient of dominant action of the MIMO lead filter only the non-homogeneous MIMO RED estimate is selected by the hybrid estimation scheme.

In this case, considering the same controller parameters as before, if only the MIMO lead filter is used to estimate the derivatives of the plant output ($\alpha = 1$), the tracking errors converge only to a residual set as shown in Fig. 4c. On the other hand, if only the non-homogeneous MIMO RED is used ($\alpha = 0$) the system becomes unstable (see Fig. 4d). This results further justify the importance of the hybrid switching scheme.

6.1 Comparison with Bilinear MRAC

Further understanding of the technique advantages can be provided by comparing the GRED-BMRAC with the bilinear MRAC (Hsu et al. 2014), which uses normalized gradient adaptation. A performance comparison between the BMRAC and matrix factorization methods can be found in Hsu et al. (2015) for plants of relative degree one, where a faster tracking and parameter convergence is obtained for the BMRAC. The results for the bilinear MRAC with reference signal r_1 are shown in Fig. 5, where it is possible to note a slower conver-

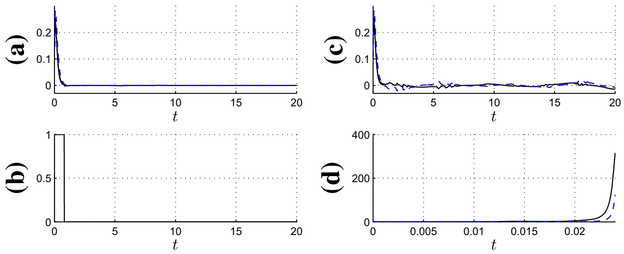

Fig. 4 Output tracking errors (—) e_1, (- -) e_2, for reference signal $r = r_2$ and BMRAC control system based on: **a** GRED estimate, **c** only lead filter estimate, **d** only RED estimate. The graphic of the switching function α for the GRED-BMRAC case is presented in **b**

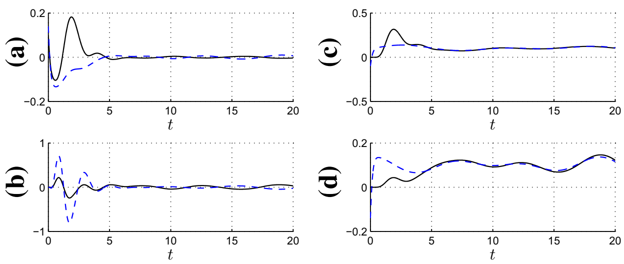

Fig. 5 Bilinear MRAC performance for reference signal $r = r_1$: **a** output tracking errors (—) e_1, (- -) e_2; **b** control signals (—) u_1, (- -) u_2; **c** (—) plant output y_1, (- -) model output y_{m1}; **d** (—) plant output y_2, (- -) model output y_{m2}

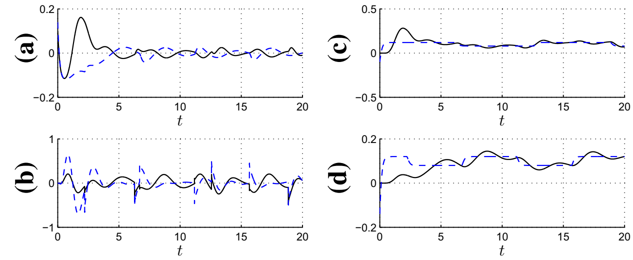

Fig. 6 Bilinear MRAC performance for reference signal $r = r_3$: **a** output tracking errors (—) e_1, (- -) e_2; **b** control signals (—) u_1, (- -) u_2; **c** (—) plant output y_1, (- -) model output y_{m1}; **d** (—) plant output y_2, (- -) model output y_{m2}

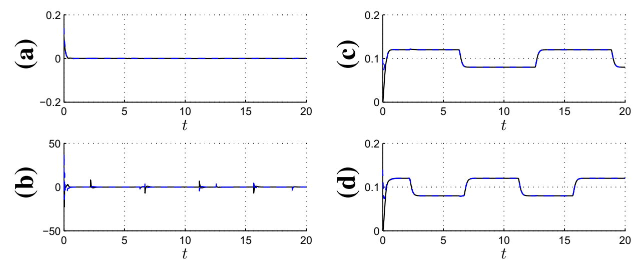

Fig. 7 GRED-BMRAC performance for reference signal $r = r_3$: **a** output tracking errors (—) e_1, (- -) e_2; **b** control signals (—) u_1, (- -) u_2; **c** (—) plant output y_1, (- -) model output y_{m1}; **d** (—) plant output y_2, (- -) model output y_{m2}

gence in comparison with GRED-BMRAC. The advantages are more evident if we consider a square wave input

$$r_3(t) = \left[0.1 + 0.02\,\mathrm{sqw}(0.5t),\ 0.1 + 0.02\,\mathrm{sqw}(0.7t + 1.6)\right]^T,$$

where $\mathrm{sqw}(2\pi t/T)$ denotes the unity square wave of period T. The results for the bilinear MRAC are shown in Fig. 6 and show a slow convergence in constrast to the results of GRED-BMRAC, as shown in Fig. 7. Note that the GRED-BMRAC displays control signals with higher amplitude due to the use of an adaptation law with projection that allows the use of higher gains in comparison with the bilinear MRAC. In practical applications, the input could be saturated to avoid an undesirably large control signal without deteriorating performance, as noted in Battistel et al. (2019).

7 Conclusions

This paper presents an extension to the multivariable binary model reference adaptive control (BMRAC) to deal with uncertain plants with non-uniform arbitrary relative degree. We present a new version of the global robust differentiator (GRED), an estimator that provides global and exact derivatives in finite time by switching a linear filter with a nonlinear one based on non-homogeneous robust exact differentiators. This estimator is then used to circumvent the relative degree obstacle, such that global exact output tracking for uncertain linear plants is obtained with robustness and predictable transient performance. This technique also does not require stringent symmetry assumptions on the high frequency gain. Simulation results show that the new technique outperforms a Lyapunov-based adaptive control scheme and a bilinear MRAC based on normalized gradient adaptation laws.

Appendix A Proofs of Theorems

A.1 Proof of Lemma 1

Applying the change of variables $\upsilon_i = \zeta_i - f^{(i)}(t)$, $i = 0, 1, \ldots, n$ to the system (30), it follows that

$$\begin{cases} \dot{\upsilon}_0 = -\lambda_0 |\upsilon_0|^{n/(n+1)}\,\mathrm{sgn}\,(\upsilon_0) - \mu_0\upsilon_0 + \upsilon_1 \\ \vdots \\ \dot{\upsilon}_i = -\lambda_i |\upsilon_i - \dot{\upsilon}_{i-1}|^{(n-i)/(n-i+1)}\,\mathrm{sgn}\,(\upsilon_i - \dot{\upsilon}_{i-1}) - \\ +\mu_i(\upsilon_i - \dot{\upsilon}_{i-1}) + \upsilon_{i+1} \\ \vdots \\ \dot{\upsilon}_n = -\lambda_n\,\mathrm{sgn}\,(\upsilon_n - \dot{\upsilon}_{n-1}) - \mu_n(\upsilon_n - \dot{\upsilon}_{n-1}) - f^{(n+1)} \end{cases}$$

Considering that $x = |x|\,\mathrm{sgn}\,(x)$ and $\mathrm{sgn}\,(\upsilon_i - \dot{\upsilon}_{i-1}) = \mathrm{sgn}\,(\upsilon_{i-1} - \dot{\upsilon}_{i-2})$ for $i = 2, \ldots, n$, and thus $\mathrm{sgn}\,(\upsilon_i - \dot{\upsilon}_{i-1}) =$

$\mathrm{sgn}(\upsilon_0)$, for $i = 1, 2, \ldots, n$, the system can be rewritten as

$$\begin{cases} \dot{\upsilon}_0 = -\left[\lambda_0|\upsilon_0|^{n/(n+1)} + \mu_0|\upsilon_0|\right]\mathrm{sgn}(\upsilon_0) + \upsilon_1 \\ \vdots \\ \dot{\upsilon}_i = -\left[\lambda_i|\upsilon_i - \dot{\upsilon}_{i-1}|^{\frac{(n-i)}{(n-i+1)}} + \mu_i|\upsilon_i - \dot{\upsilon}_{i-1}|\right]\mathrm{sgn}(\upsilon_0) + \\ +\upsilon_{i+1} \\ \vdots \\ \dot{\upsilon}_n = -\left[\lambda_n + \mu_n|\upsilon_n - \dot{\upsilon}_{n-1}|\right]\mathrm{sgn}(\upsilon_0) - f^{(n+1)} \end{cases}$$

It follows by mathematical induction that the non-recursive form of the system is given by

$$\begin{cases} \dot{\upsilon}_0 = -\left[\phi_0(|\upsilon_0|) + \psi_0|\upsilon_0|\right]\mathrm{sgn}(\upsilon_0) + \upsilon_1 \\ \vdots \\ \dot{\upsilon}_i = -\left[\phi_i(|\upsilon_0|) + \psi_i|\upsilon_0|\right]\mathrm{sgn}(\upsilon_0) + \upsilon_{i+1} \\ \vdots \\ \dot{\upsilon}_n = -\left[\phi_n(|\upsilon_0|) + \psi_n|\upsilon_0|\right]\mathrm{sgn}(\upsilon_0) - f^{(n+1)} \end{cases} \tag{43}$$

where

$$\begin{cases} \phi_0 = \lambda_0|\upsilon_0|^{n/(n+1)}, \\ \psi_0 = \mu_0 \\ \phi_i = \lambda_i(\phi_{i-1} + \psi_{i-1}|\upsilon_0|)^{(n-i)/(n-i+1)} + \mu_i\phi_{i-1}, \\ \psi_i = \mu_i\psi_{i-1}, \quad i = 1, 2, \ldots, n \end{cases}$$

Note that ψ_i is constant, and $\phi_i(|\upsilon_0|)$ obeys the following conditions

$$\begin{cases} \phi_i(|\upsilon_0|) \le \kappa_1^{[i]} \ \text{ se } |\upsilon_0| \le 1 \\ \dfrac{\phi_i(|\upsilon_0|)}{|\upsilon_0|} \le \kappa_2^{[i]} \ \text{ se } |\upsilon_0| > 1 \end{cases}$$

for $i = 0, 1, \ldots, n$ and some positive constants $\kappa_1^{[i]}$ and $\kappa_2^{[i]}$. The first inequality follows from the fact that ϕ_i is bounded, for $|\upsilon_0| \le 1$, if ϕ_{i-1} is also bounded. Since ϕ_0 is bounded under these conditions, then the first inequality follows by mathematical induction. The second inequality can be demonstrated considering that

$$\begin{cases} \dfrac{\phi_0}{|\upsilon_0|} = \lambda_0 \dfrac{1}{|\upsilon_0|^{\frac{1}{n+1}}} \\ \dfrac{\phi_i}{|\upsilon_0|} = \dfrac{\lambda_i}{|\upsilon_0|^{\frac{1}{n-i+1}}}\left(\dfrac{\phi_{i-1}}{|\upsilon_0|} + \psi_{i-1}\right)^{\frac{n-i}{n-i+1}} + \mu_i\dfrac{\phi_{i-1}}{|\upsilon_0|}, \\ \quad i = 1, 2, \ldots, n \end{cases}$$

and the fact that, for $|\upsilon_0| > 1$, $\frac{\phi_i}{|\upsilon_0|}$ is bounded if $\frac{\phi_{i-1}}{|\upsilon_0|}$ is also bounded. Since $\frac{\phi_0}{|\upsilon_0|}$ is bounded under these conditions, then the second inequality follows by mathematical induction.

The equations $\dot{\upsilon}_i = -\left[\phi_i(|\upsilon_0|) + \psi_i|\upsilon_0|\right]\mathrm{sgn}(\upsilon_0) + \upsilon_{i+1}$, $i = 0, 1, \ldots, n-1$ can be rewritten as

$$\dot{\upsilon}_i = -a_i(\upsilon_0)\upsilon_0 - b_i(\upsilon_0) + \upsilon_{i+1}$$

where

$$a_i(\upsilon_0) = \begin{cases} \psi_i, & |\upsilon_0| \le 1 \\ \psi_i + \dfrac{\phi_i(|\upsilon_0|)}{|\upsilon_0|}, & |\upsilon_0| > 1 \end{cases},$$

$$b_i(\upsilon_0) = \begin{cases} \phi_i(|\upsilon_0|)\,\mathrm{sgn}(\upsilon_0), & |\upsilon_0| \le 1 \\ 0, & |\upsilon_0| > 1 \end{cases}$$

The equation $\dot{\upsilon}_n = -\left[\phi_n(|\upsilon_0|) + \psi_n|\upsilon_0|\right]\mathrm{sgn}(\upsilon_0) - f^{(n+1)}(t)$ can be rewritten as

$$\dot{\upsilon}_n = -a_n(\upsilon_0)\upsilon_0 - b_n(\upsilon_0)$$

where

$$a_n(\upsilon_0) = \begin{cases} \psi_n, & |\upsilon_0| \le 1 \\ \psi_n + \dfrac{\phi_n(|\upsilon_0|)}{|\upsilon_0|}, & |\upsilon_0| > 1 \end{cases},$$

$$b_n(\upsilon_0) = \begin{cases} \phi_n(|\upsilon_0|)\,\mathrm{sgn}(\upsilon_0) + f^{(n+1)}(t), & |\upsilon_0| \le 1 \\ f^{(n+1)}(t), & |\upsilon_0| > 1 \end{cases}$$

Note that, once $\left|f^{(n+1)}(t)\right| \le K_{n+1}\ \forall t$, then $|a_i(\upsilon_0)| \le K_{a_i}$ and $|b_i(\upsilon_0)| \le K_{b_i}$, for $i = 0, 1, \ldots, n$ and some positive constants K_{a_i} and K_{b_i}. Defining the complete state vector as $\Upsilon = \left[\,\upsilon_0\ \upsilon_1\ \ldots\ \upsilon_n\,\right]^T$, the system (43) can be rewritten as

$$\dot{\Upsilon} = A(\Upsilon)\Upsilon + b(\Upsilon)$$

where

$$A(\Upsilon) = \begin{bmatrix} -a_0(\upsilon_0) & 1\ 0 \ldots 0 \\ -a_1(\upsilon_0) & 0\ 1 \ldots 0 \\ \vdots & \vdots\ \vdots \ddots \vdots \\ -a_{n-1}(\upsilon_0) & 0\ 0 \ldots 1 \\ -a_n(\upsilon_0) & 0\ 0 \ldots 0 \end{bmatrix} \quad b(\Upsilon) = \begin{bmatrix} -b_0(\upsilon_0) \\ -b_1(\upsilon_0) \\ \vdots \\ -b_{n-1}(\upsilon_0) \\ -b_n(\upsilon_0) \end{bmatrix}$$

and it follows that $\|b(\Upsilon)\| \le c_1$ and $\|A(\Upsilon)\| \le c_2$ for some positive constants c_1 and c_2.

Consider the function

$$V(\Upsilon(t)) = \Upsilon^T(t)\Upsilon(t)$$

It can be verified that

$$\dot{V}(\Upsilon) \le 2c_2V(\Upsilon) + 2c_1\sqrt{V(\Upsilon)}$$

Consider the comparison equation $\dot{V}_c(\Upsilon) = 2c_2 V_c(\Upsilon) + 2c_1\sqrt{V_c(\Upsilon)}$. If $V_c(0) = V(0)$, then $V(t) \leq V_c(t), \forall t \geq 0$. Introducing the new variable $\chi^2 = V_c$, it follows that

$$\chi\dot{\chi} = c_2\chi^2 + c_1\chi$$

Considering $\chi \neq 0$, then $\frac{\mathrm{d}\chi}{\mathrm{d}t} = c_2\chi + c_1$. In this case, it can be shown that

$$\frac{1}{c_2}\ln\left(\frac{2c_2\sqrt{V_c(t)} + 2c_1}{2c_2\sqrt{V_c(0)} + 2c_1}\right) = t$$

and

$$V(t) \leq \left[\left(\sqrt{V(0)} + \frac{c_1}{c_2}\right)e^{c_2 t} - \frac{c_1}{c_2}\right]^2$$

Then the function $V(t)$ cannot escape in finite time for any finite constant K_{n+1}, and thus $\upsilon \in \mathcal{L}_{\infty e}$. Furthermore, since the signals $f^{(i)}(t)$, $i = 0, 1, \ldots, n$, are bounded, then one can conclude that the state ζ cannot escape in finite time. □

A.2 Proof of Theorem 2

The estimate given by the MIMO lead filter and the MIMO RED could be related to ξ_y in (19) as follows:

$$\hat{\xi}_l = \xi_y + \varepsilon_l, \qquad \hat{\xi}_r = \xi_y + \varepsilon_r\,, \tag{44}$$

where ε_l and ε_r are estimation errors. From (44), equation (38) can be rewritten as

$$\hat{\xi}_g = \xi_y + \varepsilon_g, \quad \varepsilon_g \quad = \alpha(\tilde{\nu}_{rl})\varepsilon_l + \left[1 - \alpha(\tilde{\nu}_{rl})\right]\varepsilon_r\,. \tag{45}$$

From (39), the estimation error $\varepsilon_g(t)$ can be rewritten as:

$$\varepsilon_g = \varepsilon_l + \beta_\alpha(\tilde{\nu}_{rl}(t))\,, \tag{46}$$

where by design $\beta_\alpha(\tilde{\nu}_{rl}(t))$ is uniformly bounded by

$$\|\beta_\alpha(\tilde{\nu}_{rl}(t))\| < \varepsilon_M, \text{ with } \varepsilon_M = \tau K_R\,.$$

Substituting (45),(46) into (40)–(42), it can be seen that GRED adaptive law is equivalent to lead adaptive law (26)–(28) with an output disturbance $\|\beta_\alpha(\tilde{\nu}_{rl}(t))\| \leq \varepsilon_M$.

Therefore, Theorem 1 holds if all signals of the GRED-BMRAC system belong to $L_{\infty e}$. In order to demonstrate that the condition is true, we only have to show that all signals in the MIMO RED system are $L_{\infty e}$. This property can be proved by contradiction. Suppose that the maximal interval of finiteness of the signals in the MIMO RED is $[0, T_M)$. During this interval, all conditions of Theorem 1 hold, and thus, all signals of the remaining subsystems of the GRED-BMRAC are bounded by a constant, and in particular, $\left|y_j^{(i)}(t)\right|$, $i = 0, \ldots, \rho_j$, $j = 1, \ldots, M$, from Corollary 2. This leads to a contradiction with Lemma 1, whereby the signals in the MIMO RED could not diverge unboundedly as $t \to T_M$. As a consequence of the continuation theorem for differential equations (in Filippov's theory), T_M must be ∞, which means that all signals are defined $\forall t \geq 0$. Thus, Theorem 1 is valid for the GRED-MRAC system and the closed-loop error system with state z is GEpS with respect to a residual set.

Now, we will analyze the ultimate convergence of the GRED-BMRAC. According to Corollary 1, for sufficiently small τ and sufficiently large γ the error state z is steered to an invariant compact set $D_R := \{z : |z(t)| < R\}$ in some finite time $T_1 \geq 0$. Consider the following Lyapunov candidate

$$V = x_\varepsilon^T P_2 x_\varepsilon \tag{47}$$

whose time derivative is

$$\dot{V} = -\frac{1}{\tau}x_\varepsilon^T Q_2 x_\varepsilon + 2x_\varepsilon^T P_2 B_\varepsilon \dot{\xi}_y$$

following the previous steps

$$\dot{V} = -\frac{1}{\tau}x_\varepsilon^T Q_2 x_\varepsilon + 2x_\varepsilon^T Q_3 x_e + 2x_\varepsilon^T Q_4 X_m + 2x_\varepsilon^T Q_5 r \tag{48}$$

$$\dot{V} \leq -\frac{k_2}{\tau}\|x_\varepsilon\|^2 + k_3\|x_\varepsilon\|\|x_e\| + k_4\|x_\varepsilon\| \tag{49}$$

Within D_R, x_e can be upper bounded by $\|x_e\| \leq R$ such that

$$\dot{V} \leq -\frac{k_2}{\tau}\|x_\varepsilon\|^2 + \tau k_5 \tag{50}$$

Thus, it is possible to show that

$$\|x_\varepsilon(t)\| \leq c_\varepsilon e^{-a(t-t_0)}\|x_\varepsilon(t_0)\| + \tau k_6 \tag{51}$$

Since $\|\varepsilon_l\| \leq \|x_\varepsilon\|$, it is straightforward to show that for some finite $T_2 \geq T_1$, $\|\varepsilon_l\| \leq \bar{\varepsilon}_l$, where $\bar{\varepsilon}_l = \tau K_l$.

Since the MIMO RED is time-invariant, its initial conditions can be considered to be at $t = T_1$. From Lemma 1, the initial conditions are finite. If the parameters λ_i^j are adjusted properly, then from Levant (2009) the estimation error $\varepsilon_r(t)$ converges to zero in some finite time $T_3 > T_1$.

Since K_R is chosen such that $\varepsilon_M > \bar{\varepsilon}_l + \Delta$ and from (39), it follows that after some finite time $\bar{T} = \max\{T_2, T_3\}$ the estimation of σ becomes exact and being made exclusively by the MIMO RED ($\alpha(\tilde{\nu}_{rl}) = 0$), which implies that $\varepsilon_g(t) = 0$, $\forall t \geq \bar{T}$.

In this case, the overall error system can be described by

$$\dot{x}_e = A_c x_e + B_c K_p[u - u^*]\,, \quad \bar{e}_L = \bar{L}\bar{H}x_e\,, \tag{52}$$

since this system has uniform relative degree one we can apply the result obtained in Yanque et al. (2012). Thus, it is possible to conclude that $e(t) \to 0$. □

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