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Fuzzy Observer Stabilization for Discrete-Time Takagi–Sugeno Uncertain Systems with *k*-Samples Variations

Ali Bouyahya[1] · Yassine Manai[1] · Joseph Haggège[1]

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Abstract

In this paper, we analyze the problem of the stabilization for discrete-time Takagi–Sugeno fuzzy parametric uncertain systems. The stabilization conditions of these systems are investigated with two observers and two different Lyapunov functions: nonquadratic and delayed nonquadratic. The stabilization conditions are analyzed between k and $k + t$ sample variations in the Lyapunov function. The obtained stabilization results represent an extension of previous works with one-sample variation in discrete time. All the results are obtained in the form of linear matrix inequalities which are solved by using various convex optimization algorithms. Two theorems are proposed, and comparison via simulation is given to demonstrate the robustness of the proposed approaches. Nevertheless, this paper shows that the second proposed observer gives the less conservative results (less restrictive). These reduced conservative results are demonstrated by a larger feasible area of stabilization (stabilization domain) and a fast convergence of estimation errors compared to the first.

Keywords Linear matrix inequalities · Observers · Robust control · Stability analysis · Takagi–Sugeno model · Convex optimization algorithms

1 Introduction

Takagi–Sugeno (T–S) fuzzy systems, called sector nonlinearity approach (Takagi and Sugeno 1985), allow us to describe a nonlinear model as collection of linear time-invariant models that are blended together with nonlinear positive functions sharing the convex sum property. The obtained T–S model is not unique for a given nonlinear model, and the number of linear models exponentially increases with the number of nonlinearities involved in the model (Taniguchi et al. 2001).

One of the most important issues in the study of T–S fuzzy systems is the analysis of the stability/stabilization of these systems in both continuous and discrete time with or without uncertainties, with state feedback or outputs feedback controllers and Lyapunov functions (Chadli and Guerra 2012; Wang et al. 2015; Guerra and Vermeiren 2004; Jia et al. 2015; Jo et al. 2015; Guerra et al. 2012; Cao and Frank 2001). Via various approaches, a great number of stability/stabilization results for T–S fuzzy systems have been reported in the literature (Mozelli et al. 2009; Lee and Kim 2009; Latrach et al. 2015).

Many researches have been investigated with nonquadratic Lyapunov functions with T–S nonlinear systems (Lin et al. 2006; Tanaka et al. 2003; Manai and Benrejeb 2012a; Hui et al. 2015) to reduce the conservatism of the quadratic Lyapunov function. Conservatism comes from different sources: the type of T–S fuzzy model, the way the membership functions (MFs) are dropped off to obtain LMI expressions (Lin et al. 2006; Tanaka et al. 2003), the integration of MFs information (Koo et al. 2011; Fang et al. 2006) and the choice of Lyapunov function (Lee and Kim 2009; Tanaka and Wang 2001; Kruszewski et al. 2008; Manai and Benrejeb 2012b).

Several candidates of Lyapunov function have been proposed in the literature attempting to obtain the less conservative results and have received increasing attention as they attempt to relax the conservatism of stability and stabilization results like the piecewise Lyapunov functions (Ke

✉ Ali Bouyahya
ali.bouyahya@enit.rnu.tn

Yassine Manai
yacine.manai@enit.rnu.tn

Joseph Haggège
joseph.haggege@enit.rnu.tn

[1] Laboratory of Research in Automatic Control, National Engineering School of Tunis, University of Tunis el Manar, BP 37, 1002 Belvédère, Tunis, Tunisia

et al. 2011; Feng 2004, 2006; Zhang and Feng 2008; Qiu et al. 2010; Johansson et al. 1999). Others introduce decision variables (slack variables) in order to provide additional degrees of freedom to the LMI problem (Sala and Ariño 2007; Bouyahya et al. 2016, 2017). Some recent works investigate a delayed nonquadratic Lyapunov function. They proved that a little modification in the Lyapunov function gives a huge feasible area of stabilization (Guerra et al. 2012; Kruszewski et al. 2008; Daafouz and Bernussou 2001; Lendek et al. 2015; Xie et al. 2014). All these works have been investigated with the asymptotic stability/stabilization problems which represent the aim of this paper. Other works are recently investigated with the pointwise-in-time eigenvalue condition (Yueying et al. 2019) and novel time-dependent Lyapunov functional (Jun and Huai-Ning 2018; Eksin 2004) for exponential stability/stabilization analysis of T–S systems. Many researchers investigate the problems of stability/stabilization with controllers and observers.

In past decades, there have been many approaches investigating on observer synthesis for T–S fuzzy systems, such as Ma et al. (1998), Xie et al. (2014) and Zhang et al. (2012). An observer for fault estimation for discrete-time T–S fuzzy systems was investigated in Dang and Wang (2011) in which fuzzy full-order and reduced-order fault estimation observers were taken into account. In Boulkroune et al. (2014), an augmented technique was used to design an observer for a T–S fuzzy system in the presence of disturbances. With T–S fuzzy uncertain systems, several approaches have been developed also. Among of them, the observer-based controller was necessary, where the observer and controller have been built simultaneously (Gassara et al. 2010; Kchaou et al. 2010). However, the uncertainties in these studies must be bounded and satisfy some assumptions. Other work investigates the sliding mode observer to estimate the state variables for a T–S fuzzy system with uncertainties (Dang et al. 2012), but their uncertainties must be bounded and the conditions of the uncertainties must be given from the beginning. Furthermore, the influence of the uncertainties was handled by adding extra parts to the conventional Luenberger observer in Dang et al. (2012). Recently, a new approach for the observer synthesis based on the unknown input method was developed in Yeh et al. (2015) to estimate the state variables of an uncertain T–S fuzzy system. Their method not only eliminates the influence of the uncertainties but also guarantees that the estimation error converges asymptotically to the equilibrium point zero. The works presented in this paper is focused about the design of two new fuzzy observers for the discrete-time parametric T–S systems. In this work, these two observers are based on the interpolation of classical Luenberger observer (1966) with the addition of terms to overcome the uncertainties. The additional terms in these observers represent the addition of the uncertainties in the observers itself, and the gains are also modified compared to the initial structure of Luenberger.

These two observers represent an extension of two existing observers in the literature (Guerra et al. 2012), and the stabilization of this class of systems with these observers is studied with two Lyapunov functions: nonquadratic and delayed nonquadratic, each one is connected to one observer. The study of the stability/stabilization with Lyapunov theory in discrete time is done only by the study of the variation of the Lyapunov function with one-sample variation or k-samples variation if we want our study to be more precise. By considering the variation with k-samples, we obtain a large set of solutions in sense of LMI (feasible area of stabilization or stabilization domain) and gives a better solutions and better matrices controllers (fast convergence and decrease the amplitude of signal control) than one-sample variation. The stabilization with one-sample variation is done in Bouyahya et al. (2015). This paper investigates the stabilization with k-samples variation, and we consider the stabilization of the estimation errors for uncertain parametric discrete-time Takagi–Sugeno (T–S) models. The stabilization conditions by the two proposed observers were formulated as two theorems. The comparison between the two conditions of stabilization shows that the second is more effective than the first and the second observer can be considered as a relaxed fuzzy observer.

This paper is organized as follows. In the first section, the problem formulation and the mathematical tools are presented. The second section is dedicated to the description of a summary of the proposed observers and their stabilization conditions with one-sample variation. In the third section, we present the new stabilization conditions with k-samples variations. Fourth section is dedicated to the simulation results, and comparison between the proposed theorems obtained with each Lyapunov function is discussed. We finish by conclusion.

2 System Description and Preliminaries

The discrete-time T–S fuzzy parametric uncertain model is described by fuzzy «IF–THEN» rules, whose collection represents the approximation of the nonlinear system. The ith rule of the T–S fuzzy model is of the following form.

$$\begin{aligned}&\text{If } z_1(t) \text{ is } M_{i1} \text{ and} \ldots \text{and } z_p(t) \text{ is } M_{ip} \text{ then}\\ &\begin{cases} x(k+1) = (A_i + \Delta A_i)x(k) + (B_i + \Delta B_i)u(k) \\ y(k) = (C_i + \Delta C_i)x(k) \end{cases}\\ &\text{for } i = \{1 \ldots r\}\end{aligned} \tag{1}$$

where $M_{ij}(i = 1, 2 \ldots r,\ j = 1, 2 \ldots p)$ is a fuzzy set and r is the number of model rules, $x(k) \in \Re^n$ is the states vector; $u(k) \in \Re^m$ is the input vector; $A_i \in \Re^{n\times n}$ represent the state matrices; $B_i \in \Re^{n\times m}$ represent the control matrices;

$C_i \in \Re^{n\times m}$ represent the output matrices; and $z_1(k), \ldots, z_p(k)$ are known premise variables. ΔA_i, ΔB_i, ΔC_i are time-varying matrices representing parametric uncertainties in the T–S model. These uncertainties are bounded in norm and structured.

The final outputs of the fuzzy parametric uncertain T–S system are

$$\begin{cases} x(k+1) = \sum\limits_{i=1}^{r} h_i(z(k))\{(A_i + \Delta A_i)x(k) + (B_i + \Delta B_i)u(k)\} \\ y(k) = \sum\limits_{i=1}^{r} h_i(z(k))(C_i + \Delta C_i)x(k) \end{cases} \tag{2}$$

where

$$z(k) = \left[z_1(k)\ z_2(k)\ \ldots z_p(k)\right] \tag{3}$$

$$h_i(z(k)) = \frac{w_i(z(k))}{\sum_{i=1}^{r} w_i(z(k))} \tag{4}$$

$$w_i(z(k)) = \prod_{j=1}^{p} M_{ij}\left(z_j(k)\right) \tag{5}$$

For each sample k, the term $M_{i1}\left(z_j(k)\right)$ is defined as the membership degree of $z_j(k)$ in M_{ij}.

Since

$$\begin{cases} \sum\limits_{i=1}^{r} w_i(z(k)) \succ 0 \\ w_i(z(k)) \geq 0 \quad i = 1\ldots r \end{cases}, \tag{6}$$

we have

$$\begin{cases} \sum\limits_{i=1}^{r} h_i(z(k)) = 1 \\ h_i(z(k)) \geq 0 \quad i = 1, 2\ldots r \end{cases}$$
$$\text{for all } k \tag{7}$$

Lemmas 1–5 present the techniques and tools used through the development of the next theorems. Each of these techniques is important in the development of the stabilization condition.

Lemmas 1 and 2 represent a simplification technique of the quadratic form for some matrix representations, and Lemmas 3–5 represent some techniques of relaxation (complexity's reduction) for the stability and stabilization form.

The uncertainties can be replaced to matrices under the following form.

$$\begin{cases} \Delta A = H_a \Delta_{az} E_a \\ \|\Delta_{az}\|_2 \leq 1 \end{cases}, \quad \begin{cases} \Delta B = H_b \Delta_{bz} E_b \\ \|\Delta_{bz}\|_2 \leq 1 \end{cases}, \quad \begin{cases} \Delta C = H_c \Delta_{cz} E_c \\ \|\Delta_{cz}\|_2 \leq 1 \end{cases}$$

Lemma 1 (Schur Complement) (Boyd et al. 1994) *If $P \in \Re^{m\times m}$ definite positive matrix, $X \in \Re^{m\times n}$ full rank matrix in line, and $Q \in \Re^{n\times n}$ any matrix both following inequalities are equivalent.*

$$1.\ Q(s) - X^T(s)P^{-1}(s)X(s) > 0, \quad P(s) > 0 \tag{8}$$

$$2. \begin{bmatrix} Q(s) & (*) \\ X(s) & P(s) \end{bmatrix} > 0. \tag{9}$$

Lemma 2 (Wang and Mendel 1992) *Let us consider X and Y, $Q = Q^T > 0$ matrices with appropriate dimensions, the following inequality holds.*

$$XY^T + YX^T \leq XQX^T + YQ^{-1}Y^T. \tag{10}$$

Relaxation Whatever the choice of the Lyapunov function, the analysis of the stabilization leads us to the next inequality (11).

$$\sum_{k=1}^{r}\sum_{i=1}^{r}\sum_{i<j}^{r} v_i h_i h_j x^T \Upsilon_{ij}^k x \ < 0 \tag{11}$$

Lemma 3 (Tong et al. 2011) *Equation (11) for discrete time is fulfilled if the following conditions hold:*

$$\Upsilon_{ii}^k \prec 0 \quad \forall i, k \in \{1\ldots r\} \tag{12}$$

$$\frac{2}{r-1}\Upsilon_{ii}^k + \Upsilon_{ij}^k + \Upsilon_{ji}^k \prec 0 \quad \forall i, j, k \in \{1\ldots r\}, i \neq j. \tag{13}$$

Lemma 4 (Guerra et al. 2012) *Consider the symmetric matrices $\Upsilon_{i_1\ldots i_k, j_1\ldots j_k}$ hold true if the next conditions are true for each $(i_1 \ldots i_k) \in \{1 \ldots r\}^k$ and $(j_1 \ldots j_k) \in \{1 \ldots r\}^k$, such that $i_n \leq j_n$, $n \in \{1 \ldots r\}$.*

$$\sum_{l_1\in I_1}\sum_{\substack{l_1\in I_1 \\ m_1\neq l_1}} \cdots \sum_{l_k\in I_k}\sum_{\substack{l_k\in I_k \\ m_k\neq l_k}} \Upsilon_{l_1,\ldots l_k, m_1,\ldots m_k} < 0$$
$$\text{with } I_n = \{i_n, j_n\}, \quad n \in \{1\ldots k\}. \tag{14}$$

Note: define the following expression

$$\begin{aligned} &Y_{z(k-1),z(k),\ldots z(k+t-1),z(k),\ldots,z(k+t-1)} \\ &\triangleq \sum_{i_{k-1}=1}^{r} h_{i_0}(z(k)) \times (\sum_{i_0=1}^{r}\sum_{i_{k-1}=1}^{r}\cdots\sum_{i_0=1}^{r}\sum_{i_{k-1}=1}^{r} h_{i_1}(z(k)) \\ &\times h_{i_k}(z(k+t-1)) \times \cdots \times h_{j_k}(z(k+t-1))\Upsilon_{i_1,\ldots i_k, j_1,\ldots j_k}^{i_0}. \end{aligned} \tag{15}$$

As usual, the stability conditions will be depending on the definite negativity of the previous multiple sums. Then, Eq. (14) can be written under the form below.

$$Y_{z(k-1),z(k),\ldots z(k+t-1),z(k),\ldots z(k+t-1)} < 0 \tag{16}$$

Lemma 5 [Finsler's lemma (Skelton et al. 1998)] *Consider two matrices* $Q = Q^T \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times n}$ *and the vector* $x \in \mathbb{R}^n$ *such that* $\operatorname{rank}(R) < n$. *The following two expressions are equivalent*:

1. $x^T Qx \prec 0 \quad \forall x \in \{x \in \mathbb{R}^n,\ x \neq 0,\ Rx = 0\}$
2. $\exists M \in \mathbb{R}^{n\times m}$ such that $Q + MR + R^T M^T < 0$

These lemmas are required items for the stabilization analysis and the observer design. Next sections illustrate its use.

In the next, a brief presentation of the discrete T–S system stabilization is necessary for the understanding of our contribution. This presentation contains both the nonquadratic and delayed nonquadratic delayed Lyapunov functions and the two observers proposed by Guerra et al. (2012) in the discrete-time without uncertainties. All the development details can be found in Guerra et al. (2012).

Consider a class of discrete-time nonlinear system described by the following T–S fuzzy model:

If $z_1(t)$ is M_{i1} and $z_2(t)$ is $M_{21}\ldots$ and $z_p(t)$ is M_{ip} then

$$\begin{cases} x(k+1) = A_i\,x(k) + B_i\,u(k) \\ y(k) = C_i\,x(k) \end{cases} \text{for } i = \{1\ldots r\} \tag{17}$$

The overall discrete-time T–S fuzzy model is represented under the following form:

$$\begin{cases} x(k+1) = A_{z(k)}\,x(k) + B_{z(k)}\,u(k) \\ y(k) = C_{z(k)}\,x(k) \end{cases} \tag{18}$$

with

$$A_z = \sum_{i=1}^{r} h_i(z(k))A_i\ ,\quad B_z = \sum_{i=1}^{r} h_i(z(k))B_i\ ,\quad C_z = \sum_{i=1}^{r} h_i(z(k))C_i$$

where $h_i(z(k))$ denotes the ith normalized fuzzy weighting function. Recently, two fuzzy observers given by Eqs. (19) and (20) based on nonquadratic and delayed nonquadratic Lyapunov functions are designed by (Guerra et al. 2012).

$$\begin{cases} \hat{x}(k+1) = A_{z(k)}\,\hat{x}(k) + B_{z(k)}\,u(k) + G^{-1}_{z(k)}K_{z(k)}(y(k) - \hat{y}(k)) \\ \hat{y}(k) = C_{z(k)}\,\hat{x}(k) \end{cases} \tag{19}$$

and

$$\begin{cases} \hat{x}(k+1) = A_{z(k)}\,\hat{x}(k) + B_{z(k)}\,u(k) \\ \quad +G^{-1}_{z(k-1)z(k)}K_{z(k-1)z(k)}(y(k) - \hat{y}(k)) \\ \hat{y}(k) = C_{z(k)}\,\hat{x}(k) \end{cases} \tag{20}$$

The less conservative results of estimated error are given by observer in Eq. (20) than the first in Eq. (19). That means better results in the sense of feasibility of the nonlinear model and a fast convergence of the state's variables to the equilibrium point zero are obtained.

In Eqs. (19) and (20), $\hat{x}(k)$ is the estimated state, and $G_{z(k)}$, $K_{z(k)}$, $G_{z(k-1)z(k)}$, $K_{z(k-1)z(k)}$ are the observer matrices given by:

$$\begin{cases} G_{z(k)} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(k-1))h_j(z(k))G_j \\ K_{z(k)} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(k))h_j(z(k))K_j \end{cases},$$

$$\begin{cases} G_{z(k-1)z(k)} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(k-1))h_j(z(k))G_{ij} \\ K_{z(k-1)z(k)} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(k-1))h_j(z(k))K_{ij} \end{cases}$$

G_j, K_j, G_{ij}, K_{ij} are the fuzzy observer's matrices to be determined by the LMI algorithm.

For consistency, with the relaxed observer (20) the estimation error becomes:

$$e(k+1) = (A_{z(k)} - G^{-1}_{z(k-1)z(k)}K_{z(k-1)z(k)}C_{z(k)})e(k) \tag{21}$$

where $e(k) = x(k) - \hat{x}(k)$

By the use of the delayed nonquadratic Lyapunov function (Guerra et al. 2012),

$$V(e,z) = e^T(k)\left(\sum_{i=1}^{r} h_i(z(k-1))P_i\right)e(k) = e^T(k)P_{z(k-1)}e(k) \tag{22}$$

The condition of stabilization of discrete-time T–S with observer in Eq. (20) is given by the next theorem:

Corollary (Guerra et al. 2012) *The estimation error* (21) *is globally asymptotically convergent if there exist matrices* G_{ij}, K_{ij}, P_i *such that conditions* (12) *and* (13) *of Lemma* 3 *hold with* Υ^k_{ij} *defined by*:

$$\Upsilon^k_{ij} = \begin{bmatrix} -P_k & (*) \\ G_{jk}A_i - K_{jk}C_i & -G_{jk} - G^T_{jk} + P_j \end{bmatrix} < 0 \tag{23}$$

In the next section, a brief presentation of the stabilization analysis for discrete-time T–S parametric uncertain model based on a nonquadratic and delayed nonquadratic Lyapunov functions in the discrete case. The variation of the Lyapunov function is considered for one-sample variation. If the final equation of this variation is negative, we obtain a sufficient condition of the T–S parametric uncertain stabilization with the output feedback. Two theorems are proposed (Bouyahya et al. 2015), *and the two last observers are little modified to facilitate the development of the stabilization condition*.

3 Generalization for One-Sample Variation

3.1 Nonquadratic Lyapunov Function

Let us consider the following discrete-time Takagi–Sugeno parametric uncertain system:

$$\begin{cases} x(k+1) = (A_{z(k)} + \Delta A_{z(k)})x(k) + (B_{z(k)} + \Delta B_{z(k)})u(k) \\ y(k) = (C_{z(k)} + \Delta C_{z(k)})x(k) \end{cases} \tag{24}$$

With

$$A_{z(k)} = \sum_{i=1}^{r} h_i(z(k))A_i, \quad B_{z(k)} = \sum_{i=1}^{r} h_i(z(k))B_i,$$
$$C_{z(k)} = \sum_{i=1}^{r} h_i(z(k))C_i$$
$$\Delta A_{z(k)} = \sum_{i=1}^{r} h_i(z(k))\Delta A_i, \quad \Delta B_{z(k)} = \sum_{i=1}^{r} h_i(z(k))\Delta B_i,$$
$$\Delta C_{z(k)} = \sum_{i=1}^{r} h_i(z(k))\Delta C_i$$

The observer given by Eq. (25) represents an observer taking in account uncertainties in order to reduce the number of LMIs and the matrix size of the final condition of stability Υ_{ij}^k.

$$\begin{cases} \hat{x}(k+1) = (A_{z(k)} + \Delta A_{z(k)})\hat{x}(k) + (B_{z(k)} + \Delta B_{z(k)})u(k) \\ \quad +G_{z(k)}^{-1}K_{z(k)}(y(k) - \hat{y}(k)) \\ \hat{y}(k) = (C_{z(k)} + \Delta C_{z(k)})\,\hat{x}(k) \end{cases} \tag{25}$$

The observers (19) and (20) with the T–S parametric uncertain system give a very complex result in terms of LMI. So, the estimation error with the use of the nonquadratic Lyapunov function can be written in the following form:

$$e(k+1) = ((A_{z(k)} + \Delta A_{z(k)}) - G_{z(k)}^{-1}K_{z(k)}(C_{z(k)} + \Delta C_{z(k)}))e(k) \tag{26}$$

The nonquadratic Lyapunov function is written under the following form in Eq. (27).

$$V(e,z) = e^T(k)\left(\sum_{i=1}^{r} h_i(z(k))P_i\right)e(k) \; = e^T(k)P_{z(k)}e(k) \tag{27}$$

Equation (28) represents the final LMI inequality satisfied the stabilization of the considered uncertain system with non-quadratic Lyapunov function. All the development details are found in (Bouyahya et al. 2015).

$$\Upsilon_{ij}^k = \begin{bmatrix} -P_i & (*) & (*) & (*) \\ G_jE_{ai} & -\lambda I & 0 & 0 \\ K_jE_{ci} & 0 & -\tau I & 0 \\ G_jA_i - K_jC_i & 0 & 0 & -G_j^T - G_j + P_k + \lambda H_a H_a^T + \tau H_c H_c^T \end{bmatrix} \leq 0 \tag{28}$$

For this result, the next theorem was stated:

Theorem 1 (Bouyahya et al. 2015) *The estimation error* (26) *is globally asymptotically stable if there exist definite positive symmetric matrices P_i and matrices G_i , K_i $\{\mathbf{i} =\}$ and scalars λ, τ such that conditions* (12) *and* (13) *of Lemma* 3 *hold with Υ_{ij}^k defined in* (28).

In the next part of this section, relaxed conditions are proposed using a delayed nonquadratic Lyapunov function. In order to design a new fuzzy observer, this Lyapunov function is proposed.

Remark The delayed nonquadratic Lyapunov function is obtained from the classic nonquadratic form (Eq. 27) by replacing $P_{z(k)}$ by $P_{z(k-1)}$, and the relaxed observer is obtained from the first observer (Eq. 25) by replacing $G_{z(k)}$ by $G_{z(k-1)z(k)}$ and $K_{z(k)}$ by $K_{z(k-1)z(k)}$. This little modification reduces the conservatism of the first stabilization condition represented in Theorem 1.

3.2 Delayed Nonquadratic Lyapunov Function

The observer design with the use of delayed nonquadratic Lyapunov function is written under the following form.

$$\begin{cases} \hat{x}(k+1) = (A_{z(k)} + \Delta A_{z(k)})\hat{x}(k) + (B_{z(k)} + \Delta B_{z(k)})u(k) \\ \quad + G_{z(k-1)z(k)}^{-1}K_{z(k-1)z(k)}(y(k) - \hat{y}(k)) \\ \hat{y}(k) = (C_{z(k)} + \Delta C_{z(k)})\hat{x}(k) \end{cases} \tag{29}$$

In this case, the estimation error can be written in the form given by Eq. (30).

$$\begin{aligned} e(k+1) = ((A_{z(k)} + \Delta A_{z(k)}) \\ - G_{z(k-1)z(k)}^{-1}K_{z(k-1)z(k)}(C_{z(k)} + \Delta C_{z(k)}))e(k) \end{aligned} \tag{30}$$

The delayed nonquadratic Lyapunov function is written under the following form

$$V(e,z) = e^T(k)\left(\sum_{i=1}^{r} h_i(z(k-1))P_i\right)e(k) \; = e^T(k)P_{z(k-1)}e(k) \tag{31}$$

The estimation error (30) converges if the following inequality is satisfied.

With the symmetric matrix, $\Xi_{z(k-1)\ldots z(t+k-2)} > 0$ and $\Xi_{z(k-1)\ldots z(t+k-2)} = \mathrm{diag}(P_{z(k-1)} \ P_{z(k)} \ \ldots P_{z(k+t-2)})$.

$$\Upsilon_{ij}^{k} = \begin{bmatrix} -P_k & (*) & (*) & (*) \\ G_{jk}E_{ai} & -\lambda I & 0 & 0 \\ K_{jk}E_{ci} & 0 & -\tau I & 0 \\ G_{jk}A_i - K_{jk}C_i & 0 & 0 & -G_{jk}^T - G_{jk} + P_j + \lambda H_a H_a^T + \tau H_c H_c^T \end{bmatrix} \leq 0 \tag{32}$$

Therefore, the following theorem was stated:

Theorem 2 (Bouyahya et al. 2015) *The estimation error* (30) *is globally asymptotically stable if there exist symmetric definite positive matrices P_i and matrices G_{ij}, K_{ij} and scalars λ, τ such that conditions* (12) *and* (13) *of Lemma* 3 *hold with Υ_{ij}^k defined in* (32).

The proposed Theorems 1 *and* 2 *represent two sufficient conditions for the stabilization of the estimation error with one-sample variation. These results can be extended more by considering the variation of the Lyapunov function with k-samples. So in the next section, the stabilization condition of k-samples variations will be discussed.*

4 Generalization for *k*-Samples Variations

In discrete-time case, when more samples are considered, the results become less conservative in both controllers and observers. The main idea of the k-samples variation form is to replace the one-sample variation of the Lyapunov function by its variation over k-samples. Considering the k-samples variation with a nonquadratic and delayed nonquadratic Lyapunov functions represent the aim of this section. It is based on a one-sample variation using an extended state vector with the use of the past values of the state vector. It is written under the Finsler's lemma form (Skelton et al. 1998). The idea, in this case, is to use the higher-order variation in the state vector. The advantage of Finsler's lemma representation is to present a natural extension of the conditions of stabilization with observer. It provides a unified framework to solve stabilization problems. So, the state vector in this case of observer design is replaced by the estimation error vector.

For k-samples variations, the estimation error vector becomes

$$E_{0\ldots t}^T(k) = \left[e^T(k)\, e^T(k+1) \ldots e^T(k+t) \right] \tag{33}$$

By replacing the estimation error by its estimation error vector (33), the Lyapunov function becomes

$$V(E_{0\ldots t-1}, z) = E_{0\ldots t-1}^T(k) \Xi_{z(k-1)\ldots z(t+k-2)} E_{0\ldots t-1}(k) \tag{34}$$

The variation of the Lyapunov function for each k-samples is given in the following equation:

$$\begin{aligned} \Delta V(E_{0\ldots k-1}, z) &= E_{0\ldots t-1}^T(k)\, Q_{z(k-1)\ldots z(k+t-1)}\, E_{0\ldots t-1}(k) \\ &= E_{0\ldots t-1}^T(k) \left(\begin{bmatrix} 0_{n\times n} & 0_{n\times kn} \\ 0_{kn\times n} & \Xi_{z(k)\ldots z(t+k-1)} \end{bmatrix} - \begin{bmatrix} \Xi_{z(k-1)\ldots z(t+k-2)} & 0_{kn\times n} \\ 0_{n\times kn} & 0_{n\times n} \end{bmatrix} \right) E_{0\ldots t-1}(k) \end{aligned} \tag{35}$$

with $Q_{z(k-1)\ldots z(k+t-1)} = \begin{bmatrix} 0_{n\times n} & 0_{n\times kn} \\ 0_{kn\times n} & \Xi_{z(k)\ldots z(t+k-1)} \end{bmatrix} - \begin{bmatrix} \Xi_{z(k-1)\ldots z(t+k-2)} & 0_{kn\times n} \\ 0_{n\times kn} & 0_{n\times n} \end{bmatrix}$.

The dynamic of the estimation error at different samples is written under the following form.

$$\begin{bmatrix} \bar{A}_{z(k-1)z(k)} & -I & 0 & \ldots & 0 \\ 0 & \bar{A}_{z(k)z(k+1)} & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \bar{A}_{z(k+1)z(k+2)} & -I \end{bmatrix} \begin{bmatrix} e(k) \\ e(k+1) \\ \vdots \\ e(k+t) \end{bmatrix} = 0 \tag{36}$$

such as $\bar{A}_{z(k-1)z(k)} = (A_{z(k)} + \Delta A_{z(k)}) - G_{z(k)}^{-1} K_{z(k)} (C_{z(k)} + \Delta C_{z(k)})$.

In order to use Lemma 5 (Finsler's lemma), the k-samples variation of Lyapunov function is written as follows:

$$Q_{z(k-1)\ldots z(k+t-1)} + M \underbrace{\begin{bmatrix} \bar{A}_{z(k-1)z(k)} & -I & 0 & \ldots & 0 \\ 0 & \bar{A}_{z(k)z(k+1)} & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \bar{A}_{z(k+1)z(k+2)} & -I \end{bmatrix}}_{R} + (*) < 0 \tag{37}$$

where (*) represents $R^T M^T$, with M being an appropriate matrix written in the following form:

$$M_{z(k)\ldots z(k+t-1)} = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ G_{z(k)} & & \ddots & \vdots \\ 0 & G_{z(k)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & G_{z(k+t-1)} \end{bmatrix}$$

This equation can be written with the use of Lemma 2 in the form (38):

$$\begin{bmatrix} -P_{z(k)} + \Omega_0^1 & (*) & \ddots & 0 \\ G_{z(k)}A_{z(k)} - K_{z(k)}C_{z(k)} & -G_{z(k)} - G^T_{z(k)} + \Omega_0^2 & \ddots & \ddots \\ \ddots & \ddots & & \\ & & \ddots & (*) \\ 0 & \ddots & G_{z(k+t-1)}A_{z(k+t-1)} - K_{z(k+t-1)}C_{z(k+t-1)} & -G_{z(k+t-1)} - G^T_{z(k+t-1)} + P_{z(k+t-1)} + \Omega^2_{k-1} \end{bmatrix} < 0 \quad (38)$$

where $\Omega_i^2 = \tau_i H_a H_a^T + \mu_i H_c H_c^T$ $\Omega_i^1 = \tau_i^{-1} G^T_{z(k+i)} E^T_{a\,z(k+i)} E_{a\,z(k+i)} G_{z(k+i)} + \mu_i^{-1} K^T_{z(k+i)} E^T_{c_{z(k+i)}} E_{c_{z(k+i)}} K_{z(k+i)}$.

Using Lemma 1 (Schur complement), the final condition of observer stabilization is written in the following form.

$$\begin{bmatrix} -P_{z(k)} & (*) & (*) & (*) & 0 \;\cdots & & \cdots & \cdots & 0 \\ K_{z(k)}E_{cz(k)} & -\mu_0 I & 0 & 0 & & & & & \vdots \\ G_{z(k)}E_{az(k)} & 0 & -\tau_0 I & 0 & & & & & \vdots \\ G_{z(k)}A_{z(k)} - K_{z(k)}C_{z(k)} & 0 & 0 & -G^T_{z(k)} - G_{z(k)} + \Omega_0^2 & \ddots & \ddots & & & \vdots \\ 0 & & & \ddots & \ddots & & & & \vdots \\ \vdots & & & \ddots & \ddots & & 0 & 0 & 0 \\ & & & & & -G^T_{z(k)} - G_{z(k)} + \Omega^2_{k-2} & (*) & (*) & (*) \\ \vdots & & & 0 & & K_{z(k)}E_{bz(k)} & -\mu_{k-1} I & 0 & 0 \\ & & & & & G_{z(k)}E_{az(k)} & 0 & -\tau_{k-1} I & 0 \\ \vdots & & & 0 & & G_{z(k)}A_{z(k)} - K_{z(k)}C_{z(k)} & 0 & 0 & -G^T_{z(k)} - G_{z(k)} + \Omega^2_{k-1} \\ 0 & \cdots & \cdots & 0 & & & & & \end{bmatrix} < 0 \quad (39)$$

The use of Lemma 4 leads us to the following inequality

$$\Upsilon_{i_1,\ldots i_k, j_1,\ldots j_k} = \begin{bmatrix} -P_{i_1} & (*) & (*) & (*) & 0 \;\cdots & & \cdots & \cdots & 0 \\ K_{j_1}E_{ci_1} & -\mu_{i_1} I & 0 & 0 & & & & & \vdots \\ G_{j_1}E_{ai_1} & 0 & -\tau_{i_1} I & 0 & & & & & \vdots \\ G_{j_1}A_{i_1} - K_{j_1}C_{i_1} & 0 & 0 & -G^T_{j_1} - G_{j_1} + \Omega^2_{j_1} & \ddots & \ddots & & & \vdots \\ 0 & & & \ddots & \ddots & & & & \vdots \\ \vdots & & & \ddots & \ddots & & 0 & 0 & 0 \\ & & & & & -G^T_{j_{k-1}} - G_{j_{k-1}} + \Omega^2_{j_{k-1}} & (*) & (*) & (*) \\ \vdots & & & 0 & & K_{j_k}E_{ci_k} & -\mu_{i_k} I & 0 & 0 \\ & & & & & G_{j_k}E_{ai_k} & 0 & -\tau_{i_k} I & 0 \\ \vdots & & & 0 & & G_{j_k}A_{i_k} - K_{j_k}C_{i_k} & 0 & 0 & -G^T_{j_k} - G_{j_k} + P_{j_k} + \Omega^2_{j_k} \\ 0 & \cdots & \cdots & 0 & & & & & \end{bmatrix} < 0 \quad (40)$$

This inequality represents a sufficient condition of stabilization of the estimation error with observer design (25) for k-samples variations with ($k \geq 2$). Therefore, we state the following theorem.

Theorem 3 *The estimation error (26) is globally asymptotically stable with k-samples variation if there exist symmetric definite positive matrices P_i and matrices G_i, K_i and scalars λ_i, τ_i such that conditions (12) and (13) of Lemma 3 hold with $\Upsilon_{i_1,\ldots i_k, j_1,\ldots j_k}$ defined in (40).*

The relaxed observer design (29) leads us to relaxed conditions of estimation errors stabilization by using a delayed Lyapunov function, which its parameters depend on k-samples variations ($k \geq 2$) with normalized fuzzy weighting functions.

For the second observer, the dynamic of the estimation error at different samples is written under the following form.

$$\begin{bmatrix} \bar{A}_{z(k-1)z(k)} & -I & 0 & \dots & 0 \\ 0 & \bar{A}_{z(k)z(k+1)} & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \bar{A}_{z(k+1)z(k+2)} & -I \end{bmatrix} \begin{bmatrix} e(k) \\ e(k+1) \\ \vdots \\ e(k+t) \end{bmatrix} = 0 \tag{41}$$

such as $\bar{A}_{z(k-1)z(k)} = (A_{z(k)} + \Delta A_{z(k)}) - G^{-1}_{z(k-1),\, z(k)} K_{z(k-1),\, z(k)} (C_{z(k)} + \Delta C_{z(k)})$.

In order to use Lemma 5 (Finsler's lemma), the k-samples variation of Lyapunov function is written as follows:

$$Q_{z(k-1)\dots z(k+t-1)} + M \underbrace{\begin{bmatrix} \bar{A}_{z(k-1)z(k)} & -I & 0 & \dots & 0 \\ 0 & \bar{A}_{z(k)z(k+1)} & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \bar{A}_{z(k+1)z(k+2)} & -I \end{bmatrix}}_{R} + (*) < 0 \tag{42}$$

where $(*)$ represents $R^T M^T$, with M being an appropriate matrix written in the following form:

$$M_{z(k)\dots z(k+t-1)} = \begin{bmatrix} 0 & 0 & \dots 0 \\ G_{z(k-1)z(k)} & & \ddots \vdots \\ 0 & G_{z(k)z(k+1)} & \ddots \vdots \\ \vdots & \ddots & \ddots 0 \\ 0 & \dots & 0 \quad G_{z(k+t-2)z(k+t-1)} \end{bmatrix}$$

This equation can be written with the use of Lemma 2 in the form (43):

$$\begin{bmatrix} -P_{z(k)} + \Omega_0^1 & (*) & & \\ G_{z(k-1)\, z(k)} A_{z(k)} - K_{z(k-1)z(k)} C_{z(k)} & -G_{z(k-1)\, z(k)} - G^T_{z(k-1)z(k)} + \Omega_0^2 & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & (*) \\ & & G_{z(k-t-2)z(k+t-1)} A_{z(k+t-1)} - K_{z(k-t-2)z(k+t-1)} C_{z(k+t-1)} & -G_{z(k-t-2)z(k+t-1)} - G^T_{z(k-t-2)z(k+t-1)} + P_{z(k+t-1)} + \Omega_{k-1}^2 \end{bmatrix} < 0 \tag{43}$$

where $\Omega_i^2 = \tau_i H_a H_a^T + \mu_i H_c H_c^T$ and $\Omega_i^1 = \tau_i^{-1} G^T_{z(k+i-1)z(k+i)} E^T_{a\, z(k+i)} E_{a\, z(k+i)} G_{z(k+i-1)z(k+i)} + \mu_i^{-1} K^T_{z(k+i-1)z(k+i)} E^T_{c_{z(k+i)}} E_{c_{z(k+i)}} K_{z(k+i-1)z(k+i)}$.

Using Lemma 1 (Schur complement), the final condition of observer stabilization is written in the following form.

$$\begin{bmatrix} -P_{z(k)} & (*) & (*) & (*) & & & & & \\ K_{z(k-1)z(k)} E_{cz(k)} & -\mu_0 I & 0 & 0 & 0 & \dots & \dots & \dots & \\ G_{z(k-1)z(k)} E_{az(k)} & 0 & -\tau_0 I & 0 & & & & & \\ G_{z(k-1)z(k)} A_{z(k)} - K_{z(k-1)z(k)} C_{z(k)} & 0 & 0 & -G^T_{z(k-1)z(k)} - G_{z(k-1)z(k)} + \Omega_0^2 & & & & & \\ 0 & & \ddots & \ddots & \ddots\ \ddots & & & & \\ \vdots & & \ddots & \ddots & \ddots\ \ddots & \ddots & & & \\ \vdots & & & & -G^T_{z(k-1)z(k)} - G_{z(k-1)z(k)} + \Omega_{k-2}^2 & (*) & (*) & (*) & \\ \vdots & & & & K_{z(k-1)z(k)} E_{bz(k)} & & 0 & 0 & \\ \vdots & & & & G_{z(k-1)z(k)} E_{az(k)} & 0 & & 0 & \\ \vdots & & & & G_{z(k-1)z(k)} A_{z(k)} - K_{z(k-1)z(k)} C_{z(k)} & 0 & 0 & -G^T_{z(k-1)z(k)} - G_{z(k-1)z(k)} + \Omega_{k-1}^2 & \\ 0 & \dots & \dots & 0 & & & & & \end{bmatrix} < 0 \tag{44}$$

The use of Lemma 4 leads us to the following inequality

$$
\Upsilon_{i_1,\ldots i_k,j_1,\ldots.j_{k-1},j_k} =
\left[\begin{array}{ccccccccc}
-P_{i_1} & (*) & (*) & (*) & 0 \;\cdots & & \cdots & \cdots & 0 \\
K_{j_0,j_1}E_{ci_1} & -\mu_{i_1}I & 0 & 0 & & & & & \vdots \\
G_{j_0,j_1}E_{ai_1} & 0 & -\tau_{i_1}I & 0 & & & & & \vdots \\
G_{j_0,j_1}A_{i_1}-K_{j_1}C_{i_1} & 0 & 0 & -G^T_{j_0,j_1}-G_{j_0,j_1}+P_{i_1}+\Omega^2_{j_0,j_1} & & & & & \vdots \\
0 & & & \ddots & & & & & \vdots \\
\vdots & & & & \ddots & & & & \vdots \\
\vdots & & & & & & 0 & 0 & 0 \\
\vdots & & & 0 & & -G^T_{j_{k-2},j_{k-1}}-G_{j_{k-2},j_{k-1}}+P_{j_{k-1}}+\Omega^2_{j_{k-1}} & (*) & (*) & (*) \\
\vdots & & & & & K_{j_{k-1},j_k}E_{ci_k} & -\mu_{i_k}I & 0 & 0 \\
\vdots & & & 0 & & G_{j_{k-1},j_k}E_{ai_k} & 0 & -\tau_{i_k}I & 0 \\
0 & & & 0 & & G_{j_{k-1},j_k}A_{i_k}-K_{j_{k-1},j_k}C_{i_k} & 0 & 0 & -G^T_{j_{k-1},j_k}-G_{j_{k-1},j_k}+P_{j_k}+\Omega^2_{j_k}
\end{array}\right] < 0
\tag{45}
$$

This inequality represents a sufficient condition of stabilization of the estimation error with observer design (30) for k-samples variations with ($k \geq 2$). Therefore, we state the following theorem.

Theorem 4 *The estimation error (30) is globally asymptotically stable with k-samples variation if there exist symmetric definite positive matrices P_i and matrices G_{ij} , K_{ij} and scalars λ_i, τ_i such that conditions (12) and (13) of Lemma 3 hold with $\Upsilon_{i_1,\ldots i_k,j_1,\ldots.j_{k-1},j_k}$ defined in (45).*

Theorem 4 develops relaxation of new fuzzy observer design (29), which permits a reduction in the conservatism of observer (25) and to significantly increase the solution sets for nonlinear models. Under the framework of multiple matrices, a relaxed fuzzy observer and a nonquadratic delayed Lyapunov function have been proposed to obtain the less conservative results. In the next section, a comparative study illustrates these results.

5 Numerical Application and Validation of Results

Consider the T–S discrete-time uncertain system with two rules

$$
A_1 = \begin{pmatrix} 1 & -a \\ 0.2 & 1.5 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0.5 & -2 \\ 0.2 & 0.1 \end{pmatrix}
$$
$$
H_{a_1} = \begin{pmatrix} 0.1 & 0.2 \\ 0 & -0.2 \end{pmatrix} \quad H_{a_2} = \begin{pmatrix} -0.9 & 1 \\ 0.3 & 0.33 \end{pmatrix}
$$
$$
C_1 = \begin{pmatrix} -b & 0.5 \end{pmatrix} \quad C_2 = \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad E_{a_1} = \begin{pmatrix} 1 & 0.3 \end{pmatrix}
$$
$$
E_{c_1} = \begin{pmatrix} 1 & -2 \end{pmatrix}, \quad E_{c_2} = \begin{pmatrix} 1 & 0.23 \end{pmatrix}
$$
$$
H_{c_1} = \begin{pmatrix} -1 & 0.4 \end{pmatrix}, \quad H_{c_2} = \begin{pmatrix} -1.2 & 0.3 \end{pmatrix}, \quad E_{a_2} = \begin{pmatrix} 0.6 & 1 \end{pmatrix}
$$

The matrices H_{a_1}, H_{a_2}, E_{a_1}, E_{a_2}, E_{c_2}, H_{c_1} and H_{c_2} represent the uncertainties, and a, b are two scalars such that $a = [0,\ 6]$, $b = [0,\ 3]$.

For the simulation purpose, we choose

$$
h_1(y) = 1 - h_2(y) = \frac{1}{2}(1 + \cos(y))
$$

Figure 1 represents the sets of solutions with the use of theorem 1 with circle mark (°) and theorem 2 with point mark (·).

We conclude that the stabilization condition with relaxed observer [Eq. (29)] with one-sample variation gives a larger set of solution than the classical observer for T–S uncertain system given in Eq. (25).

All the simulation results for stability condition with k-samples variations are realized with $k = 2$. Consider another uncertain T–S fuzzy model represented by two linear fuzzy models $r = 2$:

$$
A_1 = \begin{pmatrix} 0.31 & -0.96 \\ -0.2 & 0.5 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0.5 & -2 \\ 0.2 & 0.1 \end{pmatrix}
$$
$$
H_{a_1} = \begin{pmatrix} 0.1 & 0.2 \\ 0 & -0.2 \end{pmatrix} \quad H_{a_2} = \begin{pmatrix} 0.3 & 0.33 \\ -0.9 & 1 \end{pmatrix}
$$
$$
E_{a_1} = \begin{pmatrix} 1 & 0.3 \\ 0.1 & 1 \end{pmatrix} \quad E_{a_2} = \begin{pmatrix} 0.6 & 1 \\ 0.1 & 1 \end{pmatrix} \quad C_1 = \begin{pmatrix} -1 & 1 \end{pmatrix}
$$
$$
C_2 = \begin{pmatrix} 1 & 1 \end{pmatrix} \quad E_{c_1} = \begin{pmatrix} 1 & -2 \end{pmatrix}
$$
$$
E_{c_2} = \begin{pmatrix} 1 & 0.23 \end{pmatrix} \quad H_{c_1} = \begin{pmatrix} -1 & 0.4 \end{pmatrix}, \quad H_{c_2} = \begin{pmatrix} -1.2 & 0.3 \end{pmatrix}
$$

The application of Theorem 3 with the previous example gives the following results:

$$
P_1 = 10^{-10} \times \begin{pmatrix} -0.0717 & -0.0333 \\ -0.0333 & 0.2961 \end{pmatrix}
$$
$$
P_2 = 10^{-10} \times \begin{pmatrix} -0.0921 & -0.0551 \\ -0.0551 & 0.1830 \end{pmatrix}
$$
$$
G_1 = 10^{-10} \times \begin{pmatrix} 0.1413 & -0.0852 \\ 0.0576 & 0.4042 \end{pmatrix}
$$
$$
G_2 = 10^{-10} \times \begin{pmatrix} 0.1791 & -0.0729 \\ -0.1269 & 0.6689 \end{pmatrix}
$$

Fig. 1 Set of solutions

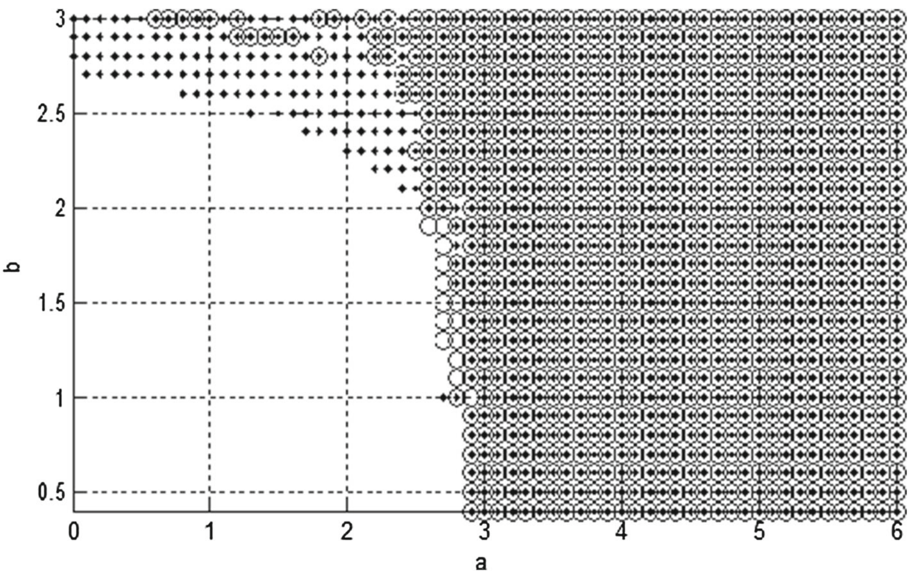
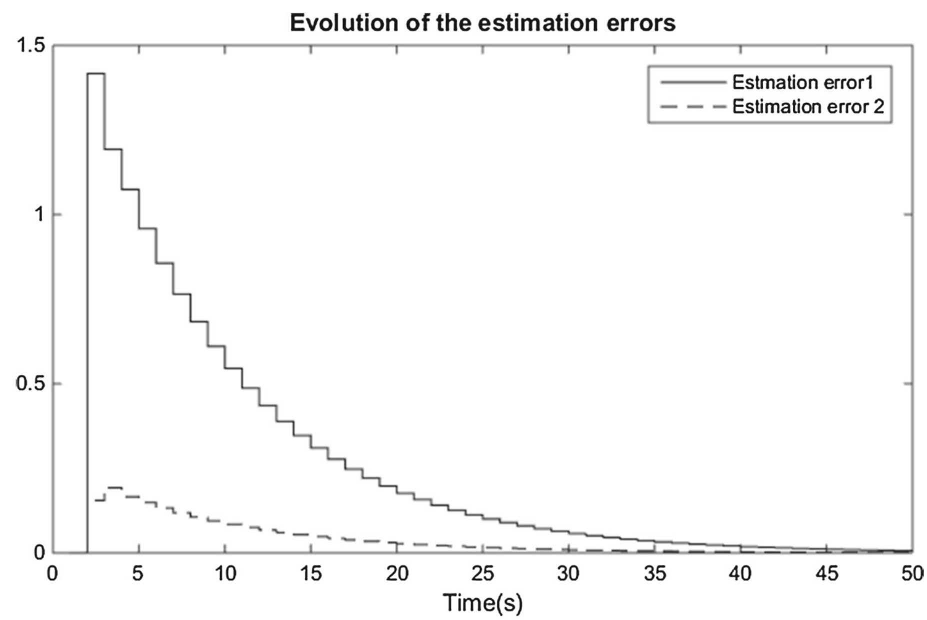


Fig. 2 Evolution of estimation error sub-model 1



$$K_1 = 10^{-11} \times \begin{pmatrix} -0.0650 \\ 0.5407 \end{pmatrix} \quad K_2 = 10^{-11} \times \begin{pmatrix} -0.4363 \\ 0.6037 \end{pmatrix}$$

$$\tau_2 = -1.2603 \times 10^{-12} \quad \lambda_1 = 4.4316 \times 10^{-11}$$

$$\tau_2 = 5.4789 \times 10^{-12} \quad \lambda_2 = 5.2112 \times 10^{-11}$$

The application of Theorem 4 with the previous example gives the following results:

$$P_1 = 10^{-9} \times \begin{pmatrix} 0.0469 & 0.0040 \\ 0.0040 & 0.1089 \end{pmatrix}$$

$$P_2 = 10^{-11} \times \begin{pmatrix} 0.5480 & 0.1911 \\ 0.1911 & 0.6204 \end{pmatrix}$$

$$G_{11} = 10^{-9} \times \begin{pmatrix} 0.0411 & -0.0007 \\ -0.0254 & 0.2312 \end{pmatrix}$$

$$G_{12} = 10^{-9} \times \begin{pmatrix} 0.0319 & -0.0127 \\ -0.0465 & 0.1805 \end{pmatrix}$$

$$G_{21} = 10^{-10} \times \begin{pmatrix} 0.5920 & -0.3800 \\ 0.1652 & 0.9015 \end{pmatrix}$$

$$G_{22} = 10^{-9} \times \begin{pmatrix} 0.0342 & -0.0087 \\ -0.0227 & 0.1455 \end{pmatrix}$$

$$K_{11} = 10^{-10} \times \begin{pmatrix} -0.0826 \\ 0.1421 \end{pmatrix} \quad K_{12} = 10^{-10} \times \begin{pmatrix} -0.1069 \\ 0.2097 \end{pmatrix}$$

$$K_{21} = 10^{-10} \times \begin{pmatrix} -0.2136 \\ -0.0675 \end{pmatrix} \quad K_{22} = 10^{-11} \times \begin{pmatrix} -0.5709 \\ 0.8883 \end{pmatrix},$$

$$\tau_1 = -1.1344 \times 10^{-11}, \quad \lambda_1 = 4.5873 \times 10^{-11},$$

$$\tau_2 = -8.2571 \times 10^{-12}, \quad \lambda_2 = 2.7114 \times 10^{-11}$$

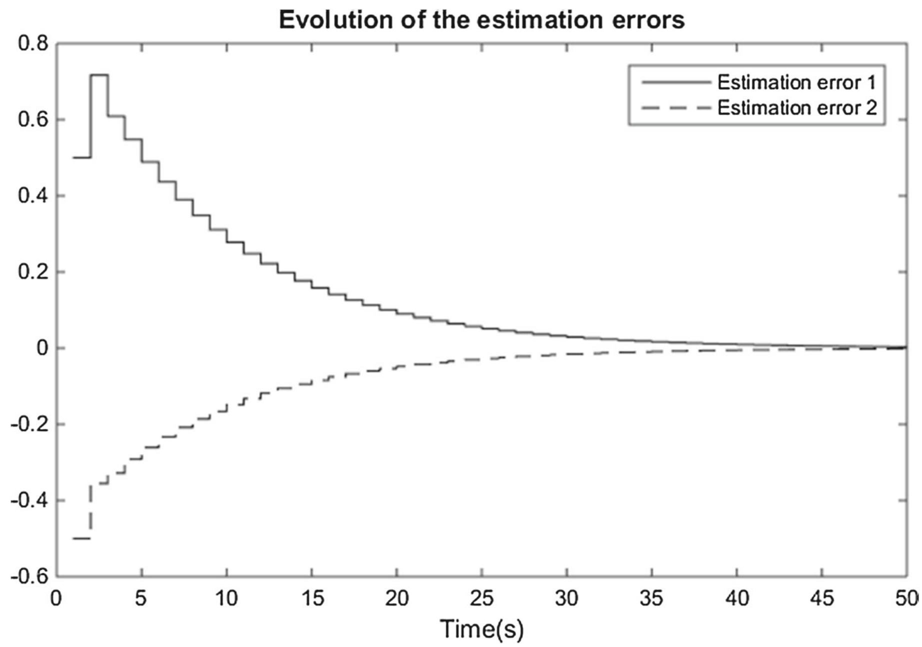


Fig. 3 Evolution of estimation error sub-model 2

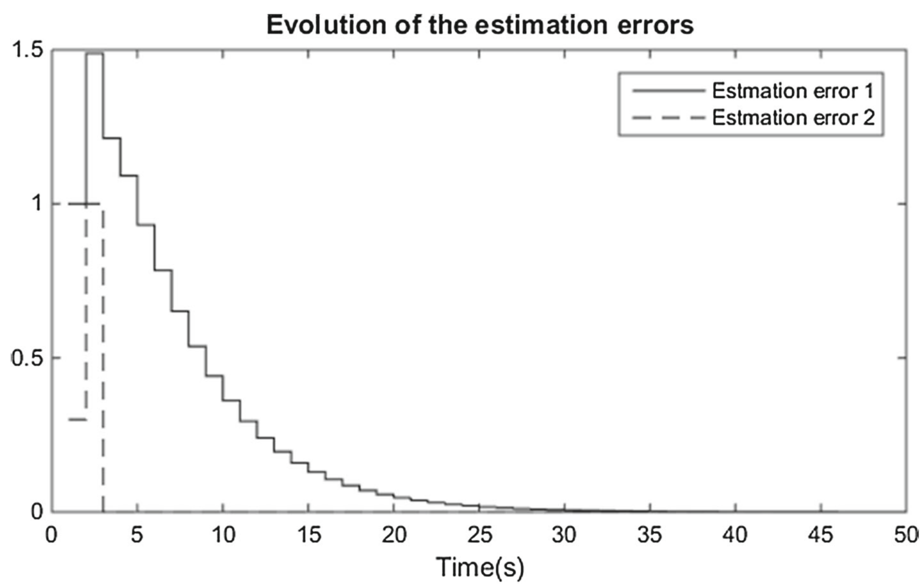


Fig. 4 Evolution of estimation error of global system

Figures 2, 3 and 4 represent the evolution of the estimation error for all sub-systems and the global system, with the observer design (25) and stabilization conditions with k-samples variations (Theorem 3).

Figures 5, 6 and 7 represent the evolution of the estimation error for all sub-systems and the global system, with the relaxed observer design (29) and stabilization conditions with k-samples variations (Theorem 4).

Remark 1 Comparing Figs. 5, 6 and 7 with Figs. 2, 3 and 4, the relaxed observer in Eq. (29) reduced the amplitudes and gives a fast convergence of the estimation errors.

Remark 2 The variation of the Lyapunov function is negative means that all the estimation errors should be convergent to the equilibrium point of systems. These figures demonstrate the estimation errors convergence to the equilibrium point zero.

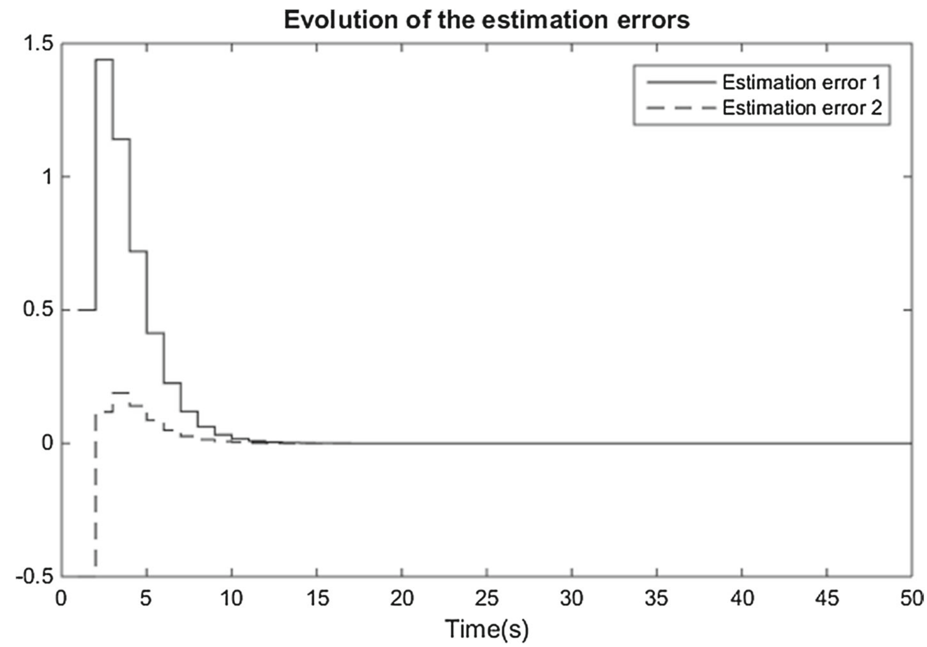


Fig. 5 Evolution of estimation error sub-model 1

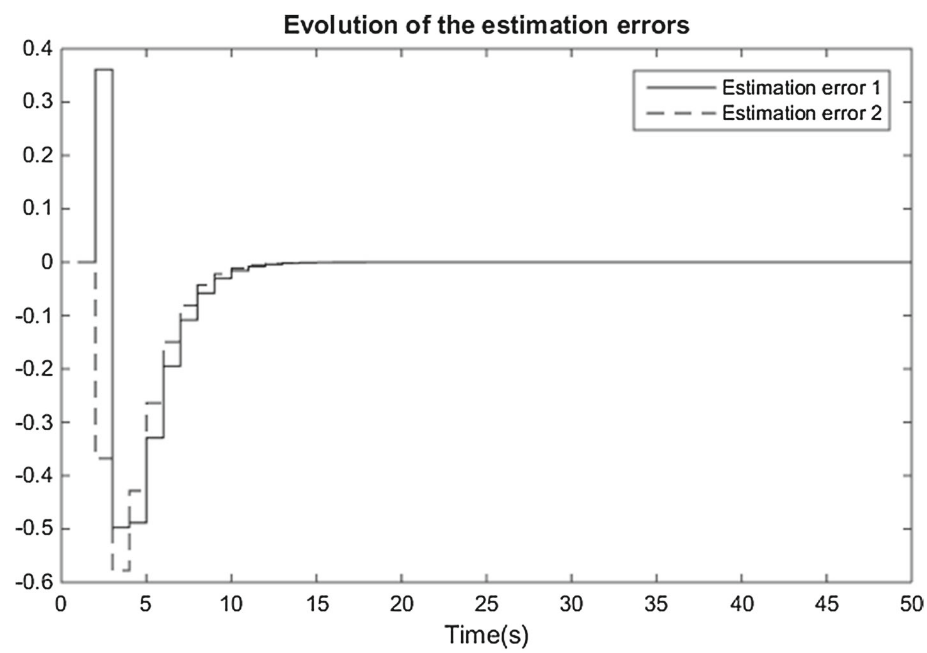


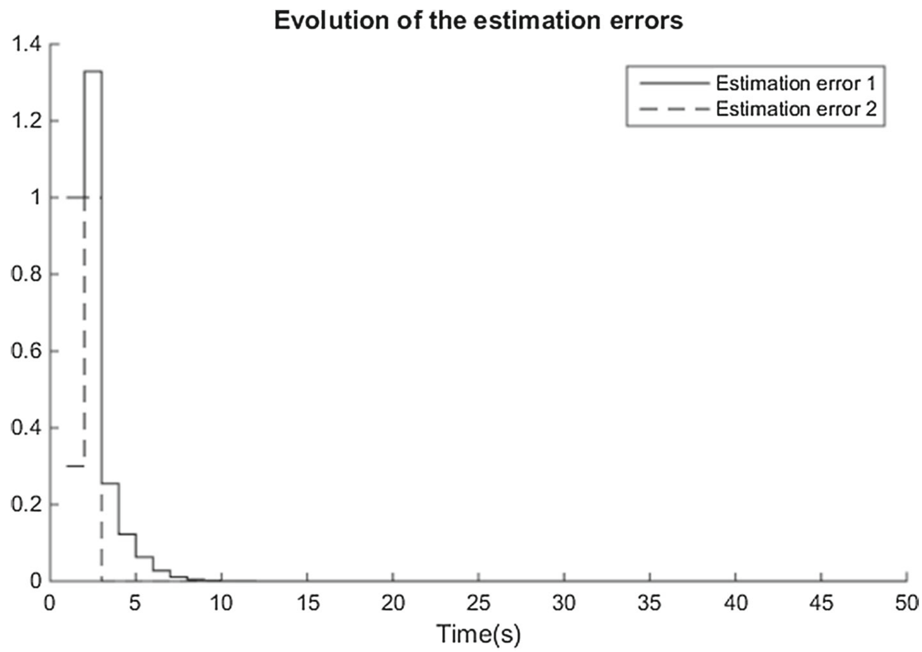
Fig. 6 Evolution of estimation error sub-model 2

The use of the delayed nonquadratic Lyapunov function gives the less conservatism results, such as a larger set of solution with one-sample variation and a fast convergence of estimation error with k-samples variation.

6 Conclusion

This paper developed relaxations of new fuzzy observer designs with the use of nonquadratic and delayed nonquadratic Lyapunov functions, where the second allowed us to reduce the conservatism of the first approach and to significantly increase the solution sets for nonlinear models. Under the framework of linear matrix inequality, a new relaxed fuzzy observer has been proposed with k-samples variations; the relaxation quality has been significantly improved. The effectiveness of the proposed approaches has been illustrated through numerical examples. Future research includes the development of design methods using delayed nonquadratic Lyapunov functions, with new fuzzy observer and fuzzy controller with discrete-time nonparametric, mixed uncertain and periodic T–S systems.

Fig. 7 Evolution of estimation error of global system



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Compliance with Ethical Standards

Conflict of interest The authors declare that there is no conflict of interests.

Ethical Approval This article does not contain any studies with human participants or animals performed by any of the authors.

References

Boulkroune, A., Tadjine, M., M'Saad, M., & Farza, M. (2014). Design of a unified adaptive fuzzy observer for uncertain nonlinear systems. *Information Sciences, 265,* 139–153.

Bouyahya, A., Manai, Y., & Haggège, J. (2015). Observer design for Takagi–Sugeno discrete time uncertain systems. In 2015 *world symposium on mechatronics engineering and applied physics, 11–13 June 2015.*

Bouyahya, A., Manai, Y., & Haggège, J. (2016). Nonquadratic Lyapunov functions for nonlinear Takagi–Sugeno discrete time uncertain systems analysis and control. *International Journal of Advanced Computer Science and Applications (IJACSA)*. https://doi.org/10.14569/IJACSA.2016.070242.

Bouyahya, A., Manai, Y., & Haggège, J. (2017). New approach of stabilization based on new Lyapunov function for nonlinear Takagi–Sugeno discrete-time uncertain systems. *Journal of Control Automation and Electrical Systems, 28*(2), 171–179.

Boyd, S., El Ghaoui, L., Feron, E., & Balakrishnan, V. (1994). *Linear matrix inequalities in system and control theory* (Vol. 15, pp. 37–47). Philadelphia: Society for Industrial and Applied Mathematics.

Cao, Y. Y., & Frank, P. M. (2001). Stability analysis and synthesis of nonlinear time-delay systems via Takagi–Sugeno fuzzy models. *Fuzzy Sets and Systems, 124*(2), 213–229.

Chadli, M., & Guerra, T. M. (2012). LMI solution for robust static output feedback control of discrete Takagi–Sugeno fuzzy models. *IEEE Transactions on Fuzzy Systems, 20*(6), 1160–1165.

Daafouz, J., & Bernussou, J. (2001). Parameter dependent Lyapunov functions for discrete time systems with time varying parameter uncertainties. *System and Control Letters, 43*(5), 355–359.

Dang, T. V., & Wang, W. J. (2011). Observer synthesis for discrete-time T–S fuzzy system with output disturbance. In *International conference on system science and engineering, Macau, China* (pp. 229–233).

Dang, T. V., Wang, W. J., Luoh, L., & Sun, C. H. (2012). Adaptive observer design for the uncertain Takagi–Sugeno fuzzy system with output disturbance. *IET Control Theory & Applications, 6*(10), 1351–1366.

Fang, C. H., Liu, Y. S., Kau, S. W., Hong, L., & Lee, C. H. (2006). A new LMI-based approach to relaxed quadratic stabilization of T–S fuzzy control systems. *IEEE Transactions on Fuzzy Systems, 14*(3), 386–397.

Feng, G. (2004). Stability analysis of discrete-time fuzzy dynamic systems based on piecewise Lyapunov functions. *IEEE Transactions Fuzzy Systems, 12*(1), 22–28.

Feng, G. (2006). A survey on analysis and design of model-based fuzzy control systems. *IEEE Transactions Fuzzy Systems, 14*(5), 676–697.

Gassara, H., Hajjaji, A. E., & Chaabane, M. (2010). Observer-based robust H∞ reliable control for uncertain T–S fuzzy systems with state time delay. *IEEE Transactions on Fuzzy Systems, 18*(6), 1027–1040.

Guerra, T. M., Kerkeni, H., Lauber, J., & Vermeiren, L. (2012). An efficient Lyapunov function for discrete T–S models: observer design. *IEEE Transactions on Fuzzy Systems, 20*(1), 187–196.

Guerra, T. M., & Vermeiren, L. (2004). LMI based relaxed non quadratic stabilization conditions for nonlinear systems in the Takagi–Sugeno's form. *Automatica, 40*(5), 823–829.

Hui, J. J., Zhang, H. X., & Kong, X. Y. (2015). Delay-dependent non-fragile H∞ control for linear systems with interval time-varying delay. *International Journal of Automation and Computing, 12*(1), 109–116.

Jia, Q., Li, H., Zhang, Y., & Chen, X. (2015). Robust observer-based sensor fault reconstruction for discrete-time systems via a descriptor system approach. *International Journal of Control, Automation and Systems, 13*(2), 274–283.

Jo, H. W., Choi, H. L., & Lim, J. T. (2015). Dynamic gain output feedback control of a class of uncertain feedforward nonlinear systems with uncertain time-delays and sensor noise. *International Journal of Control, Automation and Systems, 13*(3), 567–574.

Johansson, M., Rantzer, A., & Arzen, K. (1999). Piecewise quadratic stability of fuzzy systems. *IEEE Transactions Fuzzy Systems, 7*(6), 713–722.

Jun, W. W., & Huai-Ning, W. (2018). Exponential pointwise stabilization of semilinear parabolic distributed parameter systems via the Takagi–Sugeno fuzzy PDE model. *IEEE Transactions on Fuzzy Systems, 26*(1), 155–173.

Kchaou, M., Souissi, M., & Toumi, A. (2010). Robust H2 observer-based control design for discrete-time-delay fuzzy systems. *International Journal of Information and Systems Sciences, 6*(1), 35–48.

Ke, Z., Fuyang, J. B. C., & Xinzhe, Y. (2011). Piecewise fault estimation observer design for discrete-time Takagi–Sugeno fuzzy systems. In *Proceedings of the 30th Chinese control conference, July 22–24, Yantai, China*.

Koo, G. B., Park, J. B., & Joo, Y. H. (2011). Robust fuzzy controller for large-scale nonlinear systems using decentralized static output-feedback. *International Journal of Control, Automation and Systems, 9*(4), 649–658.

Kruszewski, A., Wang, R., & Guerra, T. M. (2008). Nonquadratic stabilization conditions for a class of uncertain nonlinear discrete time TS fuzzy models: a new approach. *IEEE Transactions on Automatic Control, 53*(2), 606–611.

Latrach, C., Kchaou, M., Rabhi, A., & Hajjaji, A. E. (2015). Decentralized networked control system design using Takagi–Sugeno (TS) fuzzy approach. *International Journal of Automation and Computing, 12*(2), 125–133.

Lee, H. J., & Kim, D. W. (2009). Robust stabilization of T–S fuzzy systems: fuzzy static output feedback under parametric uncertainty. *International Journal of Control, Automation and Systems, 7*(5), 731–736.

Lendek, Z., Guerra, T. M., & Lauber, J. (2015). Controller design for TS models using delayed nonquadratic Lyapunov functions. *IEEE Transactions on Cybernetics, 45*(3), 453–464.

Lin, C., Wang, Q. G., & Lee, T. H. (2006). Delay-dependent LMI conditions for stability and stabilization of T–S fuzzy systems with bounded time delay. *Fuzzy Sets and Systems, 157*(9), 1229–1247.

Luenberger, D. (1966). Observers for multi-variable systems. *IEEE Transactions on Automatics Control, 11,* 190–197.

Ma, X. J., Sun, Z. Q., & He, Y. Y. (1998). Analysis and design of fuzzy controller and fuzzy observer. *IEEE Transaction on Fuzzy Systems, 6*(1), 41–51.

Manai, Y., & Benrejeb, M. (2012a). New condition of stabilisation for continuous Takagi–Sugeno fuzzy system based on fuzzy Lyapunov function. *International Journal of Control and Automation, 4*(3), 51–64.

Manai, Y., & Benrejeb, M. (2012b). New condition of stabilization of uncertain continuous Takagi–Sugeno fuzzy system based on fuzzy Lyapunov function. *International Journal of Intelligent Systems and Applications, 4,* 19–25.

Mozelli, L. A., Palhares, R. M., Souza, F. O., & Mendes, E. M. (2009). Reducing conservativeness in recent stability conditions of TS fuzzy systems. *Automatica, 45,* 1580–1583.

Qiu, J., Feng, G., & Gao, H. (2010). Fuzzy-model-based piecewise H∞ static output-feedback controller design for networked nonlinear systems. *IEEE Transactions Fuzzy Systems, 18*(5), 919–934.

Sala, A., & Ariño, C. (2007). Asymptotically necessary and sufficient conditions for stability and performance in fuzzy control: Applications of Polya's theorem. *Fuzzy Sets and Systems, 158*(24), 2671–2686.

Skelton, R. E., Iwasaki, T., & Grigoriadis, K. (1998). *A unified algebraic approach to linear control design*, Taylor and Francis Series in Systems and Control. New York (pp 205–222).

Takagi, T., & Sugeno, M. (1985). Fuzzy identification of systems and its application to modeling and control. *IEEE Transactions on System, Man and Cybernetics, 15*(1), 116–132.

Tanaka, K., Hori, T., & Wang, H. O. (2003). A multiple Lyapunov function approach to stabilization of fuzzy control systems. *IEEE Transactions on Fuzzy Systems, 11*(4), 582–589.

Tanaka, K., & Wang, H. O. (2001). *Fuzzy control systems design and analysis*. New York: Wiley.

Taniguchi, T., Tanaka, K., & Wang, H. O. (2001). Model construction, rule reduction and robust compensation for generalized form of Takagi–Sugeno fuzzy systems. *IEEE Transactions on Fuzzy Systems, 9*(4), 525–537.

Tong, S. C., Li, Y. M., Feng, G., & Li, T. S. (2011). Observer-based adaptative fuzzy backstepping dynamic surface control for a class of MIMO nonlinear systems. *IEEE Transactions on Fuzzy Systems, Man and Cybernetics, Part B, 41*(4), 1124–1135.

Ustoglu, E. Yesil, M. Guzelkaya, & Eksin, I. (2004). Exponential stability of discrete-time Takagi–Sugeno fuzzy systems. In *Proceedings World automation congress*, *2004* (Vol. 17, pp. 519–524).

Wang, L. X., & Mendel, J. M. (1992). Fuzzy basis functions, universal approximation and orthogonal least-squares. *IEEE Transactions on Neural Networks, 3*(5), 807–814.

Wang, Y., Zhang, H., Zhang, J., & Wang, Y. (2015). An SOS-based observer design for discrete-time polynomial fuzzy systems. *International Journal of Fuzzy Systems, 17*(1), 94–104.

Xie, X. P., Yang, D. S., & Lin, X. Z. (2014a). Relaxed observer design of discrete-timeT–S fuzzy systems via a novel multi-instant fuzzy observer. *Signal Processing, 102,* 296–303.

Xie, X. P., Yang, D. S., & Ma, H. J. (2014b). Observer design of discrete-time T–S fuzzy systems via multi-instant homogenous matrix polynomials. *IEEE Transaction on Fuzzy Systems, 22*(6), 1714–1719.

Yeh, S. J., Chang, W., & Wang, W. J. (2015). Unknown input based observer synthesis for uncertain T–S fuzzy systems. *IET Control Theory and Applications, 9*(5), 729–735.

Yueying, W., Yuanqing, X., Choon, K. A., & Yanzheng, Z. (2019). Exponential stabilization of Takagi–Sugeno fuzzy systems with aperiodic sampling: An aperiodic adaptive event-triggered method. *IEEE Transactions on Systems, Man, and Cybernetics: Systems, 49*(2), 444–454.

Zhang, H., & Feng, G. (2008). Stability analysis and H∞ controller design of discrete-time fuzzy large-scale systems based on piecewise Lyapunov functions. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics), 38*(5), 1390–1401.

Zhang, K., Jiang, B., & Shi, P. (2012). Fault estimation observer design for discrete-time Takagi–Sugeno fuzzy systems based on piecewise Lyapunov functions. *IEEE Transactions on Fuzzy Systems, 20*(1), 192200.