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Sliding Mode Disturbance Observer Control Based on Adaptive Hybrid Projective Compound Combination Synchronization in Fractional-Order Chaotic Systems

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Abstract

In this paper, the hybrid projective compound combination synchronization (HPCCS) in a class of commensurate fractional-order chaotic Genesio–Tesi system with unknown disturbance has been investigated. To deal with the problem of bounded disturbance, the nonlinear HPCCS is proposed for the fractional-order chaotic system. Further, by choosing the appropriate control gain parameters, the nonlinear fractional-order disturbance observer can approximate the disturbance efficiently. Based on the sliding mode control (SMC) method, a simple sliding mode surface has been introduced. Moreover, by using the Lyapunov stability theory, the designed adaptive SMC method establishes that the states of the three master and two chaotic slave systems are synchronized expeditiously. Finally, some numerical simulation results are illustrated to visualize the effectiveness and the utility of the developed approach on the considered system in the presence of the external unknown bounded disturbances using MATLAB.

Keywords Compound combination synchronization · Genesio–Tesi system · Hybrid projective synchronization · Adaptive SMC · Fractional-order disturbance observer

1 Introduction

Chaos control and synchronization of chaotic systems have been a very attractive field for researchers in recent times. Chaos synchronization has many applications in different areas such as ecology, biological structures, chemical systems, physical systems, electrical circuits, information processing, secure communication, networking systems, etc. (Yildirim and Eski 2010; Vaidyanathan and Azar 2016; Vaidyanathan 2015; Moghadasianx et al. 2012; Das and Pan 2012; Yuanqing and Renquan 2017). The history of the chaotic dynamics goes back to the times when in the late eighteenth century, a great French mathematician and physicist Henri Poincare tried to evaluate the celestial three-body problem, which includes the earth, sun, and moon having a mutual gravitational pull. He showed that in orbital dynamics, the three-body problem has a complicated behavior, which is known as the chaos, and later this characteristic was known as sensitive dependence to the initial state, which is the primary feature of chaos (Russell 1967). Chaos theory is a branch of mathematics focusing on the behavior of nonlinear dynamical systems. The chaotic synchronization, trajectories of identical and nonidentical chaotic systems (master and slave) having different initial conditions, the synchronization error converges to zero. Despite the observation made by Poincare, the first introduction of chaos in a deterministic system was given by Lorenz (1963). Pecora and Carroll (1990) introduced the concept of chaos synchronization using master-slave configuration, which was unprecedented before the last three decades. Later on, researchers extended the pioneering work of Pecora and Carroll and established that synchronization is also possible for non-identical systems having entirely different properties. Till now, several control techniques to attain chaos synchronization and control have been explored in the literature. These are active control (Bhalekar 2014), adaptive control (Shao et al. 2016), sliding mode control (SMC) (Vaidyanathan and Sampath 2012), adaptive SMC (Khan and Tyagi 2017a), optimal control (Khan and Tyagi 2017b), robust adaptive sliding mode

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control (Khan and Singh 2018a), anti disturbance rejection control (ADRC) (Guo and Zhou 2014), feedback control method (Chen and Han 2003), time-delayed feed-back control (Soukkou et al. 2018) etc.

Models based on fractional-order (FO) differential systems introduced the concept of chaos control and chaos synchronization in FO chaotic systems. It has been noticed that the behavior of FO and integer-order differential systems has many disparities. The conclusions made on the stabilization for the integer-order chaotic systems may not be applicable to FO chaotic systems. The stability regions of FO differential systems differing from the integer order system are the main difficulty. This gives rise to the different stability criteria for FO chaotic systems. Various techniques to achieve chaotic synchronization have been presented for the FO chaotic systems such as complete synchronization (Mahmoud and Mahmoud 2010), anti-synchronization (Li and Zhou 2007), projective synchronization (Ding and Shen 2016), hybrid synchronization (Vaidyanathan 2016), hybrid projective synchronization (Khan and Tyagi 2018), compound synchronization (Sun et al. 2014; Prajapati et al. 2018), compound combination synchronization (Sun et al. 2016), combination-combination synchronization (Khan and Singh 2018b), dual combination synchronization (Singh et al. 2017b), dual combination combination (Khan et al. 2017), dual compound synchronization (Khan et al. 2018), double compound synchronization (Zhang and Deng 2014), and sampled-data synchronization (Yuanqing et al. 2017). Hybrid synchronization is very interesting because it possesses the coexistence of complete synchronization and anti-synchronization techniques for achieving hybrid projective synchronization between two different chaotic systems, and hybrid projective in chaotic nonlinear systems is described in Delavari and Mohadeszadeh (2018) and Manfeng et al. (2008). The different excellent approach in dealing with the disturbance is the ADRC technique. The ADRC, as an unusual design approach, was first introduced by Han (2009). ADRC method to eliminate the disturbance by creating a robust gain estimated was discussed in Guo and Zhou (2014), Guo and Liu (2014) and Guo and Jin (2013). But the SMC approach is an efficacious instrument for designing robust control law for complex nonlinear systems. The vital advantage of SMC is low sensitivity to the disturbance and uncertain parameters, which completely destroyed the prerequisite accurate modeling (Vaidyanathan and Azar 2015; Singh et al. 2017a). Aforesaid, in most studies of the research works, the effect of external bounded disturbance(BD) has been discussed, which has the consequent impact such that energy variation, modeling ingratitude, etc. These BD will diminish the achievement of the system and even destabilizing the system. There are two types of disturbances, such as matched disturbance and mismatched disturbance. The matched disturbance is classified for a class of operations; the uncertainties and disturbances perform the matched state, that is, the uncertainties and disturbances influence the system via the same passage with the control input. For a mismatched disturbance, the uncertainties and disturbances do not fulfill the matched state. In Shi et al. (2019), Pashaei and Badamchizadeh (2016) and Yang et al. (2012), mismatched disturbance is discussed using the SMC only. Disturbance cannot be easily ignored in practical applications. A nonlinear fractional-order disturbance observer (FODO) was designed to control the disturbance, and then adaptive SMC law was introduced for FO chaotic system in Khan and Tyagi (2017a) and Mofid et al. (2019). However, these techniques are not feasible for the system with mismatched disturbance.

Motivated by the above-mentioned studies, we present the adaptive SMC hybrid projective compound combination synchronization (HPCCS) scheme in the Genesio–Tesi system. The important highlights of this research are summarized as follows:

- The proposed HPCCS scheme deals with five identical fractional-order Genesio–Tesi systems.
- To the best of our knowledge, we are the first to propose two slave systems with unknown external bounded disturbance using adaptive SMC technique.
- An adaptive SMC technique with fast convergence is designed for the synchronization of FO chaotic systems.
- A more extensive type of nonlinear FODO fulfills the convergence of the disturbance estimation error to the origin.
- Simulation result with a comparison example shows the effectiveness of the introduced method.

The rest of the paper is structured as follows: Sect. 2 contains preliminaries and basic properties of fractional calculus. Problem formulation in which a general scheme of HPCCS is proposed in Sect. 3 and the system description of FO Genesio–Tesi systems is discussed in Sect. 4. The design methodology of the nonlinear FODO of a chaotic system is discussed in Sect. 5. An example of HPCCS of identical Genesio–Tesi system using adaptive SMC is investigated in Sect. 6. The numerical simulations and comparison results are discussed in Sect. 7. In Sect. 8, the application of the suggested HPCCS scheme in secure communication is presented. The results are concluded in Sect. 9.

2 Preliminaries

FO calculus is acknowledged as an expansion to the classical integer order calculus. In the literature, there are several definitions of FO derivatives available such as Riemann–Liouville's, Caputo's derivative, Grunwald–Letnikov's etc.

(Podlubny 1999). Out of these, Caputo's derivative definition is the most generally used.

Definition 1 (Podlubny 1998) The Caputo's derivative for function $f(t)$ with FO α is defined by:

$$cD_y^{\alpha} f(y) = \frac{1}{\gamma(n-\alpha)} \int_c^y \frac{f^n(x)}{(y-x)^{\alpha-n+1}} \mathrm{d}x$$

where $n-1 < \alpha < n, n \in N$ and $\gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} \mathrm{d}x$ is the gamma function.

Due to the wide range of applications of Caputo's derivative, we have also used it in our proposed research work. Some basic properties, including lemmas and assumptions of Caputo's derivative of FO, are used to prove our results, which are listed below.

Property 1 (Li and Deng 2007) If $f_1(t)$ is a constant function and the order $\alpha > 0$, the Caputo FO derivative satisfies the given condition:

$$D^{\alpha} f_1(t) = 0$$

Property 2 (Li and Deng 2007) The Caputo fractional derivative satisfies the following linear property:

$$D^{\alpha}[af_1(t) + bg_1(t)] = aD^{\alpha} f_1(t) + bD^{\alpha} g_1(t)$$

where $f_1(t)$ and $g_1(t)$ are functions of t and a and b are constants.

Lemma 1 (Aguila-Camacho et al. 2014) *Suppose $\Phi(t) \in R$ be a continuous derivable function. Then, we have*

$$\frac{1}{2} D^{\alpha} \Phi(t)^2 \leq \Phi(t) D^{\alpha} \Phi(t)$$

for any time $t \geq t_0$ and $0 < \alpha < 1$.

Lemma 2 (Li and Sun 2015) *Let the FO system satisfies*

$$D^{\alpha} C(t) \leq -b_0 C(t) + b_1$$

Then, there exists a constant $t_1 > 0$ in which for all $t \in (t_1, \infty)$, we get

$$\|C(t)\| \leq \frac{2b_1}{b_0}$$

where $b_0 > 0$, $b_1 > 0$ are constants and $C(t)$ is the state variable.

Assumption 1 Let $\Phi_i(t)$ be the unknown external disturbance for $i = 1, 2, \ldots, n$, Then, the Caputo derivative of $\Phi_i(t)$ is bounded, i.e, $|D^{\alpha} \Phi_i(t)| \leq \xi_i$, where $\xi_i > 0$ is an unknown positive constant. The upper bound of disturbance needs to be known in advance.

3 Problem Formulation

(Manfeng et al. 2008; Ojo et al. 2015) Hybrid projective compound combination synchronization scheme.

We consider the following FO chaotic systems as the three master systems.

$$D^{\alpha} x_1(t) = Ax_1(t) + g(x_1(t)) \quad (1)$$
$$D^{\alpha} x_2(t) = Ax_2(t) + g(x_2(t)) \quad (2)$$
$$D^{\alpha} x_3(t) = Ax_3(t) + g(x_3(t)) \quad (3)$$

where $A \in R^{m \times m}$ denotes a constant matrix, $x_1(t) = (x_{11}, x_{12}, \ldots, x_{1m})^T \in R^m$, $x_2(t) = (x_{21}, x_{22}, \ldots, x_{2m})^T \in R^m$, and $x_3(t) = (x_{31}, x_{32}, \ldots, x_{3m})^T \in R^m$ represent the state vectors of the master systems (1), (2) and (3), respectively, and $g(x_1(t)) = (g_1(x_1(t)), g_2(x_1(t)), \ldots, g_m(x_1(t))) \in R^m$, $g(x_2(t)) = (g_1(x_2(t)), g_2(x_2(t)), \ldots, g_m(x_2(t))) \in R^m$, and $g(x_3(t)) = (g_1(x_3(t)), g_2(x_3(t)), \ldots, g_m(x_3(t))) \in R^m$ are the nonlinear function vector.

Corresponding slave systems are defined as follows:

$$D^{\alpha} y_1(t) = Ay_1(t) + g(y_1(t)) + \Phi_1 + u_1(t) \quad (4)$$
$$D^{\alpha} y_2(t) = Ay_2(t) + g(y_2(t)) + \Phi_2 + u_2(t) \quad (5)$$

where $y_1(t) = (y_{11}, y_{12}, \ldots, y_{1m})^T \in R^m$, $y_2(t) = (y_{21}, y_{22}, \ldots, y_{2m})^T \in R^m$ represent the state vectors of slave systems (4) and (5), respectively, $g(y_1(t)) = (g_1(y_1(t)), g_2(y_1(t)), \ldots, g_m(y_1(t))) \in R^m$ and $g(y_2(t)) = (g_1(y_2(t)), g_2(y_2(t)), \ldots, g_m(y_2(t))) \in R^m$ are the nonlinear function vector of systems (4) and (5), respectively, $\Phi_1(t) = (\Phi_{11}, \Phi_{12}, \ldots, \Phi_{1m})^T \in R^m$ and $\Phi_2(t) = (\Phi_{21}, \Phi_{22}, \ldots, \Phi_{2m})^T \in R^m$ are the unknown external BD of systems (4) and (5), respectively, and $u_1(t) = (u_{11}, u_{12}, \ldots, u_{1m})^T \in R^m$ and $u_2(t) = (u_{21}, u_{22}, \ldots, u_{2m})^T \in R^m$ represent controllers of slave systems which are to be designed.

The HPCCS scheme is achieved if there exist controllers $u_j = u_{1j} + u_{2j}$, $j = 1, 2, \ldots, m$ and constants $\rho = \rho_j$, $j = 1, 2, \ldots, m$.

Definition 2 (Ojo et al. 2015; Manfeng et al. 2008) If the order of the master and the slave systems is the same and there exist a scaling matrix $\rho \in R$ such that

$$\lim_{t \to \infty} \|e\| = \lim_{t \to \infty} \|(y_1(t) + y_2(t)) - \rho x_1(t)(x_2(t) + x_3(t))\| = 0$$

where $\|.\|$ express the matrix norm, then the master systems (1)–(3) and the slave systems (4)–(5) achieved hybrid projective compound combination synchronization.

The error dynamics is:

$$
\begin{aligned}
D^{\alpha}e(t) =& D^{\alpha}y_1(t) + D^{\alpha}y_2(t) - \rho D^{\alpha}x_1(t)(x_2(t)+x_3(t)) \\
&- \rho x_1(t)(D^{\alpha}x_2(t) + D^{\alpha}x_3(t)) \\
D^{\alpha}e(t) =& Ay_1(t) + g(y_1(t)) + \Phi_1 + u_1(t) + Ay_2(t) \\
&+ g(y_2(t)) + \Phi_2 + u_2(t) \\
&- \rho(Ax_1(t) + g(x_1(t)))(x_2(t)+x_3(t)) \\
&- \rho x_1(t)(Ax_2(t) + g(x_2(t)) + Ax_3(t) + g(x_3(t)))
\end{aligned} \tag{6}
$$

Here the design of controller $u_1(t) + u_2(t)$ is given by

$$
\begin{aligned}
u_1(t) + u_2(t) =& - Ay_1(t) - g(y_1(t)) - Ay_2(t) - g(y_2(t)) \\
&+ \rho(Ax_1(t) + g(x_1(t)))(x_2(t)+x_3(t)) \\
&+ \rho x_1(t)(Ax_2(t) + g(x_2(t)) \\
&+ Ax_3(t) + g(x_3(t))) \\
&- \eta s - \hat{K} * \operatorname{sign}(s) - \hat{\Phi}_1 - \hat{\Phi}_2
\end{aligned} \tag{7}
$$

where $\operatorname{sign}(s) = \frac{|s|}{s}$ and $\eta > 0$ are constants. $\hat{K}$ is the estimated value of K. $\hat{\Phi}_1$ and $\hat{\Phi}_1$ are the estimated values of Φ_1, and Φ_2, respectively.

The HPCCS scheme can be described as follows.

- (Zhang et al. 2019; Khan and Tyagi 2018) Disturbances that exist in various fields like economic, physical, biological, mechanical, and medical systems may create an unfavorable influence on system stability. Thus, the impact of disturbances should be dismissed efficiently.
- Disturbance Φ_{ij} $(i = 1, 2;\ j = 1, 2, 3)$ is entirely unknown in the slave systems with fractional derivative, so it is not possible to formulate a synchronized mechanism of three master systems and two slave systems. To solve this difficulty, we introduce a nonlinear FODO designed for compensating the effect of external unknown BD. The external BD can be estimated correctly by the suitable nonlinear FODO.
- Further, we apply an adaptive SMC method so that the synchronized error becomes globally asymptotically stable.

4 System Description

In compound combination synchronization, we take five identical Genesio–Tesi systems.

Consider the Genesio–Tesi FO chaotic system as a master system (Park et al. 2007; Genesio and Tesi 1992; Sambas et al. 2016) .

$$
\begin{aligned}
D^{\alpha}x_{11}(t) &= x_{12} \\
D^{\alpha}x_{12}(t) &= x_{13} \\
D^{\alpha}x_{13}(t) &= -b_1x_{11} - b_2x_{12} - b_3x_{13} + b_4x_{11}^2
\end{aligned} \tag{8}
$$

where x_{11}, x_{12}, x_{13} are state variables and $b_1 = 1.0, b_2 = 1.1, b_3 = 0.4, b_4 = 1.0$ are parameters of system and order $\alpha = 0.99$. Initial condition of Genesio–Tesi system $(x_{11}, x_{12}, x_{13}) = (-0.3, 0.1, -0.2)$, so that the system displays the suspected chaotic behavior. The suggested Genesio–Tesi system producing design may have significant application value in the area of information technology such as secure communication.

Corresponding to three master systems, we take two slave systems

$$
\begin{aligned}
D^{\alpha}y_{11}(t) &= y_{12} + \Phi_{11} + u_{11} \\
D^{\alpha}y_{12}(t) &= y_{13} + \Phi_{12} + u_{12} \\
D^{\alpha}y_{13}(t) &= -b_1y_{11} - b_2y_{12} - b_3y_{13} + b_4y_{11}^2 + \Phi_{13} + u_{13}
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
D^{\alpha}y_{21}(t) &= y_{22} + \Phi_{21} + u_{21} \\
D^{\alpha}y_{22}(t) &= y_{23} + \Phi_{22} + u_{22} \\
D^{\alpha}y_{23}(t) &= -b_1y_{21} - b_2y_{22} - b_3y_{23} + b_4y_{21}^2 + \Phi_{23} + u_{23}
\end{aligned} \tag{10}
$$

5 Fractional-Order Disturbance Observer (FODO) Design

In Sect. 5, we construct a nonlinear FODO to approximate the unknown external BD in the slave systems (9–10). As BD Φ_{ij} $(i = 1, 2;\ j = 1, 2, 3)$, which are developed in the slave system (9–10), is entirely unknown, so it is not possible to formulate a synchronized mechanism of three master systems and two slave systems. To undertake this difficulty, we create a nonlinear FODO to evaluate the unknown BD. Nonlinear FODO is constructed for compensating the effect of unknown external BD. To estimate the unknown external BD in the slave systems, we design a nonlinear FODO. For designing the FODO, we propose an auxiliary variable that is based on the integer-order disturbance observer as follows (Chen et al. 2014).

$$
\begin{aligned}
\Theta_1(t) &= \Phi_{11}(t) + \Phi_{21}(t) - \sigma_1(y_{11}(t) + y_{21}(t)) \\
\Theta_2(t) &= \Phi_{12}(t) + \Phi_{22}(t) - \sigma_2(y_{12}(t) + y_{22}(t)) \\
\Theta_3(t) &= \Phi_{13}(t) + \Phi_{23}(t) - \sigma_3(y_{13}(t) + y_{13}(t))
\end{aligned} \tag{11}
$$

where $\sigma_1, \sigma_2, \sigma_3 > 0$ are constants to be determined.

In Eq. (11), applying Caputo derivative and using Eqs. (9) and (10), we get

$$
\begin{aligned}
D^{\alpha}\Theta_1(t) = & D^{\alpha}\Phi_{11}(t) + D^{\alpha}\Phi_{21}(t) - \sigma_1(y_{12} + y_{22} \\
&+ \Theta_1 + \sigma_1(y_{11} + y_{21}))
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \sigma_1(u_{11} + u_{21}) \\
D^\alpha \Theta_2(t) &= D^\alpha \Phi_{12}(t) + D^\alpha \Phi_{22}(t) - \sigma_2(y_{13} + y_{23} \\
&\quad + \Theta_2 + \sigma_2(y_{12} + y_{22}) \\
&\quad - \sigma_2(u_{12} + u_{22}) \\
D^\alpha \Theta_3(t) &= D^\alpha \Phi_{13}(t) + D^{q_3} \Phi_{23}(t) - \sigma_3(-b_1(y_{11} \\
&\quad + y_{21}) - b_2(y_{12} + y_{22}) \\
&\quad - b_3(y_{13} + y_{23}) + b_4(y_{11}^2 + y_{21}^2) \\
&\quad + \Theta_3(t) + \sigma_3(y_{13} + y_{23}) \\
&\quad - \sigma_3(u_{13} + u_{23})
\end{aligned} \tag{12}
$$

We estimate $\Theta_j(t)(\mathrm{j} = 1, 2, 3)$ in order to calculate the external unknown disturbances, which is described as:

$$
\begin{aligned}
D^\alpha \hat{\Theta}_1(t) &= -\sigma_1(y_{12} + y_{22} + \sigma_1(y_{11} + y_{21})) \\
&\quad - \sigma_1 \hat{\Theta}_1(t) - \sigma_1(u_{11} + u_{21}) \\
D^\alpha \hat{\Theta}_2(t) &= -\sigma_2(y_{13} + y_{23} + \sigma_2(y_{12} + y_{22})) \\
&\quad - \sigma_2 \hat{\Theta}_2(t) - \sigma_2(u_{12} + u_{22}) \\
D^\alpha \hat{\Theta}_3(t) &= -\sigma_3(-b_1(y_{11} + y_{21}) - b_2(y_{12} + y_{22}) \\
&\quad - b_3(y_{13} + y_{23}) \\
&\quad + b_4(y_{11}^2 + y_{21}^2)) - \sigma_3 \hat{\Theta}_3(t) \\
&\quad - \sigma_3(u_{13} + u_{23})
\end{aligned} \tag{13}
$$

where $\hat{\Theta}_j(t)$ is the estimate of Θ_j

Using Eq. (11), $\Phi_{1j}(t)$ and $\Phi_{2j}(t)$, $j = 1, 2, 3$ are written in the form

$$
\begin{aligned}
\hat{\Phi}_{11}(t) + \hat{\Phi}_{21}(t) &= \hat{\Theta}_1(t) + \sigma_1(y_{11} + y_{21}) \\
\hat{\Phi}_{12}(t) + \hat{\Phi}_{22}(t) &= \hat{\Theta}_2(t) + \sigma_2(y_{12} + y_{22}) \\
\hat{\Phi}_{13}(t) + \hat{\Phi}_{23}(t) &= \hat{\Theta}_3(t) + \sigma_3(y_{13} + y_{23})
\end{aligned} \tag{14}
$$

Define disturbance estimation error as:

$$
\begin{aligned}
\tilde{\Phi}_{11}(t) + \tilde{\Phi}_{21}(t) &= \Phi_{11} + \Phi_{21} - (\hat{\Phi}_{11} + \hat{\Phi}_{21}) \\
\tilde{\Phi}_{12}(t) + \tilde{\Phi}_{22}(t) &= \Phi_{12} + \Phi_{22} - (\hat{\Phi}_{12} + \hat{\Phi}_{22}) \\
\tilde{\Phi}_{13}(t) + \tilde{\Phi}_{23}(t) &= \Phi_{13} + \Phi_{23} - (\hat{\Phi}_{13} + \hat{\Phi}_{23})
\end{aligned} \tag{15}
$$

From Eqs. (11) and (14), we obtain

$$
\begin{aligned}
\tilde{\Theta}_1(t) &= \Theta_1(t) - \hat{\Theta}_1(t) = (\Phi_{11}(t) + \Phi_{21}(t)) \\
&\quad - (\hat{\Phi}_{11}(t) + \hat{\Phi}_{21}(t)) = \tilde{\Phi}_{11}(t) + \tilde{\Phi}_{21}(t) \\
\tilde{\Theta}_2(t) &= \Theta_2(t) - \hat{\Theta}_2(t) = (\Phi_{12}(t) + \Phi_{22}(t)) \\
&\quad - (\hat{\Phi}_{12}(t) + \hat{\Phi}_{22}(t)) = \tilde{\Phi}_{12}(t) + \tilde{\Phi}_{22}(t) \\
\tilde{\Theta}_3(t) &= \Theta_3(t) - \hat{\Theta}_3(t) = (\Phi_{13}(t) + \Phi_{23}(t)) \\
&\quad - (\hat{\Phi}_{13}(t) + \hat{\Phi}_{23}(t)) = \tilde{\Phi}_{13}(t) + \tilde{\Phi}_{23}(t)
\end{aligned} \tag{16}
$$

By subtracting Eq. (13) from (12), the Caputo derivatives $\tilde{\Theta}_j$ are expressed in the following form:

$$
\begin{aligned}
D^\alpha \tilde{\Theta}_1(t) &= -\sigma_1 \tilde{\Theta}_1(t) + D^\alpha(\phi_{11}(t) + \phi_{21}(t)) \\
D^\alpha \tilde{\Theta}_2(t) &= -\sigma_2 \tilde{\Theta}_2(t) + D^\alpha(\phi_{12}(t) + \phi_{22}(t)) \\
D^\alpha \tilde{\Theta}_3(t) &= -\sigma_3 \tilde{\Theta}_3(t) + D^\alpha(\phi_{13}(t) + \phi_{23}(t))
\end{aligned} \tag{17}
$$

Based on the above discussion, to check the convergence of approximated disturbance estimation error, a Lyapunov function can be selected as (Shao et al. 2016).

$$
\begin{aligned}
V_{\Phi_{1j}(t)+\Phi_{2j}(t)} &= 0.5(\tilde{\Phi}_{1j}(t) + \tilde{\Phi}_{2j}(t))^2 \\
&= 0.5\tilde{\Theta}_j^{\,2}(t), (j = 1, 2, 3)
\end{aligned} \tag{18}
$$

Apply the Caputo derivative in Eq. (18) and using Lemma 1, we get

$$
D^\alpha V_{\Phi_{1j}(t)+\Phi_{2j}(t)} < \tilde{\Theta}_j(t) D^\alpha \tilde{\Theta}_j(t) \tag{19}
$$

Substituting Eq. (17) into (19), the following equation can be attained:

$$
\begin{aligned}
D^\alpha V_{\Phi_{1j}(t)+\Phi_{2j}(t)} &\le \tilde{\Theta}_j(t)(-\sigma_j \tilde{\Theta}_j(t) + D^\alpha(\Phi_{1j}(t) \\
&\quad + \Phi_{2j}(t))), \ j = 1, 2, 3 \\
&\le -\sigma_j \tilde{\Theta}_j(t)^2 + \tilde{\Theta}_j(t) D^\alpha(\Phi_{1j}(t) \\
&\quad + \Phi_{2j}(t))
\end{aligned} \tag{20}
$$

We notice that

$$
\begin{aligned}
&(\tilde{\Theta}_j(t) - D^\alpha(\Phi_{1j}(t) + \Phi_{2j}(t)))^2 \\
&\quad = (\tilde{\Theta}_j(t))^2 + (D^\alpha(\Phi_{1j}(t) + \Phi_{2j}(t)))^2 \\
&\qquad - 2\tilde{\Theta}_j(t) D^\alpha(\Phi_{1j}(t) + \Phi_{2j}(t)) > 0
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
&\tilde{\Theta}_j(t) D^\alpha(\Phi_{1j}(t) + \Phi_{2j}(t)) \\
&\quad < \frac{1}{2}(\tilde{\Theta}_j(t))^2 + \frac{1}{2}(D^\alpha(\Phi_{1j}(t) + \Phi_{2j}(t)))^2
\end{aligned} \tag{22}
$$

Using Eq. (22) in Eq. (20)

$$
\begin{aligned}
D^\alpha V_{\Phi_{1j}(t)+\Phi_{2j}(t)} &\le -\sigma_j \tilde{\Theta}_j^{\,2}(t) + \frac{1}{2}(\tilde{\Theta}_j(t))^2 \\
&\quad + \frac{1}{2}(D^\alpha(\Phi_{1j}(t) + \Phi_{2j}(t)))^2
\end{aligned} \tag{23}
$$

Now using Assumption 1 in Eq. (23), we obtain

$$
\begin{aligned}
D^\alpha V_{\Phi_{1j}(t)+\Phi_{2j}(t)} &\le -\sigma_j \tilde{\Theta}_j^{\,2}(t) + 0.5\tilde{\Theta}_j^{\,2}(t) + 0.5\xi_j^{\,2} \\
&= -(\sigma_j - 0.5)\tilde{\Theta}_j^{\,2}(t) + 0.5\xi_j^{\,2}
\end{aligned}
$$

$$= -(2\sigma_j - 1)\frac{1}{2}\tilde{\Theta}_j{}^2(t) + 0.5\xi_j{}^2$$
$$\leq \Omega_0 V_{\Phi_{1j}(t)+\Phi_{2j}(t)} + \Omega_1, \, with \ j = 1, 2, 3 \tag{24}$$

where $\Omega_0 = 2\sigma_j - 1$, and $\Omega_1 = 0.5\xi_j^2$. If $\sigma_j > 0.5$, this confirms that estimated errors are bounded. Using Lemma 2 and Eq. (24), then we have.

$$|V_{\Phi_{1j}(t)+\Phi_{2j}(t)}| \leq \frac{2\Omega_1}{\Omega_0} \tag{25}$$

Substitute the value of Ω_0 and Ω_1 in equation(25), we get

$$|V_{\Phi_{1j}(t)+\Phi_{2j}(t)}| \leq \frac{\xi_j^2}{2(\sigma_j - 0.5)} \tag{26}$$

Using Eq. (18) in Eq. (26), we have

$$|\frac{1}{2}(\tilde{\Phi}_{1j}(t) + \tilde{\Phi}_{2j}(t))^2| \leq \frac{\xi_j{}^2}{2(\sigma_j - 0.5)} \tag{27}$$

$$|\tilde{\Phi}_{1j}(t) + \tilde{\Phi}_{2j}(t)| \leq \sqrt{\frac{\xi_j{}^2}{\sigma_j - 0.5}} \tag{28}$$

According to (28), indicate that $\tilde{\Phi}_{1j}(t) + \tilde{\Phi}_{2j}(t)$ is bounded above, where $\tilde{\Phi}_{1j}(t) + \tilde{\Phi}_{2j}(t)$ is known as disturbance estimation error. Thus, for the external disturbance $\Phi_{1j}(t) + \Phi_{2j}(t)$, the disturbance approximation error $|\tilde{\Phi}_{1j}(t) + \tilde{\Phi}_{2j}(t)| \leq K_j$, where $K_j > 0$ is unknown constant. In real genuine approach, it is very hard to find the upper bounds of $|\tilde{\Phi}_{1j}(t) + \tilde{\Phi}_{2j}(t)|$, therefore, we have presented estimated value $\hat{K}_j$ of $K_j (j = 1, 2, 3)$. Thus, from the above study, for the slave systems (9) and (10), the nonlinear FODO designed in (13) and (14), the disturbance estimated error $\tilde{\Phi}_{1j}(t) + \tilde{\Phi}_{2j}(t)$ of the suggested nonlinear FODO is bounded.

6 Numerical Example of Adaptive Sliding Mode Synchronization

(Khan and Tyagi 2017a; Chen and Jing 2015) This section illustrates the validity of the proposed hybrid projective compound combination synchronization (HPCCS) scheme using adaptive sliding mode control. The following FO Genesio–Tesi system is considered as master systems.

$$\begin{aligned}
D^\alpha x_{11}(t) &= x_{12}\\
D^\alpha x_{12}(t) &= x_{13}\\
D^\alpha x_{13}(t) &= -b_1 x_{11} - b_2 x_{12} - b_3 x_{13} + b_4 x_{11}^2 \qquad (29)\\
D^\alpha x_{21}(t) &= x_{22}\\
D^\alpha x_{22}(t) &= x_{23}\\
D^\alpha x_{23}(t) &= -b_1 x_{21} - b_2 x_{22} - b_3 x_{23} + b_4 x_{21}^2 \qquad (30)\\
D^\alpha x_{31}(t) &= x_{32}\\
D^\alpha x_{32}(t) &= x_{33}\\
D^\alpha x_{33}(t) &= -b_1 x_{31} - b_2 x_{32} - b_3 x_{33} + b_4 x_{31}^2 \qquad (31)
\end{aligned}$$

Corresponding two slave systems are described as:

$$\begin{aligned}
D^\alpha y_{11}(t) &= y_{12} + \text{sign}(\sin(t+1)) + u_{11}\\
D^\alpha y_{12}(t) &= y_{13} + 5\cos 6t + u_{12}\\
D^\alpha y_{13}(t) &= -b_1 y_{11} - b_2 y_{12} - b_3 y_{13} + b_4 y_{11}^2\\
&\quad + 5(\sin 4t + \cos 4t) + u_{13} \qquad (32)
\end{aligned}$$

where $\Phi_{11} = \text{sign}\sin(t+1), \Phi_{12} = 5\cos 6t, \Phi_{13} = 5(\sin 4t + \cos 4t)$ are the disturbances of the system, and u_{11}, u_{12}, u_{13} are the controllers.

$$\begin{aligned}
D^\alpha y_{21}(t) &= y_{22} + \text{sign}(\sin(2(t+1))) + u_{21}\\
D^\alpha y_{22}(t) &= y_{23} + 6\cos 5t + u_{22}\\
D^\alpha y_{23}(t) &= -b_1 y_{21} - b_2 y_{22} - b_3 y_{23} + b_4 y_{21}^2\\
&\quad + 4(\sin 5t + \cos 5t) + u_{23} \qquad (33)
\end{aligned}$$

where $\Phi_{21} = \text{sign}(\sin(2(t+1))), \Phi_{22} = 6\cos 5t, \Phi_{23} = 4(\sin 5t + \cos 5t)$ are the disturbances of the system, and u_{21}, u_{22}, u_{23} are the controllers.

In HPCCS, error state can be defined as (Khan and Tyagi 2018; Ojo et al. 2015)

$$\begin{aligned}
e_{11}(t) &= y_{11} + y_{21} - \alpha_1 x_{11}(x_{21} + x_{31})\\
e_{12}(t) &= y_{12} + y_{22} - \alpha_2 x_{12}(x_{22} + x_{32})\\
e_{13}(t) &= y_{13} + y_{23} - \alpha_3 x_{13}(x_{23} + x_{33}) \qquad (34)
\end{aligned}$$

Then, the error dynamics takes the form

$$\begin{aligned}
D^\alpha e_{11}(t) &= D^\alpha y_{11} + D^\alpha y_{21} - \alpha_1 D^\alpha x_{11}(x_{21} + x_{31})\\
&\quad - \alpha_1 x_{11}(D^\alpha x_{21} + D^\alpha x_{31})\\
D^\alpha e_{12}(t) &= D^\alpha y_{12} + D^\alpha y_{22} - \alpha_1 D^\alpha x_{12}(x_{22} + x_{32})\\
&\quad - \alpha_2 x_{12}(D^\alpha x_{22} + D^\alpha x_{32})\\
D^\alpha e_{13}(t) &= D^\alpha y_{13} + D^\alpha y_{23}\\
&\quad - \alpha_3 D^\alpha x_{13}(x_{23} + x_{33}) - \alpha_3 x_{13}(D^\alpha x_{23} + D^\alpha x_{33}) \qquad (35)
\end{aligned}$$

Using the slave systems (32–33) and master systems (29–30–31), the error dynamics becomes:

$$\begin{aligned}
D^\alpha e_{11}(t) = \; & e_{12} + (x_{22} + x_{32})(\alpha_2 x_{12}\\
& - \alpha_1 x_{11}) - \alpha_1 x_{12}(x_{21} + x_{31})\\
& + \text{sign}(\sin(t+1)) + \text{sign}(\sin(2(t+1)))\\
& + u_{11} + u_{21} \qquad (36)\\
D^\alpha e_{12}(t) = \; & e_{13} + (x_{23} + x_{33})(\alpha_3 x_{13} - \alpha_2 x_{12})\\
& - \alpha_2 x_{13}(x_{22} + x_{32})
\end{aligned}$$

$$+5\cos 6t+6\cos 5t+u_{12}+u_{22} \tag{37}$$

$$\begin{aligned}D^{\alpha}e_{13}(t)= & -b_3e_{13}-b_1(y_{11}+y_{21}-\alpha_3x_{11}(x_{21}+x_{31})\\ & -\alpha_3x_{11}(x_{23}+x_{33}))\\ & -b_2(y_{12}+y_{22}-\alpha_3x_{12}(x_{23}+x_{33})\\ & -\alpha_3x_{13}(x_{22}+x_{32}))\\ & +b_3\alpha_3(x_{11}(x_{23}+x_{33})+b_4(y_{11}^2+y_{21}^2\\ & -\alpha_3x_{13}^2(x_{23}+x_{33})\\ & -\alpha_3x_{13}(x_{21}^2+x_{31}^2))+5(\sin 4t+\cos 4t)\\ & +4(\sin 5t+\cos 5t)\\ & +u_{13}+u_{23}\end{aligned} \tag{38}$$

To achieve the goal of stabilization of FO error system, the sliding mode surface (SMS) is considered as (Shao et al. 2016):

$$s_j(t)=e_{1j}(t),\ j=1,2,3 \tag{39}$$

Applying the Caputo derivative in (39), we have:

$$D^{\alpha}s_j(t)=D^{\alpha}e_{1j}(t),\ j=1,2,3 \tag{40}$$

Synchronization controller input is composed as:

$$\begin{aligned}u_{11}(t)+u_{21}(t)= & -e_{12}-(x_{22}+x_{32})(\alpha_2x_{12}-\alpha_1x_{11})\\ & +\alpha_1x_{12}(x_{21}+x_{31})\\ & -\eta_1s_1-\hat{K}_1\mathrm{sign}(s_1(t))-\hat{\Phi}_{11}-\hat{\Phi}_{21}\end{aligned} \tag{41}$$

$$\begin{aligned}u_{12}(t)+u_{22}(t)= & -e_{13}-(x_{23}+x_{33})(\alpha_3x_{13}-\alpha_2x_{12})\\ & +\alpha_2x_{13}(x_{22}+x_{32})\\ & -\eta_2s_2-\hat{K}_2\mathrm{sign}(s_2(t))-\hat{\Phi}_{12}-\hat{\Phi}_{22}\end{aligned} \tag{42}$$

$$\begin{aligned}u_{13}(t)+u_{23}(t)= & \ b_3e_{13}+b_1(y_{11}+y_{21}-\alpha_3x_{11}(x_{23}+x_{33})\\ & -\alpha_3x_{13}(x_{21}+x_{31}))\\ & +b_2(y_{12}+y_{22}-\alpha_3x_{12}(x_{23}+x_{33})\\ & -\alpha_3x_{13}(x_{22}+x_{32}))\\ & -b_3\alpha_3x_{13}(x_{23}+x_{33})-b_4(y_{11}^2\\ & +y_{21}^2-\alpha_3x_{13}^2(x_{23}+x_{33})\\ & -\alpha_3x_{13}(x_{21}^2+x_{31}^2))-\eta_3s_3\\ & -\hat{K}_3\mathrm{sign}(s_3(t))-\hat{\Phi}_{13}-\hat{\Phi}_{23}\end{aligned} \tag{43}$$

where $\mathrm{sign}(s)=\frac{|s|}{s}$ and $\eta_j>0$ are constants. $\hat{K}_j$ is estimated value of K_j updated by

$$\begin{aligned}D^{\alpha}\hat{K}_1&=\gamma_1(|s_1(t)|-\hat{K}_1)\\ D^{\alpha}\hat{K}_2&=\gamma_2(|s_2(t)|-\hat{K}_2)\\ D^{\alpha}\hat{K}_3&=\gamma_3(|s_3(t)|-\hat{K}_3)\end{aligned} \tag{44}$$

where $\gamma_j>0$, $j=1,2,3$ are constants. Using equations (41–42–43) in equations (36–37–38), the error dynamics can be written in the form.

$$\begin{aligned}D^{\alpha}e_{11}(t)= & -\eta_1s_1-\hat{K}_1\mathrm{sign}(s_1(t))+(\Phi_{11}+\Phi_{21})\\ & -(\hat{\Phi}_{11}+\hat{\Phi}_{11})\\ D^{\alpha}e_{12}(t)= & -\eta_2s_2-\hat{K}_2\mathrm{sign}(s_2(t))+(\Phi_{12}+\Phi_{22})\\ & -(\hat{\Phi}_{12}+\hat{\Phi}_{12})\\ D^{\alpha}e_{13}(t)= & -\eta_3s_3-\hat{K}_3\mathrm{sign}(s_3(t))+(\Phi_{13}+\Phi_{23})-(\hat{\Phi}_{13}\\ & +\hat{\Phi}_{23})\end{aligned} \tag{45}$$

The SMS $s_j(t)$ is bounded and stable if the hybrid projective compound combination synchronization controllers are calculated as (41–42–43) for the error dynamics (36–37–38). Thus, we get:

$$|s_j(t)|\le\ell,\ \text{where } \ell>0 \tag{46}$$

Using Eq. (39) and (46), we obtain

$$e_{1j}(t)\le\ell,\ j=1,2,3 \tag{47}$$

According to the above discussions, it is clear that if the SMS $s_j(t)$ is bounded, then the error $e_{1j}(t)$ is also bounded. Therefore, the nonlinear FODO based adaptive SMC synchronization technique for FO chaotic systems with unknown BD can be compiled in the following theorem and will be achieved by applying the FO Lyapunov method.

Theorem 1 (Shao et al. 2016; Khan and Tyagi 2017a, 2018) *For HPCCS error system (36–37–38) with $0<\alpha<1$, if the SMS is designed according to (39) and unknown external BD is approximated by using the developed nonlinear FODO (13), and (14), then HPCCS error $e(t)$ is eventually bounded and stable under the adaptive SMC as (41–42–43) and (44).*

Proof The Lyapunov function $V(t)$ is selected for the convergence of synchronization error $e(t)$ as:

$$\begin{aligned}V(t)= & \sum_{j=1}^{3}0.5s_j(t)^2+\sum_{j=1}^{3}0.5(\tilde{\Phi}_{1j}(t)+\tilde{\Phi}_{2j}(t))^2\\ & +\sum_{j=1}^{3}0.5\left(\frac{1}{\sqrt{\gamma_j}}(\hat{K}_j-K_j)\right)^2\end{aligned} \tag{48}$$

Using Property 2 in Eq. (48), we get

$$\begin{aligned}D^{\alpha}V(t)= & \ 0.5(\sum_{j=1}^{3}D^{\alpha}s_j(t)^2+\sum_{j=1}^{3}D^{\alpha}(\tilde{\Phi}_{1j}(t)+\tilde{\Phi}_{2j}(t))^2\\ & +\sum_{j=1}^{3}D^{\alpha}\left(\frac{1}{\sqrt{\gamma_j}}(\hat{K}_j-K_j)\right)^2\end{aligned} \tag{49}$$

Substituting $\tilde{K}_j = \hat{K}_j - K_j$ and Lemma 1 in Eq. (49), we have

$$D^\alpha V(t) \leq \sum_{j=1}^{3} 0.5 s_j(t) D^\alpha s_j(t) + \sum_{j=1}^{3} 0.5 D^\alpha (\tilde{\Phi}_{1j}(t) + \tilde{\Phi}_{2j}(t))^2 + \sum_{j=1}^{3} \frac{1}{\sqrt{\gamma_j}} \tilde{K}_j D^\alpha \left(\frac{1}{\sqrt{\gamma_j}} \tilde{K}_j \right) \tag{50}$$

On applying Property 2 in Eq. (50), we achieve

$$D^\alpha V(t) \leq \sum_{j=1}^{3} s_j(t) D^\alpha s_j(t) + \sum_{j=1}^{3} 0.5 D^\alpha (\tilde{\Phi}_{1j}(t) + \tilde{\Phi}_{2j}(t))^2 + \sum_{j=1}^{3} \frac{1}{\gamma_j} \tilde{K}_j D^\alpha \tilde{K}_j \tag{51}$$

Using Eq. (40) and substituting the value of Eq. (45) in (51), we obtain

$$D^\alpha V(t) \leq \sum_{j=1}^{3} s_j(t)(-\eta_j s_j + \tilde{\Phi}_{1j}(t) + \tilde{\Phi}_{2j}(t) - \tilde{K}_j \operatorname{sign}(s_j(t))) + \sum_{j=1}^{3} 0.5 D^\alpha (\tilde{\Phi}_{1j}(t) + \tilde{\Phi}_{2j}(t))^2 + \sum_{j=1}^{3} \frac{1}{\gamma_j} \tilde{K}_j D^\alpha \tilde{K}_j \tag{52}$$

Applying Property 1 and using $\tilde{K}_j = \hat{K}_j - K_j$ $(j = 1, 2, 3)$, we get

$$D^\alpha \tilde{K}_j = D^\alpha \hat{K}_j. \tag{53}$$

where K_j is a constant parameter.

In view of Eqs. (41) and (53), we obtain

$$\begin{aligned}\sum_{j=1}^{3} \frac{1}{\gamma_j} \tilde{K}_j D^\alpha \tilde{K}_j &= \sum_{j=1}^{3} \tilde{K}_j (|s_j(t)| - \hat{K}_j) \\ &= \sum_{j=1}^{3} \tilde{K}_j |s_j(t)| - \sum_{j=1}^{3} \tilde{K}_j (\tilde{K}_j + K_j) \\ &= \sum_{j=1}^{3} \tilde{K}_j |s_j(t)| - \frac{1}{2} \sum_{j=1}^{3} \tilde{K}_j^2 - \frac{1}{2} \sum_{j=1}^{3} \tilde{K}_j^2 - \sum_{j=1}^{3} \tilde{K}_j K_j \\ &\leq \sum_{j=1}^{3} \tilde{K}_j |s_j(t)| - \frac{1}{2} \sum_{j=1}^{3} \tilde{K}_j^2 + \frac{1}{2} \sum_{j=1}^{3} K_j{}^2\end{aligned} \tag{54}$$

After substituting the above inequality equation (54) in (52), we obtain

$$D^\alpha V(t) \leq \sum_{j=1}^{3} s_j(t)(-\eta_j s_j(t) + \tilde{\Phi}_{1j}(t) + \tilde{\Phi}_{2j}(t) - \hat{K}_j \operatorname{sign}(s_j(t))) + \sum_{j=1}^{3} 0.5 D^\alpha (\tilde{\Phi}_{1j}(t) + \tilde{\Phi}_{2j}(t))^2 + \sum_{j=1}^{3} \tilde{K}_j |s_j(t)| - \sum_{j=1}^{3} 0.5 \tilde{K}_j{}^2 + \sum_{j=1}^{3} 0.5 K_j{}^2 \tag{55}$$

We notice that Eq. (55) can be rewritten as:

$$D^\alpha V(t) \leq -\sum_{j=1}^{3} \eta_j s_j^2(t) + \sum_{j=1}^{3} |s_j(t)||\tilde{\Phi}_{1j} + \tilde{\Phi}_{2j}| - \sum_{j=1}^{3} \hat{K}_j |s_j(t)| - \sum_{j=1}^{3} 0.5 \tilde{K}_j{}^2 + \sum_{j=1}^{3} 0.5 K_j{}^2 + \sum_{j=1}^{3} \tilde{K}_j |s_j(t)| + \sum_{j=1}^{3} 0.5 D^\alpha (\tilde{\Phi}_{1j}(t) + \tilde{\Phi}_{2j}(t))^2 \tag{56}$$

We observe that

$$\sum_{j=1}^{3} \tilde{K}_j |s_j(t)| - \sum_{i=j}^{3} \hat{K}_j |s_j(t)| = -\sum_{j=1}^{3} K_j |s_j(t)| \tag{57}$$

Using Eq. (57) and inequality $|\Phi_{1j}(t) + \Phi_{2j}(t)| \leq K_j$ in Eq. (56), we obtain

$$D^\alpha V(t) \leq -\sum_{j=1}^{3} \eta_j s_j^2(t) - \sum_{j=1}^{3} 0.5 \tilde{K}_j{}^2 + \sum_{j=1}^{3} 0.5 K_j{}^2 + \sum_{j=1}^{3} 0.5 D^\alpha (\tilde{\Phi}_{1j}(t) + \tilde{\Phi}_{2j}(t))^2 \tag{58}$$

Considering Eq. (24) in last term of Eq. (58), we get

$$\sum_{j=1}^{3} 0.5D^{\alpha}(\tilde{\Phi}_{1j}(t)+\tilde{\Phi}_{2j}(t))^2 = \sum_{j=1}^{3} -(\sigma_j-0.5)(\tilde{\Phi}_{1j}(t)+\tilde{\Phi}_{2j}(t))^2+\sum_{j=1}^{3} 0.5\xi_j^2 \quad (59)$$

Substituting Eq. (59) in (58), we have

$$\begin{aligned} D^{\alpha}V(t) &\leq \sum_{j=1}^{3} -\eta_j s_j^2(t) - \sum_{j=1}^{3} 0.5\tilde{K_j}^2 + \sum_{j=1}^{3} 0.5K_j^2 \\ &\quad + \sum_{j=1}^{3} -(\sigma_j-0.5)(\tilde{\Phi}_{1j}(t)+\tilde{\Phi}_{2j}(t))^2+\sum_{j=1}^{3} 0.5\xi_j^2 \\ &\leq -C_2V(t)+C_3 \end{aligned} \quad (60)$$

where $C_2 = \min(2\eta_j, 1, 2\eta_j - 1)$ and $C_3 = \sum_{j=1}^{3} 0.5\xi_j^2 + \sum_{j=1}^{3} 0.5K_j^2$.

On selecting the parameters $\eta_j > 0$ and $\sigma_j > 0.5$, the HPCCS error is bounded. Using Lemma 2 in Eq. (60), we get

$$\begin{aligned} |V(t)| &\leq \frac{2C_3}{C_2} \\ &= \frac{\sum_{j=1}^{3}\xi_j^2+\sum_{j=1}^{3}K_j^2}{C_2}. \end{aligned} \quad (61)$$

This implies that

$$\|s(t)\| \leq \sqrt{\frac{2(\sum_{j=1}^{3}\xi_j^2+\sum_{j=1}^{3}K_j^2)}{C_2}} \quad (62)$$

From Eq. (62), when $t \to \infty$ the error $e(t)$ and SMS $s(t)$ are bounded. Thus, from Lemma 2, the error dynamics (36–37–38) are bounded and stable. Hence, HPCCS between three master systems (29–30–31) and two slave systems (32–33) is attained successfully. This completes the proof. □

7 Numerical Simulations

The initial states of the master systems (29–30–31) and slave systems (32–33) are, respectively, taken as $(x_{11}, x_{12}, x_{13}) = (-0.3, 0.1, -0.2)$, $(x_{21}, x_{22}, x_{23}) = (-0.5, 0.4, -0.3)$, $(x_{31}, x_{32}, x_{33}) = (-0.7, 0.7, -0.4)$, $(y_{11}, y_{12}, y_{13}) = (-0.9, 1, -0.5)$, $(y_{21}, y_{22}, y_{23}) = (-1.1, 1.3, -0.6)$, and parameter values are $(b_1, b_2, b_3, b_4) = (1.0, 1.1, 0.4, 1.0)$ and $\alpha = 0.99$ with step size h=0.001.Therefore , the initial states for the error systems are $(e_{11}, e_{12}, e_{13}) = (3.9, 2.6, -0.6)$ and $(\hat{\Theta}_1(0), \hat{\Theta}_2(0), \hat{\Theta}_3(0)) = (.1, .1, .1)$ and $(\hat{K}_1(0), \hat{K}_2(0), \hat{K}_3(0)) = (.5, .5, .5)$, the designed parameters $(\sigma_1, \sigma_2, \sigma_3) = (120, 120, 120)$, $(\gamma_1, \gamma_2, \gamma_3) = (.1, .1, .1)$,$(\eta_1, \eta_2, \eta_3) = (90, 90, 90)$, $\alpha_1 = 1, \alpha_2 = -1, \alpha_3 = -1$, disturbance assumed as $\Phi_{11} = \text{sign}(\sin(t+1))$, $\Phi_{12} = 5\cos 6t$, $\Phi_{13} = 5(\sin 4t + \cos 4t)$, $\Phi_{21} = \text{sign}\sin(2(t+1))$, $\Phi_{22} = 6\cos 5t$, $\Phi_{23} = 4(\sin 5t + \cos 5t)$. Phase portrait of Genesio–Tesi FO systems for $\alpha = 0.99$ in 2-D and 3-D is depicted in Fig 1. HPCCS between signals $(y_{11} + y_{21})$ and $(x_{11})(x_{21} + x_{31})$ and signals $(y_{12} + y_{22})$ and $x_{12}(x_{22}+x_{32})$ and $(y_{13}+y_{23})$ and $x_{13}(x_{23}+x_{33})$ is shown in Fig 2(a), 2(b), and 2(c), respectively. Synchronization errors $e_{11}(t), e_{12}(t), e_{13}(t)$ are converging to zero in Fig. 2d. Time trajectories of the BD $\Phi_{1j} + \Phi_{2j}$ and disturbance estimation $\hat{\Phi}_{1j} + \hat{\Phi}_{2j}$ are illustrated in Fig. 3a–c. Disturbance estimation errors converging to zero as shown in Fig. 4a–c. Control inputs are displayed in Fig. 5a–c. Hence, simulation results display that the suggested HPCCS scheme is suitable for considered FO chaotic systems disturbances.

7.1 Comparison of Given HPCCS Scheme with Previous Published Literature

Khan and Tyagi (2017a) studied FODO-based adaptive SMC synchronization in the Genesio–Tesi system. They attained error synchronization in approximately the minimum time $t = 0.04$ when parameters (b_1, b_2, b_3, b_4)=(1.0, 1.1, −0.232, 1.0) and order $\alpha = 0.8$ are taken is illustrated in Fig. 6b, whereas in the present scheme, we had considered HPCCS by a FODO-based adaptive SMC in which we have taken five identical Genesio–Tesi systems (3 masters, two slaves). In our studies, the error has been synchronized approx at $t = 0.025$, as shown in Fig. 6a with the same parameters and order as given above. Comparatively, the present scheme takes less time to synchronize error trajectories. Till date, the above-discussed approach is not considered by any other researchers. This shows the novelty of our results.

8 A Secure Communication Scheme Based on HPCCS

(Xiangjun et al. 2012; Khan and Nigar 2019) An essential application of HPCCS is secure communication. The secure communication system includes the evolution of a signal that carries the information that is to endure undetectable by interceptors within a transmitter signal. We can guarantee the protection of this information by injecting it into a chaotic signal that is transmitted to a designated receiver that would be able to recognize and collect the data from the chaotic message signal. In a masking method, the message information signal $I(t) = I_1(t)(I_2(t) + I_3(t))$ is added to the chaotic master information signals and transmitted sig-

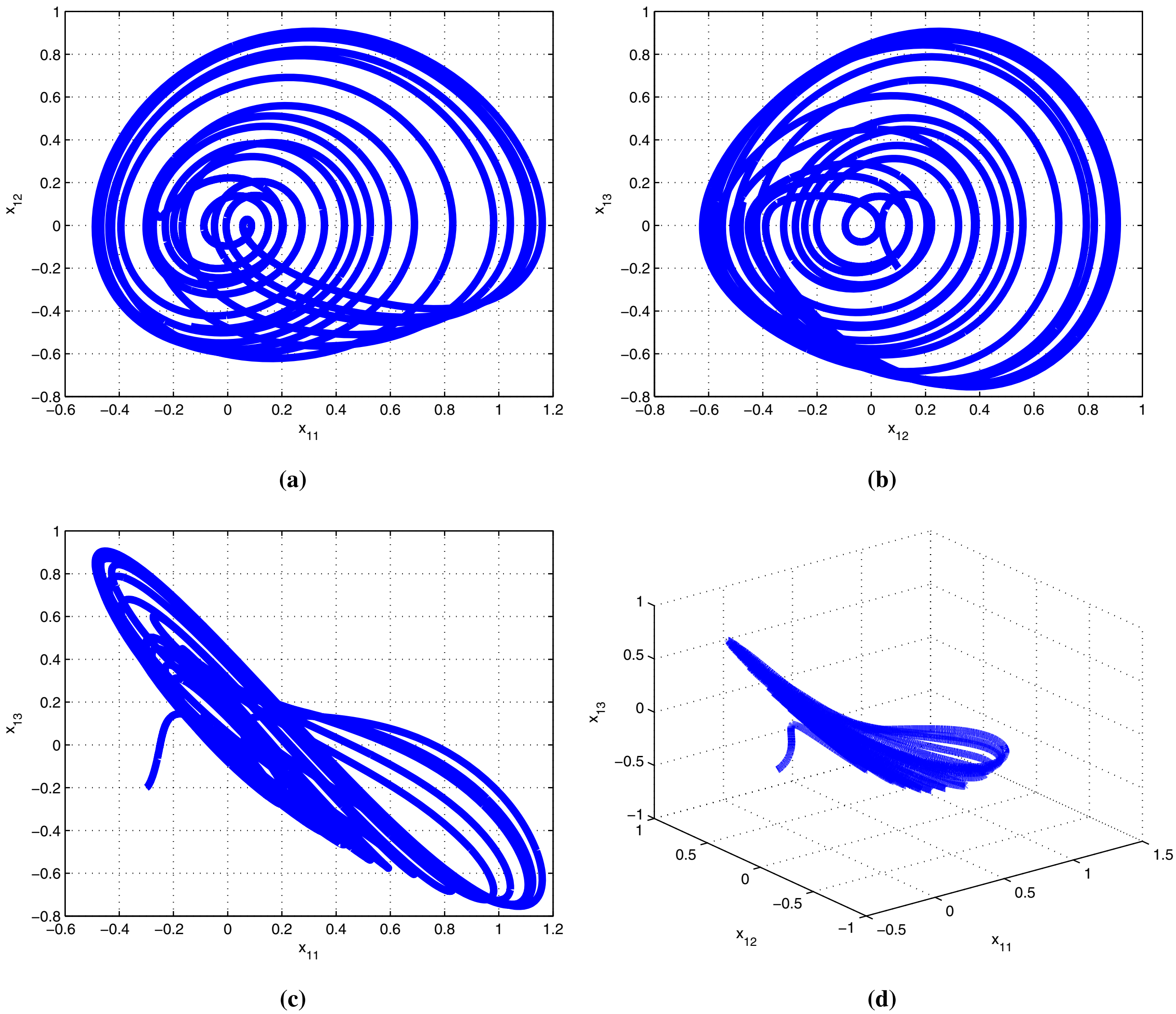

Fig. 1 Phase portrait of Genesio–Tesi system. **a** $x_{11}(t)$–$x_{12}(t)$ plane, **b** $x_{12}(t)$–$x_{13}(t)$ plane, **c** $x_{11}(t)$–$x_{13}(t)$ plane, **d** $x_{11}(t)$–$x_{12}(t)$–$x_{13}(t)$ space

nal $P(t) = I(t) + x_{12} * (x_{22} + x_{32})$ is added to a slave signal. The recovered signal $\hat{I}(t)$ is obtained when the chaotic signal $y_{12} + y_{22}$ is subtracted from the transmitted signal, i.e., $\hat{I}(t) = P(t) - \alpha_2 * (y_{12} + y_{22})$. We select the message information signals in the form of $I_1(t) = \sin 2t$, $I_2(t) = 2\sin 2t$ and $I_3(t) = 4\sin 2t$. The message signal $I(t) = 6\sin 2t^2$ and transmitted signal $P(t)$ are demonstrated in Fig. 7a, b, respectively. Figure 7c illustrates the recovered signal $\tilde{I}(t)$. The error among the initial message signal and the recovered message one is displayed in Fig. 7d. From Fig. 7d, it is simple to see that the message signal $I(t)$ is recovered correctly after a little transient.

9 Conclusion

In this paper, we use the Caputo fractional-order derivative because it has the significant benefits that it allows the initial and boundary conditions in the formulation of the problem, which is the essential requirement of applied problems. So HPCCS has been achieved in five identical FO chaotic systems using Caputo FO, wherein three master systems and two slave systems are considered. In this research work, a nonlinear FODO-based adaptive SMC HPCCS scheme has been investigated with unknown external BD of identical FO Genesio–Tesi system. The considered adaptive SMC technique verifies that the states of three master systems and two slave systems with unknown BD are synchronized rapidly by using the Lyapunov stability criterion. As the upper bound of disturbance is unknown, an adaptive SMC technique has been

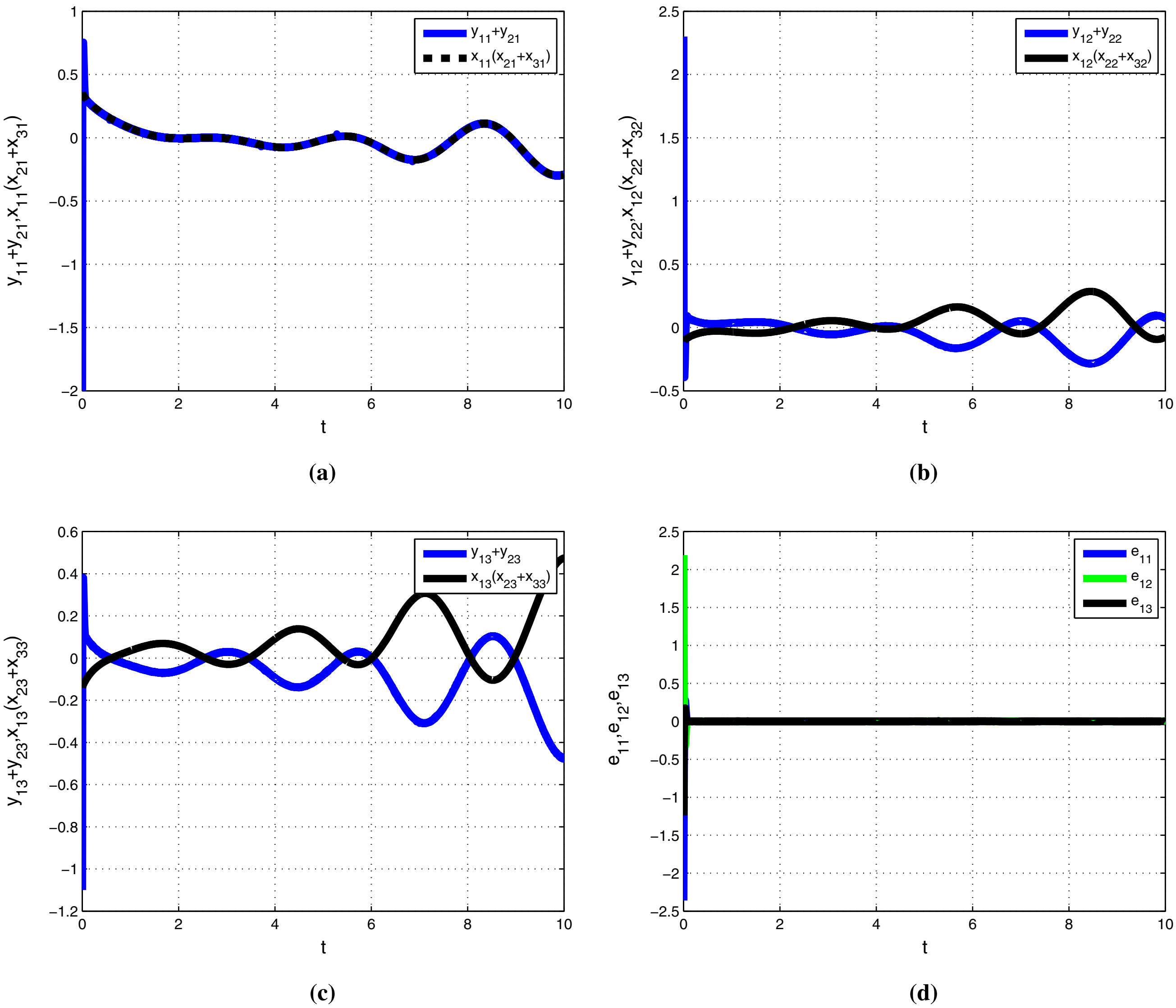

Fig. 2 Hybrid projective compound combination synchronized as under FO adaptive sliding mode control. **a** Between $y_{11}(t) + y_{21}(t)$ and $x_{11}(t)(x_{21}(t) + x_{31}(t))$. **b** Between $y_{12}(t) + y_{22}(t)$ and $x_{12}(t)(x_{22}(t) + x_{32}(t))$. **c** Between $y_{13}(t) + y_{23}(t)$ and $x_{13}(t)(x_{23}(t) + x_{33}(t))$. **d** The error state curve between systems(3 master, 1 slave) tends to 0

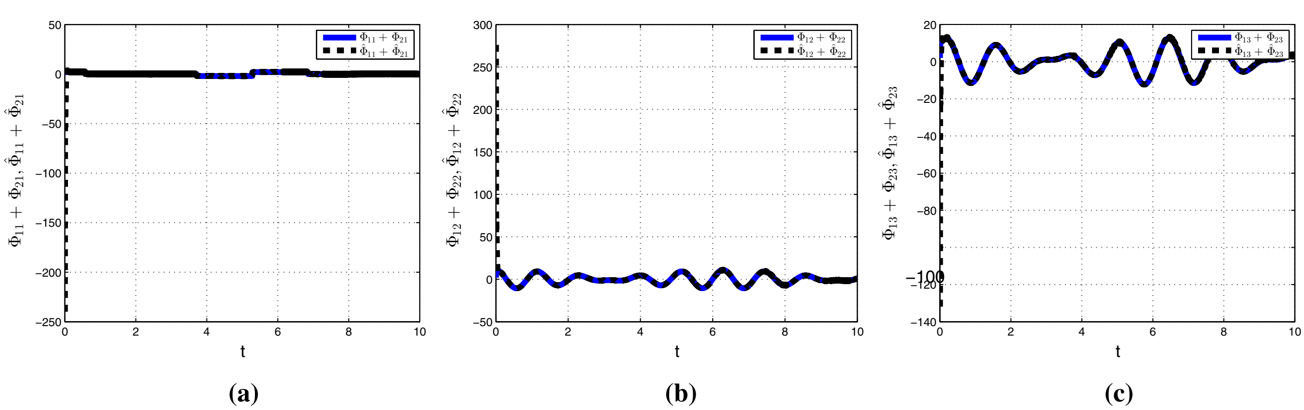

Fig. 3 Time trajectories of BD and disturbance estimation synchronization results. **a** $\Phi_{11}(t) + \Phi_{21}(t)$ and $\hat{\Phi}_{11}(t) + \hat{\Phi}_{21}(t)$, **b** $\Phi_{12}(t) + \Phi_{22}(t)$ and $\hat{\Phi}_{12}(t) + \hat{\Phi}_{22}(t)$, **c** $\Phi_{13}(t) + \Phi_{23}(t)$ and $\hat{\Phi}_{13}(t) + \hat{\Phi}_{23}(t)$

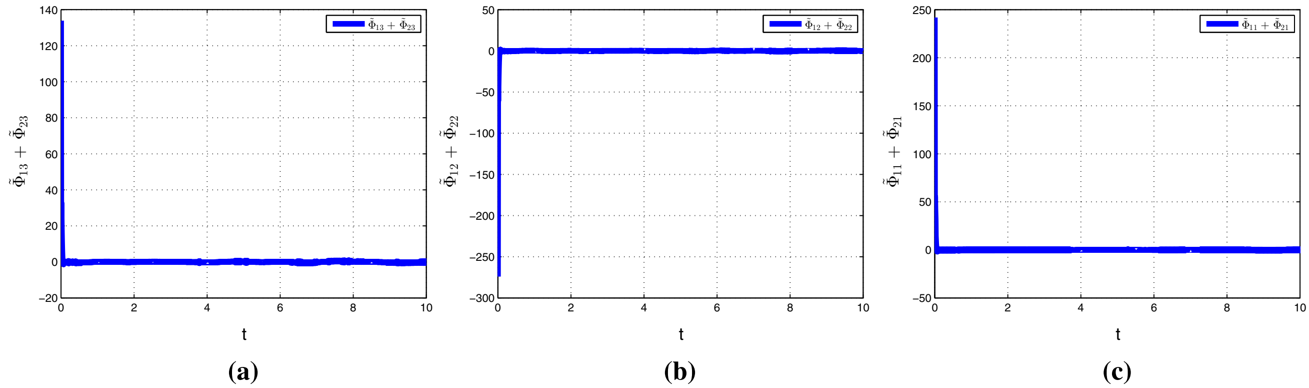

Fig. 4 Observer errors. **a** $\tilde{\Phi}_{11}(t)+\tilde{\Phi}_{21}(t)$, **b** $\tilde{\Phi}_{12}(t)+\tilde{\Phi}_{22}(t)$, **c** $\tilde{\Phi}_{13}(t)+\tilde{\Phi}_{23}(t)$

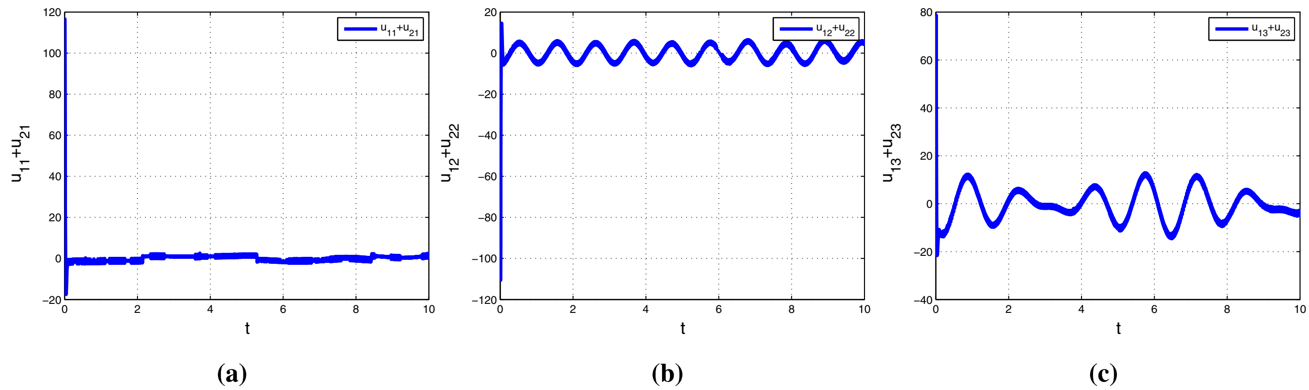

Fig. 5 Control inputs signals. **a** $u_{11}+u_{21}$, **b** $u_{12}+u_{22}$, **c** $u_{13}+u_{23}$

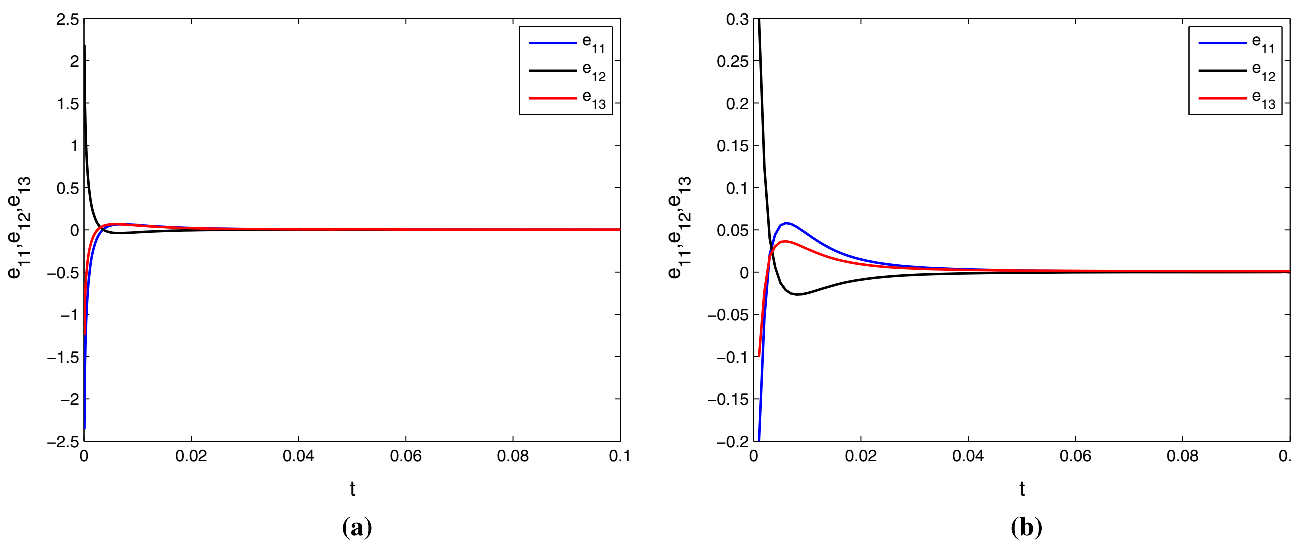

Fig. 6 **a** HPCCS error using adaptive SMC, **b** synchronization between two identical Genesio–Tesi system using adaptive SMC

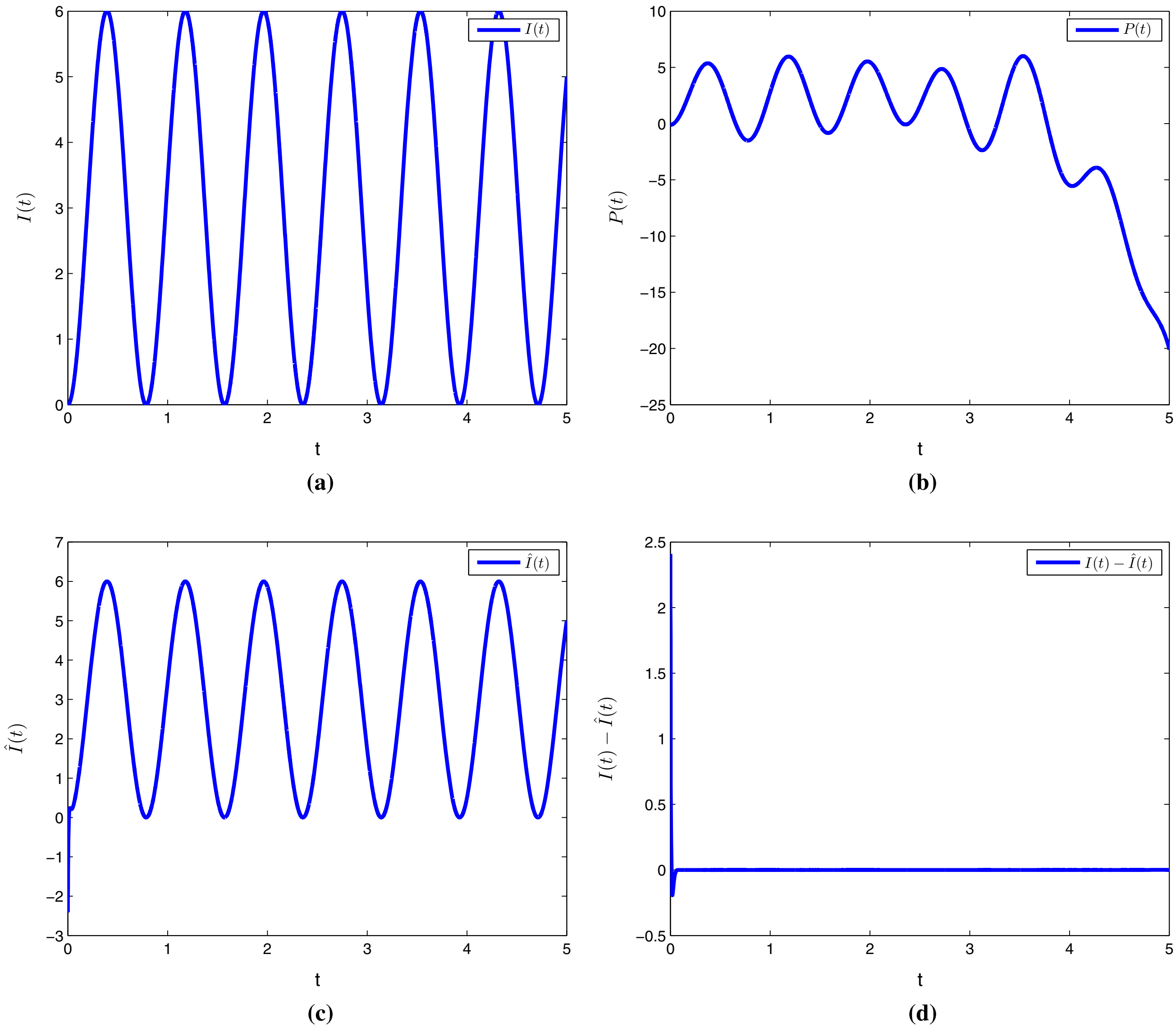

Fig. 7 **a** The information message signal $I(t)$; **b** transmitted signal $P(t)$; **c** recovered signal $\hat{I}(t)$; **d** error message signal $I(t) - \hat{I}(t)$

proposed to estimate them. The designed nonlinear FODO realizes disturbance approximation error. In our studies, the time taken by the synchronization error converging to zero is less in comparison with earlier published results. We have considered five identical systems (3 masters, 2 slaves). It is noticed that our technique is efficient and effective. Also, with the growing requirement for security of transmission, we design a proper application in the area of secure communication. Further, in the future direction, we can study on systems interrupted by model uncertainties and disturbance in FO complex chaotic systems with matched disturbance and mismatched disturbance using the adaptive SMC. To the author's knowledge, the study of HPCCS using adaptive sliding mode control with BD has not yet been explored.

References

Aguila-Camacho, N., Duarte-Mermoud, M. A., & Gallegos, J. A. (2014). Lyapunov functions for fractional order systems. *Communications in Nonlinear Science and Numerical Simulation*, *19*(9), 2951–2957.

Bhalekar, S. (2014). Synchronization of non-identical fractional order hyperchaotic systems using active control. *World Journal of Modelling and Simulation*, *10*(1), 60–68.

Chen, M., Chen, W. H., & Wu, Q. X. (2014). Adaptive fuzzy tracking control for a class of uncertain MIMO nonlinear systems using disturbance observer. *Science China Information Sciences*, *57*(1), 1–13.

Chen, M., & Han, Z. (2003). Controlling and synchronizing chaotic genesio system via nonlinear feedback control. *Chaos, Solitons & Fractals*, *17*(4), 709–716.

Chen, M., & Jing, Y. (2015). Disturbance observer-based adaptive sliding mode control for near-space vehicles. *Nonlinear Dynamics*, *82*(4), 1671–1682.

Das, S., & Pan, I. (2012). Fractional order statistical signal processing. In *Fractional order signal processing* (pp. 83–96). Springer.

Delavari, H., & Mohadeszadeh, M. (2018). Hybrid complex projective synchronization of complex chaotic systems using active control technique with nonlinearity in the control input. *Journal of Control Engineering and Applied Informatics*, *20*(1), 67–74.

Ding, Z., & Shen, Y. (2016). Projective synchronization of nonidentical fractional-order neural networks based on sliding mode controller. *Neural Networks*, *76*, 97–105.

Genesio, R., & Tesi, A. (1992). Harmonic balance methods for the analysis of chaotic dynamics in nonlinear systems. *Automatica*, *28*(3), 531–548.

Guo, B.-Z., & Jin, F.-F. (2013). The active disturbance rejection and sliding mode control approach to the stabilization of the Euler–Bernoulli beam equation with boundary input disturbance. *Automatica*, *49*(9), 2911–2918.

Guo, B.-Z., & Liu, J.-J. (2014). Sliding mode control and active disturbance rejection control to the stabilization of one-dimensional Schrödinger equation subject to boundary control matched disturbance. *International Journal of Robust and Nonlinear Control*, *24*(16), 2194–2212.

Guo, B.-Z., & Zhou, H.-C. (2014). The active disturbance rejection control to stabilization for multi-dimensional wave equation with boundary control matched disturbance. *IEEE Transactions on Automatic Control*, *60*(1), 143–157.

Han, J. (2009). From PID to active disturbance rejection control. *IEEE Transactions on Industrial Electronics*, *56*(3), 900–906.

Khan, A., & Nigar, U. (2019). Adaptive sliding mode disturbance observer control base synchronization in a class of fractional order Chua's chaotic system. *Bloomsbury India*, 107.

Khan, A., Budhraja, M., & Ibraheem, A. (2018). Multi-switching dual compound synchronization of chaotic systems. *Chinese Journal of Physics*, *56*(1), 171–179.

Khan, A., Khattar, D., & Prajapati, N. (2017). Dual combination combination multi switching synchronization of eight chaotic systems. *Chinese Journal of Physics*, *55*(4), 1209–1218.

Khan, A., & Singh, S. (2018a). Generalization of combination-combination synchronization of n-dimensional time-delay chaotic system via robust adaptive sliding mode control. *Mathematical Methods in the Applied Sciences*, *41*(9), 3356–3369.

Khan, A., & Singh, S. (2018b). Chaotic analysis and combination-combination synchronization of a novel hyperchaotic system without any equilibria. *Chinese Journal of Physics*, *56*(1), 238–251.

Khan, A., & Tyagi, A. (2017a). Fractional order disturbance observer based adaptive sliding mode synchronization of commensurate fractional order Genesio–Tesi system. *AEU-International Journal of Electronics and Communications*, *82*, 346–357.

Khan, A., & Tyagi, A. (2017b). Analysis and hyper-chaos control of a new 4-D hyper-chaotic system by using optimal and adaptive control design. *International Journal of Dynamics and Control*, *5*(4), 1147–1155.

Khan, A., & Tyagi, A. (2018). Fractional order disturbance observer based adaptive sliding mode hybrid projective synchronization of fractional order Newton–Leipnik chaotic system. *International Journal of Dynamics and Control*, *6*(3), 1136–1149.

Li, C., & Deng, W. (2007). Remarks on fractional derivatives. *Applied Mathematics and Computation*, *187*(2), 777–784.

Li, L., & Sun, Y. (2015). Adaptive fuzzy control for nonlinear fractional-order uncertain systems with unknown uncertainties and external disturbance. *Entropy*, *17*(8), 5580–5592.

Li, G.-H., & Zhou, S.-P. (2007). Anti-synchronization in different chaotic systems. *Chaos, Solitons & Fractals*, *32*(2), 516–520.

Lorenz, E. N. (1963). Deterministic nonperiodic flow. *Journal of the Atmospheric Sciences*, *20*(2), 130–141.

Mahmoud, G. M., & Mahmoud, E. E. (2010). Complete synchronization of chaotic complex nonlinear systems with uncertain parameters. *Nonlinear Dynamics*, *62*(4), 875–882.

Manfeng, H., Yang, Y., Zhenyuan, X., & Guo, L. (2008). Hybrid projective synchronization in a chaotic complex nonlinear system. *Mathematics and Computers in Simulation*, *79*(3), 449–457.

Mofid, O., Mobayen, S., & Khooban, M.-H. (2019). Sliding mode disturbance observer control based on adaptive synchronization in a class of fractional-order chaotic systems. *International Journal of Adaptive Control and Signal Processing*, *33*(3), 462–474.

Moghadasianx, M., Betin, F., Yazidi, A., Capolino, G. A., & Kianinezhad, R. (2012). Position control of six-phase induction machine using fractional-order controller. In *2012 XXth international conference on electrical machines* (pp. 1048–1054). IEEE.

Ojo, K. S., Njah, A. N., & Olusola, O. I. (2015). Compound-combination synchronization of chaos in identical and different orders chaotic systems. *Archives of Control Sciences*, *25*(4), 463–490.

Park, J. H., Lee, S. M., & Kwon, O. M. (2007). Adaptive synchronization of Genesio–Tesi chaotic system via a novel feedback control. *Physics Letters A*, *371*(4), 263–270.

Pashaei, S., & Badamchizadeh, M. (2016). A new fractional-order sliding mode controller via a nonlinear disturbance observer for a class of dynamical systems with mismatched disturbances. *ISA Transactions*, *63*, 39–48.

Pecora, L. M., & Carroll, T. L. (1990). Synchronization in chaotic systems. *Physical Review Letters*, *64*(8), 821.

Podlubny, I. (1998). Fractional derivatives and integrals. *Fractional Differential Equations*, *198*, 41–117.

Podlubny, I. (1999). Fractional-order systems and PI/sup/spl lambda//D/sup/spl mu//-controllers. *IEEE Transactions on Automatic Control*, *44*(1), 208–214.

Prajapati, N., Khan, A., & Khattar, D. (2018). On multi switching compound synchronization of non identical chaotic systems. *Chinese Journal of Physics*, *56*(4), 1656–1666.

Russell, M. (1967). Henri poincaré and the quantum theory. *Isis*, *58*(1), 37–55.

Sambas, A., Vaidyanathan, S., Mamat, M., Sanjaya, W. S. M., & Prastio, R. P. (2016). Design, analysis of the Genesio–Tesi chaotic system and its electronic experimental implementation. *International Journal of Control Theory and Applications*, *9*(1), 141–149.

Shao, S., Chen, M., & Yan, X. (2016). Adaptive sliding mode synchronization for a class of fractional-order chaotic systems with disturbance. *Nonlinear Dynamics*, *83*(4), 1855–1866.

Shi, S.-L., Li, J.-X., & Fang, Y.-M. (2019). Fractional-disturbance-observer-based sliding mode control for fractional order system with matched and mismatched disturbances. *International Journal of Control, Automation and Systems*, *17*(5), 1184–1190.

Singh, S., Azar, A. T., Ouannas, A., Zhu, Q., Zhang, W., & Na, J. (2017a). Sliding mode control technique for multi-switching synchronization of chaotic systems. In *2017 9th international conference on modelling, identification and control (ICMIC)* (pp. 880–885). IEEE.

Singh, A. K., Yadav, V. K., & Das, S. (2017b). Dual combination synchronization of the fractional order complex chaotic systems. *Journal of Computational and Nonlinear Dynamics*, *12*(1), 011017.

Soukkou, A., Boukabou, A., & Goutas, A. (2018). Generalized fractional-order time-delayed feedback control and synchronization designs for a class of fractional-order chaotic systems. *International Journal of General Systems*, *47*(7), 679–713.

Sun, J., Wang, Y., Wang, Y., Cui, G., & Shen, Y. (2016). Compound-combination synchronization of five chaotic systems via nonlinear control. *Optik-International Journal for Light and Electron Optics*, *127*(8), 4136–4143.

Sun, J., Yin, Q., & Shen, Y. (2014). Compound synchronization for four chaotic systems of integer order and fractional order. *EPL (Europhysics Letters)*, *106*(4), 40005.

Vaidyanathan, S. (2015). Adaptive biological control of generalized Lotka–Volterra three-species biological system. *International Journal of PharmTech Research*, *8*(4), 622–631.

Vaidyanathan, S. (2016). Hybrid synchronization of the generalized Lotka–Volterra three-species biological systems via adaptive control. *International Journal of PharmTech Research*, *9*(1), 179–192.

Vaidyanathan, S., & Azar, A. T. (2015). Hybrid synchronization of identical chaotic systems using sliding mode control and an application to vaidyanathan chaotic systems. In *Advances and applications in sliding mode control systems* (pp. 549–569). Springer.

Vaidyanathan, S., & Azar, A. T. (2016). Generalized projective synchronization of a novel hyperchaotic four-wing system via adaptive control method. In *Advances in chaos theory and intelligent control* (pp. 275–296). Springer.

Vaidyanathan, S., & Sampath, S. (2012). Anti-synchronization of four-wing chaotic systems via sliding mode control. *International Journal of Automation and Computing*, *9*(3), 274–279.

Xiangjun, W., Wang, H., & Hongtao, L. (2012). Modified generalized projective synchronization of a new fractional-order hyperchaotic system and its application to secure communication. *Nonlinear Analysis: Real World Applications*, *13*(3), 1441–1450.

Yang, J., Li, S., & Xinghuo, Y. (2012). Sliding-mode control for systems with mismatched uncertainties via a disturbance observer. *IEEE Transactions on Industrial Electronics*, *60*(1), 160–169.

Yildirim, S., & Eski, I. (2010). Noise analysis of robot manipulator using neural networks. *Robotics and Computer-Integrated Manufacturing*, *26*(4), 282–290.

Yuanqing, W., & Renquan, L. (2017). Output synchronization and *l*_2-gain analysis for network systems. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, *48*(12), 2105–2114.

Yuanqing, W., Renquan, L., Shi, P., Hongye, S., & Zheng-Guang, W. (2017). Sampled-data synchronization of complex networks with partial couplings and T–S fuzzy nodes. *IEEE Transactions on Fuzzy Systems*, *26*(2), 782–793.

Zhang, B., & Deng, F. (2014). Double-compound synchronization of six memristor-based Lorenz systems. *Nonlinear Dynamics*, *77*(4), 1519–1530.

Zhang, S., Kong, L., Qi, S., Jing, P., He, W., & Bin, X. (2019). Adaptive neural control of unknown non-affine nonlinear systems with input deadzone and unknown disturbance. *Nonlinear Dynamics*, *95*(2), 1283–1299.