




# Approximate Models of Singularly Perturbed Time-Varying Systems: A Bond Graph Approach

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## Abstract

A bond graph model in an integral causality assignment (BGI) for a singularly perturbed linear time-varying (LTV) system is proposed. The LTV constitutive relations of the elements and MTF and MGY elements modulated by LTV functions of the BGI are considered. A new bond graph called singularly perturbed varying bond graph (SPVBG) for determining the quasi-steady-state model is presented. This SPVBG has the property that the storage elements for the slow and fast dynamics have an integral and derivative causality assignment, respectively. In order to apply the proposed methodology, a case study of an electromechanical system is modelled by bond graphs and approximated models are obtained. Finally, simulation results for the exact and approximated solutions are shown.

**Keywords** Bond graph · Singular perturbations · LTV systems · Approximate models

## 1 Introduction

The linear time-varying (LTV) systems have been receiving increasing attention by system and control community, since they appear frequently in practical engineering areas such as aerospace control systems (D'Angelo 1970). While important, LTV systems are very hard to investigate despite the fundamental stability problem. However, numerous important progresses, including but not limited to Okano et al. (2006), Desoer (1969), Amato et al. (1993) and Jetto and Orsini (2009), have been achieved through the effort of

researchers. They more or less all rely on the use of a linear time-invariant plant as an approximation of the LTV system and ensuring that the influence of the approximation is not excessive. The main advantage of this frozen time method is the possibility of exploiting the great deal of tools which have been developed for linear time-invariant (LTI) systems (Yao et al. 2012).

Methods for exact analysis of LTV systems are relatively few and tend to be either extremely difficult to apply or limited in application to a small class of systems. The fundamental difficulty underlying the analysis of linear time-varying systems becomes apparent when one reviews some of the common methods for exact analysis. However, various methods for obtaining exact solutions as not complementary to each other and difficulty in obtaining exact solutions with one method are usually sufficient indication that difficulty will be encountered with another (D'Angelo 1970).

The stability for LTV systems is reported in Pradeep and Shrivastava (1988). The possibility problem for LTV systems is established in Forbes and Damaren (2010). The structural controllability of LTI systems is extended to LTV systems (Hartung et al. 2013). A tracking control design for LTV systems is proposed in Chen (1998). The stability and stabilizability giving the necessary and sufficient conditions of LTV systems are proposed in Amato et al. (2010).

A novel approach to the identification of LTV systems based on the concept of duality is introduced in Rapisarda

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(2018). A methodology for the synthesis of time-varying output feedback gains that guarantee the asymptotic uniform stability of continuous-time LTV systems is presented in Agultari and Peres (2019). A characterization of strong structural input and state observability for LTV systems is studied in Gracy et al. (2017). A new methodology to determine the controllability and the observability matrices of linear time-varying systems modelled by bond graphs is proposed in Frih et al. (2018).

Nevertheless, the realistic representation of many systems that have the presence of some parasitic parameters, such as small time constants, resistances, inductances, capacitances, moments of inertia and Reynolds number, is often the source for the increased order and stiffness of these systems. The stiffness, attributed to the simultaneous occurrence of slow and fast phenomena, gives rise to time scales. The systems in which the suppression of a small parameter is responsible for the degeneration (or reduction) of dimension (or order) of the system are labelled as singularly perturbed systems, which are a special representation of the general class of time scale systems (Subbaram Naidu and Calise 2001).

There are several references of singular perturbations methods; some of them are Kokotovic et al. (1986) and Khalil (2002). Also, singular perturbation and time scale problems in aerospace systems are described in Subbaram Naidu and Calise (2001). A common approximation used in the analysis of power systems is the neglect of the dynamic saliency in synchronous machines. In Pekarek et al. (2002), it is shown that eliminating the error accomplished with neglecting dynamic saliency can be accomplished with the addition of singular perturbation into the machine model. Verhulst (2007) describes geometric singular perturbation theory and to review a number of non-hyperbolic cases.

The decoupling of the slow and fast dynamics of a LTI system to LTV systems by the Chang transformation is extended in Yang and Zhu (2010). The stability conditions of LTV systems with singular perturbations are introduced in Javid (1978). An observer applied to LTV singularly perturbed systems is found in Javid (1982).

The bond graph is a useful and important tool for physical system modelling. This is based on power representation; it enables the description of the system through energy storage and dissipative elements (Karnopp et al. 2016). Bond graph can represent a variety of energy types, whose junction structure can give a valuable information of the properties of the physical system (Borutzky 2011).

The bond graph methods can be applied to the following papers: A new snap oscillator based on diode nonlinearity is introduced in Tchinda et al. (2019). Two methods for estimating parameters of friction models and two methods for friction compensation applied to an inverted pendulum are analysed in Teixeira et al. (2013). State space models for loosely coupled inductive power transfer systems

with five different compensation topologies are considered in Maddalenna and Godoy (2017). An algorithm to describe the mathematical modelling of electrical grid signals, especially during transients, is presented in Batista et al. (2016). These papers include LTV systems in a general form.

We can find some papers applying bond graph to singular perturbation methods. Sueur and Dauphin-tanguy (1991); Sueur and Dauphin-Tanguy (1991) describe how the bond graph model is a helpful tool for system analysis in the special case of simplifying the modelling of two time scale systems. The fast and slow dynamics of bond graph models can be estimated by determination of causal loop gains. In Dauphin-Tanguy et al. (1985), the notion of a reciprocal system, with singular perturbations techniques, can obtain more accuracy on the fast time scale behaviour of the system. Also, in Gonzalez and Barrera (2013) the quasi-steady-state model of a LTI system with singular perturbations by fixing the causality of the storage field of the corresponding bond graph model is determined. The analysis of a class of nonlinear systems with singular perturbations in bond graph is presented in Gonzalez and Padilla (2018). Approximate bond graph models for two time scales systems are proposed in Gonzalez and Padilla (2016). The analysis of a singular perturbed system with a feedback and observer in the physical domain is presented in Gonzalez (2016).

In this paper, a bond graph in a preferred integral causality (BGI) of a linear time-varying (LTV) system with singular perturbations is defined. The storage elements of the corresponding BGI can have integral or derivative causality assignment. From the BGI, a new bond graph called singularly perturbed varying bond graph (SPVBG) is proposed. This SPVBG has properties such that the storage elements of the slow dynamics with integral causality maintain this causality and the storage elements of the fast dynamics have a derivative causality assignment. Hence, the quasi-steady-state model based on SPVBG can be obtained. The stability and bounded conditions of a LTV system with singular perturbations modelled by bond graphs are introduced. Thus, a theorem to determine the approximate models of both dynamics in the physical domain is proposed.

Section 2 describes the singular perturbation model. Section 3 proposes the modelling of a LTV system with two time scale in the physical domain. In Sect. 4, a quasi-steady-state model and approximate models with a bond graph approach are presented. The proposed methodology to an example is applied in Sect. 5. Finally, Sect. 6 gives our conclusions.

## 2 The Standard Singular Perturbation Model

The general model of systems with singular perturbations has been studied by Tikhonov (1948), Tikhonov (1952),

Levinson (1950), Vasil'eva (1963), Wasow (1965), O'Malley (1971), etc. This model can be described by

$$\dot{x}_1 = f(x_1, x_2, \varepsilon, t), \quad x_1(t_0) = x_1^0 \in \mathbb{R}^n \quad (1)$$

$$\varepsilon \dot{x}_2 = g(x_1, x_2, \varepsilon, t), \quad x_2(t_0) = x_2^0 \in \mathbb{R}^m \quad (2)$$

where  $f$  and  $g$  are assumed to be sufficiently many times continuously differentiable functions for their arguments,  $x_1, x_2, \varepsilon, t$ . Also,  $\varepsilon$  is a small positive parameter associated with the presence of parasitic elements (e.g. small inductances, capacitances, inertias, etc.) which serve as a measure of the time scale separation (Kokotovic et al. 1986, p. 2).

In control and systems theory, the model defined by (1) and (2) is a way towards the determination of reduced-order models, which is a common engineering task. A parameter perturbation, called singular  $\varepsilon$ , permits to obtain the order reduction. When we set  $\varepsilon = 0$ , the dimension of the state space of (1) and (2) reduces from  $n + m$  to  $n$  because the differential equation (2) degenerates into the algebraic equation

$$0 = g(\bar{x}_1, \bar{x}_2, 0, t) \quad (3)$$

where the bar indicates that the variables belong to a system with  $\varepsilon = 0$  (Kokotovic et al. 1986, p. 3). Also, in a domain of interest (3) can have  $k \geq 1$  distinct real roots

$$\bar{x}_2 = \bar{\phi}_i(\bar{x}_1, t), \quad i = 1, 2, \dots, k \quad (4)$$

This assumption ensures that a well-defined  $n$ -dimensional reduced model will correspond to each root (4). The reduced model is obtained by substituting (4) into (1).

$$\dot{\bar{x}}_1 = f(\bar{x}_1, \bar{\phi}_i(\bar{x}_1, t), 0, t), \quad \bar{x}_1(t_0) = x_1^0 \quad (5)$$

and keep the same initial condition for the state variable  $\bar{x}_1(t)$  as for  $x_1(t)$ . In the sequel, we shall drop the subscript  $i$  and rewrite (5) more compactly as

$$\dot{\bar{x}}_1 = f(\bar{x}_1, t), \quad \bar{x}_1(t_0) = x_1^0 \quad (6)$$

This model is called a quasi-steady-state model, because  $x_2$  may rapidly converge to a root of (3), which is the quasi-steady-state form of (2).

## 2.1 Linear Time-Varying Systems

Linear time-invariant models are of interest in local or small signal approximations of more realistic nonlinear models of dynamic systems (Kokotovic et al. 1986, pp. 75–76). Consider a LTI system to study two time scale properties of the following form:

$$\dot{x}_1 = A_{11}(t)x_1 + A_{12}(t)x_2 + B_1(t)u, \quad x_1 \in \mathbb{R}^n \quad (7)$$

$$\varepsilon \dot{x}_2 = A_{21}(t)x_1 + A_{22}(t)x_2 + B_2(t)u, \quad x_2 \in \mathbb{R}^m \quad (8)$$

The slow reduced model is obtained by setting  $\varepsilon = 0$  in (8); then,

$$\bar{x}_2 = -A_{22}^{-1}(t)A_{21}(t)\bar{x}_1 - A_{22}^{-1}(t)B_2(t)u; \quad (9)$$

substituting (9) into (7), we have (Kokotovic et al. 1986),

$$\dot{\bar{x}}_1 = A_0(t)\bar{x}_1 + B_0(t)u \quad (10)$$

where

$$A_0(t) = A_{11}(t) - A_{12}(t)A_{22}^{-1}(t)A_{21}(t) \quad (11)$$

$$B_0(t) = B_1(t) - A_{12}(t)A_{22}^{-1}(t)B_2(t) \quad (12)$$

## 2.2 State Approximations

For singular perturbed systems, the submatrix  $A_{22}(t)$  is required to satisfy the stability conditions; then, boundary layers will be asymptotically stable and the time scale decomposition will be useful.

Consider the system

$$\varepsilon \dot{x}_2 = A_{22}(t)x_2, \quad t \in T, \quad x_2 \in \mathbb{R}^m \quad (13)$$

where the time interval  $T$  could be a finite interval  $[t_0, t_f]$  or an infinite interval  $[t_0, \infty)$ .

The parameters  $c_1$ ,  $c_2$  and  $c_3$  are assumed to be known and constant on the following assumptions:

**Assumption 1** (Kokotovic et al. 1986, p. 202).

$$\operatorname{Re} \lambda(A_{22}(t)) \leq -c_1 < 0, \quad \forall t \in T \quad (14)$$

**Assumption 2** (Kokotovic et al. 1986, p. 202).

$$\|A_{22}(t)\| \leq c_2, \quad \forall t \in T \quad (15)$$

**Assumption 3** (Kokotovic et al. 1986, p. 203).

$$\left\| \dot{A}_{22}(t) \right\| \leq c_3, \quad \forall t \in T \quad (16)$$

Consider the singularly perturbed system defined by (7) and (8) where the input  $u$  is a continuous bounded function of  $t \in T$  and the initial state is given by  $x_1(t_0) = x_1^0$  and  $x_2(t_0) = x_2^0$ .

**Theorem 1** (Kokotovic et al. 1986, p. 212) Assume that  $A_{22}(t)$  satisfies Assumptions 1–3 that  $A_{ij}(t)$  are continuously differentiable and bounded on  $T$  and that  $\dot{A}_{12}, \dot{A}_{21}$

and  $\dot{A}_{22}$  are bounded on  $T$ ; then, there exists an  $\varepsilon^* > 0$  such that for all  $\varepsilon \in (0, \varepsilon^*)$  there exist continuously differentiable matrices  $L$  and  $H$  on  $T$ , satisfying  $\varepsilon \dot{L} = A_{22}L - A_{21} - \varepsilon L(A_{11} - A_{12}L)$  and  $\varepsilon \dot{H} = HA_{22} - A_{12} + \varepsilon HLA_{12} - \varepsilon(A_{11} - A_{12}L)H$ . Moreover,  $L(t) = L_0(t) + O(\varepsilon)$  and  $H(t) = H_0(t) + O(\varepsilon)$ .

The basic idea of using time scale analysis in obtaining low-order models is to decouple the slow and fast dynamics. Hence, the matrices  $L(t)$  and  $H(t)$  permit to obtain a complete separation of the fast and slow states of the system (7) and (8) and the first approximation is defined by

$$L_0(t) = A_{22}^{-1}(t) A_{21}(t) \quad (17)$$

$$H_0(t) = A_{12}(t) A_{22}^{-1}(t) \quad (18)$$

Theorem 1 guarantees the possibility of applying perturbation methods to LTV systems. Also, Theorem 1 gives the result for first-order approximations.

Also, the next theorem defines the solution of states for slow and fast dynamics of LTV systems.

**Theorem 2** (Kokotovic et al. 1986, pp. 227–228) Suppose for all  $t \in [t_0, t_f]$  that the assumptions of Theorem 1 hold and  $B_i(t)$ ,  $i = 1, 2$ , and  $u$  are continuous. Then, there exists and  $\varepsilon^* > 0$  such that for all  $\varepsilon \in (0, \varepsilon^*)$  the following expressions hold uniformly on  $[t_0, t_f]$ :

$$x_1 = \bar{x}_1 + O(\varepsilon) \quad (19)$$

$$x_2 = -A_{22}^{-1}(t) A_{21}(t) \bar{x}_1 + x_2^f + O(\varepsilon) \quad (20)$$

where  $\dot{x}_2^f = A_{22}(t) x_2^f + B_2(t) u$ .

Theorem 2 allows to obtain reduced-order systems for LTV systems with singular perturbations. Hence, the states for slow dynamics are determined from the quasi-steady-state model given by (10) and the states for fast dynamics are obtained from the reduced fast model and the interconnection between slow and fast dynamics given by  $-A_{22}^{-1}(t) A_{21}(t)$ .

In the next section, a LTV system with two time scales using a bond graph model is proposed.

### 3 Modelling in Bond Graph of a Singularly Perturbed LTV System

Consider the following scheme of a bond graph model with an integral causality assignment (BGI) for a LTV system with singular perturbations which includes the key vectors of Fig. 1.

The block diagram of Fig. 1 contains:

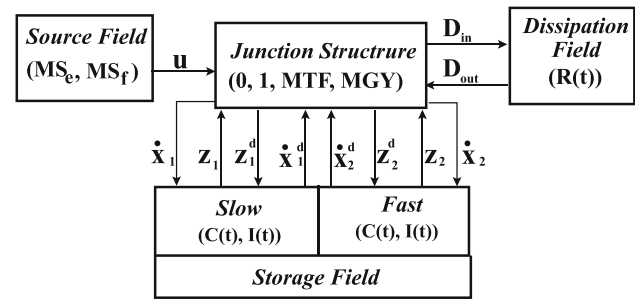


Fig. 1 Key vectors of a bond graph

- Source field denoted by  $(MS_e, MS_f)$  that determines the plant input  $u \in \mathbb{R}^p$ .
- Junction structure denoted by  $(0, 1, \text{MTF}, \text{MGY})$  with 0 and 1 junctions, transformers MTF and gyrators MGY which are modulated by time or constant functions.
- Energy storage field denoted by  $(C(t), I(t))$  that defines energy variables  $q$  and  $p$  associated with  $C(t)$  and  $I(t)$  elements divided by:
  - The states for the slow dynamics  $x_1 \in \mathbb{R}^n$  and  $x_1^d \in \mathbb{R}^{n_d}$  associated with storage elements in integral and derivative causality, respectively.
  - The co-energy vectors for the slow dynamics  $z_1 \in \mathbb{R}^n$  and  $z_1^d \in \mathbb{R}^{n_d}$  of the storage elements in integral and derivative causality, respectively.
  - The states for the fast dynamics  $x_2 \in \mathbb{R}^m$  and  $x_2^d \in \mathbb{R}^{m_d}$  associated with storage elements in integral and derivative causality, respectively.
  - The co-energy vectors for the fast dynamics  $z_2 \in \mathbb{R}^m$  and  $z_2^d \in \mathbb{R}^{m_d}$  of the storage elements in integral and derivative causality, respectively.
- Energy dissipation field denoted by  $R(t)$  that defines  $D_{in} \in \mathbb{R}^r$  and  $D_{out} \in \mathbb{R}^r$  as a mixture of power variables  $e$  and  $f$  indicating the energy exchanges between the dissipation field and the junction structure.

For systems of high order, the bond graph methodology permits to know the dynamics in an easy and direct way whose the fast and slow dynamics of a bond graph model can be estimated by determination of causal loop gains (Dauphin-Tanguy et al. 1985; Orbak et al. 2003).

The constitutive relations of the storage field for the slow dynamics are

$$z_1 = F_1(t) x_1 \quad (21)$$

$$z_1^d = F_1^d(t) x_1^d \quad (22)$$

for the fast dynamics are

$$z_2 = F_2(t) x_2 \quad (23)$$

$$z_2^d = F_2^d(t) x_2^d \quad (24)$$

and for the dissipation field are

$$D_{\text{out}} = L(t) D_{\text{in}} \quad (25)$$

The relations of the junction structure are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ D_{\text{in}} \\ z_1^d \\ z_2^d \end{bmatrix} = \begin{bmatrix} S_{11}^{11}(t) & S_{11}^{12}(t) & | & S_{12}^{11}(t) & | & S_{13}^{11}(t) & | & S_{14}^{11}(t) & | & 0 \\ S_{11}^{21}(t) & S_{11}^{22}(t) & | & S_{12}^{21}(t) & | & S_{13}^{21}(t) & | & 0 & | & S_{14}^{22}(t) \\ \hline S_{21}^{11}(t) & S_{21}^{12}(t) & | & S_{22}(t) & | & S_{23}(t) & | & 0 & | & 0 \\ \hline S_{31}^{11}(t) & 0 & | & 0 & | & 0 & | & 0 & | & 0 \\ 0 & S_{31}^{22}(t) & | & 0 & | & 0 & | & 0 & | & 0 \end{bmatrix} \begin{bmatrix} z_1^d \\ z_1^d \\ D_{\text{out}} \\ u \\ x_1^d \\ x_2^d \end{bmatrix}^T \quad (26)$$

The entries of  $S(t)$  take values inside the set  $\{0, \pm 1, \pm m(t), \pm n(t)\}$  where  $m(t)$  and  $n(t)$  are transformer and gyrator modules for the LTV systems. Also, the properties  $S_{11}^{11}$ ,  $S_{11}^{22}$  and  $S_{22}$  are square skew-symmetric matrices, and  $S_{12}^{11}$ ,  $S_{12}^{21}$ ,  $S_{11}^{12}$ ,  $S_{14}^{11}$ ,  $S_{21}^{22}$ ,  $S_{21}^{12}$ ,  $S_{11}^{21}$ ,  $S_{31}^{22}$  are matrices each other negative transpose, respectively.

In order to determine the state equation for a singularly perturbed model, from (7) and (8) the matrices using the bond graph approach are:

$$\begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} = \begin{bmatrix} E_1(t) & 0 \\ 0 & E_2(t) F_2(t) \end{bmatrix}^{-1} \begin{bmatrix} S_{11}^{11}(t) + S_{12}^{11}(t) M(t) S_{21}^{11}(t) \\ S_{11}^{21}(t) + S_{12}^{21}(t) M(t) S_{21}^{11}(t) \\ S_{11}^{12}(t) + S_{12}^{11}(t) M(t) S_{21}^{12}(t) \\ S_{11}^{22}(t) + S_{12}^{21}(t) M(t) S_{21}^{12}(t) \end{bmatrix} \begin{bmatrix} F_1(t) & 0 \\ 0 & F_2(t) \end{bmatrix} + \begin{bmatrix} S_{14}^{11}(t) \frac{d}{dt} \left\{ [F_1^d(t)]^{-1} S_{31}^{11}(t) F_1(t) \right\} \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ S_{14}^{22}(t) \frac{d}{dt} \left\{ [F_2^d(t)]^{-1} S_{31}^{22}(t) F_2(t) \right\} \end{bmatrix} \quad (27)$$

$$\begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix} = \begin{bmatrix} E_1(t) & 0 \\ 0 & E_2(t) F_2(t) \end{bmatrix}^{-1} \begin{bmatrix} S_{13}^{11}(t) + S_{12}^{11}(t) M(t) S_{23}(t) \\ S_{13}^{21}(t) + S_{12}^{21}(t) M(t) S_{23}(t) \end{bmatrix} \quad (28)$$

where

$$E_1(t) = I - S_{14}^{11}(t) [F_1^d(t)]^{-1} S_{31}^{11}(t) F_1(t) \quad (29)$$

$$E_2(t) = I - S_{14}^{22}(t) [F_2^d(t)]^{-1} S_{31}^{22}(t) F_2(t) \quad (30)$$

$$M(t) = L(t) [I - S_{22}(t) L(t)]^{-1} \quad (31)$$

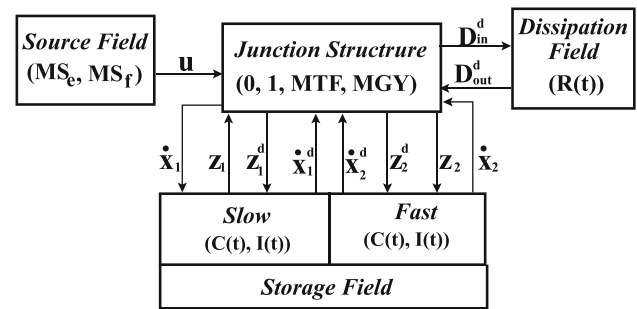


Fig. 2 Key vectors of a bond graph

In the next section, the approximate models of LTV systems using bond graph models are proposed.

#### 4 Analysis of LTV Systems with Singular Perturbations in a Bond Graph Approach

The direct determination of the real roots from algebraic equation and a quasi-steady-state model for LTV systems modelled by bond graphs is presented. A scheme to analyse the properties of a LTV singularly perturbed system represented in a bond graph called singularly perturbed varying bond graph (SPVBG) is shown in Fig. 2.

It is important to distinguish that from the key vectors in Fig. 2:

- For the storage elements of the fast dynamics, which are linearly independent  $(\dot{x}_2, z_2)$ , we have changed the causality and a derivative causality has been assigned.
- For the dissipation field in this bond graph we have  $(D_{\text{in}}^d, D_{\text{out}}^d)$  in order to have a correct causality for the junctions into the bond graph and the constitutive relation is given by

$$D_{\text{out}}^h = L^h(t) D_{\text{in}}^h \quad (32)$$

- For the storage elements of the slow dynamics we maintain an integral causality assignment.

The junction structure of the SPVBG on the following lemma is presented.

**Lemma 1** Consider a bond graph of a singularly perturbed LTV system whose storage elements have integral and derivative causality assignments to slow and fast dynamics, respectively, shown in the scheme of Fig. 2. The structure of the system is defined by



$$\begin{bmatrix} \dot{x}_1 \\ z_2 \\ z_2^d \\ D_{in}^h \\ z_1^d \end{bmatrix} = \begin{bmatrix} H_{11}^{11}(t) & H_{11}^{12}(t) & H_{11}^{13}(t) & | & H_{12}^{11}(t) & | & H_{13}^{11}(t) & | & H_{14}^{11}(t) \\ H_{11}^{21}(t) & H_{11}^{22}(t) & H_{11}^{23}(t) & | & H_{12}^{21}(t) & | & H_{13}^{21}(t) & | & 0 \\ H_{11}^{31}(t) & H_{11}^{32}(t) & H_{11}^{33}(t) & | & H_{12}^{31}(t) & | & H_{13}^{31}(t) & | & 0 \\ \hline H_{21}^{11}(t) & H_{21}^{12}(t) & H_{21}^{13}(t) & | & H_{22}^{11}(t) & | & H_{23}^{11}(t) & | & 0 \\ H_{41}^{11}(t) & 0 & 0 & | & 0 & | & 0 & | & 0 \end{bmatrix} \begin{bmatrix} \dot{z}_1 & \dot{x}_2 & \dot{x}_2^d & | & D_{out}^h & | & u & | & \dot{x}_1^d \end{bmatrix}^T \quad (33)$$

the real roots of the algebraic equation from differential equation of fast state variables by setting are

$$\widetilde{x}_2 = \widetilde{A}_{21}(t) \widetilde{x}_1 + \widetilde{B}_2(t) u \quad (34)$$

where

$$\widetilde{A}_{21}(t) = F_2^{-1}(t) \left[ H_{11}^{21}(t) + H_{12}^{21}(t) M^h(t) H_{21}^{11}(t) \right] F_1(t) \quad (35)$$

$$\widetilde{B}_2(t) = F_2^{-1}(t) \left[ H_{13}^{21}(t) + H_{12}^{21}(t) M^h(t) H_{23}^{11}(t) \right] \quad (36)$$

and the quasi-steady-state model is defined by

$$\dot{\widetilde{x}}_1 = \widetilde{A}_{11}(t) \widetilde{x}_1 + \widetilde{B}_1(t) u \quad (37)$$

where

$$\widetilde{A}_{11}(t) = [\widetilde{E}_1(t)]^{-1} \left[ \widetilde{A}_{11}^h(t) + \widetilde{A}_{11}^d(t) \right] \quad (38)$$

with

$$\widetilde{A}_{11}^h(t) = \left[ H_{11}^{11}(t) + H_{12}^{11}(t) M^h(t) H_{21}^{11}(t) \right] F_1(t) \quad (39)$$

$$\begin{aligned} \widetilde{A}_{11}^d(t) &= H_{14}^{11}(t) \\ &\left\{ \frac{d[F_1^d(t)]^{-1}}{dt} H_{41}^{11}(t) + [F_1^d(t)]^{-1} \frac{dH_{41}^{11}(t)}{dt} \right\} F_1(t) \\ &+ H_{14}^{11}(t) [F_1^d(t)]^{-1} H_{41}^{11}(t) \frac{dF_1(t)}{dt} \end{aligned} \quad (40)$$

$$\widetilde{B}_1(t) = [\widetilde{E}_1(t)]^{-1} \left[ H_{13}^{11}(t) + H_{12}^{11}(t) M^h(t) H_{23}^{11}(t) \right] \quad (41)$$

being

$$\widetilde{E}_1(t) = I - H_{14}^{11}(t) [F_1^d(t)]^{-1} H_{41}^{11}(t) F_1(t) \quad (42)$$

$$M^h(t) = L^h(t) \left[ I - H_{22}(t) L^h(t) \right]^{-1} \quad (43)$$

The proof is presented in “Appendix A”.

Assumptions 1–3 described in the traditional approach can be defined in a bond graph approach in Assumptions 4–6. Also, Assumptions 7 and 8 consider the boundary conditions required for Theorem 1.

**Assumption 4** The subsystem stability conditions for the fast dynamics of LTV systems modelled by bond graphs based on the junction structure are expressed by

$$\begin{aligned} S_{11}^{22}(t) \geq 0; \quad S_{22}(t) \geq 0; \quad \left[ S_{31}^{22}(t) \frac{dS_{31}^{22}(t)}{dt} \right] \geq 0, \quad \forall t \in T \\ \frac{dF_2(t)}{dt} \geq 0, \quad \frac{d[F_2^d(t)]}{dt} \geq 0 \end{aligned} \quad (44)$$

satisfying Assumption 1,  $\text{ReRe}\lambda(A_{22}(t)) \leq c_1 < 0, \forall t \in T$

**Assumption 5** Boundary conditions of the state matrix for the slow dynamics of LTV systems using the junction structure of the bond graph model are given by

$$\begin{aligned} \|S_{11}^{22}(t)\| \leq c_2; \quad \|S_{14}^{22}(t)\| \leq c_2; \quad \|S_{12}^{21}(t)\| \leq c_2; \\ \|S_{21}^{12}(t)\| \leq c_2, \quad \forall t \in T \\ \|F_2(t)\| \leq c_2; \quad \|[E_2(t) F_2(t)]^{-1}\| \leq c_2; \\ \left\| \frac{d[F_2^d(t)]^{-1} S_{31}^{22}(t) F_2(t)}{dt} \right\| \leq c_2 \end{aligned} \quad (45)$$

satisfying Assumption 2,  $\|A_{22}(t)\| \leq c_2 \forall t \in T$

**Assumption 6** The boundary conditions of the derivative of the state matrix for the slow dynamics of LTV systems represented by the junction structure of the bond graph are defined by

$$\begin{aligned} \left\| \frac{dS_{11}^{22}(t) F_2(t)}{dt} \right\| \leq c_3; \\ \left\| \frac{dS_{12}^{21}(t) M(t) S_{21}^{12}(t) F_2(t)}{dt} \right\| \leq c_3, \quad \forall t \in T \\ \left\| \frac{dS_{14}^{22}(t)}{dt} \right\| \leq c_3; \quad \left\| \frac{d^2[S_{12}^{21}(t) M(t) S_{21}^{12}(t)]}{dt^2} \right\| \leq c_3; \\ \left\| \frac{d[E_2(t) F_2(t)]^{-1}}{dt} \right\| \leq c_3 \end{aligned} \quad (46)$$

satisfying Assumption 3,  $\|\dot{\widetilde{A}}_{22}(t)\| \leq c_3 \forall t \in T$ .

**Assumption 7** The boundary conditions of the derivative of the state matrix from the fast to slow dynamics for LTV systems modelled by bond graphs and using the junction structure are given by

$$\begin{aligned}
\|E_1^{-1}(t)\| &\leq c_4; \quad \left\| \frac{dE_1^{-1}(t)}{dt} \right\| \leq c_4; \\
\left\| \frac{dF_2(t)}{dt} \right\| &\leq c_4; \quad \forall t \in T \\
\left\| \frac{d[S_{11}^{12}(t) + S_{12}^{11}(t)M(t)S_{21}^{12}(t)]}{dt} \right\| \\
&\leq c_4 \quad \|S_{12}^{11}(t)\| \leq c_4; \\
\|S_{21}^{12}(t)\| &\leq c_4;
\end{aligned} \quad (47)$$

satisfying that  $\|\dot{A}_{12}(t)\| \leq c_4 \forall t \in T$ .

**Assumption 8** The boundary conditions of the derivative of the state matrix from the slow to fast dynamics for LTV systems modelled by bond graphs and based on the junction structure are expressed by

$$\begin{aligned}
\|S_{12}^{21}(t)\| &\leq c_5; \quad \|S_{21}^{11}(t)\| \leq c_5; \\
\|F_1(t)\| &\leq c_5; \quad \forall t \in T \\
\|S_{11}^{21}(t)\| &\leq c_5; \\
\left\| \frac{d[S_{11}^{21}(t) + S_{12}^{21}(t)M(t)S_{21}^{11}(t)]}{dt} \right\| &\leq c_5
\end{aligned} \quad (48)$$

satisfying that  $\|\dot{A}_{21}(t)\| \leq c_5 \forall t \in T$ .

The proofs of Assumptions 4–6 are presented in “Appendix B”. Based on Assumptions 4–8 and Lemma 1, reduced models for slow and fast dynamics of a LTV system with singular perturbations can be obtained by the following theorem.

**Theorem 3** Consider a linear time-varying system with singular perturbations modelled by bond graphs; a bond graph (SVPBG) is built whose storage elements for the slow dynamics have an integral causality assignment, and for the fast dynamics a derivative causality is assigned; another bond graph called reduced fast bond graph (RFBGI) based on the storage elements for the fast dynamics in an integral causality assignment and removing the storage elements for the slow dynamics and all the elements and junctions connected to these dynamics is constructed. Also, Assumptions 4–8 for all  $t \in [t_0, t_f]$  are satisfied that  $A_{ij}(t)$  are continuously differentiable and bounded in  $T$  and  $B_i(t)$ ,  $i = 1, 2$  and  $u$  are continuous. Then, there exists  $\varepsilon^* > 0$  such that for all  $\varepsilon \in (0, \varepsilon^*)$  the following expressions hold uniformly on  $[t_0, t_f]$ :

$$x_1 = \bar{x}_1 + O(\varepsilon) \quad (49)$$

$$x_2 = \widetilde{A_{21}}(t)\bar{x}_1 + x_2^f + O(\varepsilon) \quad (50)$$

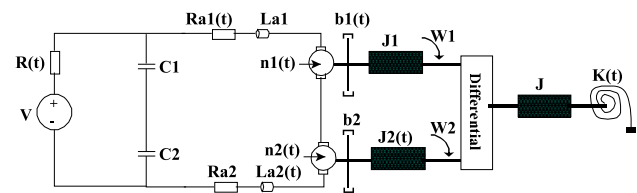


Fig. 3 DC motor scheme connected a load

where the quasi-steady-state model from the SVPBG is

$$\dot{\bar{x}}_1 = \widetilde{A_{11}}(t)\bar{x}_1 + \widetilde{B_1}(t)u \quad (51)$$

defined by Lemma 1 and the reduced fast model from RFBGI is

$$\dot{x}_2^f = A_{22}(t)x_2^f + B_2(t)u \quad (52)$$

The proof of Theorem 3 is presented in “Appendix C”.

In the next section, the methodology to obtain the quasi-steady-state model for a LTV system represented in bond graph models is applied.

## 5 Case Study

Consider a load driven by two separately DC motors connected by a differential shown in Fig. 3. A voltage source  $V$  supplies the energy for the two DC motors through two capacitors  $C_1$  and  $C_2$ , the power in the armature circuits ( $R_{a1}(t)$ ,  $L_{a1}$ ) and ( $R_{a2}$ ,  $L_{a2}(t)$ ) is transduced to shafts power, and the field port establishes a magnetic field, which provides the coupling between electric and mechanical variables by  $n_1(t)$  and  $n_2(t)$  for the individual conductors on the rotors. The mechanical parts are connected to a load formed by a spring  $K(t)$  and an inertia  $J$  through two inertias  $J_1$  and  $J_2(t)$  with their frictions  $b_1(t)$  and  $b_2$  corresponding to each motor.

A bond graph model in an integral causality assignment of the DC motors is shown in Fig. 4. Practically, in all well-designed DC motors,  $L_{a1}$  and  $L_{a2}$  are small and the capacitors  $C_1$  and  $C_2$  are small too, which determines the parameter  $\varepsilon = \text{diag} \left\{ \frac{1}{L_{a1}}, \frac{1}{C_1}, \frac{1}{L_{a2}(t)}, \frac{1}{C_2} \right\}$ .

In Fig. 4,  $R(t)$ ,  $R_{a1}(t)$ ,  $L_{a2}(t)$ ,  $n_1(t)$ ,  $n_2(t)$ ,  $b_2(t)$ ,  $J_2(t)$  and  $K(t)$  are LTV elements and the rest of the elements:  $C_1$ ,  $C_2$ ,  $L_{a1}$ ,  $R_{a2}$ ,  $R_{b1}$ ,  $J_1$  and  $J$  are LTI. The key vectors and the constitutive relations of the bond graph are

$$D_{\text{in}} = \begin{bmatrix} f_{20} \\ f_{19} \\ f_{12} \\ f_{11} \\ e_2 \end{bmatrix}; \quad D_{\text{out}} = \begin{bmatrix} e_{20} \\ e_{19} \\ e_{12} \\ e_{11} \\ f_2 \end{bmatrix}; \quad u = e_1$$

$$L = \text{diag} \left\{ b_1(t), b_2, R_{a1}(t), R_{a2}, \frac{1}{R(t)} \right\} \quad (53)$$

for slow dynamics

$$x_1 = \begin{bmatrix} p_{17} \\ p_{18} \\ q_{24} \end{bmatrix}; \dot{x}_1 = \begin{bmatrix} e_{17} \\ e_{18} \\ f_{24} \end{bmatrix}; z_1 = \begin{bmatrix} f_{17} \\ f_{18} \\ e_{24} \end{bmatrix}; \begin{matrix} x_1^d = p_{25} \\ \dot{x}_1^d = e_{25} \\ z_1^d = f_{25} \end{matrix}$$

$$F_1 = \text{diag} \left\{ \frac{1}{J_1}, \frac{1}{J_2(t)}, \frac{1}{K(t)} \right\} \quad (54)$$

$$F_1^d = \frac{1}{J} \quad (55)$$

for fast dynamics

$$x_2 = \begin{bmatrix} p_9 \\ q_5 \\ p_{10} \\ q_6 \end{bmatrix}; \dot{x}_2 = \begin{bmatrix} e_9 \\ f_5 \\ e_{10} \\ f_6 \end{bmatrix}; z_2 = \begin{bmatrix} f_9 \\ e_5 \\ f_{10} \\ e_6 \end{bmatrix}$$

$$F_2 = \text{diag} \left\{ \frac{1}{L_{a1}}, \frac{1}{C_1}, \frac{1}{L_{a2}(t)}, \frac{1}{C_2} \right\} \quad (56)$$

and the junction structure matrix is given by

$$\begin{bmatrix} e_{17} & e_{18} & f_{24} \parallel e_9 & f_5 & e_{10} & f_6 \parallel f_{20} & f_{19} & f_{12} & f_{11} & e_2 \parallel f_{25} \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & 0 & -1 \parallel n_1(t) & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 \parallel 0 & 0 & 0 & n_2(t) & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 \parallel 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -n_1(t) & 0 & 0 \parallel 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \parallel -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -n_2(t) & 0 \parallel 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 \parallel 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 \parallel 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 \parallel 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \parallel 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \parallel 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \parallel 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 0 \parallel 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\times [f_{17} \ f_{18} \ e_{24} \parallel f_9 \ e_5 \ f_{10} \ e_6 \parallel e_{20} \ e_{19} \ e_{12} \ e_{11} \ f_2 \parallel e_1 \ e_{25}]^T \quad (57)$$

From (53) to (57) with (27), (29) and (31), the slow state equation is

$$E_1(t) \dot{x}_1 = \begin{bmatrix} \frac{-b_1(t)}{J_1} & 0 & \frac{-1}{K(t)} \\ 0 & \frac{-b_2(t)}{J_2(t)} & \frac{-1}{K(t)} \\ \frac{1}{J_1} & \frac{1}{J_2(t)} & 0 \end{bmatrix} x_1$$

$$+ \begin{bmatrix} \frac{n_1(t)}{L_{a1}} & 0 & 0 & 0 \\ 0 & 0 & \frac{n_2(t)}{L_{a2}} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x_2 \quad (58)$$

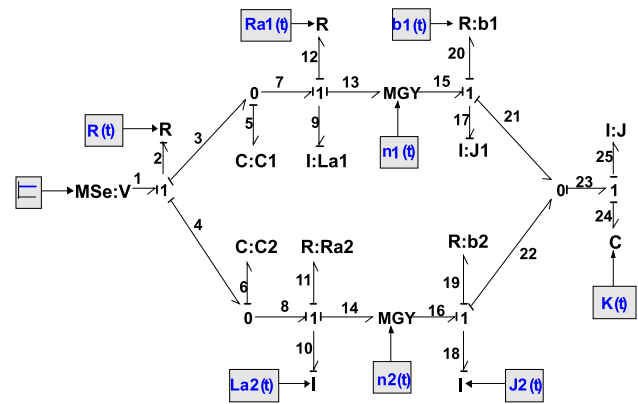


Fig. 4 BGI of the DC motors

where

$$E_1(t) = \begin{bmatrix} 1 + \frac{J}{J_1} & \frac{J}{J_2(t)} & 0 \\ \frac{J}{J_1} & 1 + \frac{J}{J_2(t)} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (59)$$

and the fast state equation is obtained by substituting (53), (56) and (57) into the second line of (27)

$$\varepsilon \dot{x}_2 = \begin{bmatrix} \frac{-n_1(t)L_{a1}}{J_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{-n_2(t)L_{a2}(t)}{J_2(t)} & 0 \end{bmatrix} x_1$$

$$+ \begin{bmatrix} -R_{a1}(t) & \frac{L_{a1}}{C_1} & 0 & 0 \\ \frac{-C_1}{L_{a1}} & \frac{-1}{R(t)} & 0 & \frac{-C_1}{R(t)C_2} \\ 0 & 0 & -R_{a2} & \frac{L_{a2}(t)}{C_2} \\ 0 & \frac{-C_2}{R(t)C_1} & \frac{-C_2}{L_{a2}(t)} & \frac{-1}{R(t)} \end{bmatrix} x_2 + \begin{bmatrix} \frac{C_1}{R(t)} \\ 0 \\ 0 \\ \frac{C_2}{R(t)} \end{bmatrix} u \quad (60)$$

The dynamic performance of the LTV elements of the system is shown in Figs. 5, 6 and 7.

It is clear that the dissipation elements  $b_1(t)$ ,  $R_{a1}(t)$  are LTV elements according to Fig. 5. Also, in this case study, the storage elements  $J_2(t)$ ,  $L_{a2}(t)$  and  $K(t)$  are LTV functions and the electromechanical conversion of the two DC motors  $n_1(t)$  and  $n_2(t)$  is assumed to be LTV functions which are shown in Figs. 6 and 7.

In the case of LTV systems, the variation in model parameters is an important property of the system and Assumptions defined in Sect. 4 have to be satisfied in order to determine the approximate models.

From (57),

$$S_{11}^{22}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \geq 0; \quad (61)$$



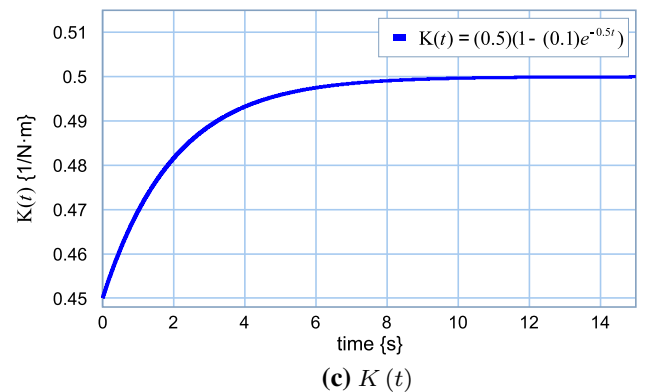
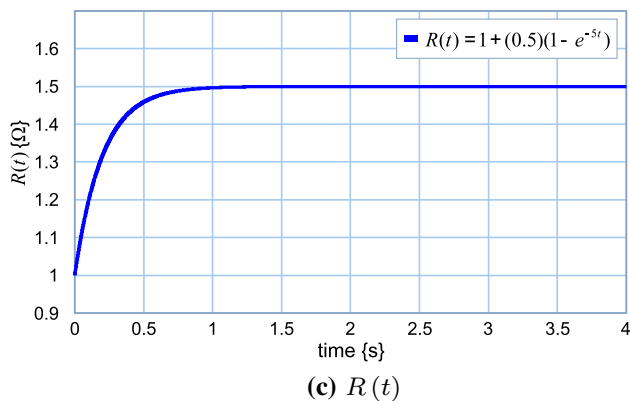
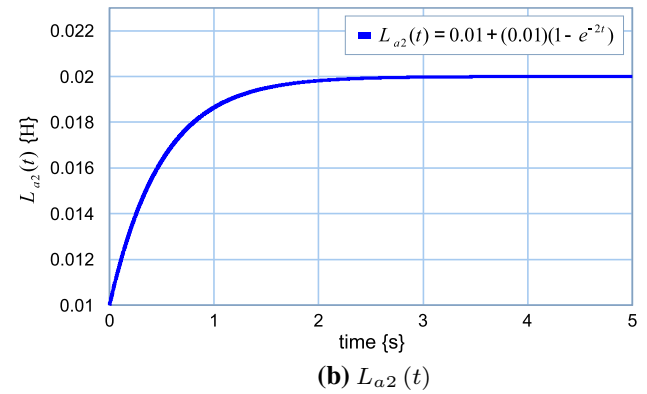
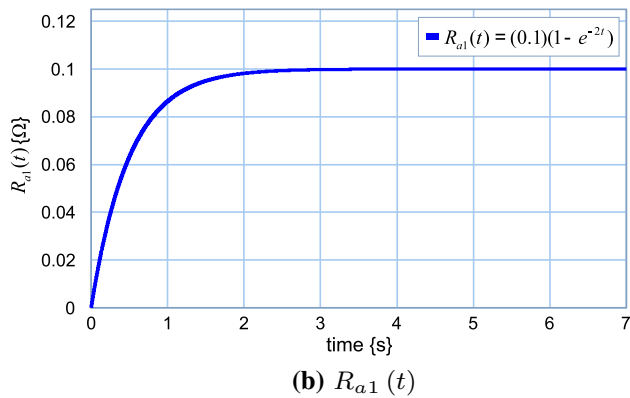
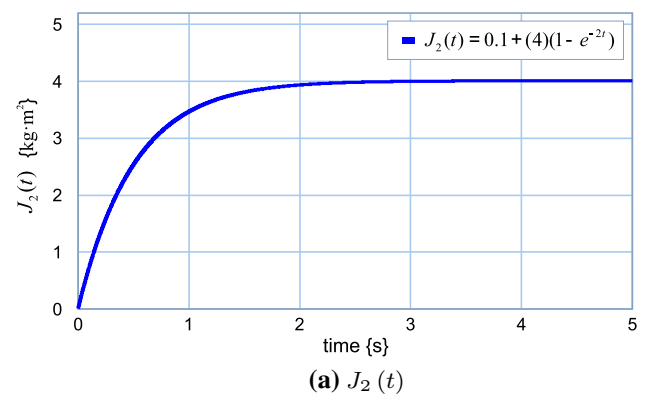
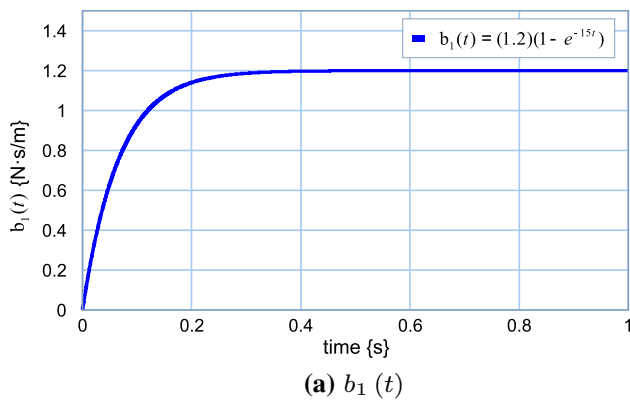


Fig. 5 Dissipation elements

Fig. 6 Storage elements

$$F_2(t) = \begin{bmatrix} \frac{1}{L_{a1}} & 0 & 0 & 0 \\ 0 & \frac{1}{C_1} & 0 & 0 \\ 0 & 0 & \frac{1}{L_{a2}(t)} & 0 \\ 0 & 0 & 0 & \frac{1}{C_2} \end{bmatrix} \geq 0 \quad (62)$$

$$S_{22}(t) = 0; \quad S_{22}^{31}(t) = 0 \text{ and } \frac{dF_2(t)}{dt} \geq 0 \quad (63)$$

and considering (62) and (63) Assumption 4 is satisfied.

From (62) and (57),

$$\|S_{21}^{12}(t)\| = \left\| \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \right\| \leq c_2;$$

$$\|F_2(t)\| = \left\| \begin{bmatrix} \frac{1}{L_{a1}} & 0 & 0 & 0 \\ 0 & \frac{1}{C_1} & 0 & 0 \\ 0 & 0 & \frac{1}{L_{a2}(t)} & 0 \\ 0 & 0 & 0 & \frac{1}{C_2} \end{bmatrix} \right\| \leq c_2$$

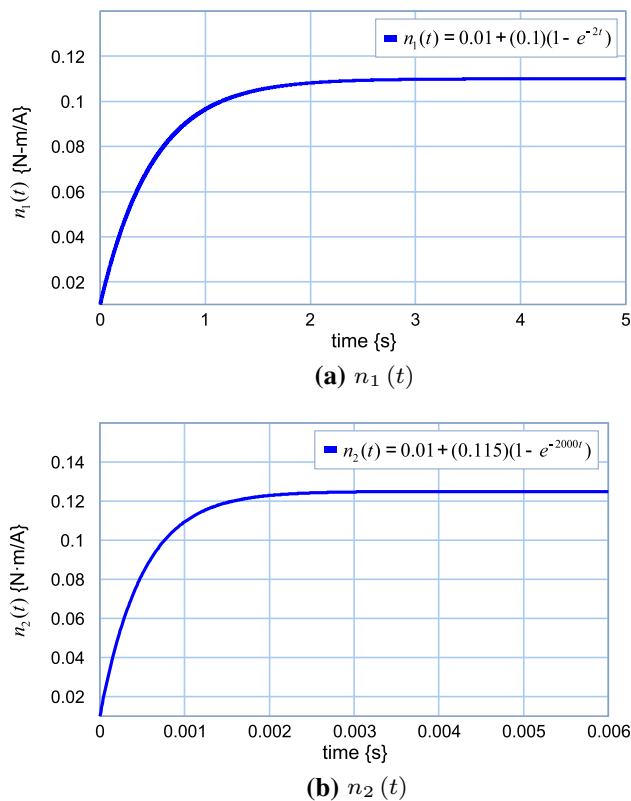


Fig. 7 MGY elements

$$\|F_2^{-1}(t)\| = \left\| \begin{bmatrix} L_{a1} & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 \\ 0 & 0 & L_{a2}(t) & 0 \\ 0 & 0 & 0 & C_2 \end{bmatrix} \right\| \leq c_2$$

also  $F_2^d(t) = 0$ ; then,  $\left\| \frac{d[F_2^d(t)]^{-1} S_{31}^{22}(t) F_2(t)}{dt} \right\| \leq c_2$  and Assumption 5 is satisfied.

From (56) and (57),

$$\begin{aligned} & \left\| \frac{d}{dt} \begin{bmatrix} 0 & \frac{1}{C_1} & 0 & 0 \\ \frac{-1}{L_{a1}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{C_2} \\ 0 & 0 & \frac{-1}{L_{a2}(t)} & 0 \end{bmatrix} \right\| \leq c_3; \\ & \left\| \frac{d}{dt} \begin{bmatrix} \frac{-R_{a1}(t)}{L_{a1}} & \frac{1}{C_1} & 0 & 0 \\ \frac{-1}{L_{a1}} & \frac{R(t)C_1}{R(t)C_1} & 0 & \frac{-1}{R(t)C_2} \\ 0 & 0 & \frac{-R_{a2}}{L_{a2}(t)} & \frac{1}{C_2} \\ 0 & \frac{-1}{R(t)C_1} & \frac{-1}{L_{a2}(t)} & \frac{-1}{R(t)C_2} \end{bmatrix} \right\| \leq c_3 \\ & \left\| \frac{d^2}{dt^2} \begin{bmatrix} -R_{a1}(t) & 1 & 0 & 0 \\ -1 & \frac{-1}{R(t)} & 0 & \frac{-1}{R(t)} \\ 0 & 0 & -R_{a2} & 1 \\ 0 & \frac{-1}{R(t)} & -1 & \frac{-1}{R(t)} \end{bmatrix} \right\| \leq c_3; \end{aligned} \quad (64)$$

$$\left\| \frac{d}{dt} \begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 \\ 0 & 0 & L_2 & 0 \\ 0 & 0 & 0 & C_2 \end{bmatrix} \right\| \leq c_3 \quad (65)$$

and applying (64) and (65), Assumption 6 is satisfied.

From (55)–(57),

$$\begin{aligned} & \left\| \begin{bmatrix} \frac{J_1}{\Delta} [J + J_2(t)] & \frac{-JJ_1}{\Delta} & 0 \\ \frac{-JJ_2(t)}{\Delta} & \frac{J_2(t)}{\Delta} [J + J_1] & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\| \leq c_4; \\ & \left\| \frac{d}{dt} \begin{bmatrix} \frac{J_1}{\Delta} [J + J_2(t)] & \frac{-JJ_1}{\Delta} & 0 \\ \frac{-JJ_2(t)}{\Delta} & \frac{J_2(t)}{\Delta} [J + J_1] & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\| \leq c_4 \end{aligned}$$

where  $\Delta = JJ_1 + JJ_2(t) + J_1J_2(t)$

$$\begin{aligned} & \left\| \frac{d}{dt} \begin{bmatrix} \frac{1}{La_1} & 0 & 0 & 0 \\ 0 & \frac{1}{C_1} & 0 & 0 \\ 0 & 0 & \frac{1}{La_2(t)} & 0 \\ 0 & 0 & 0 & \frac{1}{C_2} \end{bmatrix} \right\| \leq c_4; \\ & \left\| \frac{d}{dt} \begin{bmatrix} -R_{a1}(t) & 1 & 0 & 0 \\ -1 & \frac{-1}{R(t)} & 0 & \frac{-1}{R(t)} \\ 0 & 0 & -R_{a2} & 1 \\ 0 & \frac{-1}{R(t)} & -1 & \frac{-1}{R(t)} \end{bmatrix} \right\| \leq c_4 \end{aligned}$$

$$\left\| \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\| \leq c_4;$$

$$\left\| \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \right\| \leq c_4$$

and Assumption 7 is satisfied.

Finally, from (55) and (57)

$$\left\| \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right\| \leq c_5; \quad \left\| \begin{bmatrix} -n_1(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -n_2(t) & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\| \leq c_5$$

in this case  $S_{12}^{21}(t)M(t)S_{21}^{11}(t) = 0$ ; then,  $\|S_{11}^{21}(t) + S_{12}^{21}(t)M(t)S_{21}^{11}(t)\| = \|S_{11}^{21}(t)\|$

$$\left\| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\| \leq c_5; \quad \left\| \begin{bmatrix} \frac{1}{J_1} & 0 & 0 \\ 0 & \frac{1}{J_2(t)} & 0 \\ 0 & 0 & \frac{1}{K(t)} \end{bmatrix} \right\| \leq c_5$$

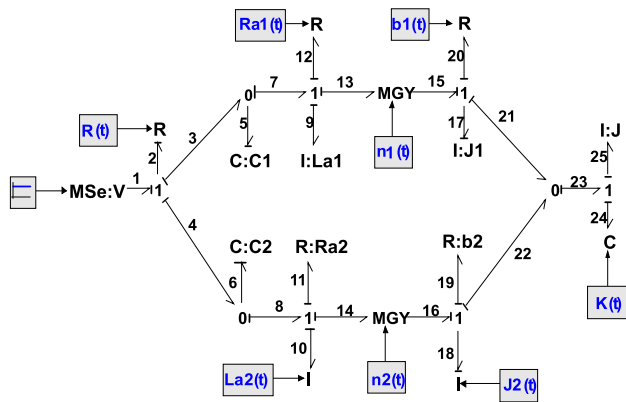


Fig. 8 SPVBG of the DC motors with load

Assumption 8 is satisfied.

In order to determine the quasi-steady-state model, SPVBG is shown in Fig. 8. The difference between BGI and SPVBG is that the storage elements for the fast dynamics  $I : L_{a1}$ ,  $I : L_{a2}$ ,  $C : C_1$  and  $C : C_2$  have a derivative causality assignment.

In this case, the key vectors for dissipation field of the SPVBG respect to BGI are the same  $D_{in}^h = D_{in}$ ,  $D_{out}^h = D_{out}$  and  $L^h = L$ , and their junction structure is

In this case, the submatrix  $H_{22}(t) \neq 0$ , and from (43),  $M^h(t)$  is given by

$$M^h(t) = \begin{bmatrix} \Lambda b_1(t) & 0 & 0 & 0 & 0 \\ 0 & \Lambda b_2 & 0 & 0 & 0 \\ 0 & 0 & R_{a1}(t)[R(t) + R_{a2}] & -R_{a1}(t)R_{a2} & R_{a1}(t) \\ 0 & 0 & -R_{a1}(t)R_{a2} & R_{a2}[R(t) + R_{a1}(t)] & R_{a2} \\ 0 & 0 & -R_{a1}(t) & -R_{a2} & 1 \end{bmatrix} \frac{1}{\Lambda} \quad (67)$$

where  $\Lambda = R(t) + R_{a1}(t) + R_{a2}$ .

From (34)–(36) with (54), (56), (66) and (67), the algebraic solutions for the fast dynamics are defined by

$$\bar{x}_2 = \begin{bmatrix} \frac{-n_1(t)L_{a1}}{J_1} & \frac{-n_2(t)L_{a1}}{J_2(t)} & 0 \\ \frac{n_1(t)[R(t) + R_{a2}]C_1}{J_1} & \frac{-n_2(t)R_{a1}(t)}{J_2(t)} & 0 \\ \frac{-n_1(t)L_{a2}(t)}{J_1} & \frac{-n_2(t)L_{a2}(t)}{J_2(t)} & 0 \\ \frac{-n_1(t)R_{a2}C_2}{J_1} & \frac{n_2(t)[R(t) + R_{a1}(t)]}{J_2(t)} & 0 \end{bmatrix} \frac{1}{\Delta} \bar{x}_1 - \begin{bmatrix} L_{a1} \\ C_1 \\ L_{a2}(t) \\ C_2 \end{bmatrix} \frac{1}{\Delta} u \quad (68)$$

where  $\Delta = R(t) + R_{a1}(t) + R_{a2}$ .

$$\begin{bmatrix} e_{17} \\ e_{18} \\ f_{24} \\ f_9 \\ e_5 \\ f_{10} \\ e_6 \\ f_{20} \\ f_{19} \\ f_{12} \\ f_{11} \\ e_2 \\ f_{25} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 & -n_1(t) & 0 & 0 & -1 & 0 & 0 & 0 & n_1(t) & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & -n_2(t) & 0 & -1 & 0 & 0 & n_2(t) & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ n_1(t) & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & n_2(t) & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -n_1(t) & -n_2(t) & 0 & -1 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{17} \\ f_{18} \\ e_{24} \\ e_9 \\ f_5 \\ e_{10} \\ f_6 \\ e_{20} \\ e_{19} \\ e_{12} \\ e_{11} \\ f_2 \\ e_1 \\ e_{25} \end{bmatrix} \quad (66)$$

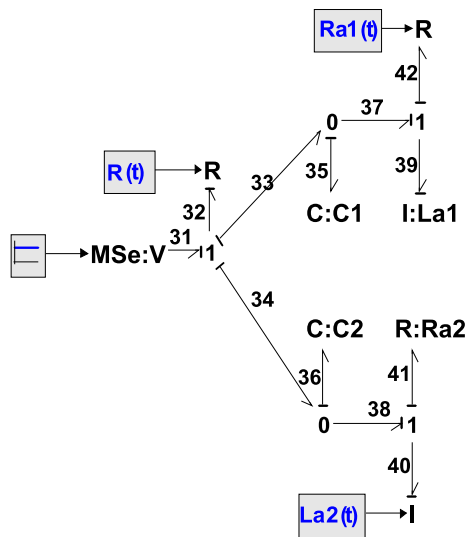


Fig. 9 Reduced fast dynamics model

From (37), (38), (41) and (42) with (54), (66) and (67), the quasi-steady-state mode is

$$\dot{\bar{x}}_1 = \begin{bmatrix} \frac{-1}{J_1} \left[ b_1(t) + \frac{n_1^2(t)}{\Delta} \right] & \frac{-n_1(t)n_2(t)}{J_2(t)\Delta} & \frac{-1}{K} \\ \frac{-n_1(t)n_2(t)}{J_1\Delta} & \frac{-1}{J_2(t)} \left[ b_2 + \frac{n_2^2(t)}{\Delta} \right] & \frac{-1}{K} \\ \frac{1}{J_1} & \frac{1}{J_2(t)} & 0 \end{bmatrix} \bar{x}_1 + \begin{bmatrix} n_1(t) \\ n_2(t) \\ 0 \end{bmatrix} \frac{1}{\Delta} u \quad (69)$$

In order to get the approximated model for the fast dynamics, the reduced fast dynamics model is shown in Fig. 9.

By applying Theorem 3, Fig. 10 shows the connection of the bond graphs of Figs. 8 and 9 which determines the solution of the approximate models defined by (49) and (50).

In order to verify the approximate models for the slow and fast dynamics of the case study defined by bond graph models, simulation results are shown. The numerical parameters of the system for LTI elements are:  $V_a = 10V$ ,  $L_{a1} = 0.01H$ ,  $C_1 = 0.01F$ ,  $C_2 = 0.01F$ ,  $J_1 = 4N-m-s^2$ ,  $J = 4N-m-s^2$ ,  $R_{a2} = 0.1\Omega$ ,  $b_2 = 1.2N-s/m$ . It is instructive to observe the dynamic performance of the system with singular perturbations which is shown in Fig. 11.

The system formed by two DC motors has three slow dynamics, and the dynamic behaviour is illustrated in Fig. 11a where the time scale is (0, 100s). Also, the time scale for the fast dynamics is (0, 3s) which is shown in Fig. 11b. Hence, the given numerical parameters describe a system with two time scale.

Figure 12 shows the exact and approximate models for the slow dynamics of this case study.

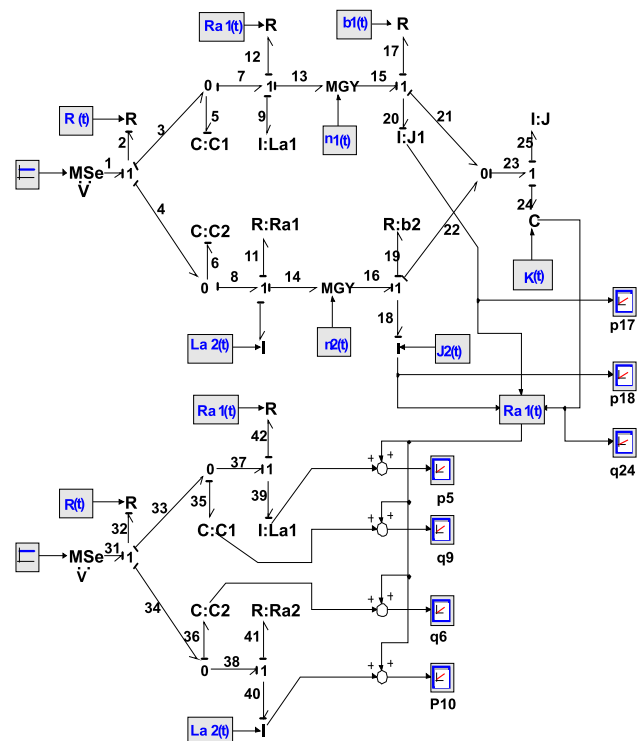
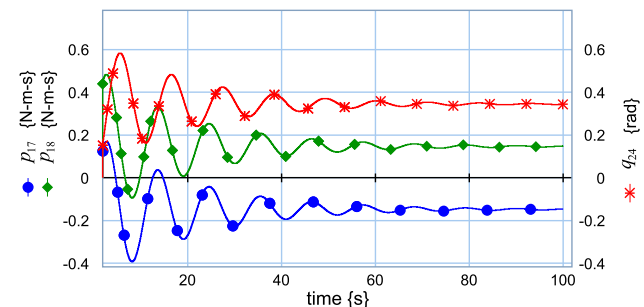
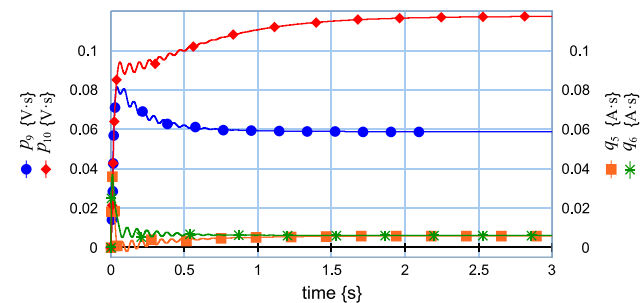


Fig. 10 Complete reduced fast dynamics model



(a) Slow dynamics



(b) Fast dynamics.

Fig. 11 Response for the system dynamics

The comparison between the approximate and exact slow dynamics which is shown in Fig. 12 gives the numerical difference of both models. However, the approximate slow

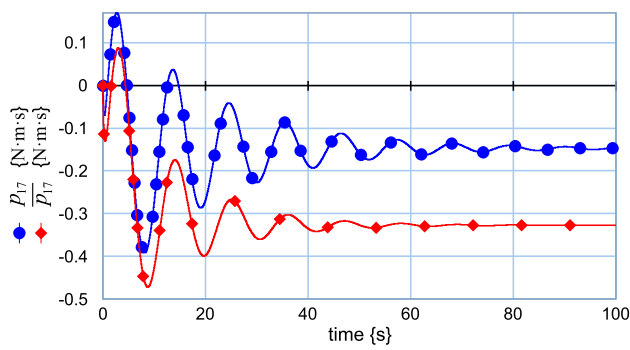
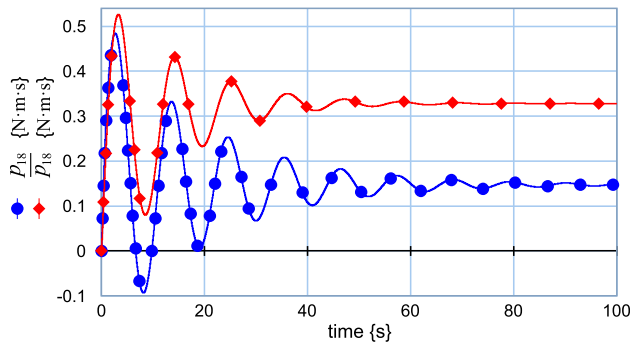
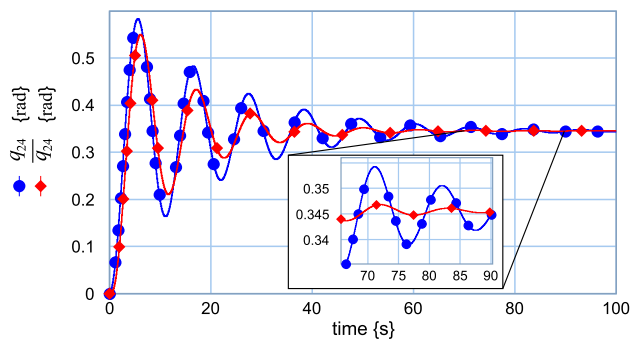
(a)  $p_{17}$  and  $\bar{p}_{17}$ (b)  $p_{18}$  and  $\bar{p}_{18}$ (c)  $q_{24}$  and  $\bar{q}_{24}$ 

Fig. 12 Response of the exact and approximate slow dynamics

dynamics determine a reduced-order model that for analysis and/or synthesis is simpler.

The performance of the exact and approximate models for the fast dynamics is shown in Figs. 13 and 14.

It can be seen that behaviour for the approximate and exact fast dynamics is very close according to Figs. 13 and 14. However, the approximate fast dynamics are obtained from a reduced-order model

In this case study, we want to show that the proposed methodology is direct and simple. Note that the proposed methodology allows to get the quasi-steady-state model changing the causality to the storage elements that represents the fast states and the reduced models for both dynamics are obtained.

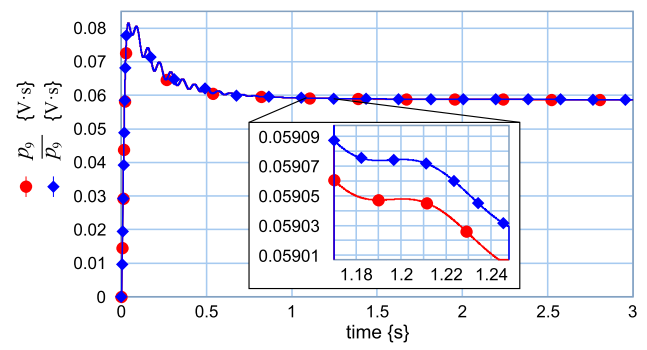
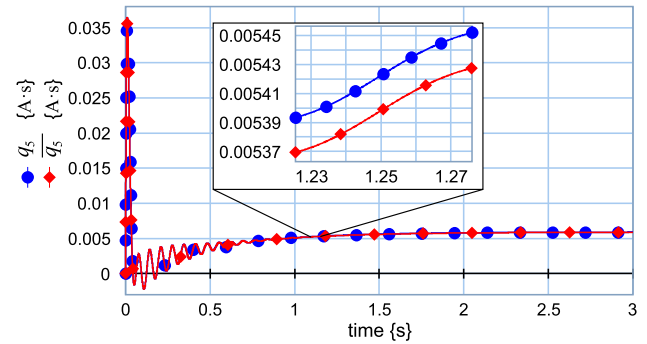
(a)  $p_9$  and  $\bar{p}_9$ (b)  $q_5$  and  $\bar{q}_5$ 

Fig. 13 Response of exact and approximate fast dynamics

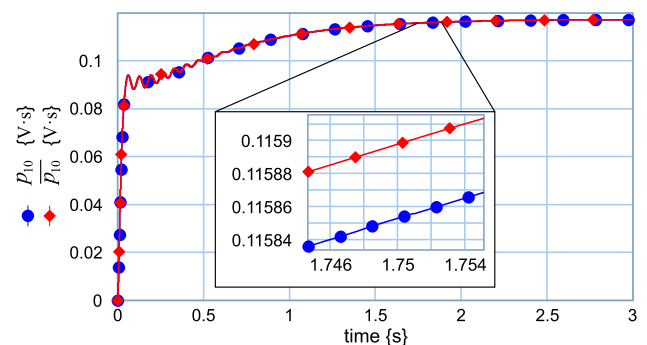
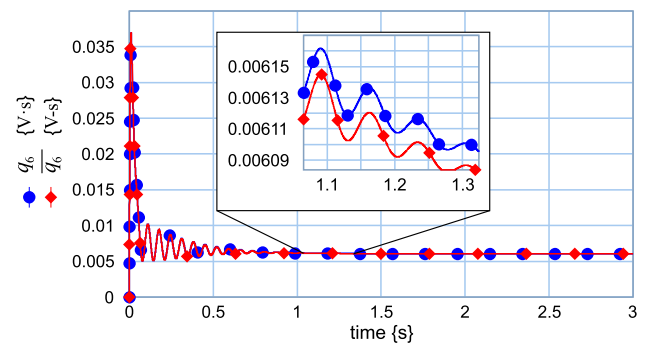
(a)  $p_9$  and  $\bar{p}_9$ (b)  $q_6$  and  $\bar{q}_6$ 

Fig. 14 Response of exact and approximate fast dynamics



## 6 Conclusions

A bond graph model of a singularly perturbed LTV system has been presented. A new bond graph called singularly perturbed varying bond graph (SPVBG) to obtain the quasi-steady-state model is presented. By removing the elements connected to the storage elements that represent the slow dynamics, a reduced fast bond graph model (RFBGI) is got. From the connection of SPVBG and RFBG, approximate bond graph models for both dynamics are determined. Finally, the proposed methodology to a case study showing the simulation results has been applied.

## Proof of Lemma 1

Proof. From the second line of (33) with (32)

$$D_{in}^h = \left[ I - H_{22}(t) L^h(t) \right]^{-1} \times \left[ H_{21}^{11}(t) z_1 + H_{21}^{12}(t) \dot{x}_2 + H_{21}^{13}(t) \dot{x}_2^d + H_{23}(t) u \right] \quad (70)$$

from the derivative of the fifth line of (33) with (21) and (22)

$$\dot{x}_1^d = \left\{ \frac{d[F_1^d(t)]^{-1}}{dt} H_{41}^{11}(t) + [F_1^d(t)]^{-1} \dot{H}_{41}^{11}(t) \right\} F_1(t) x_1 + [F_1^d(t)]^{-1} H_{41}^{11} \dot{F}_1(t) x_1 + [F_1^d(t)]^{-1} H_{41}^{11} F_1(t) \dot{x}_1 \quad (71)$$

by substituting (21), (70) and (71) into the first line of (33) with (43),

$$\begin{aligned} \dot{x}_1 = & H_{11}^{11}(t) z_1 + H_{11}^{12}(t) \dot{x}_2 + H_{11}^{13}(t) \dot{x}_2^d + H_{13}^{11} u \\ & + H_{12}^{11}(t) M_h(t) [H_{21}^{11}(t) z_1 + H_{21}^{12}(t) \dot{x}_2 + H_{21}^{13}(t) \dot{x}_2^d \\ & + H_{23}(t) u] + H_{14}^{11}(t) \\ & \times \left\{ \frac{d[F_1^d(t)]^{-1}}{dt} H_{41}^{11}(t) + [F_1^d(t)]^{-1} \dot{H}_{41}^{11}(t) \right\} F_1(t) x_1 \\ & + H_{14}^{11}(t) [F_1^d(t)]^{-1} H_{41}^{11} \dot{F}_1(t) x_1 + H_{14}^{11}(t) \\ & \times [F_1^d(t)]^{-1} H_{41}^{11} F_1(t) \dot{x}_1 \end{aligned} \quad (72)$$

simplifying (72) with (42) we have

$$\begin{aligned} \widetilde{E}_1(t) \dot{x}_1 = & [H_{11}^{11}(t) + H_{12}^{11} M_h(t) H_{21}^{11}(t)] F_1(t) x_1 \\ & + [H_{11}^{12}(t) + H_{12}^{11}(t) + M_h(t) H_{21}^{12}(t)] \dot{x}_2 \end{aligned}$$

$$\begin{aligned} & + [H_{11}^{13}(t) + H_{12}^{11}(t) M_h(t) H_{21}^{13}(t)] \dot{x}_2^d \\ & + H_{14}^{11}(t) [F_1^d(t)]^{-1} H_{41}^{11} F_1(t) \dot{x}_1 \\ & + [H_{13}^{11}(t) + H_{12}^{11}(t) M_h(t) H_{23}(t)] u \\ & + H_{14}^{11}(t) \left\{ \frac{d[F_1^d(t)]^{-1}}{dt} H_{41}^{11}(t) \right. \\ & \left. + [F_1^d(t)]^{-1} \dot{H}_{41}^{11}(t) F_1(t) x_1 \right\} \end{aligned} \quad (73)$$

with (38)–(40) and (41), (73) being written by

$$\dot{x}_1 = \widetilde{A}_{11}(t) x_1 + \widetilde{A}_{12}(t) \dot{x}_2 + \widetilde{B}_1(t) u + \widetilde{A}_{13}(t) \dot{x}_2^d \quad (74)$$

where

$$\widetilde{A}_{12}(t) = [\widetilde{E}_1(t)]^{-1} [H_{11}^{12}(t) + H_{12}^{11}(t) M_h(t) H_{21}^{12}(t)] \quad (75)$$

$$\widetilde{A}_{13}(t) = [\widetilde{E}_1(t)]^{-1} [H_{11}^{13}(t) + H_{12}^{11}(t) M_h(t) H_{21}^{13}(t)] \quad (76)$$

From the second line of (33) and taking (70) with (43),

$$\begin{aligned} z_2 = & [H_{11}^{21}(t) + H_{12}^{21}(t) M_h(t) H_{21}^{11}(t)] F_1(t) x_1(t) \\ & + [H_{11}^{22}(t) + H_{12}^{21}(t) M_h(t) H_{21}^{12}(t)] \dot{x}_2 \\ & + [H_{11}^{23}(t) + H_{12}^{21}(t) M_h(t) H_{21}^{13}(t)] \dot{x}_2^d \\ & + [H_{13}^{21}(t) + H_{12}^{21}(t) M_h(t) H_{23}(t)] u \end{aligned} \quad (77)$$

with (23), (35) and (36), (77) being written by

$$x_2 = \widetilde{A}_{21}(t) x_1 + \widetilde{A}_{22}(t) \dot{x}_2 + \widetilde{A}_{23}(t) \dot{x}_2^d + \widetilde{B}_2 u \quad (78)$$

where

$$\widetilde{A}_{22}(t) = F_2^{-1}(t) [H_{11}^{22}(t) + H_{12}^{21}(t) M_h(t) H_{21}^{12}(t)] \quad (79)$$

$$\widetilde{A}_{23}(t) = F_2^{-1}(t) [H_{11}^{23}(t) + H_{12}^{21}(t) M_h(t) H_{21}^{13}(t)] \quad (80)$$

For the steady-state response,  $\dot{x}_2 = 0$  and  $\dot{x}_2^d = 0$ ; from (78), the real roots of (34) for the fast dynamics are proved.

From (78) with the linearly independent state variables

$$\dot{x}_2 = -\widetilde{A}_{22}^{-1}(t) \widetilde{A}_{21}(t) x_1 + \widetilde{A}_{22}^{-1}(t) x_2 - \widetilde{A}_{22}^{-1}(t) \widetilde{B}_2 u \quad (81)$$

and comparing with (8), the relationships between these models are given by

$$\widetilde{A}_{22}(t) = A_{22}^{-1}(t) \varepsilon \quad (82)$$

$$\widetilde{A}_{21}(t) = -A_{22}^{-1}(t) A_{21}(t) \quad (83)$$

$$\widetilde{B}_2(t) = -A_{22}^{-1}(t) B_2(t) \quad (84)$$

by substituting (81) into (78)

$$\begin{aligned} \dot{x}_1 = & \left[ \widetilde{A}_{11}(t) - \widetilde{A}_{12}(t) \widetilde{A}_{22}^{-1}(t) \widetilde{A}_{21}(t) \right] x_1 \\ & + \left[ \widetilde{A}_{12}(t) \widetilde{A}_{22}^{-1}(t) \right] x_2 \\ & + \left[ \widetilde{B}_1(t) - \widetilde{A}_{12}(t) \widetilde{A}_{22}^{-1}(t) \widetilde{B}_2(t) \right] u \end{aligned} \quad (85)$$

comparing (7) with (85)

$$\widetilde{A}_{12}(t) = A_{12}(t) A_{22}^{-1}(t) \varepsilon \quad (86)$$

$$\widetilde{A}_{11}(t) = A_{11}(t) - A_{12}(t) A_{22}^{-1}(t) A_{21}(t) = A_0(t) \quad (87)$$

$$\widetilde{B}_1(t) = B_1(t) - A_{12}(t) A_{22}^{-1}(t) B_2(t) = B_0(t) \quad (88)$$

it can be seen that the quasi-steady-state model is determined by  $x_2 = 0$  in (85) with (86)–(88) and (10), (37) being proved.

## Proof of Assumptions

### Assumption 4

From the second line of (27)

$$\begin{aligned} A_{22}(t) = & [E_2(t) F_2(t)]^{-1} \\ & \times \left\{ \left[ S_{11}^{22}(t) F_2(t) + S_{11}^{21}(t) M(t) S_{21}^{12}(t) F_2(t) \right] \right. \\ & \left. + S_{14}^{22}(t) \frac{d}{dt} \left\{ \left[ F_2^d(t) \right]^{-1} S_{31}^{22}(t) F_2(t) \right\} \right\} \end{aligned} \quad (89)$$

and

$$\begin{aligned} A_{22}^T(t) = & \left\{ F_2^T(t) \left[ S_{11}^{21}(t) \right]^T \right. \\ & + F_2^T(t) \left[ S_{21}^{12}(t) \right]^T M^T(t) \left[ S_{11}^{21}(t) \right]^T \\ & \times \frac{d}{dt} \left\{ F_2(t) \left[ S_{31}^{22}(t) \right]^T \left[ F_2^d(t) \right]^{-1} \right\} \left[ S_{14}^{22}(t) \right]^T \\ & \left. \times [E_2(t) F_2(t)]^{-T} \right\} \end{aligned} \quad (90)$$

the system will be stable if

$$A_{22}(t) + A_{22}^T(t) \leq 0 \quad (91)$$

the properties for the junction structure are established as

$$S_{11}^{22}(t) = - \left[ S_{11}^{22}(t) \right]^T;$$

$$S_{14}^{22}(t) = - \left[ S_{31}^{22}(t) \right]^T$$

$$S_{12}^{21}(t) = - \left[ S_{21}^{12}(t) \right]^T$$

$$S_{22}(t) = - [S_{22}(t)]^T$$

and

$$F_2 = F_2^T; \quad L = L^T; \quad F_2^d = (F_2^d)^T$$

by substituting (89) and (90) into (91)

$$\begin{aligned} & -F_2^{-1}(t) E_2^{-1}(t) \left[ S_{11}^{22}(t) \right]^T - F_2(t) \left[ S_{11}^{22}(t) \right] \\ & \times F_2^{-1}(t) E_2^{-1}(t) - F_2^{-1}(t) E_2^{-1}(t) \left[ S_{11}^{22}(t) \right]^T \\ & \times \left[ L^{-1}(t) + S_{22}^T(t) \right]^{-1} S_{21}^{12}(t) F_2(t) \\ & - F_2(t) \left[ S_{12}^{21}(t) \right] \left[ L^{-1}(t) + S_{22}^T(t) \right]^{-1} \\ & \times \left[ S_{12}^{21}(t) \right]^T - F_2^{-1}(t) E_2^{-1}(t) \left[ S_{31}^{22}(t) \right]^T \\ & \times \frac{d}{dt} \left\{ \left[ F_2^d(t) \right]^{-1} S_{31}^{22}(t) F_2(t) \right\} \\ & - \frac{d}{dt} \left\{ F_2(t) \left[ S_{31}^{22}(t) \right]^T \left[ F_2^d(t) \right]^{-1} \right\} \\ & \times \left[ S_{14}^{22}(t) \right]^T F_2^{-1}(t) E_2^{-1}(t) \leq 0 \end{aligned} \quad (92)$$

from (92), the stability conditions given in (44) are proved.

### Assumption 5

From the second line of (27) and applying the norm to this expression

$$\begin{aligned} \|A_2(t)\| = & \left\| [E_2(t) F_2(t)]^{-1} \right. \\ & \times \left\{ \left[ S_{11}^{22}(t) F_2(t) + S_{11}^{21}(t) M(t) S_{21}^{12}(t) F_2(t) \right] \right. \\ & \left. + S_{14}^{22}(t) \frac{d}{dt} \left\{ \left[ F_2^d(t) \right]^{-1} S_{31}^{22}(t) F_2(t) \right\} \right\} \left. \right\| \end{aligned} \quad (93)$$

this can be written by

$$\begin{aligned} \|A_2(t)\| \leq & \left\| [E_2(t) F_2(t)]^{-1} \right\| \\ & \times \left\{ \left\| S_{11}^{22}(t) \right\| + \left\| S_{11}^{21}(t) \right\| \times \|M(t)\| \right. \end{aligned}$$

$$\begin{aligned} & \times \|S_{21}^{12}(t)\| \times \|F_2(t)\| \\ & \times \|S_{14}^{22}(t)\| \times \left\| \frac{d}{dt} \left\{ [F_2^d(t)]^{-1} S_{31}^{22}(t) F_2(t) \right\} \right\| \end{aligned} \quad (94)$$

from (94) with (45), Assumption 5 is proved.

### Assumption 6

By obtaining the derivative of the second line of (27)

$$\begin{aligned} \dot{A}_{22}(t) = & \frac{d[E_2(t) F_2(t)]^{-1}}{dt} \left\{ [S_{11}^{22}(t) F_2(t) \right. \\ & + S_{11}^{21}(t) M(t) S_{21}^{12}(t) F_2(t)] \\ & + S_{14}^{22}(t) \frac{d}{dt} \left\{ [F_2^d(t)]^{-1} S_{31}^{22}(t) F_2(t) \right\} \\ & + [E_2(t) F_2(t)]^{-1} \left\{ \frac{dS_{11}^{22}(t) F_2(t)}{dt} \right. \\ & + \frac{dS_{11}^{21}(t) M(t) S_{21}^{12}(t) F_2(t)}{dt} \\ & \left. \left. + \frac{d}{dt} S_{14}^{22}(t) \frac{d}{dt} \left\{ [F_2^d(t)]^{-1} S_{31}^{22}(t) F_2(t) \right\} \right\} \right\} \end{aligned} \quad (95)$$

and applying the norm

$$\begin{aligned} \|\dot{A}_{22}(t)\| \leq & \left\| \frac{d[E_2(t) F_2(t)]^{-1}}{dt} \right\| \\ & \times \left[ \|S_{11}^{22}(t)\| + \|S_{11}^{21}(t)\| \times \|M(t)\| \times \|S_{21}^{12}(t)\| \right] \\ & \times \|F_2(t)\| + \left\| [E_2(t) F_2(t)]^{-1} \right\| \\ & \times \left[ \left\| \frac{dS_{11}^{22}(t) F_2(t)}{dt} \right\| + \left\| \frac{dS_{11}^{21}(t) M(t) S_{21}^{12}(t) F_2(t)}{dt} \right\| \right. \\ & + \left\| \frac{dS_{14}^{22}(t)}{dt} \right\| \times \left\| \frac{d[F_2^d(t)]^{-1} S_{31}^{22}(t) F_2(t)}{dt} \right\| \\ & + \|S_{14}^{22}(t)\| \times \left\| \frac{d^2[F_2^d(t)]^{-1} S_{31}^{22}(t) F_2(t)}{dt^2} \right\| \\ & \left. + \|S_{14}^{22}(t)\| \times \left\| \frac{d^2[F_2^d(t)]^{-1} S_{31}^{22}(t) F_2(t)}{dt^2} \right\| \right] \end{aligned} \quad (96)$$

from (96) and (46), Assumption 6 is proved.

### Assumption 7

By obtaining the derivative of the first line of (27) corresponding to  $A_{12}(t)$

$$\begin{aligned} \dot{A}_{12}(t) &= \frac{dE_1^{-1}(t)}{dt} [S_{11}^{12}(t) + S_{12}^{11}(t) M(t) S_{21}^{12}(t)] F_2(t) \\ &+ E_1^{-1}(t) \frac{d[S_{11}^{12}(t) + S_{12}^{11}(t) M(t) S_{21}^{12}(t)] F_2(t)}{dt} \end{aligned} \quad (97)$$

and applying the norm to  $\dot{A}_{22}(t)$  giving

$$\begin{aligned} \|\dot{A}_{12}(t)\| \leq & \left\| \frac{dE_1^{-1}(t)}{dt} \right\| \left[ \|S_{11}^{12}(t)\| + \|S_{12}^{11}(t)\| \right] \\ & \times \|M(t)\| \times \|S_{21}^{12}(t)\| \times \|F_2(t)\| \\ & + \|E_1^{-1}(t)\| \times \left\| \frac{d[S_{11}^{12}(t) + S_{12}^{11}(t) M(t) S_{21}^{12}(t)]}{dt} \right\| \\ & \times \|F_2(t)\| + \|E_1^{-1}(t)\| \times [\|S_{11}^{12}(t)\| + \|S_{12}^{11}(t)\|] \\ & \times \|M(t)\| \times \|S_{21}^{12}(t)\| \times \left\| \frac{dF_2(t)}{dt} \right\| \end{aligned} \quad (98)$$

from (98) and (47), Assumption 7 is proved.

### Assumption 8

The derivative of the second line of (27) corresponding to  $A_{21}(t)$  is

$$\begin{aligned} \dot{A}_{21}(t) = & \frac{d[E_2(t) F_2(t)]^{-1}}{dt} \\ & \times [S_{11}^{21}(t) + S_{12}^{21}(t) M(t) S_{21}^{11}(t)] F_1(t) \\ & + [E_2(t) F_2(t)]^{-1} \\ & \times \frac{d[S_{11}^{21}(t) + S_{12}^{21}(t) M(t) S_{21}^{11}(t)] F_1(t)}{dt} \end{aligned} \quad (99)$$

and the norm of this component is given by

$$\begin{aligned} \|\dot{A}_{21}(t)\| = & \left\| \frac{d[E_2(t) F_2(t)]^{-1}}{dt} \right\| \\ & \times [\|S_{11}^{21}(t)\| + \|S_{12}^{21}(t)\| \times \|M(t)\| \times \|S_{21}^{11}(t)\|] \\ & \times \|F_1(t)\| + \left\| [E_2(t) F_2(t)]^{-1} \right\| \\ & \times \left\| \frac{d[S_{11}^{21}(t) + S_{12}^{21}(t) M(t) S_{21}^{11}(t)]}{dt} \right\| \times \|F_1(t)\| \\ & + \left\| [E_2(t) F_2(t)]^{-1} \right\| \times [\|S_{11}^{21}(t)\| + \|S_{12}^{21}(t)\|] \\ & \times \|M(t)\| \times \|S_{21}^{11}(t)\| \times \left\| \frac{dF_1(t)}{dt} \right\| \end{aligned} \quad (100)$$

from (100) and (48), Assumption 8 is proved.

## Proof of Theorem 3

Assumptions 4–7 are equivalent to Assumptions 1–3; then, we can consider a continuously differentiable and bounded linear time-varying system on  $T$  modelled by bond graphs (BGI). This BGI represents a LTV system with singular perturbations. Also, the reduced fast model in a bond graph approach is stable according to Assumption 4. Theorem 1 gives the conditions in order to have the result for first-order approximations and Theorem 2 proposes the state approximations for a LTV singularly perturbed system. From (10) with (11) and (12),  $\bar{x}_1$  is obtained, in a bond graph approach Lemma 1; from (37) with (38) and (41),  $\bar{x}_1$  is determined and (49) is proved.

From the reduced fast model  $\dot{x}_2^f = A_{22}(t)x_2^f + B_2(t)u$ , the state  $x_2^f$  is obtained and from (9) the approximation state for the fast dynamics is got by (20). In a bond graph approach, the model for the fast reduced RFBGI is built by removing the elements, bonds and junctions of the slow dynamics for the complete bond graph (BGI). Hence, RFBGI determines  $x_2^f$ , and from SVPBG and (35) of Lemma 1, (50) is proved.

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