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Modulus Synchronization in Non-identical Hyperchaotic Complex Systems and Hyperchaotic Real System Using Adaptive Control

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Abstract

This paper presents a new modulus combination–combination synchronization (MCCS) scheme using the adaptive control technique. MCCS scheme is performed between complex hyperchaotic (HC) systems and real hyperchaotic (HC) systems. The HC complex Lorenz and Lu are taken as master systems, and the HC Chen system and Newton–Leipnik are taken as slave systems. Based on the Lyapunov stability theory, adaptive control and parameter update law are obtained from making the MCCS. According to the appropriateness of modulus synchronization as a persuasive explication for secure communication, we then explored the application of the suggested adaptive MCCS design. Also, the complexity of master systems improves the protection of stable transmission. Technical investigation and conclusion of simulations verify the performance of the suggested technique using MATLAB.

Keywords HC complex system · Modulus synchronization · Combination–combination (C–C) · Adaptive control · Lyapunov stability theory · Secure communication

1 Introduction

Chaos is a ubiquitous event in nonlinear mathematics and physics. Chaos is defined as the random and unpredictable phenomenon, or the behavior of a complex system, where little changes in the origin positions can lead to significant differences over the period. In general practice, chaos means complete disorder and disorganization, but in science, it implies that the equations expressing nonlinear systems are very sensitive to initial inputs. Although such systems usually exhibit some consistency, it is impossible to prophesy their future behavior with a high degree of certainty. Henri Poincare finds the chaotic deterministic system (Russell 1967), which placed the establishment of modern chaos theory in the eighteenth century. Later on, Lorenz (1963) gives the first chaotic attractor, named the Lorenz attractor.

Many methods have been applied to examine the chaotic behavior. Firstly by drawing phase portrait, secondly by drawing bifurcation, thirdly by drawing the Poincare section, and the last one is by finding Lyapunov exponents. The Lyapunov exponent is the most significant milestone in the theory of chaotic systems and has proved to be an immensely useful technique for analysis. A HC system is defined as an attractor with more than one positive Lyapunov exponents. The minimal dimension for HC is four. Rossler (1979) proposed the first HC system in 1979 to a defined chemical reaction. Later, various researchers suggested different HC and HC complex systems for distinct applications. HC complex systems can take a larger message signal and improve the protection of information because they have real parts and imaginary parts. Recently, these systems find use in many physical cases, such as detuned lasers (Mahmoud and AL-Harthi 2020), rotating fluids, and electronic circuits (Vaidyanathan et al. 2019).

Pecora and Carroll (1990) were continuing in this field and instigate the study chaos synchronization. Synchronization of chaos is an event that may happen when two or more chaotic systems are coupled, and synchronization error converges to zero, and it plays a significant role in several different contexts such as biological models (Vaidyanathan 2015), robotics (DRK et al. 2018), information processing (Das and Pan 2011), secure communication (Mahmoud et al. 2013), neural networks (Wang et al. 2017), image process-

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ing (Volos et al. 2013), and finance models (Xin and Zhang 2015).

The example of synchronization schemes currently available in the literature includes complete synchronization (Mahmoud and Mahmoud 2010a), anti-synchronization (Li and Zhou 2007), hybrid synchronization (Vaidyanathan 2016), projective synchronization (Ding and Shen 2016), modified projective synchronization (Hamri and Ouahabi 2017), hybrid complex projective synchronization, modulus synchronization (Li et al. 2019), combination synchronization (Runzi et al. 2011), combination projective synchronization (Khan and Nigar 2020a), dual combination synchronization, C–C synchronization (Khan and Nigar 2019a; Khan and Singh 2018a), etc. Synchronization of a complex two coupled dynamics, dynamical properties of a new complex system, HC complex Lorenz system, modified projective synchronization in complex Chen and Lu system and complex complete synchronization are interesting issue discussed by Mahmoud et al. (2007, 2008), Mahmoud and Mahmoud (2010b), Mahamoud and Ahmed (2011) and Mahmoud (2014).

With the increasing application of synchronization of chaotic systems, various control methods have been introduced for the control of chaos, which includes active control (Bhalekar 2014), adaptive control (Khan and Tyagi 2017a), sliding mode control (Wang et al. 2012), adaptive sliding mode control (Khan and Tyagi 2017b; Khan and Nigar 2019b, 2020b), feedback control (Liu and Liu 2018), optimal control (Khan and Tyagi 2017a), etc. Out of these methods, adaptive control is one of the most proper methods to achieve synchronization. Hubler (1989) was the first who investigate the chaos synchronization using adaptive control in which Lyapunov stability is applied to derived control adaptation laws. Accordingly, many researchers have produced an adaptive control method for synchronizing of chaotic systems (Liao and Tsai 2000; Yassen 2003; Li et al. 2011; Wang and Sun 2011; Aghababa 2012). Adaptive control incorporates a set of methods that gives an orderly procedure for automatic adjustment of the controllers in real-time to achieve or to support the wanted level of appearance of the control system when the parameters of the system model are entirely unknown and change in time. For example, as an aircraft flies, its mass will steadily reduce due to fuel loss, a control law that regulates itself to such developing conditions. Thus, the adaptive controller's derivation for the synchronization of chaotic systems in the appearance of parameter uncertainty is a significant challenge. When the parameters of the chaotic (HC) systems vary unpredictably in time, these circumstances happen due to the parameters denoting time-varying. To achieve the right level of the control system, a general procedure is to build an adaptive controller followed by parameter update laws for synchronization of two chaotic (HC) systems. In the chaotic dynamical system, the state of the slave's system is evolving over time, guided by the adaptive controller, and error dynamics obtain this adaptive controller. The error dynamics are the results of both master and slave systems.

In the Lyapunov stability theory (LST), we define a Lyapunov function $V(x)$ is a positive definite for the system under study obtained with the help of the system's error system and parameters. We need to show the time derivative of a Lyapunov function along with the error dynamics is negative definite, i.e., $\dot{V}(x) \leq 0$ using the adaptive control law and parameters update law (Yoshizawa 1966; Rouche et al. 1977; Sastry 2013). In Zhang et al. (2018) and Al-Mahbashi et al. (2019), the author investigates finite synchronization and finite time-lag synchronization for a complex dynamical network. Complex modified hybrid function projective synchronization between complex system variable was examined in Liu et al. (2016). Moreover, the author Khan and Nigar (2019a) investigates adaptive hybrid complex projective C–C synchronization between a complex HC system. Modified projective synchronization and modified function projective synchronization of a real chaotic system and the chaotic complex system were discussed in Sun et al. (2014).

The previously mentioned synchronization has been obtained for real master systems, and real slave systems or both master and slave systems are complex systems. It is an essential and fascinating problem to the design synchronization for a complex master system and real slave system. To the best of my knowledge, this idea is so far to be introduced. This kind of synchronization is extra winning and challenging. An exciting and attractive result related to this topic has been discussed, such as novel synchronization technique to achieve a hybrid module phase in the complex hyperchaotic system using adaptive control investigated by Wang and Luo (2013), whereas in Nian et al. (2010), Fuzhong et al. introduced the idea of module phase synchronization in which modules varied in a particular field after the synchronization . Moreover, hybrid module synchronization with time delay among two complex HC systems using adaptive control was presented in Chao (2016) by Luochao. Hence, we can say, modulus synchronization of a complex system variable plays a vital role because of its application in stable transmission was discussed in Li et al. (2019).

According to these mentioned analyses, we present MCCS scheme using an adaptive control technique. A suitable adaptive controller is designed to perform synchronization among HC complex master systems and HC real slave system. Some of the difficulties arise in solving the MCCS method. We used four non-identical chaotic systems in the MCCS scheme, which have a different–different parameter. LST is typically used to derive adaptive control laws and show convergence. For displaying the derivative of the Lyapunov function is negative definite, we have to calculate all the adaptive parameters and adaptive controllers, which is very difficult to calculate. According to Theorems 1 and 2, the adaptive control technique is applied to estimate unknown parameters. When adaptive controllers are structured, special consideration is necessary for convergence and robustness issues.

The essential highlights of this research are summarized as follows.

- This paper firstly proposed modulus combination–combination synchronization (MCCS) to deal with the HC complex system and HC real system.
- An adaptive control technique with fast convergence is designed for the modulus synchronization.
- This paper discusses a secure communication design based on modulus combination–combination synchronization.

The rest of the manuscript is structured as follows: Sect. 2 presents the synchronization principle of MCCS using the adaptive control technique, and Sect. 3 contains system descriptions of chaotic systems. Section 4 contains the numerical example of testing the adaptive control's analytical method to perform the MCCS. Section 5 provides the numerical simulation of the MCCS by using MATLAB. We obtain a proper arrangement among mathematical treatments and simulation outcomes for our suggested MCCS, and comparative investigations are also discussed in Sect. 6. We received an application of MCCS using the chaos masking method for message information, that we could still recover after decryption of the message in Sect. 7. The conclusion is finally declared in the last Sect. 8.

2 Synchronization Principle of Modulus Combination–Combination (MCCS) Using Adaptive Control Technique

This section investigates the principle of MCCS (Li et al. 2019; Sun et al. 2013). Two non-identical complex HC master systems can be described as follows:

$$\dot{x} = h_1(x)\Omega_1 + g_1(x) \tag{1}$$

$$7\dot{y} = h_2(y)\Omega_2 + g_2(y). \tag{2}$$

The two non-identical real chaotic(or HC) slave systems are defined as:

$$\dot{w} = h_3(w)\Omega_3 + g_3(w) + \rho_1 \tag{3}$$

$$\dot{z} = h_4(x)\Omega_4 + g_4(z) + \rho_2 \tag{4}$$

where $x = (x_1, x_2, \ldots, x_N)^T \in C$, $y = (y_1, y_2, \ldots, y_N)^T \in C$ represents the state vector of master systems (1) and (2). We mention that x and y can be expressed as $x = x^r + jx^i$, $y = y^r + jy^i$, with $j = \sqrt{-1}$ and r represent as real parts and i represents imaginary parts. Assume $x_1 = x_{11m} + jx_{12m}$, $x_2 = x_{13m} + jx_{14m}$,...,$x_N = x_{1(N-1)m} + jx_{1Nm}$, then $x^r = (x_{11m}, x_{13m}, \ldots, x_{1(N-1)m})^T$, $x^i = (x_{12m}, x_{14m}, \ldots, x_{1Nm})^T$. $y_1 = y_{11m} + jy_{12m}$, $y_2 = y_{13m} + jy_{14m}$,...,$y_n = y_{1(N-1)m} + jy_{1Nn}$, then $y^r = (y_{11m}, y_{13m}, \ldots, y_{1(N-1)m})^T$, $y^i = (y_{12m}, y_{14m}, \ldots, y_{1Nm})^T$. $h_1(x)$ and $h_2(x)$ presents $N \times N$ matrix function and $g_1(x)$, $g_2(x)$ are $N \times 1$ continuous vector function, Ω_1, Ω_2 are $N \times 1$ real nonlinear parameter. $w = (w_{11s}, w_{12s}, \ldots, w_{1Ms})^T \in R$, $z = (z_{11s}, z_{12s}, \ldots, z_{1Ms})^T \in R$ are the state vectors of slave systems (3) and (4).$h_3(x)$ and $h_4(x)$ presents $M \times M$ matrix function and $g_3(x)$, $g_4(x)$ are $M \times 1$ continuous vector function, Ω_3, Ω_4 are $M \times 1$ real nonlinear parameter. $\rho_1 = (\rho_{11}, \rho_{12}, \ldots, \rho_{1M}) \in R^M$, $\rho_2 = (\rho_{21}, \rho_{22}, \ldots, \rho_{2M}) \in R^M$ are the adaptive control inputs.

Definition 2.1 (Sun et al. 2013; Li et al. 2019) For the complex master systems (1) and (2), and real slave systems (3) and (4), our aim is to synchronize the trajectory of $|x + y|$ with that of $w + z$ which can be expressed as.

$$lim_{t\to\infty}\|e\| = lim_{t\to\infty}\|w + z - |x + y|\| = 0 \tag{5}$$

where $|.|$ represents the modulus of complex variable and $\|.\|$ is the matrix norm.

Remark 1 If $(x^r, x^i) \neq (0, 0)$ and $(y^r, y^i) \neq (0, 0)$, then error system in Eq. (5) can be rewritten as follows:

$$\begin{aligned} lim_{t\to\infty}\|e\| &= lim_{t\to\infty}\|w + z \\ &\quad - |x^r + jx^i + y^r + jy^i|\| = 0 \\ lim_{t\to\infty}\|e\| &= lim_{t\to\infty}\|w + z \\ &\quad - \sqrt{(x^r + y^r)^2 + (x^i + y^i)^2}\| = 0. \end{aligned} \tag{6}$$

The error dynamics obtain from Eq. (6) such as:

$$\begin{aligned} \dot{e}(t) &= \dot{w} + \dot{z} - \frac{(x^r + y^r)(\dot{x^r} + \dot{y^r})}{\sqrt{(x^r + y^r)^2 + (x^i + y^i)^2}} \\ &\quad - \frac{(x^i + y^i)(\dot{x^i} + \dot{y^i})}{\sqrt{(x^r + y^r)^2 + (x^i + y^i)^2}}. \end{aligned} \tag{7}$$

Substitute Eqs. (1), (2), (3), and (4) in Eq. (7), we get the error dynamics:

$$\begin{aligned} \dot{e}(t) &= h_3(w)\Omega_3 + g_3(w) + \rho_1 + h_4(z)\Omega_4 + g_4(z) + \rho_2 \\ &\quad - \frac{(x^r + y^r)(h_1(x^r)\Omega_1 + g_1(x^r) + h_2(y^r)\Omega_2 + g_2(y^r))}{\sqrt{(x^r + y^r)^2 + (x^i + y^i)^2}} \\ &\quad - \frac{(x^i + y^i)(h_1(x^i)\Omega_1 + g_1(x^i) + h_2(y^i)\Omega_2 + g_2(y^i))}{\sqrt{(x^r + y^r)^2 + (x^i + y^i)^2}}. \end{aligned} \tag{8}$$

Now, MCCS can be performed by design the relevant adaptive control $\rho_1 + \rho_2$ and parameter update laws Ω_i.

where k denotes a positive real constant, $\hat{\Omega}_1(t)$, $\hat{\Omega}_2(t)$, $\hat{\Omega}_3(t)$, $\hat{\Omega}_4(t)$ *are the estimated values of* Ω_1, $\Omega_2(t)$, $\Omega_3(t)$, $\Omega_4(t)$, *respectively, and the adaptive laws of parameters are considered as follows.*

$$\begin{cases} \dot{\hat{\Omega}}_1 = -[\frac{(x^r+y^r)h_1(x^r)\hat{\Omega}_1 e}{\sqrt{(x^r+y^r)^2+(x^i+y^i)^2}} + \frac{(x^i+y^i)h_1(x^i)\hat{\Omega}_1 e}{\sqrt{(x^r+y^r)^2+(x^i+y^i)^2}}]^T e + k_{\Omega_1}\tilde{\Omega}_1 \\ \dot{\hat{\Omega}}_2 = -[\frac{(x^r+y^r)h_2(y^r)\hat{\Omega}_2 e}{\sqrt{(x^r+y^r)^2+(x^i+y^i)^2}} + \frac{(x^i+y^i)h_2(y^i)\hat{\Omega}_2 e}{\sqrt{(x^r+y^r)^2+(x^i+y^i)^2}}]^T e + k_{\Omega_2}\tilde{\Omega}_2 \\ \dot{\hat{\Omega}}_3 = [h_3(w)]^T e + k_{\Omega_3}\tilde{\Omega}_3 \\ \dot{\hat{\Omega}}_4 = [h_4(z)]^T e + k_{\Omega_4}\tilde{\Omega}_4 \end{cases} \tag{10}$$

Remark 2 If $w = 0$, $x = 0$, $z \neq 0$ and $y \neq 0$ or $z = 0$, $y = 0$, $w \neq 0$ and $x \neq 0$, then MCCS turned into modulus synchronization.

$lim_{t\to\infty}\|e\| = lim_{t\to\infty}\|z - |y|\| = 0.$

Remark 3 If $w = 0$ and,$z \neq 0$ or $z = 0$ and $w \neq 0$, then MCCS turned into modulus combination synchronization.

$lim_{t\to\infty}\|e\| = lim_{t\to\infty}\|z - |x + y|\| = 0.$

Remark 4 If $(y^r, y^i) = (0, 0)$, and $z \neq 0$ or $z = 0$ and $w \neq 0$, then Eq. 6 turned into modulus combination synchronization.

$lim_{t\to\infty}\|e\| = lim_{t\to\infty}\|w + z - \sqrt{x^{r2} + x^{i2}}\| = 0.$

Remark 5 If $x^r \neq 0$, $x^i = 0$ and $y^r \neq 0$ and $y^i = 0$, then MCCS turned into an absolute C–C synchronization of real systems.

$lim_{t\to\infty}\|e\| = lim_{t\to\infty}\|w + z - |x^r + y^r|\| = 0.$

Theorem 1 *If* $(x^r, x^i) \neq (0, 0)$ *and* $(y^r, y^i) \neq (0, 0)$, *then error system can be written as* $e(t) = w + z - \sqrt{(x^r + y^r)^2 + (x^i + y^i)^2}$. *Then, the two real slave systems* (3) *and* (4) *can modulus synchronized with the complex master systems* (1) *and* (2) *are globally asymptotically stable, if the suitable adaptive controllers* $\rho_1 + \rho_2$ *is considered as follows* (Li et al. 2011; Khan and Nigar 2019a).

$$\begin{aligned} \rho_1 + \rho_2 = & -h_3(w)\hat{\Omega}_3 - g_3(w) - h_4(z)\hat{\Omega}_4 - g_4(z) \\ & + \frac{(x^r+y^r)(h_1(x^r)\hat{\Omega}_1 + g_1(x^r) + h_2(y^r)\hat{\Omega}_2 + g_2(y^r))}{\sqrt{(x^r+y^r)^2+(x^i+y^i)^2}} \\ & + \frac{(x^i+y^i)(h_1(x^i)\hat{\Omega}_1 + g_1(x^i) + h_2(y^i)\hat{\Omega}_2 + g_2(y^i))}{\sqrt{(x^r+y^r)^2+(x^i+y^i)^2}} \\ & - ke \end{aligned} \tag{9}$$

Proof Substitute Eq. (9) in (8), we obtain error dynamics as follows:

$$\begin{aligned} \dot{e}(t) = & h_3(w)\tilde{\Omega}_3 + h_4(z)\tilde{\Omega}_4 \\ & - \frac{(x^r+y^r)(h_1(x^r)\tilde{\Omega}_1 + h_2(y^r)\tilde{\Omega}_2)}{\sqrt{(x^r+y^r)^2+(x^i+y^i)^2}} \\ & - \frac{(x^i+y^i)(h_1(x^i)\tilde{\Omega}_1 + h_2(y^i)\tilde{\Omega}_2)}{\sqrt{(x^r+y^r)^2+(x^i+y^i)^2}} - ke. \end{aligned} \tag{11}$$

Theorem 1 is defined for complex master systems and real slave systems for MCCS, which demonstrates that MCCS is globally and exponentially synchronized. That is, we have to show modulus error $e(t) \to 0$ exponentially as $t \to \infty$ for all initial values. And hence, the theorem one always follows when the time derivative of a Lyapunov function is negative definite, as defined by the LST using the adaptive control law (9) and parameters update law (10).

Now we design Lyapunov function $V(t)$ for convergence such as (Li et al. 2011):

$$V(t) = \frac{1}{2}(e^T e + \tilde{\Omega}_1^T\tilde{\Omega}_1 + \tilde{\Omega}_2^T\tilde{\Omega}_2 + \tilde{\Omega}_3^T\tilde{\Omega}_3 + \tilde{\Omega}_4^T\tilde{\Omega}_4). \tag{12}$$

Finding the derivative of V, we get:

$$\dot{V}(t) = \dot{e}^T e + \tilde{\Omega}_1^T\dot{\tilde{\Omega}}_1 + \tilde{\Omega}_2^T\dot{\tilde{\Omega}}_2 + \tilde{\Omega}_3^T\dot{\tilde{\Omega}}_3 + \tilde{\Omega}_4^T\dot{\tilde{\Omega}}_4$$

where $\tilde{\Omega}_1 = \Omega_1 - \hat{\Omega}_1$, $\tilde{\Omega}_2 = \Omega_2 - \hat{\Omega}_2$, $\tilde{\Omega}_3 = \Omega_3 - \hat{\Omega}_3$, $\tilde{\Omega}_4 = \Omega_4 - \hat{\Omega}_4$,
which implies $\dot{\tilde{\Omega}}_1 = -\dot{\hat{\Omega}}_1$, $\dot{\tilde{\Omega}}_2 = -\dot{\hat{\Omega}}_2$, $\dot{\tilde{\Omega}}_3 = -\dot{\hat{\Omega}}_3$, $\dot{\tilde{\Omega}}_4 = -\dot{\hat{\Omega}}_4$

$$\begin{aligned} \dot{V}(t) = & \dot{e}^T e + \tilde{\Omega}_1^T(-\dot{\hat{\Omega}}_1) + \tilde{\Omega}_2^T(-\dot{\hat{\Omega}}_2) + \tilde{\Omega}_3^T(-\dot{\hat{\Omega}}_3) \\ & + \tilde{\Omega}_4^T(-\dot{\hat{\Omega}}_4). \end{aligned} \tag{13}$$

Using Eqs. (10) and (11) in Eq. (13), we get :

$$
\begin{aligned}
\dot{V}(t) = {} & e[h_3(w)\tilde{\Omega}_3 + h_4(z)\tilde{\Omega}_4 \\
& - \frac{(x^r + y^r)(h_1(x^r)\tilde{\Omega}_1 + h_2(y^r)\tilde{\Omega}_2)}{\sqrt{(x^r + y^r)^2 + (x^i + y^i)^2}} \\
& - \frac{(x^i + y^i)(h_1(x^i)\tilde{\Omega}_1 + h_2(y^i)\tilde{\Omega}_2)}{\sqrt{(x^r + y^r)^2 + (x^i + y^i)^2}} - ke]^T \\
& - \tilde{\Omega}_1^T[-[\frac{(x^r + y^r)h_1(x^r)e}{\sqrt{(x^r + y^r)^2 + (x^i + y^i)^2}} \\
& + \frac{(x^i + y^i)h_1(x^i)e}{\sqrt{(x^r + y^r)^2 + (x^i + y^i)^2}}]^T e + k_{\Omega_1}\tilde{\Omega}_1] \\
& - \tilde{\Omega}_2^T[-[\frac{(x^r + y^r)h_1(y^r)e}{\sqrt{(x^r + y^r)^2 + (x^i + y^i)^2}} \\
& + \frac{(x^i + y^i)h_2(y^i)e}{\sqrt{(x^r + y^r)^2 + (x^i + y^i)^2}}]^T e + k_{\Omega_2}\tilde{\Omega}_2] \\
& - \tilde{\Omega}_3^T[h_3(w)]^T e - \tilde{\Omega}_4^T[h_4(z)]^T e \\
= {} & h_3(w)e\tilde{\Omega}_3 + h_4(z)e\tilde{\Omega}_4 \\
& - e\frac{(x^r + y^r)h_1(x^r)\tilde{\Omega}_1}{\sqrt{(x^r + y^r)^2 + (x^i + y^i)^2}} \\
& - e\frac{(x^r + y^r)h_1(x^i)\tilde{\Omega}_1}{\sqrt{(x^r + y^r)^2 + (x^i + y^i)^2}} \\
& - e\frac{(x^r + y^r)h_2(y^r)\tilde{\Omega}_2}{\sqrt{(x^r + y^r)^2 + (x^i + y^i)^2}} \\
& - e\frac{(x^r + y^r)h_2(y^i)\tilde{\Omega}_2}{\sqrt{(x^r + y^r)^2 + (x^i + y^i)^2}} - ke^T e \\
& + \tilde{\Omega}_1^T[\frac{(x^r + y^r)h_1(x^r)}{\sqrt{(x^r + y^r)^2 + (x^i + y^i)^2}} \\
& + \frac{(x^i + y^i)h_1(x^i)}{\sqrt{(x^r + y^r)^2 + (x^i + y^i)^2}}]^T e - k_{\Omega_1}\tilde{\Omega}_1^{T\tilde{\Omega}_1} \\
& + \tilde{\Omega}_2^T[\frac{(x^r + y^r)h_1(y^r)}{\sqrt{(x^r + y^r)^2 + (x^i + y^i)^2}} \\
& + \frac{(x^i + y^i)h_2(y^i)}{\sqrt{(x^r + y^r)^2 + (x^i + y^i)^2}}]^T e - k_{\Omega_2}\tilde{\Omega}_2^T\tilde{\Omega}_2 \\
& - \tilde{\Omega}_3^T[h_3(w)]^T e - k_{\Omega_3}\tilde{\Omega}_3^T\tilde{\Omega}_3 \\
& - \tilde{\Omega}_4^T[h_4(z)]^T e - k_{\Omega_4}\tilde{\Omega}_4^T\tilde{\Omega}_4 \\
= {} & -ke^T e - k_{\Omega_1}\tilde{\Omega}_1^T\tilde{\Omega}_1 - k_{\Omega_2}\tilde{\Omega}_2^T\tilde{\Omega}_2 \\
& - k_{\Omega_3}\tilde{\Omega}_3^T\tilde{\Omega}_3 - k_{\Omega_4}\tilde{\Omega}_4^T\tilde{\Omega}_4 \\
& \leq 0
\end{aligned}
$$

Hence, we observe that $V(t)$ is positive definite, and $\dot{V}(t)$ is negative definite. According to the LST, we see it $lim_{t\to\infty}\|e\| = 0$, which indicates that the master systems (1) and (2) will perform MCCS with the slave systems (3) and (4). □

3 System Descriptions of HC Complex System and HC Real System

Consider a HC complex Lorenz system proposed by Mahmoud et al. (2008):

$$
\begin{cases}
\dot{x}_1 = \Omega_{11}(x_2 - x_1) + (1+i)x_4 \\
\dot{x}_2 = \Omega_{13}x_1 - x_2 - x_1x_3 \\
\dot{x}_3 = \frac{1}{2}(\bar{x}_1x_2 + x_1\bar{x}_2) - \Omega_{12}x_3 \\
\dot{x}_4 = \frac{1}{2}(\bar{x}_1x_2 + x_1\bar{x}_2) - \Omega_{14}x_4
\end{cases}
\tag{14}
$$

where $\Omega_{11}, \Omega_{12}, \Omega_{13}, \Omega_{14}$ denote the real parameters, while $x_1 = x_{11m} + ix_{12m}$ and $x_2 = x_{13m} + ix_{14m}$ denote complex variable and $x_3 = x_{15m}$ and $x_4 = x_{16m}$ are real variable.

$$
\begin{cases}
\dot{x}_{11m} + i\dot{x}_{12m} = \Omega_{11}(x_{13m} + ix_{14m} - (x_{11m} + ix_{12m})) + (1+i)x_{16} \\
\dot{x}_{13m} + i\dot{x}_{14m} = \Omega_{13}(x_{11m} + ix_{12m}) - (x_{13m} + ix_{14m}) \\
\qquad -(x_{11m} + ix_{12m})x_{15m} \\
\dot{x}_{15m} = \frac{1}{2}((x_{11m} - ix_{12m})(x_{13m} + ix_{14m}) \\
\qquad +(x_{11m} + ix_{12m})(x_{13m} - ix_{14m})) - \Omega_{12}x_{15m} \\
\dot{x}_{16m} = \frac{1}{2}((x_{11m} - ix_{12m})(x_{13m} + ix_{14m}) \\
\qquad +(x_{11m} + ix_{12m})(x_{13m} - ix_{14m})) - \Omega_{14}x_{16m}
\end{cases}
\tag{15}
$$

.

The real form of system (14) is written as follows:

$$
\begin{cases}
\dot{x}_{11m} = \Omega_{11}(x_{13m} - x_{11m}) + x_{16m} \\
\dot{x}_{12m} = \Omega_{11}(x_{14m} - x_{12m}) + x_{16m} \\
\dot{x}_{13m} = \Omega_{13}x_{11m} - x_{11m}x_{15m} - x_{13m} \\
\dot{x}_{14m} = \Omega_{13}x_{12m} - x_{12m}x_{15m} - x_{14m} \\
\dot{x}_{15m} = x_{11m}x_{13m} + x_{12m}x_{14m} - \Omega_{12}x_{15m} \\
\dot{x}_{16m} = x_{11m}x_{13m} + x_{12m}x_{14m} - \Omega_{14}x_{16m}
\end{cases}
\tag{16}
$$

. Consider HC complex Lu system proposed by Wang et al. (2016)

$$
\begin{cases}
\dot{y}_1 = \Omega_{21}(y_{12m} - y_{11m}) \\
\dot{y}_2 = -y_{11m}y_{13m} + \Omega_{22}y_{12m} - \Omega_{23}(4 + 0.03y_{14m})y_{11m} \\
\dot{y}_3 = \frac{1}{2}(\bar{y}_{11m}y_{12m} + y_{11m}\bar{y}_{12m}) - \Omega_{24}y_{13m} \\
\dot{y}_4 = \frac{1}{2}(y_{11m} + \bar{y}_{11m})
\end{cases}
\tag{17}
$$

where Ω_{21}, Ω_{22}, Ω_{23}, Ω_{24} denote the real parameters, while $y_1 = y_{11m} + iy_{12m}$ and $y_2 = y_{13m} + iy_{14m}$ denote complex variable and $y_3 = y_{15m}$ and $y_4 = y_{16m}$ are real variable.

$$\begin{cases} \dot{y}_{11m} + i\dot{y}_{12m} = \Omega_{21}(y_{13m} + iy_{14m} - (y_{11m} + iy_{12m})) \\ \dot{y}_{13m} + i\dot{y}_{14m} = -(y_{11m} + iy_{12m})y_{15m} + \Omega_{22}(y_{13m} + iy_{14m}) \\ \qquad -\Omega_{23}(4 + 0.03y_{16m})(y_{11m} + iy_{12m}) \\ \dot{y}_{15m} = \frac{1}{2}((y_{11m} - iy_{12m})(y_{13m} + iy_{14m}) \\ \qquad +(y_{11m} + iy_{12m})(y_{13m} - iy_{14m})) - \Omega_{24}y_{15m} \\ \dot{y}_{16m} = \frac{1}{2}(y_{11m} + iy_{12m} + y_{11m} - iy_{12m}) \end{cases} \tag{18}$$

The real form of system (17) is written as follows:

$$\begin{cases} \dot{y}_{11m} = \Omega_{21}(y_{13m} - y_{11m}) \\ \dot{y}_{12m} = \Omega_{21}(y_{14m} - y_{12m}) \\ \dot{y}_{13m} = -y_{11m}y_{15m} + \Omega_{22}y_{13m} - \Omega_{23}(4 + 0.03y_{16m}^2)y_{11m} \\ \dot{y}_{14m} = -y_{12m}y_{15m} + \Omega_{22}y_{14m} - \Omega_{23}(4 + 0.03y_{16m}^2)y_{12m} \\ \dot{y}_{15m} = y_{11m}y_{13m} + y_{12m}y_{14m} - \Omega_{24}y_{15m} \\ \dot{y}_{16m} = y_{11m} \end{cases} \tag{19}$$

Two HC systems are taken as:

HC Chen system (Li et al. 2005)

$$\begin{cases} \dot{w}_{11s} = \Omega_{31}(w_{12s} - w_{11s}) + w_{13s} \\ \dot{w}_{12s} = \Omega_{32}w_{11s} + \Omega_{33}w_{12s} - w_{11s}w_{13s} \\ \dot{w}_{13s} = -\Omega_{34}w_{13s} + w_{11s}w_{12s} \\ \dot{w}_{14s} = \Omega_{35}w_{14s} + w_{12s}w_{13s} \end{cases} \tag{20}$$

where Ω_{31}, Ω_{32}, Ω_{33}, Ω_{34}, Ω_{35} denote the real parameters, while w_{11s}, w_{12s}, w_{13s}, w_{14s} denote real variable.

HC Newton–Leipnik chaotic system (Ghosh and Bhattacharya 2010)

$$\begin{cases} \dot{z}_{11s} = -\Omega_{41}z_{11s} + z_{12s} + 10z_{12s}z_{13s} + z_{14s} \\ \dot{z}_{12s} = -z_{11s} - 0.4z_{12s} + 5z_{11s}z_{13s} \\ \dot{z}_{13s} = \Omega_{42}z_{13s} - 5z_{11s}z_{12s} \\ \dot{z}_{14s} = -\Omega_{43}z_{11s}z_{13s} + \Omega_{44}z_{14s} \end{cases} \tag{21}$$

where Ω_{41}, Ω_{42}, Ω_{43}, Ω_{44} denote the real parameters, while z_{11s}, $z_{12s}z_{13s}$, z_{14s} denote real variable.

4 Example of Modulus Combination–Combination Synchronization

In the following, Eqs. (16) and (19) act as the master system and Eqs. (22) and (23) are selected as a slave system with controller written as.

$$\begin{cases} \dot{w}_{11s} = \Omega_{31}(w_{12s} - w_{11s}) + w_{13s} + \rho_{11} \\ \dot{w}_{12s} = \Omega_{32}w_{11s} + \Omega_{33}w_{12s} - w_{11s}w_{13s} + \rho_{12} \\ \dot{w}_{13s} = -\Omega_{34}w_{13s} + w_{11s}w_{12s} + \rho_{13} \\ \dot{w}_{14s} = \Omega_{35}w_{14s} + w_{12s}w_{13s} + \rho_{14} \end{cases} \tag{22}$$

$$\begin{cases} \dot{z}_{11s} = -\Omega_{41}z_{11s} + z_{12s} + 10z_{12s}z_{13s} + z_{14s} + \rho_{21} \\ \dot{z}_{12s} = -z_{11s} - 0.4z_{12s} + 5z_{11s}z_{13s} + \rho_{22} \\ \dot{z}_{13s} = \Omega_{42}z_{13s} - 5z_{11s}z_{12s} + \rho_{23} \\ \dot{z}_{14s} = -\Omega_{43}z_{11s}z_{13s} + \Omega_{44}z_{14s} + \rho_{24} \end{cases} \tag{23}$$

where ρ_{11}, ρ_{12}, ρ_{13} , ρ_{14}, ρ_{21}, ρ_{22}, ρ_{23} ,ρ_{24} are the appropriate design adaptive controllers.

Definition 1 shows that the error system can be obtain as:

$$\begin{cases} e_{11} = (w_{11s} + z_{11s}) - \sqrt{(x_{11m} + y_{11m})^2 + (x_{12m} + y_{12m})^2} \\ e_{12} = (w_{12s} + z_{22s}) - \sqrt{(x_{13m} + y_{13m})^2 + (x_{14m} + y_{14m})^2} \\ e_{13} = (w_{13s} + z_{23s}) - \sqrt{(x_{15m} + y_{15m})^2} \\ e_{14} = (w_{14s} + z_{24s}) - \sqrt{(x_{16m} + y_{16m})^2} \end{cases} . \tag{24}$$

We find the derivative of error system of Eq. (24), we get

$$\begin{cases} \dot{e_{11}} = \dot{w_{11}} + \dot{z_{11}} - \frac{(x_{11m} + y_{11m})(\dot{x_{11m}} + \dot{y_{11m}})}{\sqrt{(x_{11m} + y_{11m})^2 + (x_{12m} + y_{12m})^2}} \\ \qquad - \frac{(x_{12m} + y_{12m})(\dot{x_{12m}} + \dot{y_{12m}})}{\sqrt{(x_{11m} + y_{11m})^2 + (x_{12m} + y_{12m})^2}} \\ \dot{e_{12}} = \dot{w_{12}} + \dot{z_{12}} - \frac{(x_{13m} + y_{13m})(\dot{x_{13m}} + \dot{y_{13m}})}{\sqrt{(x_{13m} + y_{13m})^2 + (x_{14m} + y_{14m})^2}} \\ \qquad - \frac{(x_{14m} + y_{14m})(\dot{x_{14m}} + \dot{y_{14m}})}{\sqrt{(x_{11m} + y_{11m})^2 + (x_{12m} + y_{12m})^2}} \\ \dot{e_{13}} = \dot{w_{13}} + \dot{z_{13}} - \frac{(x_{15m} + y_{15m})(\dot{x_{15m}} + \dot{y_{15m}})}{\sqrt{(x_{15m} + y_{15m})^2}} \\ \dot{e_{14}} = \dot{w_{14}} + \dot{z_{14}} - \frac{(x_{16m} + y_{16m})(\dot{x_{16m}} + \dot{y_{16m}})}{\sqrt{(x_{16m} + y_{16m})^2}} \end{cases} \tag{25}$$

Using Eqs. (16), (19), (22), and (23) in the error dynamics (25) such that:

$$
\begin{cases}
\dot{e_{1}1} = \Omega_{31}(w_{12s} - w_{11s}) + w_{13s} - \Omega_{41} z_{11s} + z_{12s} + 10 z_{12s} z_{13s} + z_{14s} + \rho_{11} + \rho_{21} \\
\quad - \dfrac{(x_{11m} + y_{11m})(\Omega_{11}(x_{13m} - x_{11m}) + x_{16m} + \Omega_{21}(y_{13m} - y_{11m}))}{\sqrt{(x_{11m} + y_{11m})^2 + (x_{12m} + y_{12m})^2}} \\
\quad - \dfrac{(x_{12m} + y_{12m})(\Omega_{11}(x_{14m} - x_{12m}) + x_{16m} + \Omega_{21}(y_{14m} - y_{12m}))}{\sqrt{(x_{11m} + y_{11m})^2 + (x_{12m} + y_{12m})^2}} \\
\dot{e_{1}2} = \Omega_{32} w_{11s} + \Omega_{33} w_{12s} - w_{11s} w_{13s} + \rho_{12} + \rho_{22} \\
\quad - \dfrac{(x_{13m} + y_{13m})(\Omega_{13} x_{11m} - x_{11m} x_{15m} - x_{13m})}{\sqrt{(x_{13m} + y_{13m})^2 + (x_{14m} + y_{14m})^2}} \\
\quad - \dfrac{(x_{13m} + y_{13m})(-y_{11m} y_{15m} + \Omega_{22} y_{13m} - \Omega_{23}(4 + 0.03 y_{16m}^2) y_{11m})}{\sqrt{(x_{13m} + y_{13m})^2 + (x_{14m} + y_{14m})^2}} \\
\quad - \dfrac{(x_{14m} + y_{14m})(c x_{12m} - x_{12m} x_{15m} - x_{14m})}{\sqrt{(x_{13m} + y_{13m})^2 + (x_{14m} + y_{14m})^2}} \\
\quad - \dfrac{(x_{14m} + y_{14m})(-y_{12m} y_{15m} + \Omega_{22} y_{14m} - \Omega_{23}(4 + 0.03 y_{16m}^2) y_{12m})}{\sqrt{(x_{13m} + y_{13m})^2 + (x_{14m} + y_{14m})^2}} \\
\dot{e_{1}3} = -\Omega_{34} w_{13s} + w_{11s} w_{12s} + \Omega_{42} z_{13s} - 5 z_{11s} z_{12s} + \rho_{13} + \rho_{23} \\
\quad - \dfrac{(x_{15m} + y_{15m})(x_{11m} x_{13m} + x_{12m} x_{14m} - \Omega_{12} x_{15m})}{\sqrt{(x_{15m} + y_{15m})^2}} \\
- \dfrac{(x_{15m} + y_{15m})(y_{11m} y_{13m} + y_{12m} y_{14m} - \Omega_{24} y_{15m})}{\sqrt{(x_{15m} + y_{15m})^2}} \\
\dot{e_{1}4} = \Omega_{35} w_{14s} + w_{12s} w_{13s} - \Omega_{43} z_{11s} z_{13s} + \Omega_{44} z_{14s} + \rho_{14} + \rho_{24} \\
\quad - \dfrac{(x_{16m} + y_{16m})(x_{11m} x_{13m} + x_{12m} x_{14m} - \Omega_{14} x_{16m} + y_{11m})}{\sqrt{(x_{16m} + y_{16m})^2}}
\end{cases}
\tag{26}
$$

To achieve the control goal, we introduce the following theorem.

Theorem 2 *In, error system* $e(t) = w+z-\sqrt{(x^r + y^r)^2 + (x^i + y^i)^2}$ *and* $(x^r, x^i) \neq (0, 0)$ *and* $(y^r, y^i) \neq (0, 0)$*. The master systems (16) and (19) will achieve modulus combination–combination synchronization with slave systems (22) and (23). If the adaptive control function* $\rho_{11} + \rho_{21}$, $\rho_{12} + \rho_{22}$, $\rho_{13}+\rho_{23}$ *and* $\rho_{14}+\rho_{24}$ *are selected such that* Li et al. (2011).

$$\begin{cases} \rho_{11} + \rho_{21} = -k_1 e_{11} - \hat{\Omega}_{31}(w_{12s} - w_{11s}) - w_{13s} + \hat{\Omega}_{41} z_{11s} - z_{12s} - 10 z_{12s} z_{13s} - z_{14s} \\ \quad + \dfrac{(x_{11m} + y_{11m})(\hat{\Omega}_{11}(x_{13m} - x_{11m}) + x_{16m} + \hat{\Omega}_{21}(y_{13m} - y_{11m}))}{\sqrt{(x_{11m} + y_{11m})^2 + (x_{12m} + y_{12m})^2}} \\ \quad + \dfrac{(x_{12m} + y_{12m})(\hat{\Omega}_{11}(x_{14m} - x_{12m}) + x_{16m} + \hat{\Omega}_{21}(y_{14m} - y_{12m}))}{\sqrt{(x_{11m} + y_{11m})^2 + (x_{12m} + y_{12m})^2}} \\ \rho_{12} + \rho_{22} = -k_2 e_{12} - \hat{\Omega}_{32} w_{11s} - \hat{\Omega}_{33} w_{12s} + w_{11s} w_{13s} \\ \quad + \dfrac{(x_{13m} + y_{13m})(\hat{\Omega}_{13} x_{11m} - x_{11m} x_{15m} - x_{13m})}{\sqrt{(x_{13m} + y_{13m})^2 + (x_{14m} + y_{14m})^2}} \\ \quad + \dfrac{(x_{13m} + y_{13m})(-y_{11m} y_{15m} + \hat{\Omega}_{22} y_{13m} - \hat{\Omega}_{23}(4 + 0.03 y_{16m}^2) y_{11m})}{\sqrt{(x_{13m} + y_{13m})^2 + (x_{14m} + y_{14m})^2}} \\ \quad + \dfrac{(x_{14m} + y_{14m})(\hat{\Omega}_{13} x_{12m} - x_{12m} x_{15m} - x_{14m})}{\sqrt{(x_{13m} + y_{13m})^2 + (x_{14m} + y_{14m})^2}} \\ \quad + \dfrac{(x_{14m} + y_{14m})(-y_{12m} y_{15m} + \hat{\Omega}_{22} y_{14m} - \hat{\Omega}_{23}(4 + 0.03 y_{16m}^2) y_{12m})}{\sqrt{(x_{13m} + y_{13m})^2 + (x_{14m} + y_{14m})^2}} \\ \rho_{13} + \rho_{23} = -k_3 e_{13} + \hat{\Omega}_{34} w_{13s} - w_{11s} w_{12s} - \hat{\Omega}_{42} z_{13s} - 5 z_{11s} z_{12s} \\ \quad - \dfrac{(x_{15m} + y_{15m})(-\hat{\Omega}_{12} x_{15m} - \hat{\Omega}_{24} y_{15m})}{\sqrt{(x_{15m} + y_{15m})^2}} \\ \rho_{14} + \rho_{24} = -k_4 e_{14} + \hat{\Omega}_{35} w_{14s} - w_{12s} w_{13s} + \hat{\Omega}_{43} z_{11s} z_{13s} - \hat{\Omega}_{44} z_{14s} \\ \quad + \dfrac{(x_{16m} + y_{16m})(x_{11m} x_{13m} + x_{12m} x_{14m} - \hat{\Omega}_{14} x_{16m} + y_{11m})}{\sqrt{(x_{16m} + y_{16m})^2}} \end{cases} \tag{27}$$

and the adaptive parameters are considered as:

$$\begin{cases}
\dot{\hat{\Omega}}_{11} = -\dfrac{(x_{11m}+y_{11m})(x_{13m}-x_{11m})e_{11}}{\sqrt{(x_{11m}+y_{11m})^2+(x_{12m}+y_{12m})^2}} - \dfrac{(x_{12m}+y_{12m})(x_{14m}-x_{12m})e_{11}}{\sqrt{(x_{11m}+y_{11m})^2+(x_{12m}+y_{12m})^2}} \\
\quad +k_5(\Omega_{11}-\hat{\Omega}_{11}) \\
\dot{\hat{\Omega}}_{12} = -\dfrac{(x_{15m}+y_{15m})x_{15m}e_{13}}{\sqrt{(x_{15m}+y_{15m})^2}} + k_6(\Omega_{12}-\hat{\Omega}_{12}) \\
\dot{\hat{\Omega}}_{13} = -\dfrac{(x_{13m}+y_{13m})x_{11m}e_{12}}{\sqrt{(x_{13m}+y_{13m})^2+(x_{14m}+y_{14m})^2}} + k_7(\Omega_{13}-\hat{\Omega}_{13}) \\
\dot{\hat{\Omega}}_{14} = -\dfrac{(x_{16m}+y_{16m})x_{16m}e_{14}}{\sqrt{(x_{16m}+y_{16m})^2}} + k_8(\Omega_{14}-\hat{\Omega}_{14}) \\
\dot{\hat{\Omega}}_{21} = -\dfrac{(x_{11m}+y_{11m})(y_{13m}-y_{11m})e_{11}}{\sqrt{(x_{11m}+y_{11m})^2+(x_{12m}+y_{12m})^2}} - \dfrac{(x_{12m}+y_{12m})(y_{14m}-y_{12m})e_{11}}{\sqrt{(x_{11m}+y_{11m})^2+(x_{12m}+y_{12m})^2}} \\
\quad +k_9(\Omega_{21}-\hat{\Omega}_{21}) \\
\dot{\hat{\Omega}}_{22} = -\dfrac{(x_{13m}+y_{13m})y_{13m}e_{12}}{\sqrt{(x_{13m}+y_{13m})^2+(x_{14m}+y_{14m})^2}} - \dfrac{(x_{14m}+y_{14m})y_{14}e_{12}}{\sqrt{(x_{13m}+y_{13m})^2+(x_{14m}+y_{14m})^2}} \\
\quad +k_{10}(\Omega_{22}-\hat{\Omega}_{22})
\end{cases}$$

$$\begin{cases}
\dot{\hat{\Omega}}_{23} = \dfrac{(x_{13m}+y_{13m})(4+0.03y_{16m}^2)y_{11m}e_{12}}{\sqrt{(x_{13m}+y_{13m})^2+(x_{14m}+y_{14m})^2}} + \dfrac{(x_{14m}+y_{14m})(4+0.03y_{16m}^2)y_{14}e_{12}}{\sqrt{(x_{13m}+y_{13m})^2+(x_{14m}+y_{14m})^2}} \\
\quad +k_{11}(\Omega_{23}-\hat{\Omega}_{23}) \\
\dot{\hat{\Omega}}_{24} = -\dfrac{(x_{15m}+y_{15m})y_{11m}e_{13}}{\sqrt{(x_{15m}+y_{15m})^2}} + k_{12}((\Omega_{24}-\hat{\Omega}_{24}) \\
\dot{\hat{\Omega}}_{31} = (w_{12s}-w_{11s})e_{11} + k_{13}(\Omega_{31}-\hat{\Omega}_{31}) \\
\dot{\hat{\Omega}}_{32} = w_{11s}e_{12} + k_{14}(\Omega_{32}-\hat{\Omega}_{32}) \\
\dot{\hat{\Omega}}_{33} = w_{12s}e_{11} + k_{15}(\Omega_{33}-\hat{\Omega}_{33}) \\
\dot{\hat{\Omega}}_{34} = -w_{13s}e_{13} + k_{16}(\Omega_{34}-\hat{\Omega}_{34}) \\
\dot{\hat{\Omega}}_{35} = -w_{14s}e_{14} + k_{17}(\Omega_{35}-\hat{\Omega}_{35}) \\
\dot{\hat{\Omega}}_{41} = -z_{11s}e_{11} + k_{18}(\Omega_{41}-\hat{\Omega}_{41}) \\
\dot{\hat{\Omega}}_{42} = z_{13s}e_{13} + k_{19}(\Omega_{42}-\hat{\Omega}_{42}) \\
\dot{\hat{\Omega}}_{43} = -z_{13s}z_{13s}e_{13} + k_{20}(\Omega_{43}-\hat{\Omega}_{43}) \\
\dot{\hat{\Omega}}_{44} = z_{14s}e_{14} + k_{21}(\Omega_{44}-\hat{\Omega}_{44})
\end{cases} \tag{28}$$

where $k_i > 0$ for $i = 1, 2, \ldots, 21$ are positive real constant.

Proof Using Eq. (27) in Eq. (26), finally error dynamics is written as

$$\begin{cases} \dot{e_{11}} = \tilde{\Omega}_{31}(w_{12s} - w_{11s}) - \tilde{\Omega}_{41}z_{11s} \\ \quad -\dfrac{(x_{11m} + y_{11m})(\tilde{\Omega}_{11}(x_{13m} - x_{11m}) + \tilde{\Omega}_{21}(y_{13m} - y_{11m}))}{\sqrt{(x_{11m} + y_{11m})^2 + (x_{12m} + y_{12m})^2}} \\ \quad -\dfrac{(x_{12m} + y_{12m})(\tilde{\Omega}_{11}(x_{14m} - x_{12m}) + \tilde{\Omega}_{21}(y_{14m} - y_{12m}))}{\sqrt{(x_{11m} + y_{11m})^2 + (x_{12m} + y_{12m})^2}} - k_1e_{11} \\ \dot{e_{12}} = \tilde{\Omega}_{32}w_{11s} + \tilde{\Omega}_{33}w_{12s} - \dfrac{(x_{13m} + y_{13m})(\tilde{\Omega}_{13}x_{11m} + \tilde{\Omega}_{22}y_{13m} - \tilde{\Omega}_{23}(4 + 0.03y_{16m}^2)y_{11m})}{\sqrt{(x_{13m} + y_{13m})^2 + (x_{14m} + y_{14m})^2}} \\ \quad -\dfrac{(x_{14m} + y_{14m})(\tilde{\Omega}_{13}x_{12m} + \tilde{\Omega}_{22}y_{14m} - \tilde{\Omega}_{23}(4 + 0.03y_{16m}^2)y_{12m})}{\sqrt{(x_{13m} + y_{13m})^2 + (x_{14m} + y_{14m})^2}} - k_2e_{12} \\ \dot{e_{13}} = -\tilde{\Omega}_{34}w_{13s} + \tilde{\Omega}_{42}z_{13s} - \dfrac{(x_{15m} + y_{15m})(-\tilde{\Omega}_{12}x_{15m} - \tilde{\Omega}_{24}y_{15m})}{\sqrt{(x_{15m} + y_{15m})^2}} - k_3e_{13} \\ \dot{e_{14}} = \tilde{\Omega}_{35}w_{14s} - \tilde{\Omega}_{43}z_{11s}z_{13s} + \tilde{\Omega}_{44}z_{14s} - \dfrac{(x_{16m} + y_{16m})(\tilde{\Omega}_{14}x_{16m})}{\sqrt{(x_{16m} + y_{16m})^2}} - k_4e_{14} \end{cases} \tag{29}$$

The Lyapunov function V in the form of:

$$\begin{aligned} V = \frac{1}{2}[&e_{11}^2 + e_{12}^2 + e_{13}^2 + e_{14}^2 + \tilde{\Omega}_{11}^2 + \tilde{\Omega}_{12}^2 + \tilde{\Omega}_{13}^2 + \tilde{\Omega}_{14}^2 \\ &+ \tilde{\Omega}_{21}^2 + \tilde{\Omega}_{22}^2 + \tilde{\Omega}_{23}^2 + \tilde{\Omega}_{24}^2 + \tilde{\Omega}_{31}^2 + \tilde{\Omega}_{32}^2 + \tilde{\Omega}_{33}^2 \\ &+ \tilde{\Omega}_{34}^2 + \tilde{\Omega}_{35}^2 + \tilde{\Omega}_{41}^2 + \tilde{\Omega}_{42}^2 + \tilde{\Omega}_{43}^2 + \tilde{\Omega}_{44}^2] \end{aligned} \tag{30}$$

which is a positive definite.

Derivative of V is obtained as:

$$\begin{aligned} \dot{V} = &e_{11}\dot{e_{11}} + e_{12}\dot{e_{12}} + e_{13}\dot{e_{13}} + e_{14}\dot{e_{14}} + \tilde{\Omega}_{11}\dot{\tilde{\Omega}}_{11} + \tilde{\Omega}_{12}\dot{\tilde{\Omega}}_{12} \\ &+ \tilde{\Omega}_{13}\dot{\tilde{\Omega}}_{13} + \tilde{\Omega}_{14}\dot{\tilde{\Omega}}_{14} + \tilde{\Omega}_{21}\dot{\tilde{\Omega}}_{21} + \tilde{\Omega}_{22}\dot{\tilde{\Omega}}_{22} + \tilde{\Omega}_{23}\dot{\tilde{\Omega}}_{23} \\ &+ \tilde{\Omega}_{24}\dot{\tilde{\Omega}}_{24} + \tilde{\Omega}_{31}\dot{\tilde{\Omega}}_{31} + \tilde{\Omega}_{32}\dot{\tilde{\Omega}}_{32} + \tilde{\Omega}_{33}\dot{\tilde{\Omega}}_{33} + \tilde{\Omega}_{34}\dot{\tilde{\Omega}}_{34} \\ &+ \tilde{\Omega}_{35}\dot{\tilde{\Omega}}_{35} + \tilde{\Omega}_{41}\dot{\tilde{\Omega}}_{41} + \tilde{\Omega}_{42}\dot{\tilde{\Omega}}_{42} + \tilde{\Omega}_{43}\dot{\tilde{\Omega}}_{43} + \tilde{\Omega}_{44}\dot{\tilde{\Omega}}_{44} \end{aligned} \tag{31}$$

since $\tilde{\Omega}_{ij} = \Omega_{ij} - \hat{\Omega}_{ij}$ which implies , $\dot{\tilde{\Omega}}_{ij} = -\dot{\hat{\Omega}}_{ij}$

$$\begin{aligned} \dot{V} = \ &e_{11}\dot{e_{11}} + e_{12}\dot{e_{12}} + e_{13}\dot{e_{13}} + e_{14}\dot{e_{14}} - \tilde{\Omega}_{11}\dot{\hat{\Omega}}_{11} - \tilde{\Omega}_{12}\dot{\hat{\Omega}}_{12} \\ &- \tilde{\Omega}_{13}\dot{\hat{\Omega}}_{13} - \tilde{\Omega}_{14}\dot{\hat{\Omega}}_{14} - \tilde{\Omega}_{21}\dot{\hat{\Omega}}_{21} - \tilde{\Omega}_{22}\dot{\hat{\Omega}}_{22} - \tilde{\Omega}_{23}\dot{\hat{\Omega}}_{23} \\ &- \tilde{\Omega}_{24}\dot{\hat{\Omega}}_{24} - \tilde{\Omega}_{31}\dot{\hat{\Omega}}_{31} - \tilde{\Omega}_{32}\dot{\hat{\Omega}}_{32} - \tilde{\Omega}_{33}\dot{\hat{\Omega}}_{33} - \tilde{\Omega}_{34}\dot{\hat{\Omega}}_{34} \\ &- \tilde{\Omega}_{35}\dot{\hat{\Omega}}_{35} - \tilde{\Omega}_{41}\dot{\hat{\Omega}}_{41} - \tilde{\Omega}_{42}\dot{\hat{\Omega}}_{42} - \tilde{\Omega}_{43}\dot{\hat{\Omega}}_{43} - \tilde{\Omega}_{44}\dot{\hat{\Omega}}_{44} \end{aligned} \tag{32}$$

Using Eqs. (28) and (29) in Eq. (32). We get the derivative of V in the form of:

$$\begin{aligned} \dot{V} = \ &e_{11}(\tilde{\Omega}_{31}(w_{12s} - w_{11s}) - \tilde{\Omega}_{41}z_{11s} \\ &- \frac{(x_{11m} + y_{11m})(\tilde{\Omega}_{11}(x_{13m} - x_{11m}) + \tilde{\Omega}_{21}(y_{13m} - y_{11m}))}{\sqrt{(x_{11m} + y_{11m})^2 + (x_{12m} + y_{12m})^2}} \\ &- \frac{(x_{12m} + y_{12m})(\tilde{\Omega}_{11}(x_{14m} - x_{12m}) + \tilde{\Omega}_{21}(y_{14m} - y_{12m}))}{\sqrt{(x_{11m} + y_{11m})^2 + (x_{12m} + y_{12m})^2}} \\ &- k_1e_{11}) + e_{12}(\tilde{\Omega}_{32}w_{11s} + \tilde{\Omega}_{33}w_{12s} \\ &- \frac{(x_{13m} + y_{13m})(\tilde{\Omega}_{13}x_{11m} + \tilde{\Omega}_{22}y_{13m} - \tilde{\Omega}_{23}(4 + 0.03y_{16m}^2)y_{11m})}{\sqrt{(x_{13m} + y_{13m})^2 + (x_{14m} + y_{14m})^2}} \\ &- \frac{(x_{14m} + y_{14m})(\tilde{\Omega}_{13}x_{12m} + \tilde{\Omega}_{22}y_{14m} - \tilde{\Omega}_{23}(4 + 0.03y_{16m}^2)y_{12m})}{\sqrt{(x_{13m} + y_{13m})^2 + (x_{14m} + y_{14m})^2}} \\ &- k_2e_{12}) + e_{13}(-\tilde{\Omega}_{34}w_{13s} \\ &+ \tilde{\Omega}_{42}z_{13s} \\ &- \frac{(x_{15m} + y_{15m})(-\tilde{\Omega}_{12}x_{15m} - \tilde{\Omega}_{24}y_{15m})}{\sqrt{(x_{15m} + y_{15m})^2}} \\ &- k_3e_{13}) + e_{14}(\tilde{\Omega}_{35}w_{14s} - \tilde{\Omega}_{43}z_{11s}z_{13s} \\ &+ \tilde{\Omega}_{44}z_{14s} \\ &- \frac{(x_{16m} + y_{16m})(\tilde{\Omega}_{14}x_{16m})}{\sqrt{(x_{16m} + y_{16m})^2}} - k_4e_{14}) \\ &- \tilde{\Omega}_{11}(\frac{(x_{11m} + y_{11m})(x_{13m} - x_{11m})e_{11}}{\sqrt{(x_{11m} + y_{11m})^2 + (x_{12m} + y_{12m})^2}} \\ &- \frac{(x_{12m} + y_{12m})(x_{14m} - x_{12m})}{\sqrt{(x_{11m} + y_{11m})^2 + (x_{12m} + y_{12m})^2}} + k_5\tilde{\Omega}_{11}) - \tilde{\Omega}_{12} \\ &(-\frac{(x_{15m} + y_{15m})x_{15m}e_{13}}{\sqrt{(x_{15m} + y_{15m})^2}} + k_6\tilde{\Omega}_{12}) \\ &- \tilde{\Omega}_{13}(-\frac{(x_{13m} + y_{13m})x_{11m}e_{12}}{\sqrt{(x_{13m} + y_{13m})^2 + (x_{14m} + y_{14m})^2}} + k_7\tilde{\Omega}_{13}) - \tilde{\Omega}_{14} \end{aligned}$$

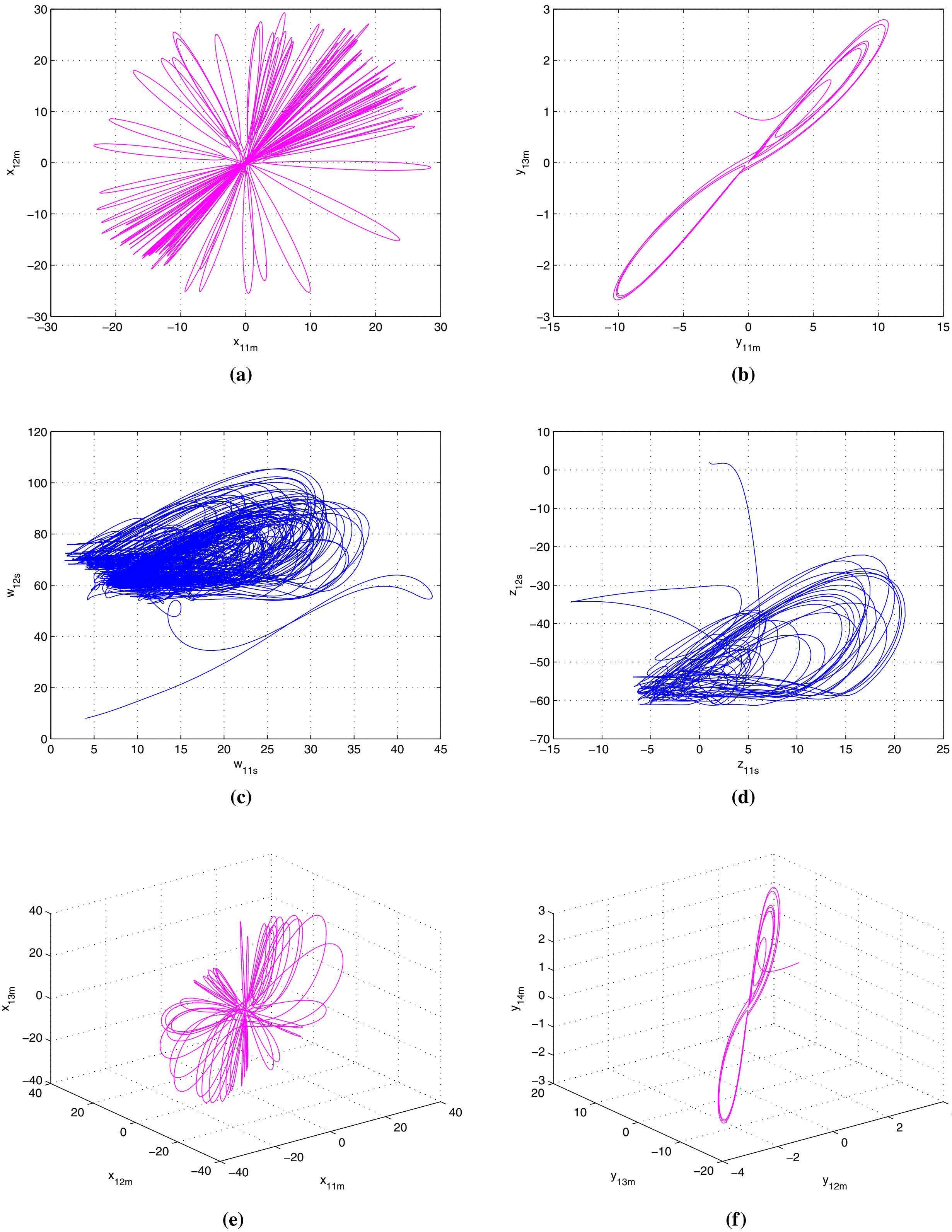


Fig. 1 Phase portraits of HC complex system and HC real system in 2D and 3D **a** complex Lorenz in $x_{11m} - x_{12m}$ plane, **b** complex Lu systems in $y_{11m} - y_{13m}$ plane, **c** Chen systems in $w_{11s} - w_{12s}$ plane, **d** Newton–Leipnik HC system in $z_{11s} - z_{12s}$ plane, **e** complex Lorenz in $x_{11m} - x_{12m} - x_{13m}$ space , **f** Lu systems in $y_{12m} - y_{13m} - y_{14m}$ space, **g** Chen in $w_{11s} - w_{13s} - w_{14s}$ space , **h** Newton–Leipnik systems in $z_{11s} - z_{12s} - z_{13s}$ space

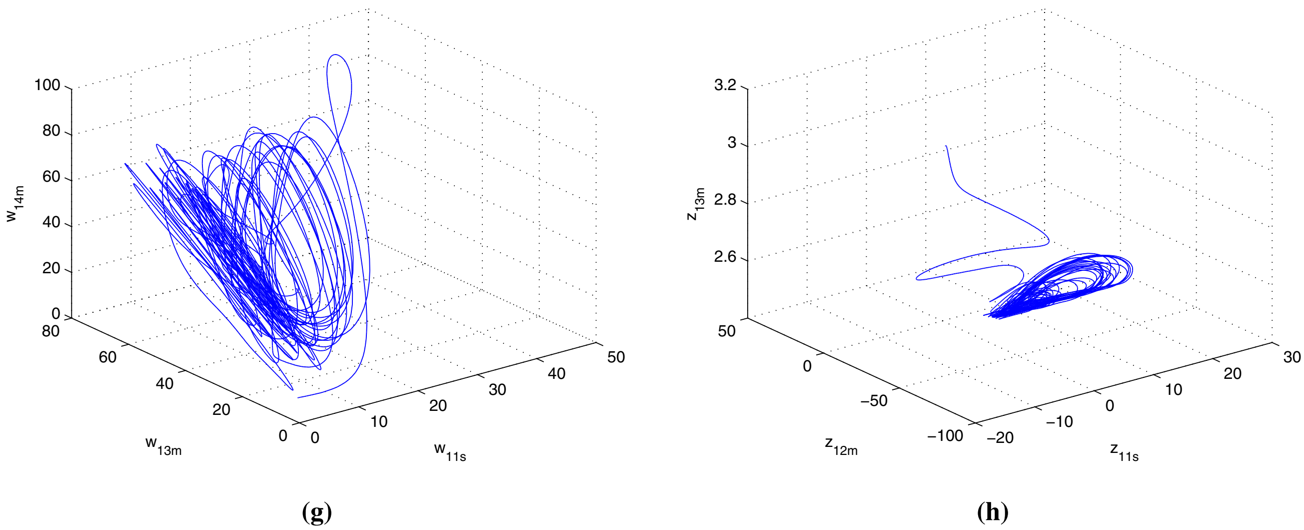


Fig. 1 continued

$$
\begin{aligned}
&(-\frac{(x_{16m}+y_{16m})x_{16m}e_{14}}{\sqrt{(x_{16m}+y_{16m})^2}}+k_8\tilde{\Omega}_{14})\\
&-\tilde{\Omega}_{21}(-\frac{(x_{11m}+y_{11m})(y_{13m}-y_{11m})e_{11}}{\sqrt{(x_{11m}+y_{11m})^2+(x_{12m}+y_{12m})^2}}\\
&-\frac{(x_{12m}+y_{12m})(y_{14m}-y_{12m})}{\sqrt{(x_{11m}+y_{11m})^2+(x_{12m}+y_{12m})^2}}+k_9\tilde{\Omega}_{21})\\
&-\tilde{\Omega}_{22}(-\frac{(x_{13m}+y_{13m})y_{13m}e_{12}}{\sqrt{(x_{13m}+y_{13m})^2+(x_{14m}+y_{14m})^2}}\\
&-\frac{(x_{14m}+y_{14m})y_{14}e_{12}}{\sqrt{(x_{13m}+y_{13m})^2+(x_{14m}+y_{14m})^2}}+k_{10}\tilde{\Omega}_{22})\\
&-\tilde{\Omega}_{23}(\frac{(x_{13m}+y_{13m})(4+0.03y_{16m}^2)y_{11m}e_{12}}{\sqrt{(x_{13m}+y_{13m})^2+(x_{14m}+y_{14m})^2}}\\
&+\frac{(x_{14m}+y_{14m})(4+0.03y_{16m}^2)y_{14}e_{12}}{\sqrt{(x_{13m}+y_{13m})^2+(x_{14m}+y_{14m})^2}}+k_{11}\tilde{\Omega}_{23})\\
&-\tilde{\Omega}_{24}(-\frac{(x_{15m}+y_{15m})y_{11m}e_{13}}{\sqrt{(x_{15m}+y_{15m})^2}}+k_{12}\tilde{\Omega}_{24})\\
&-\tilde{\Omega}_{31}((w_{12s}-w_{11s})e_{11}+k_{13}\tilde{\Omega}_{31})-\tilde{\Omega}_{32}(w_{11s}e_{12}\\
&+k_{14}\tilde{\Omega}_{32})-\tilde{\Omega}_{33}(w_{12s}e_{11}+k_{15}\tilde{\Omega}_{33})-\tilde{\Omega}_{34}(-w_{13s}e_{13}\\
&+k_{16}\tilde{\Omega}_{34})-\tilde{\Omega}_{35}(-w_{14s}e_{14}+k_{17}\tilde{\Omega}_{35})\\
&-\tilde{\Omega}_{41}(-z_{11s}e_{11}+k_{18}\tilde{\Omega}_{41})-\tilde{\Omega}_{42}(z_{13s}e_{13}+k_{19}\tilde{\Omega}_{42})\\
&-\tilde{\Omega}_{43}(-z_{13s}z_{13s}e_{13}+k_{20}\tilde{\Omega}_{43})\\
&-\tilde{\Omega}_{44}(z_{14s}e_{14}+k_{21}\tilde{\Omega}_{44}).
\end{aligned}
\tag{33}
$$

In Eq. (33), canceling out the terms, finally we get:

$$
\begin{aligned}
\dot{V}=&-k_1e_{11}^2-k_2e_{12}^2-k_3e_{13}^2-k_4e_{14}^2-k_5\tilde{\Omega}_{11}^2\\
&-k_6\tilde{\Omega}_{12}^2-k_7\tilde{\Omega}_{13}^2-k_8\tilde{\Omega}_{14}^2\\
&-k_9\tilde{\Omega}_{21}^2-k_{10}\tilde{\Omega}_{22}^2-k_{11}\tilde{\Omega}_{23}^2-k_{12}\tilde{\Omega}_{24}^2\\
&-k_{13}\tilde{\Omega}_{31}^2-k_{14}\tilde{\Omega}_{32}^2-k_{15}\tilde{\Omega}_{33}^2\\
&-k_{16}\tilde{\Omega}_{34}^2-k_{17}\tilde{\Omega}_{35}^2-k_{18}\tilde{\Omega}_{41}^2\\
&-k_{19}\tilde{\Omega}_{42}^2-k_{20}\tilde{\Omega}_{43}^2-k_{21}\tilde{\Omega}_{44}^2\\
\leq&0
\end{aligned}
$$

where $k_i > 0$ for $i = 1, 2, \ldots, 21$.

Based on the LST, the error dynamics (29) is globally asymptotically stable. It implies that complex HC master systems (16), (19) and the real HC slave systems (22), (23) are synchronized under the controller (27) and parameter update law (28); the error variables goes zeros as time t tends to infinity. □

5 Numerical Simulations

In this section, we use the fourth order Runge–Kutta method to carry a mathematical simulation to illustrate the effectiveness of the defined controller. For simulation results, we assume the parameter values of master and slave systems are selected to assure that the systems perform chaotically, these are ($\Omega_{11} = 14$, $\Omega_{12} = 5$, $\Omega_{13} = 45$, $\Omega_{14} = 5.5$), ($\Omega_{21} = 36$, $\Omega_{22} = 20$, $\Omega_{23} = 3.2$, $\Omega_{24} = 5$), ($\Omega_{31} = 35$, $\Omega_{32} = 7$, $\Omega_{33} = 12$, $\Omega_{34} = 3$, $\Omega_{35} = 0.6$), ($\Omega_{41} = 0.4$, $\Omega_{42} = 0.175$, $\Omega_{43} = 0.8$, $\Omega_{44} = 0.01$), respectively. The initial states of the master systems and the slave systems for modulus combination–combination synchronization are considered to be as $(x_{11m}, x_{12m}, x_{13m}, x_{14m}, x_{15m}, x_{16m}) = (1, 2, 3, 4, 5, 6)$, $(y_{11m}, y_{12m}, y_{13m}, y_{14m}, y_{15m}, y_{16m}) = (-1, 2, 1, 1, 2, -1)$, $(w_{11s}, w_{12s}, w_{13s}, w_{14s}) = (4, 8, 9, 3)$, $(z_{11s}, z_{12s}, z_{13s}, z_{14s}) = (1, 2, 3, 4)$, respectively. Also, the control gains are assumed to be as $k_i = 4$ for $i = 1, 2, \ldots, 21$. Initial states of synchronization errors are obtained as $(e_{11}, e_{12}, e_{13}, e_{14}) = (1, 3.596, 5, 2)$. Figure 1a–h depicts the 2D and 3D phase portraits of the master systems and slave systems, respectively. Figure 2a–d displays the time

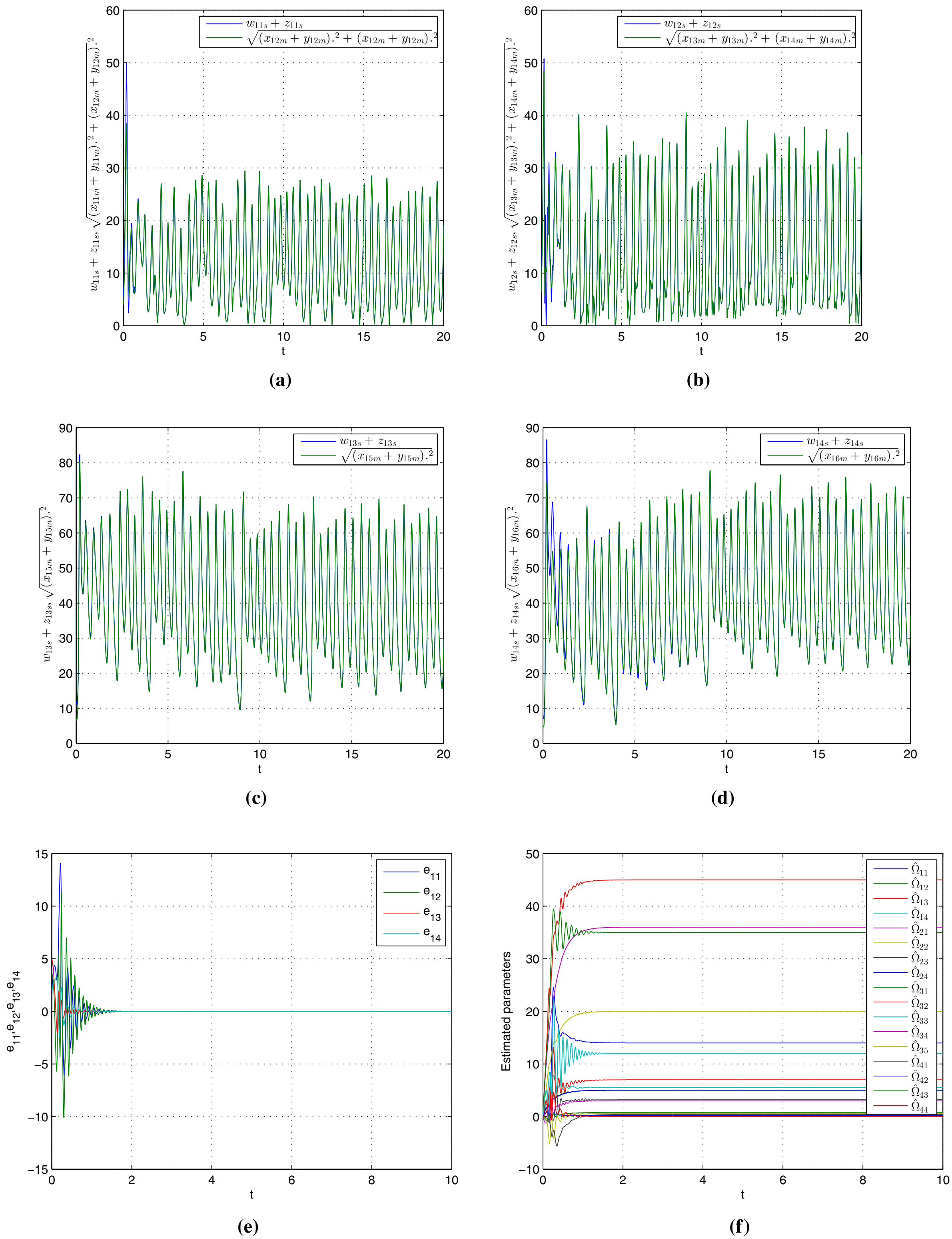


Fig. 2 Modulus combination–combination synchronization trajectories **a** $w_{11s}+z_{11s}$ and $\sqrt{(x_{11m}+y_{11m})^2+(x_{12m}+y_{12m})^2}$ versus 't', **b** $w_{12s}+z_{12s}$ and $\sqrt{(x_{13m}+y_{13m})^2+(x_{14m}+y_{14m})^2}$ versus 't', **c** $w_{13s}+z_{13s}$ and $\sqrt{(x_{15m}+y_{15m})^2}$ versus 't', **d** $w_{14s}+z_{14s}$ and $\sqrt{(x_{16m}+y_{16m})^2}$ versus 't', **e** modulus combination–combination synchronization errors, **f** the estimated values of the unknown parameters $\hat{\Omega}_{11}$, $\hat{\Omega}_{12}$, $\hat{\Omega}_{13}$, $\hat{\Omega}_{14}$, $\hat{\Omega}_{21}$, $\hat{\Omega}_{22}$, $\hat{\Omega}_{23}$, $\hat{\Omega}_{24}$, $\hat{\Omega}_{31}$, $\hat{\Omega}_{32}$, $\hat{\Omega}_{33}$, $\hat{\Omega}_{34}$, $\hat{\Omega}_{35}$, $\hat{\Omega}_{41}$, $\hat{\Omega}_{42}$, $\hat{\Omega}_{43}$, $\hat{\Omega}_{44}$

Fig. 3 **a** Modulus synchronization error using active control; **b** modulus combination–combination synchronization error using adaptive control

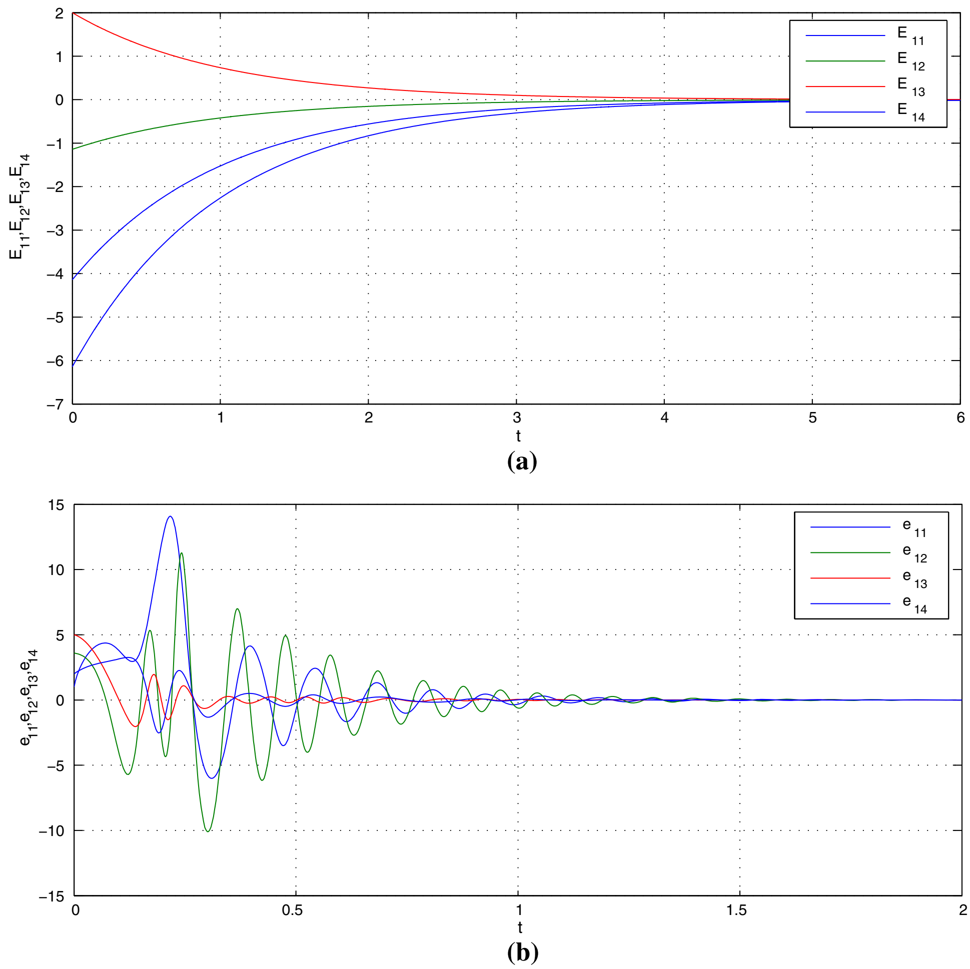
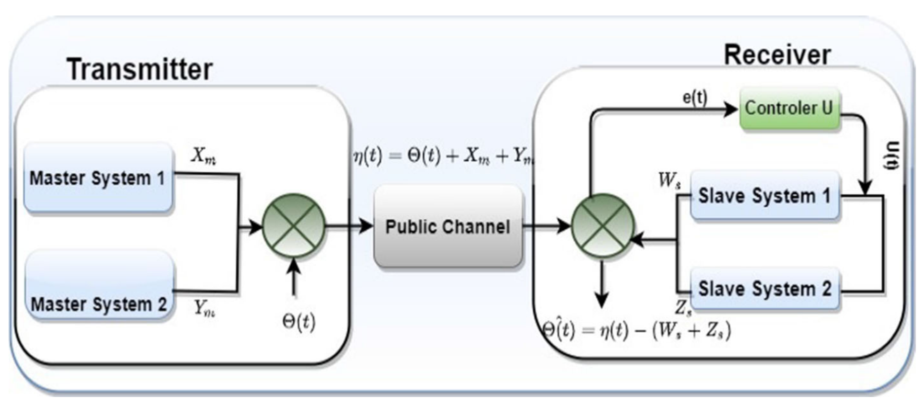


Fig. 4 MCCS-based secure communication system by chaotic masking method



response of the state variables of master systems and slave systems by using the appropriate adaptive controller. Figure 2e illustrates the time of error synchronization states and Fig. 2f indicates the estimated values of unknown parameters ($\hat{\Omega}_{11}, \hat{\Omega}_{12}, \hat{\Omega}_{13}, \hat{\Omega}_{14}, \hat{\Omega}_{21}, \hat{\Omega}_{22}, \hat{\Omega}_{23}, \hat{\Omega}_{24}, \Omega_{25}, \hat{\Omega}_{31}, \hat{\Omega}_{32}, \hat{\Omega}_{33}, \hat{\Omega}_{34}, \hat{\Omega}_{35}, \hat{\Omega}_{41}, \hat{\Omega}_{42}, \hat{\Omega}_{43}, \Omega_{44}$) converge to their initial states asymptotically with time. Hence, the considered MCCS technique among complex HC systems and HC real systems is tested numerically.

6 Comparison Observation of the Studied MCCS Method With the Earlier Issued Work

In Zhou et al. (2014), author investigates combination–combination (C–C) synchronization between four complex chaotic systems, observing that the synchronization error is attained at $t = 5$ (approx). Also, in Khan and Singh (2018a), the author used active control to the achieved C–C synchronization of a novel HC system where it is seen that the

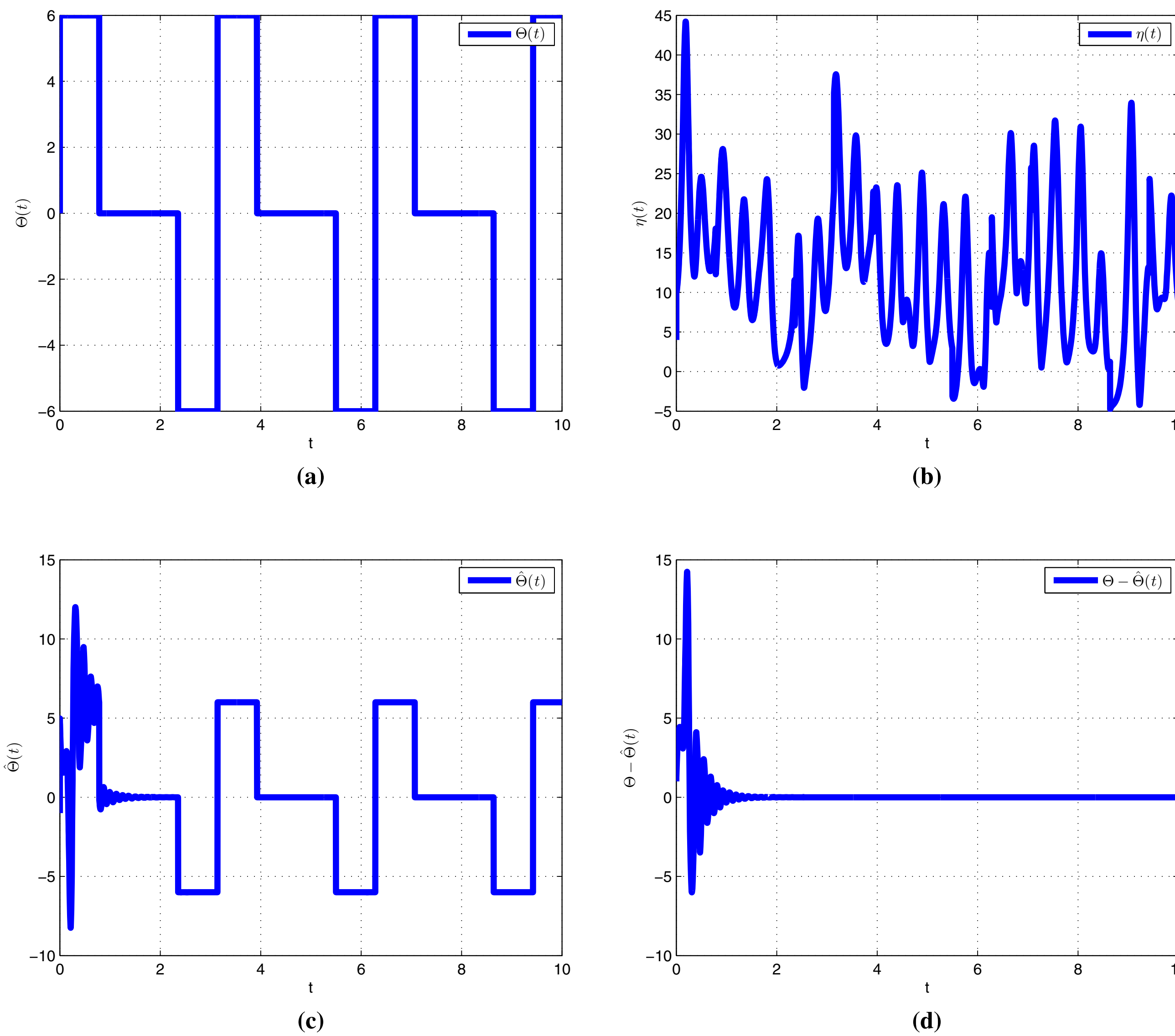


Fig. 5 **a** Information signal $\Theta(t)$, **b** the encrypted signal $\eta(t)$, **c** decrypted signal $\hat{\Theta}(t)$, **d** error between $\Theta(t) - \hat{\Theta}(t)$

synchronization error is converging to zero at $t = 5$ (approx). Further in Khan and Singh (2018b), the author studied a generalization of C–C synchronization of n-dimensional time-delay chaotic via robust adaptive sliding mode control, where it noted that the synchronization state is attained at $t = 5.1$ (approx.). Moreover, in Yadav et al. (2019), the author proposed phase synchronization among non-identical complex chaotic systems of fractional order. There, the synchronization error converges to zero at $t = 4.5$ (approx.). Further, in Khan et al. (2019), author used the adaptive control technique and achieved C–C anti-synchronization of four fractional-order HC systems. They completed the synchronization error at $t = 1.8$(approx). Lastly, in Li et al. (2019), the author investigates modulus synchronization between an HC complex system and an HC real system using an active control technique, in which modulus synchronization occurs between one master system and one slave system. They succeeded in error synchronization at $t = 4.5$ (approx), as shown in Fig. 3a, whereas in the current scheme is extended work of modulus synchronization and C–C synchronization. We had studied MCCS in non-identical HC complex systems and HC real systems (two masters, two slaves) using adaptive control. In our investigations, the error has been synchronized at $t = 1.5$ (approx), as demonstrated in Fig. 3b. Hence, the synchronization time via our investigated methodology is least among all the above-discussed methods. Enough time and energy are conserved for our later practical application. We give the application of MCCS in secure communication provided in followed section. Accordingly, it illustrates that our examined MCCS scheme is more advantageous over earlier issued work.

7 Application of Modulus Combination–Combination Synchronization for Secure Communication

This section discusses a new secure communication design based on MCCS of four non-identical HC complex systems and HC slave systems (two master systems, two slave systems) (Khan and Nigar 2019a; Xiang-Jun et al. 2011; He and Cai 2014). Here, a chaotic system signal is applied for the masking and recovery of an information signal in an application of communication. The secure communication scheme of MCCS is sketched as Fig. 4. Recently, the chaos-based method has gained a broad deal of application for secure communication. In this method, $\Theta(t) = \Theta_{11}(t) + \Theta_{12}(t) + \Theta_{21}(t) + \Theta_{22}(t)$ is the information message signal to be transmitted where $\Theta_{11}(t)$, $\Theta_{12}(t)$, $\Theta_{21}(t)$, and $\Theta_{22}(t)$ are added to the master systems (16) and (19). $\hat{\Theta}(t)$ is the decrypted message signal . The signal transmitted in the chaos masking method is $\eta(t)$, so masking process is $\eta(t) = \Theta(t) + x_{11m} + y_{11m} + x_{12m} + y_{12m}$., and $\hat{\Theta}(t)$ is the signal at the recovered end , which can be obtained as $\hat{\Theta}(t) = \eta(t) - (w_{11s} + z_{11s})$. Signal error can be found as: $e = |\Theta(t) - \hat{\Theta}(t)|$. We choose the information signals such as $\Theta_{11}(t) = sign(sin2t)$ and $\Theta_{12}(t) = sign(sin4t)$, $\Theta_{21}(t) = 2sign(sin2t)$ and $\Theta_{22}(t) = 2sign(sin4t)$. Figure 5 demonstrates that the original information message signal $\Theta(t)$ can be recovered successfully.

8 Conclusion

This paper firstly introduced modulus combination–combination synchronization (MCCS) using an adaptive control technique and application in chaotic secure communication. MCCS is achieved between complex HC systems and real HC systems. The adaptive controller is proposed based on the LST and completed MCCS. Compared with the earlier works, the synchronization error takes less time and gives the application of secure transmission. The MCCS technique is more generalized. MCCS certifies the effectiveness of our approach and thus prompts us to expand MCCS, such as projective MCCS, anti-MCCS, generalized MCCS, and hybrid MCCS. Further, in the future direction, we can study systems interrupted by model uncertainties and disturbance in HC complex systems and HC real system using modulus synchronization. To the author's knowledge, the study of modulus combination–combination synchronization using adaptive control has not yet been explored.

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