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Control with Positivity Constraint for 2D Continuous-Time Systems in Roesser Model

M. Alfidi[1] · A. Hmamed[2] · F. Tadeo[4] · A. Benzaouia[3]

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Abstract
This paper deals with the stability synthesis for a class of 2D continuous-time systems described by the Roesser model. Conditions for stability and stabilization of positive continuous-time Roesser systems are derived, for which the states take non-negative values whenever the initial conditions are non-negative. As well, the synthesis of state-feedback controllers, including the requirement of positiveness of the controllers, and its extension to uncertain plants are solved in terms of linear programming. In addition, the synthesis problem with non-symmetrical bounds and stabilization are also treated. Numerical examples are included to illustrate the proposed approach.

Keywords 2D continuous-time systems · Stabilization · Bounded controls · Positive systems · Positive control · Roesser model · Linear programming

1 Introduction

Two-dimensional (2D) dynamical systems exist in many practical applications, such as in digital data filtering, image processing Roesser (1975), thermal processes, gas absorption, modeling of partial differential equations Marszalek (1984), fractional order systems Dami et al. (2017) and stochastic systems Duan et al. (2014).

In the study of distributed parameter system, partial differential equations arise: these equations actually correspond to 2D or nD continuous-time systems Mastorakis et al. (2000), Mastorakis and Swamy (2002). Therefore, the study of 2D continuous systems is of both practical and theoretical importance. Up to date, some results for 2D continuous-time systems described by the Roesser 2D continuous systems have been given in, for instance, Badie et al. (2018a), Badie et al. (2020), Hmamed et al. (2016), Lam et al. (2004), Xu et al. (2005a, b), Duan et al. (2014).

In connection with Roesser (1975) and Fornasini and Marchesini (1978) models, some important problems such as realization, controllability, minimum energy control Kaczorek (1997), finite frequency H_∞ filtering Li and Gao (2012), dissipativity control and filtering Ki et al. (2015) have been extensively investigated. However, the stabilization problem is still not completely solved. It is also worth pointing out that all of the above-mentioned results were obtained in the context of positive 2D discrete-time systems. For positive 2D continuous-time systems, no results are available in the literature (see Alfidi and Hmamed 2007 for the corresponding 2D discrete-time systems conditions). Moreover, most of previous contributions are based on linear matrix inequalities (LMIs) techniques, with linear programming methods providing a simpler methodology, which motivates the present study.

✉ M. Alfidi
alfidi.med@gmail.com

A. Hmamed
hmamed_abdelaziz@yahoo.fr

F. Tadeo
fernando.tadeo@uva.es

A. Benzaouia
benzaouia@ucam.ac.ma

[1] Engineering, Systems, and Applications Laboratory, National School of Applied Sciences, Sidi Mohamed Ben Abdellah University, Fez, Morocco

[2] LESSI, Department of Physics Faculty of Sciences, Dhar El Mehraz, B.P 1796, 30000 Fes-Atlas, Morocco

[3] LAEPT-URAC 28, Faculty of Sciences Semlalia, University Cadi Ayyad, B.P. 2390, Marrakech, Morocco

[4] Department of Ingenieria de Sistemas y Automatica, Universidad de Valladolid, 47005 Valladolid, Spain

Thus, this paper proposes a new approach for the stabilization of 2D continuous-time positive systems by means of state feedback. Following this approach, one can solve different problems for these positive systems: nominal stability and stabilization, stabilization with restricted sign controls, robust stabilization with polytopic uncertainties and stabilization with bounded controls.

Moreover, we propose a simple numerical method for a complete treatment of the stabilization problem of positive 2D continuous-time systems in Roesser model. This method is based on a previous approach for 1D positive systems initiated in Rami and Tadeo (2005, 2006a, b) and Benzaouia and El hajjaji (2018) where some synthesis problems are solved in term of linear programming (LP) and LMIs, respectively. In addition, based on this approach, we provide LP necessary and sufficient conditions for the stabilization problem with positive controls. The robust stabilization problem in the presence of polytopic uncertainties is also investigated.

The remainder of the paper is structured as follows: In Sect. 2, the problem is formulated and some preliminary results are given. Section 3 studies the stabilization problem. Thereafter, robust stabilization problem is addressed in Sect. 4. Then, the state-feedback synthesis problem with controls having non-symmetrical bounds is tackled in Sect. 5. In Sect. 6, numerical examples illustrate the proposed results.

Notation: R_+^n denotes the nonnegative orthant of the n-dimensional real space R^n. The following descriptions of matrices are used throughout the paper: $A = [a_{ij}]$ and $B^T = [b_1^T \dots b_n^T]$. M^T denotes the transpose of the real matrix M. The notation $M > 0$ (resp. $M \geq 0$), where M is a real matrix (or a vector), means that all the components of M are strictly positive (resp. nonnegative).

2 Problem Formulation and Preliminaries

Consider the following 2D continuous-time system described by Roesser model Lodge and Fahmy (1982), Kaczorek (1985):

$$(\Sigma): \begin{bmatrix} \frac{\partial}{\partial t_1} x^h(t_1,t_2) \\ \frac{\partial}{\partial t_2} x^v(t_1,t_2) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h(t_1,t_2) \\ x^v(t_1,t_2) \end{bmatrix}, \quad (1)$$

where $A_{11} \in R^{n_1 \times n_1}$, $A_{12} \in R^{n_1 \times n_2}$, $A_{21} \in R^{n_2 \times n_1}$, $A_{22} \in R^{n_2 \times n_2}$, are given constant real matrices. The vectors $x^h(t_1,t_2) \in R^{n_1}$ and $x^v(t_1,t_2) \in R^{n_2}$ are, respectively, the horizontal and vertical states at the point (t_1,t_2). Boundary initial conditions for system (1) are given by:

$$\begin{cases} x^h(0,t_2) = g(t_2) & \forall\, t_2 \in \mathbf{R}^+, \\ x^v(t_1,0) = f(t_1) & \forall\, t_1 \in \mathbf{R}^+. \end{cases} \quad (2)$$

where $f(t_1) \in R^{n_2}$ and $g(t_2) \in R^{n_1}$ whose entries are known functions of t_1 and t_2, respectively.
Introducing parameters and vectors of (1) such as:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \; x(t_1,t_2) = \begin{bmatrix} x^h(t_1,t_2) \\ x^v(t_1,t_2) \end{bmatrix},$$
$$\dot{x}(t_1,t_2) = \begin{bmatrix} \frac{\partial}{\partial t_1} x^h(t_1,t_2) \\ \frac{\partial}{\partial t_2} x^v(t_1,t_2) \end{bmatrix}.$$

We can write (1) in the form:

$$\dot{x}(t_1,t_2) = Ax(t_1,t_2). \quad (3)$$

In the sequel, the following definition will be used.

Definition 1 Kaczorek (1996): System (1) is called positive if for any given nonnegative boundary conditions $g(t_2) \geq 0$ and $f(t_1) \geq 0$, the resulting states are also nonnegative $x(t_1,t_2) \geq 0$, for all $t_1 \geq 0, , t_2 \geq 0$.

Definition 2 Berman and Plemmon (1994): A real matrix M is called a Metzler matrix if its off-diagonal elements are nonnegative : $m_{ij} \geq 0,\ i \neq j$.

Definition 3 Horn and Johnson (1991): A real matrix M is called a Hurwitz matrix if $Re(\lambda) < 0$, for all $\lambda \in \sigma(M)$, with $\sigma(M)$ denotes the set of eigenvalues of matrix M.

There are a number of equivalent conditions for a Metzler matrix to be Hurwitz Berman and Plemmon (1994), Horn and Johnson (1991). The following result records some of these conditions which are relevant for the work of this paper.

Theorem 1 Berman and Plemmon (1994) *Let $M \in R^{n\times n}$ be Metzler matrix. Then, the following conditions are equivalent:*

(i) *M is Hurwitz.*
(ii) *There is some vector $\lambda > 0$ in R^n with $A\lambda > 0$.*
(iii) *$A^{-1} \leq 0$.*
(iv) *there is a positive diagonal matrix $P \in R^{n\times n}$ satisfying $A^T P + PA < 0$.*

Asymptotic stability for continuous nD systems has been extensively studied in the literature Jury (1978), Piekarski (1977). A well-known sufficient frequency condition for asymptotic stability is stated in the following result on the basis of the approach given in Jury (1978).

Theorem 2 Kaczorek (1985) *2D continuous-time system (1) is asymptotically stable if*

$$\begin{cases} \det\left(\begin{bmatrix} s_1 I_{n_1} - A_{11} & -A_{12} \\ -A_{21} & s_2 I_{n_2} - A_{22} \end{bmatrix}\right) \neq 0, \\ \forall (s_1,s_2) \in \{(s_1;s_2) : Re(s_1) \geq 0,\ Re(s_2) \geq 0\}. \end{cases}$$

The Lyapunov equation approach to test the stability of the 2D continuous system in (1) is proposed as follows.

Lemma 1 Piekarski (1977), Galkowski (2002) *The 2D continuous system in* (1) *is asymptotically stable if there exist matrices* $P_1 > 0$ *and* $P_2 > 0$ *satisfying the following LMI:*

$$A^T P + PA < 0$$

where $P = diag(P_1, P_2)$.

Next, we need to find under which condition the system (1) is positive. In fact, this condition is given by the following result.

Lemma 2 *The System* (3) *is positive if and only if the matrix A is Metzler.*

Proof $(Necessity)$ Letting $x(0,0) = e_j$(unit vector of the x_j axis), it follows $\dot{x}(0,0) = Ae_j = j^{th}$ column of A. But the trajectory of a positive system cannot leave the positive orthant $\mathbf{R}^n_+$, so that $\dot{x}(0,0) \geq 0$, $\forall i \neq j$. Therefore, the elements of A that are not on the diagonal must be positive or zero, then, the matrix A must be a Metzler matrix.
$(Sufficiency)$: In order to prove $x(t_1,t_2) \geq 0$, it is sufficient to check that the vector $\dot{x}(t_1,t_2)$ does not point toward the outside of $\mathbf{R}^n_+$ whenever $x(t_1,t_2)$ is on the boundary of R^n_+. This is equivalent to verifying that the components of the vector $\dot{x}(t_1,t_2) = Ax(t_1,t_2)$ corresponding to zero components of $x(t_1,t_2) \geq 0$ are nonnegative. Denoting by $\mathcal{J}$ the set of indices of such components, (i.e., $x_i(t_1,t_2) = 0$ for $i \in \mathcal{J}$), we can write

$$\dot{x}_i(t_1,t_2) = \sum_{j\mathcal{J}} a_{ij}x_j(t_1,t_2) \quad i \in \mathcal{J}.$$

So that, from the nonnegativity of the a_{ij} with $i \neq j$, it follows that $\dot{x}_i(t_1,t_2) \geq 0$. □

The above result permits to determine whether a system (1) is positive or not, by simply looking at the entries of the dynamic matrix of the system. In connection with the above Lemma, the following definition will be used.

Definition 4 (i) The system (1) [or equivalently system (3)] is said to be asymptotically stable for some set $\mathcal{X}$ of initial conditions if

$$\lim_{t_1\to+\infty, t_2\to+\infty} x(t_1,t_2) = 0.$$

(ii) The system (1) is said to be L_1-stable for some $\mathcal{X}$ of initial conditions if

$$\int_0^{+\infty}\int_0^{+\infty} x(t_1,t_2)\mathrm{d}t_1\mathrm{d}t_2 < +\infty.$$

The analysis of the stability of the system (1) is presented in the following result.

Theorem 3 *Let* $\mathcal{X} = L_1[R_+, R^{n_1}] \times [R_+, R^{n_2}]$. *Assume that the system (1) (or equivalently system (3)) is positive or equivalently the matrix A is Metzler. Then, the following conditions are equivalent:*

(i) *The system* (1) *is asymptotically stable.*
(ii) *There is some vector* $\lambda > 0$ *in* R^n *with* $A\lambda < 0$.
(iii) *A is Hurwitz.*
(iv) *There is a positive diagonal matrix* $P \in R^{n\times n}$ *satisfying* $A^T P + PA < 0$.
(v) *The system* (1) *is* $L_1 - stable$.

But, before proving the Theorem 3, we need the following Lemma:

Lemma 3 *Let* $M \in R^{n\times n}$ *be a Metzler matrix such that M is Hurwitz, then*
$\forall V,\ \exists W \quad such\ that \quad V = MW$.
Moreover, if $V \geq 0$ $(V \leq 0)$, *then* $W \leq 0$ $(W \geq 0)$.

Proof The proof is obvious by using the part (iii) of Theorem 1. □

Proof of Theorem 3 $(i) \Rightarrow (ii)$ Let $x^h(0,t_2) = g(t_2) > 0$ and $x^v(t_1,0) = f(t_1) > 0$. By integrating (1) twice from 0 to T, we obtain:

$$\begin{bmatrix} \int_0^T x^h(T,t_2)\mathrm{d}t_2 \\ \int_0^T x^v(t_1,T)\mathrm{d}t_1 \end{bmatrix} - \begin{bmatrix} \int_0^T g(t_2)\mathrm{d}t_2 \\ \int_0^T f(t_1)\mathrm{d}t_1 \end{bmatrix}.$$
$$= A\int_0^T\int_0^T x(t_1,t_2)dt_1\mathrm{d}t_2 \tag{4}$$

Since $x^h(t_1,t_2)$ and $x^v(t_1,t_2)$ go to zero (2D system (1) is asymptotically stable), one can choose T such that:

$$\begin{cases} x^h(T,t_2) - g(t_2) < -\epsilon_1, \\ x^v(t_1,T) - f(t_1) < -\epsilon_2,\ \epsilon_1 > 0,\ \epsilon_2 > 0, \end{cases} \tag{5}$$

with $\lambda = \int_0^T\int_0^T x(t_1,t_2)\mathrm{d}t_1dt_2$.
Since by continuity and $x^h(0,t_2) = g(t_2) > 0$, $x^v(t_1,0) = f(t_1) > 0$, the integrals must be strictly positive. Consequently $\lambda > 0$, this proofs that (i) implies (ii).

$(iv) \Rightarrow (i)$.

Assume that (iv) holds, then by Lemma 1, the system (1) is asymptotically stable.

$(i) \Rightarrow (v)$ is trivial and evident.

$(v) \Rightarrow (i)$ is trivial because $x(t_1,t_2) \geq 0$ and continuous and $\int_0^{+\infty}\int_0^{+\infty} x(t_1,t_2)\mathrm{d}t_1\mathrm{d}t_2 < +\infty$.

this implies that $x(t_1,t_2) \to 0\ for\ t_1 \to +\infty\ and\ t_2 \to +\infty$.

$(ii) \Rightarrow (v)$ Let $T > 0$ be arbitrary, $(g(t_2), f(t_1)) \in L_1(R_+, R^{n_1}) \times L_1(R_+, R^{n_2})$. Since A is Metzler and Hurwitz and using Lemma 3, there exists $\lambda \geq 0$ such that:

$$\begin{bmatrix} -\int_0^{+\infty} g(t_2)\mathrm{d}t_2 \\ -\int_0^{+\infty} f(t_1)\mathrm{d}t_1 \end{bmatrix} = A\lambda. \tag{6}$$

Now, from Eq. (4), we obtain:

$$\begin{bmatrix} \int_0^{+\infty} x^h(T, t_2)\mathrm{d}t_2 - \int_0^{+\infty} g(t_2)\mathrm{d}t_2 \\ \int_0^{+\infty} x^v(t_1, T)\mathrm{d}t_1 - \int_0^{+\infty} f(t_1)\mathrm{d}t_1 \end{bmatrix}.$$

$$\leq A \int_0^T \int_0^T x(t_1, t_2)\mathrm{d}t_1\mathrm{d}t_2. \tag{7}$$

By subtracting (5) from (6), we obtain:

$$\begin{bmatrix} \int_0^T x^h(T, t_2)\mathrm{d}t_2 \\ \int_0^T x^v(t_1, T)\mathrm{d}t_1 \end{bmatrix} \leq A \left\{ \int_0^T \int_0^T x(t_1, t_2)\mathrm{d}t_1\mathrm{d}t_2 - \lambda \right\} \tag{8}$$

The left hand of (7) is nonnegative, then

$$V = A \left\{ \int_0^T \int_0^T x(t_1, t_2)\mathrm{d}t_1\mathrm{d}t_2 - \lambda \right\} \geq 0,$$

by using Lemma 3, we obtain, $\forall T$

$$\int_0^T \int_0^T x(t_1, t_2)\mathrm{d}t_1\mathrm{d}t_2 \leq \lambda.$$

Since $x(t_1, t_2) \geq 0$, then:

$$\lim_{T \to +\infty} \int_0^T \int_0^T x(t_1, t_2)\mathrm{d}t_1\mathrm{d}t_2.$$

And we have: $\int_0^{+\infty} \int_0^{+\infty} x(t_1, t_2)\mathrm{d}t_1\mathrm{d}t_2 \leq \lambda$ which is finite then the system (1) is $L_1 - stable$. This completes the proof. □

3 Controller Synthesis

This section studies the stabilization problem of 2D continuous-time systems described by Roesser model:

$$\begin{bmatrix} \frac{\partial}{\partial t_1} x^h(t_1, t_2) \\ \frac{\partial}{\partial t_2} x^v(t_1, t_2) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h(t_1, t_2) \\ x^v(t_1, t_2) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t_1, t_2), \tag{9}$$

for which the control law to be investigated has the state-feedback form

$$u(t_1, t_2) = K \begin{bmatrix} x^h(t_1, t_2) \\ x^v(t_1, t_2) \end{bmatrix}. \tag{10}$$

This control law is designed in order to ensure the positivity and the asymptotic stability of the resulting closed-loop system. As it will be shown, our proposed approach does not impose any restriction on the dynamics of the governed system. For instance, the free 2D Roesser model can be possibly non-positive. In this case, our synthesis design can be interpreted as enforcing the 2D Roesser system to be positive.

Now, consider the closed-loop 2D Roesser model:

$$\begin{aligned} \begin{bmatrix} \frac{\partial}{\partial t_1} x^h(t_1, t_2) \\ \frac{\partial}{\partial t_2} x^v(t_1, t_2) \end{bmatrix} &= (A + BK) \begin{bmatrix} x^h(t_1, t_2) \\ x^v(t_1, t_2) \end{bmatrix}, \\ x^h(0, t_2) &= x_0^h(t_2), \quad \forall t_2 \in \mathbf{R}^+, \\ x^v(t_1, 0) &= x_0^v(t_1), \quad \forall t_1 \in \mathbf{R}^+, \end{aligned} \tag{11}$$

where $A := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $B := \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ are supposed to be any kind of given real matrices. In what follows, we provide the main result of this section.

Theorem 4 *The closed-loop Roesser system ((11)) is positive and asymptotically stable for any initial boundary conditions, if and only if there exist $(n_1 + n_2) + 1$ vectors $d = [d_1 \dots d_{n_1+n_2}]^T \in \boldsymbol{R}^{n_1+n_2}$ and $y_1 \dots y_{n_1+n_2} \in \boldsymbol{R}^m$ such that:*

$$\begin{cases} Ad + B(\sum_{i=1}^{n_1+n_2} y_i) < 0, \\ d > 0, \\ a_{ij}d_j + b_i y_j \geq 0, \quad for\ i \neq j, \end{cases} \tag{12}$$

with $A = [a_{ij}]$, $B^T = [b_1^T \dots b_n^T]$ and $n = n_1 + n_2$. Moreover, the gain matrix K is given by:

$$K = [d_1^{-1} y_1 \dots d_{n_1+n_2}^{-1} y_{n_1+n_2}]. \tag{13}$$

Proof Assume that condition (12) is satisfied and define the appropriate matrix $K = [k_1, \dots, k_{n_1+n_2}]$ with columns constructed as follows $k_i = d_i^{-1} y_i$ for $i = 1, \dots, n_1 + n_2$. Now, by this construction, it is easy to see that $A + BK$ is nonnegative matrix. Effectively, from the last inequalities in condition (12) we have for $i \neq j$:

$$(a_{ij}d_j + b_i y_j)d_j^{-1} = a_{ij} + b_i k_j = (A + BK)_{ij} \geq 0.$$

Next, we show the asymptotic stability under the feedback control: $u = K \begin{bmatrix} x^h \\ x^v \end{bmatrix}$. Using the previous gain, we obtain by

calculation $BKD = B(\sum_{i=1}^{n_1+n_2} y_i)$, which is used in condition (12) and leads to $(A + BK)d < 0$. Now, since $d > 0$ and $(A + BK)$ is Metzler matrix, then by using Theorem 3, we conclude that the 2D system described by the closed-loop Roesser model (11) is asymptotically stable.

The rest of the proof follows the same line of arguments. □

The importance of the above result is relevant, because it provides not only a checkable necessary and sufficient conditions but also a simple approach to address numerically the computation of the problem. Indeed, one can see that the conditions of Theorem 5 are linear. Thus, these conditions can be solved as a standard linear programming problem.

Remark 1 Theorem 5 provides necessary and sufficient conditions for 2D continuous systems (11) to be positive and asymptotically stable in terms of a linear programming problem approach. Note that in the case when system (11) is reduced to a 1D continuous system, Theorem 4 matches with the results given in Rami and Tadeo (2007) for 1D continuous-time systems. Therefore, Theorem 4 can be viewed as an extension of existing results on positiveness and asymptotic stability for 1D continuous systems to 2D case.

In the sequel, we show that the positivity of stabilizing controls can also be handled by using a similar LP approach and a similar proof as it is provided in the preceding result.

Theorem 5 *The following statements are equivalent:*

(i) *There exists a positive state-feedback law* $u = K\begin{bmatrix} x^h \\ x^v \end{bmatrix} \geq 0$ *such that the closed-loop Roesser system* (11) *is positive and asymptotically stable for any initial boundary conditions.*

(ii) *There exists a matrix* $K \in R^{m\times(n_1+n_2)}$ *such that* $K \geq 0$ *and* $A + BK$ *is both a Metzler and a Hurwitz matrix.*

(iii) *The following LP problem in the variables* $d = [d_1 \dots d_n]^T \in R^n$ *and* $y_1 \dots y_n \in R^m$ *is feasible.*

$$\begin{cases} Ad + B\left(\sum_{i=1}^{n_1+n_2} y_i\right) < 0, \\ d > 0, \\ y_i \geq 0, \ 1 \leq i \leq n_1 + n_2, \\ a_{ij}d_j + b_i y_j \geq 0, \ i \neq j, \end{cases} \tag{14}$$

with $A = [a_{ij}]$, $B^T = [b_1^T \dots b_n^T]$ *and* $n = n_1 + n_2$ *. Moreover, the gain matrix in conditions* (i) *and* (ii) *can be chosen as:*

$$K = [d_1^{-1}y_1 \dots d_n^{-1}y_n],$$

with d *and* $y_1, \dots, y_n$*, where* $n = n_1 + n_2$ *are given by any feasible solution to the above LP problem*

Now, some significant remarks are provided.

Remark 2 Note that if a negative state-feedback control law is to be considered it suffices to impose $y_i \leq 0$ instead of $y_i \geq 0$ in the previous LP formulation.

Remark 3 We emphasize that our LP formulation in Theorems 4 and 5 does not impose any restriction on the dynamics of the governed system, i.e., on the data matrices $A \in R^{(n_1+n_2)\times(n_1+n_2)}$ and $B \in R^{(n_1+n_2)\times m}$. For instance, A is not necessarily a Metzler matrix; in this case, the free system is not positive, so that the synthesis design can be viewed as enforcing the system to be positive.

Remark 4 A positive unstable 2D Roesser linear system cannot be stabilized by any positive state-feedback control law if the matrix B is nonnegative. Effectively, the existence of a stabilizing positive state-feedback control (by Theorem 5) necessarily implies that $Ad < 0, d > 0$ which is impossible since the 2D Roesser system is unstable (or equivalently A is not a Hurwitz matrix).

Remark 5 The polytopic uncertainty represents a general ways to describe physical parameter uncertainties, where the set of system parameters is a convex polyhedron Badie et al. (2018b); Chang et al. (2015). Thus, one of the main advantages of our work is the study of the robust stabilization problem for positive continuous-time Roesser systems with polytopic uncertainties, which has not been fully investigated yet and deserves more attention.

4 Synthesis with Uncertain Plant

An important issue in the control design is *robust stability*, that is, ensuring stability under uncertainty or against possible perturbations. In this section, we consider robust stabilization of 2D Roesser systems for which the dynamics are not exactly known and subject to uncertainties which are captured in a polytopic domain. Consider the following uncertain system:

$$\begin{bmatrix} \frac{\partial}{\partial t_1}x^h(t_1,t_2) \\ \frac{\partial}{\partial t_2}x^v(t_1,t_2) \end{bmatrix} = A_\alpha \begin{bmatrix} x^h(t_1,t_2) \\ x^v(t_1,t_2) \end{bmatrix} + B_\alpha u(t_1,t_2), \tag{15}$$

where the matrices $A_\alpha \in R^{(n_1+n_2)\times(n_1+n_2)}$ and $B_\alpha \in R^{(n_1+n_2)\times m}$ are supposed to be not exactly known, but it is assumed that they belong to the following convex set:

$$\left[\, A_\alpha \;\; B_\alpha \,\right] \in \mathbf{D}, \tag{16}$$

$$\mathbf{D} := \left\{ \sum_{i=1}^{l} \alpha_i \left[\, A^i \;\; B^i \,\right], \sum_{i=1}^{l} \alpha_i = 1,\ \alpha_i \geq 0 \right\},$$

where $[A^1 \;\; B^1], \ldots, [A^l \;\; B^l]$ are known matrices. Our robust synthesis design aims to find a single constant gain matrix K for which the following closed-loop system is positive and asymptotically stable for every $[A_\alpha \;\; B_\alpha] \in \mathbf{D}$:

$$\begin{bmatrix} \frac{\partial}{\partial t_1} x^h(t_1, t_2) \\ \frac{\partial}{\partial t_2} x^v(t_1, t_2) \end{bmatrix} = (A_\alpha + B_\alpha K) \begin{bmatrix} x^h(t_1, t_2) \\ x^v(t_1, t_2) \end{bmatrix}. \tag{17}$$

This kind of uncertainties in the model (17) can be directly handled by the proposed approach as is stated in the following result.

Theorem 6 *There exists a robust state-feedback law* $u = K \begin{bmatrix} x^h \\ x^v \end{bmatrix}$ *such that the resulting 2D closed-loop system* (17) *is positive and asymptotically stable for any initial boundary conditions and for every* $[A_\alpha \;\; B_\alpha] \in \mathbf{D}$*, if the following LP problem with the variables* $d = [d_1 \ldots d_{n_1+n_2}]^T \in R^{n_1+n_2}$ *and* $y_1, \ldots, y_{n_1+n_2} \in R^m$*, is feasible.*

$$\begin{cases} A^k d + B^k \left(\sum_{i=1}^{n_1+n_2} y_i \right) < 0, \;\; for\, k = 1, \ldots, l \\ d > 0, \\ a_{ij}^k d_j + b_i^k y_j \geq 0, \;\; i \neq j,\ k = 1, \ldots, l \end{cases} \tag{18}$$

with $A^k = [a_{ij}^k]$, $B^{k^T} = [b_1^{k^T} \ldots b_{n_1+n_2}^{k^T}]$, *and* $n = n_1 + n_2$, $k = 1, ..., l$. *Moreover, the gain matrix can be computed as:*

$$K = [d_1^{-1} y_1 \ldots d_n^{-1} y_n],$$

where d *and* $y_1, \ldots, y_n$*, with* $n = n_1 + n_2$ *are given by any feasible solution to the above LP problem.*

Proof By a simple convexity argument, the proof is straightforward. □

5 Synthesis With Bounded Controls

This section studies the problem of bounded controls for different kinds: bounded positive control and control not restricted in sign, but with nonsymmetric bounds. The following key role lemma will be used for our purpose.

Lemma 4 *Consider the trajectory* $x(t_1, t_2)$ *of the autonomous system* $\dot{x}(t_1, t_2) = Ax(t_1, t_2)$*; then for a given* $\bar{x} > 0$ *we have:* $0 \leq x(t_1, t_2) \leq \bar{x}$*, for any initial conditions satisfying* $0 \leq \begin{pmatrix} g(t_2) \\ f(t_1) \end{pmatrix} \leq \bar{x}$*, if and only if A is Metzler and* $A\bar{x} \leq 0$.

Proof $(Sufficiency)$ By using the transformation $z(t_1, t_2) = \bar{x} - x(t_1, t_2)$, (3) can be transformed into the following form:

$$\dot{z}(t_1, t_2) = Az(t_1, t_2) - A\bar{x}$$

by analogy with the sufficiency of the proof of Lemma 2 we can prove that

$$z(t_1, t_2) \geq 0 \;\; \forall t_1 \geq 0, \;\; \forall t_2 \geq 0, \;\; \text{i. e.}$$

$$x(t_1, t_2) \leq \bar{x} \;\; \forall t_1 \geq 0, \;\; \forall t_2 \geq 0$$

$(Necessity)$ By Lemma 2, the system is necessarily positive and then A must be Metzler. Now, let $x(0, 0) = \bar{x}$, then all the components of the derivative of the state at zero are negative because the state must satisfy $x(t_1, t_2) \leq \bar{x}$ at any time, so that we necessarily have $\dot{x}(0, 0) = A\bar{x} \leq 0$ □

A. Positive Bounded Control

This part considers the following constrained system:

$$\begin{cases} \begin{bmatrix} \frac{\partial}{\partial t_1} x^h(t_1, t_2) \\ \frac{\partial}{\partial t_2} x^v(t_1, t_2) \end{bmatrix} = \begin{bmatrix} x^h(t_1, t_2) \\ x^v(t_1, t_2) \end{bmatrix} \\ \qquad\qquad + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t_1, t_2) \\ \begin{bmatrix} x^h(t_1, t_2) \\ x^v(t_1, t_2) \end{bmatrix} \geq 0 \, and \, 0 \leq u(t_1, t_2) \leq \bar{u}. \end{cases}$$

That is, the trajectory of the system is positive and the input is constrained to be positive and bounded by any given value $\bar{u}$.

The aim here is to address the following problem:

Given $\bar{u} > 0$ find $\bar{x} > 0$ corresponding to the set of initial conditions:

$$\mathcal{X} = x(0, 0) \in R_+^n \mid 0 \leq x(0, 0) \leq \bar{x}$$

for which we can determine a positive bounded state-feedback control law $0 \leq u = Kx(t_1, t_2)$ such that the resulting closed-loop system is positive and asymptotically stable.

Now, we state the main result of this section.

Theorem 7 *Consider the following LP problem with the variables* $\bar{x} = [\bar{x}_1 \ \dots \ \bar{x}_n]^T \in R^n$ *and* $y_1, \dots, y_n \in R^p$

$$\begin{cases} A\bar{x} + B\left(\sum_{i=1}^{n_1+n_2} y_i\right) < 0, \\ \bar{x} > 0, \\ y_i \geq 0, \ 1 \leq i \leq n_1 + n_2, \\ \sum_{i=1}^{n_1+n_2} y_i \leq \bar{u}, \\ a_{ij}d_j + b_i y_j \geq 0, \ \ for\, i \neq j, \end{cases} \tag{19}$$

with $A = [a_{ij}]$, $B^T = [b_1^T \dots b_n^T]$. *Define the matrix*

$$K = [\bar{x}_1^{-1} y_1 \dots \bar{x}_n^{-1} y_n]$$

and consider the 2D closed-loop system $\dot{x}(t_1, t_2) = (A + BK)x(t_1, t_2)$ *under state feedback control* $u(t_1, t_2) = Kx(t_1, t_2)$. *Then,* $0 \leq u(t_1, t_2) \leq \bar{u}$ *for any initial state satisfying* $\bar{x} > 0$ *corresponding to the set of initial conditions:*

$$\mathcal{X} = x(0) \in R^{n\times n} \mid 0 \leq \begin{bmatrix} g(t_2) \\ f(t_1) \end{bmatrix} \leq \bar{x}.$$

Moreover, the closed-loop system is positive and asymptotically stable.

Proof The proof follows the same lines of the proof of Theorem 51 in Rami and Tadeo (2007). □

B. Non-positive Control With Non-symmetrical Bounds Consider the following constrained 2D system:

$$\begin{cases} \begin{bmatrix} \frac{\partial}{\partial t_1} x^h(t_1, t_2) \\ \frac{\partial}{\partial t_2} x^v(t_1, t_2) \end{bmatrix} = \begin{bmatrix} x^h(t_1, t_2) \\ x^v(t_1, t_2) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t_1, t_2), \\ \begin{bmatrix} x^h(t_1, t_2) \\ x^v(t_1, t_2) \end{bmatrix} \geq 0 \, and \ -\bar{u}_1 \leq u(t_1, t_2) \leq \bar{u}_2. \end{cases}$$

we will address the following problem with nonsymmetrical bounds on the control.

Given $\bar{u}_1 > 0$, $\bar{u}_2 > 0$, find $\bar{x} > 0$, corresponding to the set of initial conditions $\{\mathcal{X} = \{x(0, 0) \in R^{n\times n} \mid 0 \leq \begin{bmatrix} g(t_2) \\ f(t_1) \end{bmatrix} \leq \bar{x}\}$, which can determine a bounded state feedback control law $-\bar{u}_1 \leq u(t_1, t_2) = Kx(t_1, t_2) \leq \bar{u}_2$ such that the resulting 2D closed-loop system is positive and asymptotically stable.

Theorem 8 *Consider the following LP problem with the variables* $\bar{x} = [\bar{x}_1 \ \dots \ \bar{x}_n]^n \in R^n$ *and* $z_1^1, \dots, z_n^1 \in R^p$ *and* $z_1^2, \dots, z_n^2 \in R^p$

$$\begin{cases} A\bar{x} + B\left(\sum_{i=1}^{n_1+n_2} (z_i^1 - z_i^2)\right) < 0, \\ \bar{x} > 0, \\ z_i^1 \geq 0, \ 1 \leq i \leq n_1 + n_2, \\ \sum_{i=1}^{n_1+n_2} z_i^1 \leq \bar{u}_2, \\ z_i^2 \geq 0, \ 1 \leq i \leq n_1 + n_2, \\ \sum_{i=1}^{n_1+n_2} z_i^2 \leq \bar{u}_1, \\ a_{ij}d_j + b_i(z_j^1 - z_j^2) \geq 0, \ \ for\, i \neq j, \end{cases} \tag{20}$$

with $A = [a_{ij}]$, $B^T = [b_1^T \dots b_n^T]$. *Define the matrix:*

$$K = [\bar{x}_1^{-1}(z_1^1 - z_1^2) \dots \bar{x}_n^{-1}(z_n^1 - z_n^2)]$$

and consider the 2D closed-loop system:

$$\dot{x}(t_1, t_2) = (A + BK)x(t_1, t_2)$$

under state feedback control $u(t_1, t_2) = Kx(t_1, t_2)$. *Then,* $-\bar{u}_1 \leq u(t_1, t_2) \leq \bar{u}_2$, *for any initial state satisfying* $\bar{x} > 0$, *corresponding to the set of initial conditions* $\mathcal{X} = \{x(0, 0) \in R^{n\times n} \mid 0 \leq \begin{bmatrix} g(t_2) \\ f(t_1) \end{bmatrix} \leq \bar{x}\}$. *Moreover, the 2D closed-loop system is positive and asymptotically stable.*

Proof The proof follows the same lines of the proof of Theorem 7 by using the fact that any gain matrix K can be expressed as the difference of two positive matrices. □

6 Numerical Examples

Example 1 **(Stability synthesis)** It is well known that some dynamical processes in gas absorption water stream heating and air drying can be described by the Darboux equation Marszalek (1984):

$$\frac{\partial^2 s(x,t)}{\partial x \partial t} = a_1 \frac{\partial s(x,t)}{\partial t} + a_2 \frac{\partial s(x,t)}{\partial x} + a_0 s(x,t) + bf(x,t) \tag{21}$$

where $s(x, t)$ is an unknown function at $x(space) \in [0, \ x_f]$ and $t(time) \in [0, \ \infty]$, a_0, a_1, a_2 and b are real coefficients, and $f(x, t)$ is the input function. Let define:

$$r(x,t) = \frac{\partial s(x,t)}{\partial t} - a_2 s(x,t), \tag{22}$$

Then, the hyperbolic equation ((21)) of second order can be transformed into an equivalent system of a 2D continuous-time Roesser model of the form

$$\begin{bmatrix} \frac{\partial}{\partial x} r(x,t) \\ \frac{\partial}{\partial t} s(x,t) \end{bmatrix} = \begin{bmatrix} a_1 & a_1 a_2 + a_0 \\ 1 & a_2 \end{bmatrix} \begin{bmatrix} r(x,t) \\ s(x,t) \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} f(x,t)$$

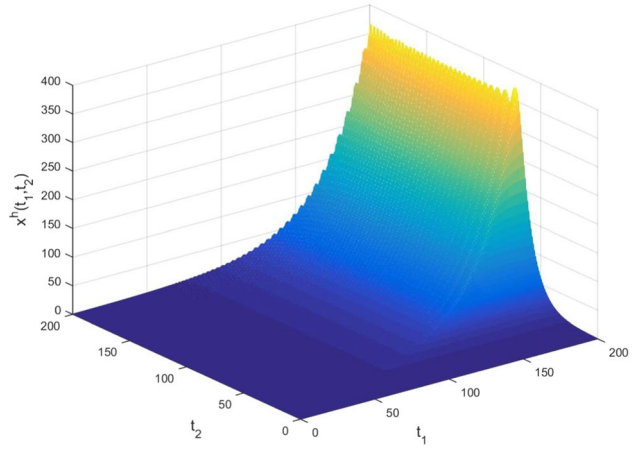


Fig. 1 Open-loop response of $x^h(t_1, t_2)$.

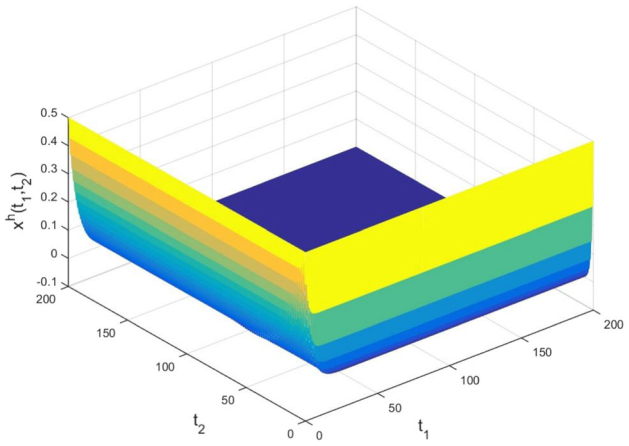


Fig. 2 Closed-loop response of $x^h(t_1, t_2)$.

Let $a_0 = 0.2,\ a_1 = -3,\ a_2 = -1\ and\ b = 0.3$. then, the 2D Roesser system (9) is described by the following matrices:

$$A = \left[\begin{array}{c|c} -3 & 3.2 \\ \hline 1 & -1 \end{array}\right],\ B = \left[\begin{array}{c} 0.3 \\ \hline 0 \end{array}\right].$$

It is easy to see that the free 2D system ($u = 0$) is positive, but not asymptotically stable. This fact is illustrated by the response of the free system: for this $x^h(t_1, t_2)$ is depicted in Fig. 1. Our purpose is to design a state-feedback controller given by (10) such that the 2D closed-loop system (11) is simultaneously positive and asymptotically stable. By applying Theorem 6, and by finding a solution fulfilling the following inequalities.

$$\begin{bmatrix} -3 & 3.2 & 0.3 & 0.3 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ y_1 \\ y_2 \end{bmatrix} < 0,$$

$$\begin{bmatrix} 0 & -3.2 & 0 & -0.3 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ y_1 \\ y_2 \end{bmatrix} < 0.$$

Since we have shown in Theorem 4 that the gain of a stabilizing control (10) is given by $K = [y_1 d_1^{-1}\ y_2 d_2^{-1}]$, the following feasible solution to the above LP problem is used:

$$\begin{bmatrix} d_1 \\ d_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 72.7450 \\ 74.6429 \\ -58.3224 \\ -222.4908 \end{bmatrix},$$

to obtain the following gain of a stabilizing controller:

$$K = [-0.8017\ \ -2.9807].$$

The corresponding 2D system in closed-loop is given by:

$$A + BK = \left[\begin{array}{c|c} -3.2405 & 2.3058 \\ \hline 1.0000 & -1.0000 \end{array}\right].$$

Hence, it suffices to look at the entries of the matrix $A + BK$, to conclude that the 2D closed-loop system (11) is positive (according to Lemma 2). In addition, according to Theorem 3, the closed-loop system is asymptotically stable (also, it can be checked that the matrix $A + BK$ has all eigenvalues in the left of imaginary axis of the complex plan, namely: $\lambda_1 = -4.0073$ and $\lambda_2 = -0.2333$). Figure 2 shows the evolution of $x^h(t_1, t_2)$ of the 2D closed-loop system, and we can see that all the states are nonnegative and converge to zeros, the other state responses are similar, and hence, omitted.

Example 2 **(Stability synthesis)** As an illustration of our results, we treat the following 2D Roesser system (9) described by the matrices:

$$A = \left[\begin{array}{c|c} 0.1 & -0.05 \\ \hline 1 & -1 \end{array}\right],\ B = \left[\begin{array}{c} 0.2 \\ \hline 0 \end{array}\right].$$

Of course, as the matrix A is not Metzler, the free system ($u = 0$) is not positive. The response of $x^v(t_1, t_2)$ of the open-loop system is shown in Fig. 3, which shows that open-loop system is unstable. Now, we are looking for a state-feedback control that stabilizes the 2D system and enforces it to be positive. For this purpose, it suffices to use the result of Theorem 4 and find a solution fulfilling the following inequalities:

$$\begin{bmatrix} 0.1 & -0.05 & 0.2 & 0.2 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ y_1 \\ y_2 \end{bmatrix} < 0,$$

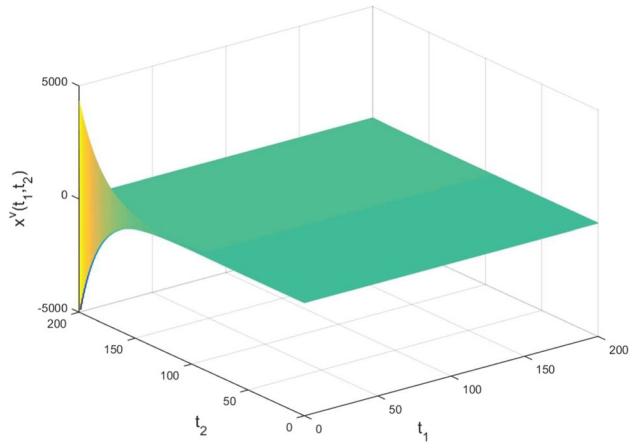


Fig. 3 Open-loop response of $x^v(t_1, t_2)$.

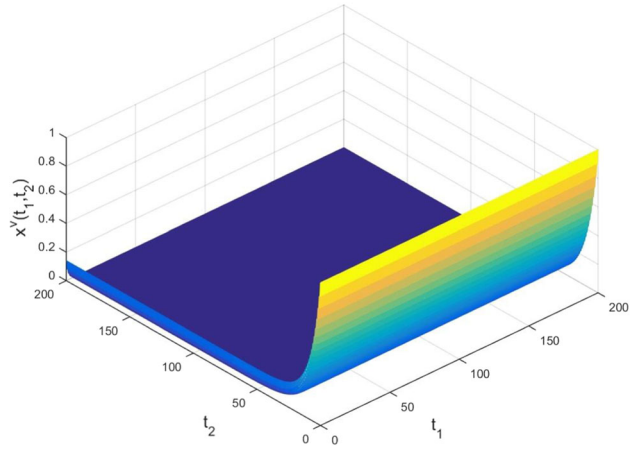


Fig. 4 Closed-loop response of $x^v(t_1, t_2)$.

$$\begin{bmatrix} 0 & 0.05 & 0 & -0.2 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ y_1 \\ y_2 \end{bmatrix} < 0.$$

Since we have shown in Theorem 4 that the gain of a stabilizing control is given by: $K = [y_1 d_1^{-1} \; y_2 d_2^{-1}]$, we have used the following feasible solution to the above LP problem:

$$\begin{bmatrix} d_1 \\ d_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 66.0689 \\ 126.4736 \\ -110.6228 \\ 33.7327 \end{bmatrix},$$

to obtain the following gain of a stabilizing controller:

$$K = [-1.6744 \; 0.2667].$$

The corresponding 2D system in closed-loop is given by:

$$A + BK = \left[\begin{array}{c|c} -0.2349 & 0.0033 \\ \hline 1.0000 & -1.0000 \end{array}\right].$$

Hence, it suffices to look at the entries of the matrix $A + BK$, to conclude that the closed-loop system is positive (according to Lemma 2). In addition, according to Theorem 3 , the 2D closed-loop system is asymptotically stable (also, it can be checked that the matrix $A + BK$ has all eigenvalues in the left of the imaginary axis of the complex plan, namely: $\lambda_1 = -0.2305$ and $\lambda_2 = -1.0043$). Figure 4 shows the evolution of the $x^v(t_1, t_2)$ 2D closed-loop system, and we can see that all the states are nonnegative and converge to zeros, the other state responses are similar, and hence, omitted.

Example 3 **(Positive feedback)** Let us consider the following Roesser system (9) described by:

$$A = \left[\begin{array}{c|c} -1 & 1.2 \\ \hline 1 & -1 \end{array}\right], B = \left[\begin{array}{c} -0.2 \\ \hline 0 \end{array}\right].$$

Note that the 2D system in open-loop is unstable (see Fig. 5), Here, our task is to use a positive state-feedback control, in order to stabilize the 2D system and enforce the state to be positive. Based on the result provided by Theorem 5, the following inequalities must be fulfilled:

$$\begin{bmatrix} -1 & 1.2 & -0.2 & -0.2 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ y_1 \\ y_2 \end{bmatrix} < 0,$$

$$\begin{bmatrix} 0 & -1.2 & 0 & 0.2 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ y_1 \\ y_2 \end{bmatrix} < 0.$$

Now, from the following feasible solution to the above LP problem:

$$\begin{bmatrix} d_1 \\ d_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 78.9685 \\ 86.3958 \\ 95.6369 \\ 93.2388 \end{bmatrix},$$

we obtain (as stated in Theorem 5) the following gain matrix of a stabilizing controller as $K = [1.2111 \; 1.0792]$ (computed as $K = [y_1 d_1^{-1} \; y_2 d_2^{-1}]$). So that the dynamic matrix of 2D closed-loop system (11) is:

$$A + BK = \left[\begin{array}{c|c} -1.2422 & 0.9842 \\ \hline 1.0000 & -1.0000 \end{array}\right].$$

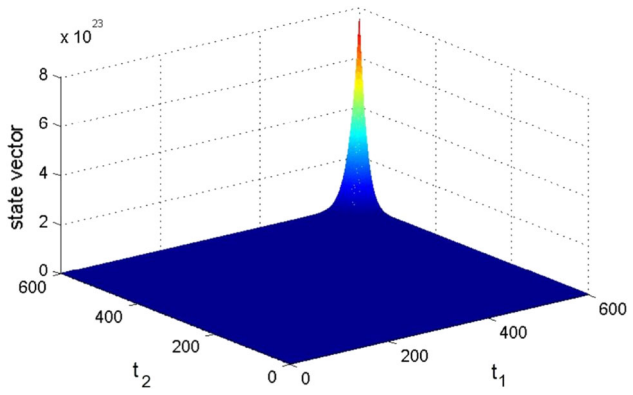


Fig. 5 Open-loop response of $x^h(t_1, t_2)$.

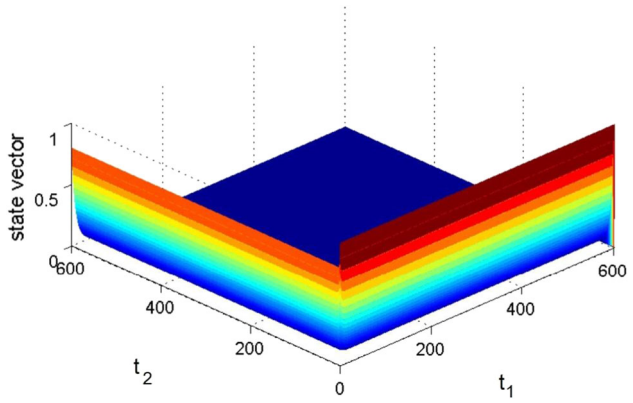


Fig. 6 Closed-loop response of $x^h(t_1, t_2)$.

It can be seen in Fig. 6, that the evolution of $x^h(t_1, t_2)$ in 2D closed-loop system is convergent to zero and asymptotically stable, it can be also checked that matrix $(A + BK)$ has all eigenvalues in the left of the imaginary axis of the complex plan, namely: $\lambda_1 = -2.1205$, and $\lambda_2 = -0.1217$.

Example 4 **(Uncertain plan)** In this example, we consider an uncertain 2D Roesser system (15) subject to a parametric perturbation as follows:

$$A_\alpha = \left[\begin{array}{c|c} -3 & 3.2 + 0.7\alpha \\ \hline 1 & -1 \end{array}\right], \quad B_\alpha = \left[\begin{array}{c} 0.3 + 0.1\alpha \\ \hline 0 \end{array}\right],$$

where $|\alpha| \leq (0.03)^{-1}$.

We are looking for a robust state-feedback control which stabilizes and enforces the positivity of all the plants between the two extreme plants ($\alpha = -(0.03)^{-1}$ and $\alpha = (0.03)^{-1}$). By applying Theorem 6, the following conditions must be satisfied.

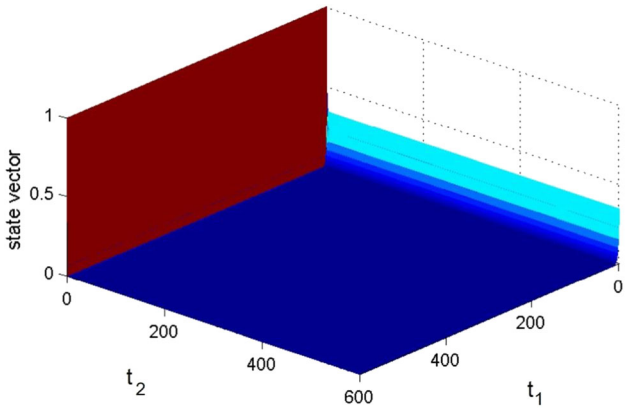


Fig. 7 Open-loop response of $x^h(t_1, t_2)$ (for $\alpha = -(0.03)^{-1}$).

$$\begin{bmatrix} -3 & 26.5333 & 3.6333 & 3.6333 \\ 1 & -1 & 0 & 0 \\ 0 & -26.5333 & 0 & -3.6333 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ y_1 \\ y_2 \end{bmatrix} < 0,$$

$$\begin{bmatrix} -3 & -20.1333 & -3.0333 & -3.0333 \\ 1 & -1 & 0 & 0 \\ 0 & 20.1333 & 0 & 3.0333 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ y_1 \\ y_2 \end{bmatrix} < 0.$$

One feasible solution to the above LP problem is:

$$\begin{bmatrix} d_1 \\ d_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 45.8119 \\ 46.2765 \\ -2.4141 \\ -322.0814 \end{bmatrix},$$

from which we calculate the gain of the robust stabilizing controller (as $K = [y_1 d_1^{-1} \;\; y_2 d_2^{-1}]$):

$$K = [-0.0527 \;\; -6.9599].$$

Hence, with this gain all the 2D closed-loop systems between the two extreme plants ($\alpha = -(0.03)^{-1}$ and $\alpha = (0.03)^{-1}$) are positive and asymptotically stable. The state evolution of the two extreme plants ($\alpha = -(0.03)^{-1}$ and $\alpha = (0.03)^{-1}$) of $x^h(t_1, t_2)$ starting from initial positive boundaries conditions are depicted in Figs. 7 and 8, the other state responses are similar, and hence, omitted.

Example 5 **(Non-positive Control With Non-symmetrical Bounds)** Consider Example 1 with non-positive Control and

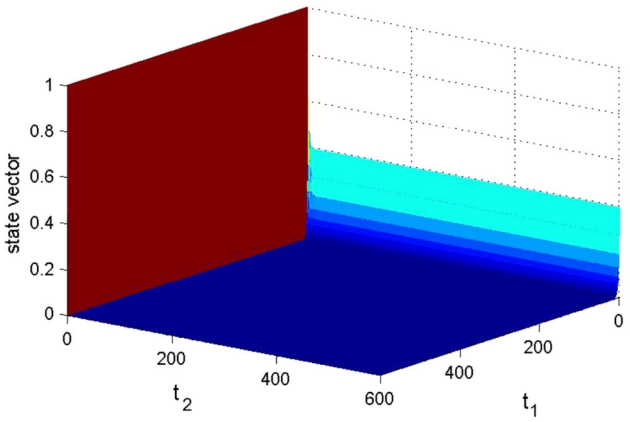


Fig. 8 Closed-loop response of $x^h(t_1, t_2)$ (for $\alpha = (0.03)^{-1}$).

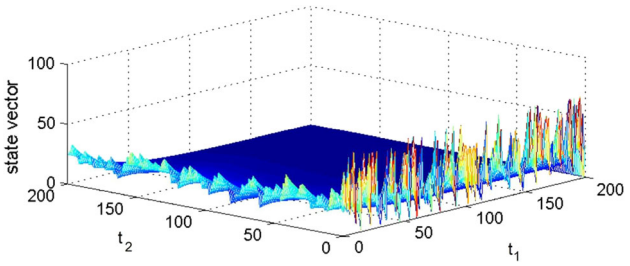


Fig. 9 Closed-loop response of $x^h(t_1, t_2)$

non-symmetrical bounds where $\bar{u}_1 = 10, \bar{u}_2 = 20$. By applying Theorem 8, we obtain:

$$K = [0.0494 \quad -0.0194].$$

Hence, the 2D closed-loop system (11) under the state-feedback control $u(t_1, t_2) = Kx(t_1, t_2)$, with $-\bar{u}_1 \leq u(t_1, t_2) \leq \bar{u}_2$ starting from initial boundaries conditions (randomly generated), is positive and asymptotically stable. The evolution of $x^h(t_1, t_2)$ is depicted in Fig. 9, the other state responses are similar, and hence, omitted.

7 Conclusion

In this paper, a novel approach for solving some synthesis problems for positive 2D continuous-time linear systems described by the Roesser model has been developed. More precisely, the stabilization problem has been considered, and necessary and sufficient conditions for its solvability have been proposed. Moreover, the obtained result was extended to the uncertain case. In addition, the synthesis problem with bounded controls has been addressed. It must be emphasized that all the proposed conditions are solvable in terms of simple linear programming problems. It is expected that this approach could be extended to solve the positive observer-based control in forthcoming work.

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