



# Stability Analysis of DC-Microgrids: A Gradient Formulation

Alejandro Garcés<sup>1</sup> 

Received: 4 January 2019 / Revised: 2 July 2019 / Accepted: 24 September 2019 / Published online: 3 October 2019  
© Brazilian Society for Automatics–SBA 2019

## Abstract

This paper presents a nonlinear stability analysis for DC-microgrids in both, interconnected and island operation with primary control. The proposed analysis is based on the fact that the dynamical model of the grid is a gradient system generated by a strongly convex function. The stability analysis is thus reduced to a series of convex optimization problems. The proposed method allows to: (i) demonstrate the existence and uniqueness of the equilibrium, (ii) calculate this equilibrium, (iii) give conditions for global stability using a Lyapunov function and (iv) estimate the region of attraction. Previous works only address one of these aspects. Numeric calculations performed in CVX and simulation results in MATLAB complement the analysis and demonstrate how to use this theoretical results in practical problems.

**Keywords** Nonlinear circuits · DC-microgrids · Stability analysis · Convex optimization · Gradient systems

## 1 Introduction

Microgrids and DC-distribution are promising technologies for integrating solar panels, batteries and fuel cells among other components that operate in DC (Meng et al. 2017). These technologies are integrated through power electronic converters and controlled to maintain a constant power (i.e., constant power generation or constant power loads). In particular, constant power loads can introduce a negative resistance effect which in turn generates transient stability problems. A constant power terminal introduces a nonlinear behavior, and hence, linear methods for stability analysis are not enough in this new context (Malik 2013).

Several stability methodologies have been proposed for systems with ad hoc controls (Inam et al. 2016). However, the conventional droop control with a constant power model is the most common approach for control and stabilization of microgrids (Dragicevic et al. 2016). Most of the stability studies on this type of controls are based on linearization (i.e., small signal stability). Transient stability in these generalized

models is still a challenge due to their nonlinear behavior; even finding the equilibrium point can be a challenge.

This paper proposes a methodology for transient stability analysis of DC-microgrids based on the study of gradient systems. Although these types of systems have a rich and general theory, the paper is focused in a particular type, namely gradient systems with a strongly convex function. Existence and uniqueness of the equilibrium are demonstrated as well as the conditions for global stability. In addition, simple methods for calculating this equilibrium and estimating the region of attraction are presented. Being a convex problem, the convergence of the algorithm is guaranteed. The model is simple enough to be tractable computationally and shows the main interaction between nonlinear components. The proposed analysis shows a surprising connection between dynamical systems and convex analysis. This connection is explored from a practical point of view, since the stability analysis is transformed into a series of convex optimization problems that can be solved numerically.

The existence of the equilibrium was analyzed in Sanchez et al. (2014) for one constant power load and generalized in Barabanov et al. (2016) for several loads. A different approach was presented in Garcés (2017) and Garcés (2018) based on the convergence of the power flow. The use of convex analysis for this type of problems was analyzed in Garcés and Montoya (2019). In these works, it was demonstrated that conventional algorithms such as Gauss and Newton's methods converge to a unique equilibrium point under well-

This work is a partial result of the project 111077657914, funded by the Colombian Administrative Department of Science, Technology, and Innovation (COLCIENCIAS), contract number 032-2018.

✉ Alejandro Garcés  
alejandro.garces@utp.edu.co

<sup>1</sup> Department of Electric Power Engineering, Universidad Tecnológica de Pereira, Carrera 27 N10-02 Los Alamos, AA: 97, Pereira 660003, Colombia

defined conditions; however, the problem of stability was not addressed in those papers.

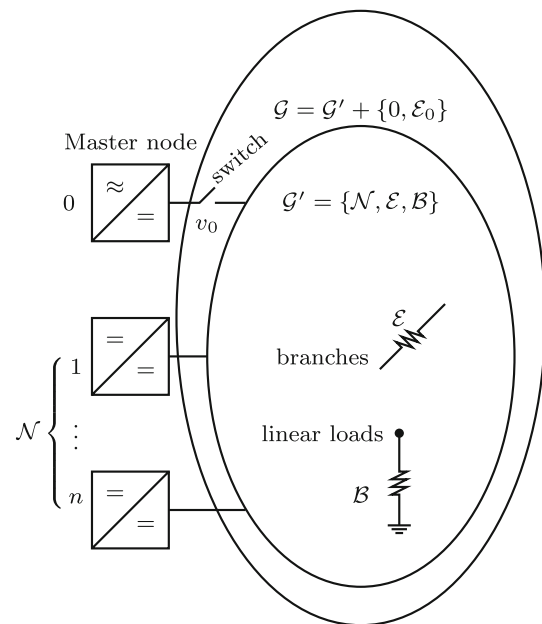
From the stability point of view, several studies have been presented for the small signal case (Majumder 2013; Radwan and Mohamed 2012; Li and Shahidehpour 2018). However, transient stability analysis is required in order to increase the accuracy of the study and consider the nonlinear behavior of the grid. A recent review of transient stability analysis in microgrids can be found in Kabalan et al. (2017). That review showed the necessity of systematic methods for large-scale stability analysis and reduced order models of the grid. The method presented here fulfills these conditions. In Su et al. (2018), a stabilization method was proposed for a DC-microgrid in which all the terminals were connected to the same bus-bar with only one equivalent constant power load. Droop controls were considered only on the sources. The method presented here, considers the topology of the grid with different constant power loads and droop control in both the constant voltage and constant power terminals. Constant power loads have been the main concern of recent stability analysis such as Liu et al. (2018) and Herrera et al. (2017). However, most of these studies were developed for ad hoc controls, requiring a detailed model of the converter. None of these approaches revealed the gradient characteristics of the model.

Convex analysis has been proposed before for the linear case. In Herrera et al. (2017), a semidefinite programming methodology was proposed by formulating a Lure problem with quadratic bounds. That formulation allowed to estimate the region of attraction in grids with constant power loads. However, the topology of the grid was limited to a unique bus-bar and the analysis was basically linear. To the best of the author's knowledge, there are not applications of the nonlinear approach proposed here.

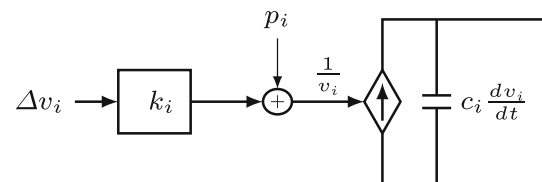
The rest of the paper is organized as follows: Section 2 presents the dynamical model of a DC-microgrid considering constant power terminals. Section 3 describes the stability analysis based on the use of gradient systems with strongly convex functions. Section 4 explains how the stability problem is transformed into convex optimization models which allow to determine the attraction region, equilibrium point and under-voltage limit. Simulation results are presented in Sect. 5 followed by conclusions, appendix and references.

## 2 Problem Definition

Let us consider a DC-microgrid with droop control that is expected to operate either grid connected or in island mode. In the first case, the grid is represented by an oriented graph  $\mathcal{G} = \{\{0, \mathcal{N}\}, \{\mathcal{E}, \mathcal{E}_0, \mathcal{B}\}\}$  where the master node is labeled by 0 and the rest of the nodes are  $\mathcal{N} = \{1, 2, \dots, n\}$ . The branches of the system are split in the branches that connect



**Fig. 1** Schematic representation of a DC-microgrid. The graph  $\mathcal{G}'$  includes all linear loads  $\mathcal{B}$  and branches between nodes with constant power terminals  $\mathcal{N}$ . The graph  $\mathcal{G}$  contains all the components of  $\mathcal{G}'$  plus the master node and the branches that connect the master node



**Fig. 2** Average model of each constant power terminal including droop control and the dynamics of the capacitor

the master node, represented by the set  $\mathcal{E}_0 \subseteq \{0\} \times \mathcal{N}$  and the rest of the branches  $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ . Linear loads are included in the graph and represented by the set  $\mathcal{B} \subseteq \mathcal{N}$ .

In island mode, the grid is represented by a reduced graph  $\mathcal{G}' = \{\mathcal{N}, \mathcal{E}, \mathcal{B}\}$  since the master node is disconnected. Figure 1 shows the main components of the graphs  $\mathcal{G}$  and  $\mathcal{G}'$ . In both cases, the grid is purely restive, and hence, a real admittance matrix  $G = [g_{ij}]$  is defined to represent the relation between nodal voltages and nodal currents. Linear loads are included in this matrix and eliminated by a classical Kron's reduction (Kron 1942). Step nodes are also eliminated by the Kron's reduction.

The master node maintains a constant voltage  $v_0$ , whereas the nodes in  $\mathcal{N}$  are constant power terminals. The model of each constant power terminal is depicted in Fig 2. It includes the capacitive effect of the converter and the droop control. This model is widely used in different applications including DC-microgrids (Montoya 2018) and multiterminal HVDC transmission (Doria-Cerezo et al. 2016; Sanchez et al. 2019).

The model of the grid is given by the dynamics of each single node  $i$  as shown in (1):

$$c_i \frac{dv_i}{dt} = \frac{p_i + k_i(v_{i\text{ref}} - v_i)}{v_i} - g_{0i}v_0 - \sum_{j=1}^n g_{ij}v_j \quad (1)$$

where  $v_i$  is the voltage at the node  $i$ ,  $v_{i\text{ref}}$  is the reference for this voltage (usually a value close to 1 pu given by the secondary/tertiary control),  $g_{ij}$  are entries of the admittance matrix and  $c_i$  is the capacitance of the converter. Notice this is a nonlinear dynamical system due to the presence of constant power devices. The model is the same for grid connected or island operation. The only difference is that under island operation, the switch is opened and hence  $g_{i0} = 0$ .

The stability analysis proposed in this paper is based in the following assumptions, valid for both  $\mathcal{G}$  and  $\mathcal{G}'$ :

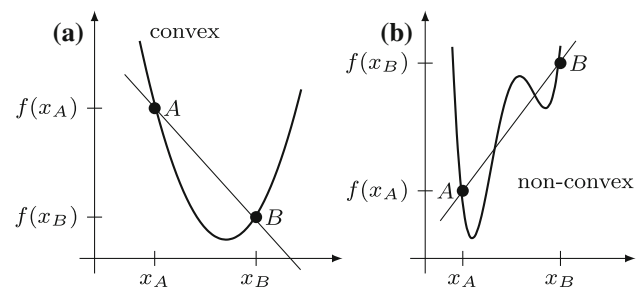
- A1 each converter is equipped with a droop control with constant  $k_i \geq 0$ .
- A2  $c_i > 0$ .
- A3 the graph is connected.
- A4 there is at least one linear load.
- A5 none of the constant power terminals are in short-circuit.

Notice that A1 implies that it is possible to have terminals that do not participate in the primary control (i.e.,  $k_i$  can be zero). A2 is a consequence of  $\mathcal{N}$  being constituted by only constant power terminals. A3 and A4 entail the following lemma whose demonstration is sketched in “Appendix A”:

**Lemma 1** Under assumptions A3 and AA, the admittance matrix  $G = [g_{ij}]$  is non-singular and positive definite ( $G > 0$ )

Finally, A5 is required since a short-circuit in a constant power terminal leads to an infinite current in the capacitor. Nevertheless, short-circuits can be studied by this model, if they occur in the linear loads or step nodes. Transients generated by disconnection of the master node or abrupt changes in generation or demand are also allowed in the model. Under these assumptions, it is clear that  $v_i > 0$  for all  $i \in \mathcal{N}$ .

It is important to notice that this model is general enough and allows to represent different components according to their functionality. For example, a solar panel is just a constant power device with  $p_i > 0$ , while a constant power load corresponds to the same model but with  $p_i < 0$ . As will be demonstrate below, constant power generation does not create stability problems, being the constant power loads the main source of instability. A simple justification of this fact is that constant power loads generate a negative resistance effect that induces instability.



**Fig. 3** Example of a convex and a non-convex function. In a convex function, all points in the interval AB are below the line segment

## 3 Convex Analysis for Gradient Systems

### 3.1 Convex Analysis

Before analyzing the proposed methodology on DC-microgrids, let us review some concepts from convex optimization theory and its application in the stability analysis of dynamical systems:

**Definition 1** A set  $\Omega \subset \mathbb{R}^n$  is convex if for any  $x, y \in \Omega$ , we have that

$$(1 - \lambda)x + \lambda y \in \Omega \quad (2)$$

for all  $\lambda \in \mathbb{R}$ ,  $0 \leq \lambda \leq 1$ .

**Definition 2** A real-valued function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is convex if its domain is convex and for any two points  $x, y \in \mathbb{R}^n$ , we have that

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \quad (3)$$

for all  $\lambda \in \mathbb{R}$  such that  $0 \leq \lambda \leq 1$ . In addition, the function is strictly convex if  $g(x) = f(x) - \mu/2 \|x\|^2$  is convex with  $\mu$  a real, positive number.

The intuition behind Definition 2 is that, in a convex function, all values in the interval  $[x_A, x_B]$  are below the segment  $\overline{AB}$ . Figure 3 shows a comparison between a convex and a non-convex function. Strictly convex functions are convex function with the additional property of not having flat regions. For example, the convex function depicted in Fig 3 is strictly convex.

**Theorem 1** (See Nesterov and Nemirovskii 1994) Consider a non-empty convex set  $\Omega \subseteq \mathbb{R}^n$  and a twice differentiable function  $W : \Omega \rightarrow \mathbb{R}$  such that

$$\frac{\partial^2 W}{\partial x^2} \geq \mu I_n \quad (4)$$

where  $\mu > 0$ ,  $\partial^2 W / \partial x^2$  is the Hessian matrix of  $W$  and  $I_n$  is the identity matrix of size  $n$ . Thus,  $W$  is a strongly convex; for

this type of function, there exists a global minimum  $\tilde{x} \in \Omega$  and this minimum is unique.

**Remark 1** There is a more general theorem which includes strictly convex functions (see Nesterov and Nemirovskii (1994) for more details). However, Theorem 1 is enough for the dynamical problem studied in this paper and avoids the proliferation of unnecessary definitions and technicalities.

**Remark 2** It is well known that convex functions have a global optimum. However, for the case of strongly convex functions, this optimizer is not only global, but the solution itself is unique.

### 3.2 Gradient Systems

Let us consider the type of systems to be studied, namely gradient systems, and their relation with convex optimization.

**Definition 3** A gradient system is a dynamical system that can be represented as (5)

$$M(x) \frac{dx}{dt} = -\frac{\partial W}{\partial x} \quad (5)$$

where  $M \in \mathbb{R}^{n \times n}$  is a non-singular matrix called inertia matrix and  $W : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  is a twice differentiable continuous function with  $\partial W / \partial x$  the gradient of the function. In addition, we say that (5) is a strongly convex gradient system if  $M \succ 0$  (i.e., positive definite) and  $W$  is a strongly convex function.

Let us analyze the dynamics of these type of systems:

**Theorem 2** Consider a strongly convex gradient system described by (5). Then, there is a unique equilibrium point  $\tilde{x} \in \Omega$  and this equilibrium is asymptotically stable. In addition,  $\Omega$  is an estimation of the attraction region.

**Proof** Consider the gradient system (5) with a strongly convex function  $W$ . The equilibrium point is given by the points in which  $\partial W / \partial x = 0$  which correspond to the global minimum of  $W$ . This minimum is also unique due to Theorem 1, guaranteeing the existence and uniqueness of the equilibrium point. For the stability analysis, consider a Lyapunov function  $\mathcal{V}(x) = W(x) - W(\tilde{x})$  which evidently fulfills the conditions for stability, namely  $\mathcal{V}(\tilde{x}) = 0$ ,  $\mathcal{V}(x) > 0$ ,  $\forall x \in \Omega - \{\tilde{x}\}$  and

$$\frac{d\mathcal{V}}{dt} = -\left(\frac{\partial W}{\partial x}\right)^T M(x)^{-1} \left(\frac{\partial W}{\partial x}\right) \leq 0 \quad (6)$$

Asymptotic stability is directly demonstrated by invoking LaSalle's invariant principle.  $\square$

This simple result allows to transform the problem of stability into a convex optimization problem. This is an advantage since the equilibrium point can be calculated numerically. It is also possible to estimate the region of attraction as will be demonstrated in the next sections.

### 4 DC-microgrids as gradient systems

DC-microgrids with assumptions A1 to A4 can be represented as strongly convex gradient systems. Let us formalize this by the following lemma:

**Lemma 2** The model of the DC-microgrid given by (1) can be represented as a gradient system with gradient given by

$$W(v) = -\sum_{i=1}^n (p_i + k_i v_{i\text{ref}}) \ln(v_i) + \sum_{i=1}^n (g_{oi} v_0 + k_i) v_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n g_{ij} v_i v_j \quad (7)$$

and  $M = \text{diag}(c_i)$

**Proof** Notice that (1) can be represented as

$$M \frac{dv}{dt} = -\frac{\partial W}{\partial v} \quad (8)$$

which is clearly a gradient system  $\square$

**Theorem 3** Under the assumptions A1 to A4, the DC-microgrid given by (1) is a strongly convex gradient system in the set  $\Omega$  given by (9) is non-empty

$$\Omega = \left\{ v \in \mathbb{R}^n : Q \text{diag}(1/v_i^2) + G \succ \mu I_n \right\} \quad (9)$$

where  $\mu > 0$  and  $Q = \text{diag}(p_i + k_i \cdot v_{i\text{ref}})$ , and the equilibrium point  $\tilde{v} \in \Omega$ . Thus, the equilibrium is unique and stable. In addition,  $\Omega$  constitutes an estimation of the region of attraction.

**Proof** The DC-microgrid given by (1) is a gradient system as was demonstrated in Lemma 2. In addition, the Hessian matrix of  $W$  is given by

$$\frac{\partial^2 W}{\partial v^2} = Q \text{diag}(1/v_i^2) + G \quad (10)$$

Then,  $W$  is strongly convex in the case in which the set  $\Omega$  given by (9) exists. In this circumstances, the equilibrium point exists and is unique due to Theorem 1 and it is stable due to Theorem 2.  $\square$

Theorem 3 allows to study the stability of a DC-microgrid using concepts from optimization. Thus, it is possible to determine the equilibrium point as a constrained optimization problem and estimate the region of attraction as will be presented in the next section. Although (9) is non-convex, it can be transformed into a convex set by defining a new variable  $x_i = 1/v_i^2$ .

In addition, it is possible to formalize the following fact, that is well known in practice:

**Corollary 1** *Under the assumptions A1 to A5, a DC-microgrid without constant power loads is stable.*

**Proof** Notice that  $p_i \geq 0$  for DC-microgrids without constant power loads, thus  $P \geq 0$ . In addition,  $Q$  and  $\text{diag}(1/v_i^2)$  are always positive semidefinite and  $G$  is positive definite due to Lemma 1.  $\square$

**Remark 3** This result agrees with the one presented in (Garces and Gutierrez 2018) where a DC-microgrid with two constant power terminals was considered. However, the analysis presented here is for any number of constant power terminals.

**Corollary 2** *If  $G$  is strictly diagonally dominant, then an inner estimation of the region of attraction is given by*

$$p_i + k_i v_{iref} > -r_i v_i^2 \quad (11)$$

$$\text{with } r_i = g_{ii} - \sum_{i \neq j} g_{ij} \geq 0$$

**Proof** Notice that if (10) is strictly diagonally dominant then it is positive definite. Then, a sufficient condition for  $W$  being strongly convex is that

$$\frac{p_i + k_i v_{iref}}{v_i^2} + g_{ii} > \sum_{i \neq j} |g_{ij}| \quad (12)$$

This expression is equivalent to (11)  $\square$

**Remark 4** Although the set  $\Omega$  given by (9) constitutes a better estimation of the region of attraction, (11) is simpler to evaluate and requires only local information of the voltage in each node  $i$ .

Theorem 3 allows to define different convex optimization models for the stability analysis as presented in the next section.

## 5 Optimization models for stability analysis of DC-microgrids

### 5.1 Equilibrium Point

The equilibrium point of (1) can be obtained as a convex optimization model as follows:

**Model 1** (equilibrium point) *The equilibrium point of a DC-microgrid with the Assumptions A1 to A5 can be calculated by solving the following optimization problem*

$$\begin{aligned} \min_{v_i} \quad & W(v) \\ \text{subject to} \quad & Q \text{diag}(1/v_i^2) + G \succ 0 \end{aligned} \quad (13)$$

The semidefinite constraint can be neglected during the optimization and evaluated after the algorithm has achieved an optimal point. If the constraint is fulfilled, then the optimum is global and unique, and the equilibrium point is asymptotically stable as was demonstrated in Theorem 3. The optimization problem can be solved by gradient or Newton's methods. The former has a linear convergence, while the latter has a quadratic convergence (Nesterov and Nemirovskii 1994).

### 5.2 Region of Attraction

The region of attraction can be calculated from the set  $\Omega$  given by Eq. (9). However, this set can be non-convex. Therefore, a new variable  $x$  is defined such that  $x_i = 1/v_i^2$ . Therefore, the region of attraction can be calculated from the following convex optimization problem.

**Model 2** (region of attraction) *The following optimization model gives an estimation of the region of attraction around the equilibrium point given by Model 1*

$$\max \alpha \quad (14)$$

$$Q \text{diag}(x_i) + G \succeq \mu I_n \quad (15)$$

$$x_i \geq \alpha \geq 0 \quad (16)$$

It is possible to return to the original values by using the transformation  $v_i = 1/\sqrt{x_i}$ . The objective is to find the minimum voltage  $v_i$  (i.e., the maximum value of  $x_i$ ) that belongs to  $\Omega$ . This value must be positive since  $x_i = 1/v_i^2$ . In practice, this value can be considered as a limit for a voltage stability analysis and as an under-voltage protection, i.e., the component must be disconnected for voltages lower than these voltages. The intuition behind this model is the following: a minimum value of voltage  $v_i$  corresponds to a maximum value of  $x_i$  since both variables are inversely proportional. Therefore, maximizing  $\alpha$  is equivalent to find the minimum value of voltage that fulfills (13). Notice that Model 2 does not depend on the voltage  $v_0$ , and hence, it can be used during voltage sags in the main grid.

Model 2 requires to solve a semidefinite programming model. However, Corollary 2 allows to estimate the region of attraction by a simple inequality. This method is defined as Model 3



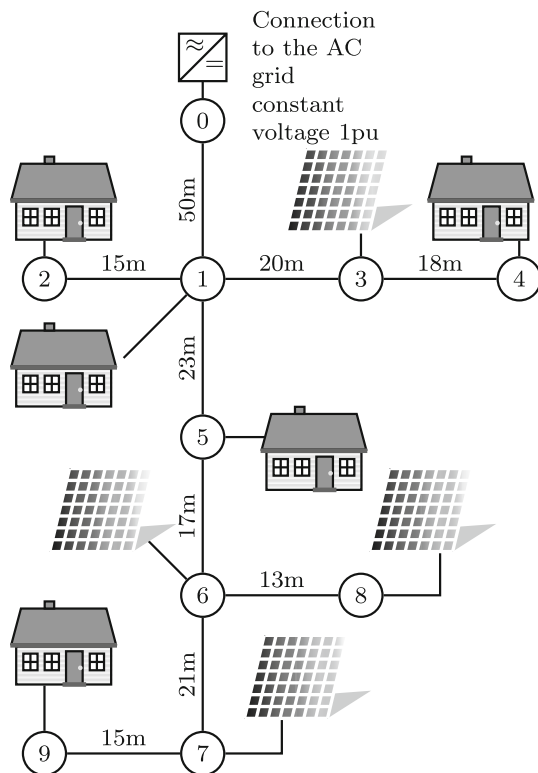
**Model 3** An inner estimation of the region of attraction, for  $G$  strictly diagonally dominant, is given by

$$v_{i(\min)}^2 > \max \left\{ 0, -\frac{p_i + k_i v_{iref}}{r_i} \right\} \quad (17)$$

with  $r_i$  as given in Corollary 2.

If  $p_i > 0$  then the minimum voltage is zero as expected due to Corollary 1. In the case of the constant power loads, the value of  $p_i$  is negative giving a  $v_{\min} > 0$ .

Notice that  $v_{\min}$  calculated from Model 3 depends only on the parameters of the grid and the voltage measured in each node. Therefore, it can be used to determine the minimum voltage for under-voltage protection. The calculation of each  $v_{\min}$  is made off-line and introduced in the control of each power electronic converter. The protection consists on disconnecting the converter when the voltage is equal or bellow  $v_{\min}$ . Dynamical results of this protection will be presented in the next section.



**Fig. 4** A 10-node microgrid constant power generation and constant power loads. The figure shows solar panels and residential users, but the model is general enough for any type of distributed resource. All terminals are DC even residential loads

## 6 Results

### 6.1 Test System

The proposed methodology was evaluated in the 10-node DC-microgrid depicted in Fig 4. Nominal values of the grid are 380 V/1 kW, and each line segment has a resistance of 1.5 mΩ/m. The rest of the parameters are given in Table 1 and the figure itself.

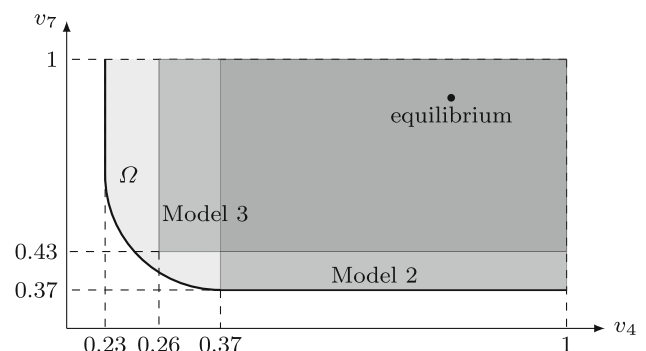
### 6.2 Grid with Two Constant Power Terminals

Let us study the case in which only  $p_4$  and  $p_7$  are connected. The motivation of such a basic case is that the dynamical model is in  $\mathbb{R}^2$  allowing graphical representations.

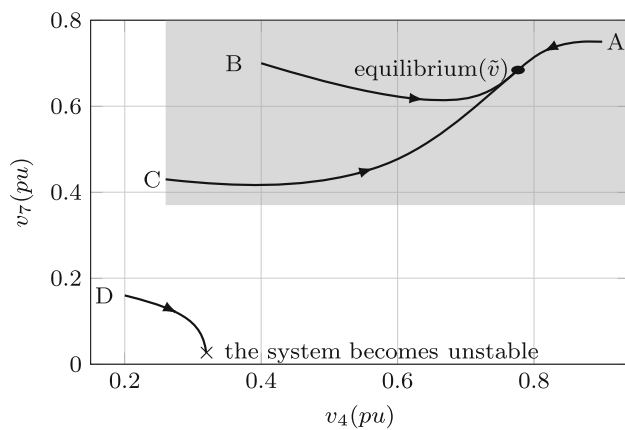
The equilibrium point was obtained by Model 1 using the Newton's method giving  $v_4 = 0.77$  and  $v_7 = 0.68$ . The region of attraction is given by  $\Omega$  as depicted in Fig 5. In this case,  $\Omega$  is convex, but this is not true in a more general case. Two inner estimations of the attraction region given by Model 2 and Model 3 were also calculated. Model 2 estimated  $v_{4\min} = v_{7\min} = 0.37$ , whereas Model 3 estimated  $v_{4\min} = 0.26$  and  $v_{7\min} = 0.43$ . These estimations are also depicted in Fig 5. Model 2 gave a more precise estimation in the second

**Table 1** Parameters of the constant power terminals and results of each model

Node	$k_i$	$p_i$ (kW)	$c_i$ (μF)	Equilibrium	$v_{\min}$
1	1	−80	690	0.9893	0.0958
2	1	−70	560	0.9851	0.0958
3	1	80	630	0.9917	0.0001
4	1	−50	500	0.9881	0.0710
5	1	−70	690	0.9954	0.0958
6	1	80	640	1.0047	0.0001
7	1	−100	600	1.0129	0.0001
8	1	50	630	1.0026	0.0001
9	1	−90	690	0.9973	0.0818



**Fig. 5** Region of attraction given by  $\Omega$ , Model 2 and Model 3. The figure is not on scale



**Fig. 6** Estimation of the region of attraction for the case of two constant power loads. Three different trajectories starting from points A, B and C converge to the equilibrium point  $\tilde{v}$ . A trajectory from an initial point in D, outside the region of attraction, goes to zero

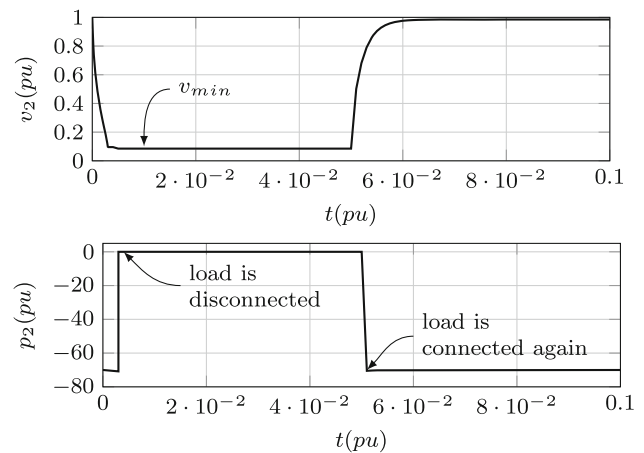
variable, while Model 3 was less precise in both variables. However, Model 3 was calculated directly, while Model 2 required to solve a semidefinite optimization problem. Time calculation for Model 2 was 0.820541 seconds, while time calculation for Model 3 was 0.001695. Both models must be calculated off-line, and hence, time calculation is not an issue in practical applications.

The dynamical system was simulated in order to evaluate these estimations of the region of attraction. Figure 6 shows the trajectories from different initial points labeled as A, B, C and D. The trajectories from A to C are inside the region of attraction, while trajectory from D was defined outside the region. As expected, the trajectories inside the region of attraction converged asymptotically to the equilibrium point. Trajectory D (outside the region of attraction) decreased until eventually reached zero increasing the current in the capacitor to infinity. The system is unstable in that case.

### 6.3 Entire Microgrid

Let us consider now the general case with 7 constant power terminals: 3 constant power generations and 4 constant power loads. The main results of each model are given in Table 1. The equilibrium point and the estimation of the region of attraction were obtained by Models 1 and 3 respectively.

The under-voltage protection was evaluated for the case of a voltage sag in the main converter; the grid started from an initial condition of  $v_0 = 1$  pu, and then, the voltage was reduced until  $v_0 = 0.2$  pu for  $t = 0.05$  s. As result, nodal voltages reached values below  $v_{\min}$ . In terminals 1 and 2, the under-voltage protection acted in that cases, disconnecting these terminals. The rest of the nodes remained connected to the grid. This simulation can be regarded as a low-voltage ride through for DC-microgrids. Figure 7 shows this type of protection for Load  $p_2$ . Notice that the region of attraction



**Fig. 7** Using the region of attraction as indicator for under-voltage protection. Due to the voltage sag in  $v_0$ , the voltage in node 2 is reduced until it reaches a value below that  $v_{\min}$ . The system disconnects the load until the voltages are again feasible. Stability is guaranteed in the entire process

was calculated considering the entire grid, but the criteria for the under-voltage protection were completely local; this means that each terminal requires only information of its own voltage. Numerical simulations are available in (Garces 2016) in order to reproduce the results and evaluate other possible configurations.

The power generated or demanded at  $v_{\min}$  is usually larger than the maximum power of the converters. Therefore, over-current protection can act before the under-voltage protection presented here. In other words, the region of attraction for constant power terminals is larger than the operative conditions.

## 7 Conclusions

A set of convex optimization models were presented for transient stability analysis of DC-microgrids. Model 1 transformed the power flow analysis into an optimization problem. Although the solution of the problem was performed by using the Newton's method, the optimization model guaranteed convergence of the algorithm and asymptotic stability in the sense of Lyapunov.

Model 2 and Model 3 were used to estimate the region of attraction of the equilibrium. Model 2 required to solve a semidefinite optimization problem, while Model 3 was a direct application of an inequality. Both Models were precise enough for practical applications although Model 3 presented a simple implementation and fast calculation.

The estimation of the region of attraction was used to calculate the minimum voltage for under-voltage protection. The proposed method allowed a simple method for under-voltage protection since it required only local infor-

mation. Simulations results demonstrated that under-voltage protection allowed to surplus a short-circuit in the grid or a low-voltage ride through in the slack. The calculation of the region of attraction was made off-line, and therefore, this type of protection can be used in practice by incorporating a disconnection capability in the control of the converter.

## A Proof of Lemma 1

Let us define  $\rho_{ij}$  as the admittance of the branch  $(ij) \in \mathcal{E}$  and  $\beta_i$  as the admittance of the linear load connected to the node  $i \in \mathcal{N}$ . Notice that  $\rho_{ij} > 0$  and  $\beta_i \geq 0$ . Moreover, there is at least one  $\beta_i > 0$  due to Assumption A4. The nodal admittance matrix is built in the following way

$$g_{ij} = \begin{cases} \beta_i + \sum_{i \neq j} \rho_{ij}, & i = j \\ -\rho_{ij}, & i \neq j \end{cases} \quad (18)$$

Notice that  $G$  is diagonally dominant since

$$g_{ii} \geq \sum_{i \neq j} |g_{ij}| \quad (19)$$

Therefore, the matrix  $G$  is positive semidefinite ( $G \geq 0$ ). This fact can be easily proof by using the Theorem of Gershgorin (Saad 2003). In order to proof that  $G > 0$ , it is enough to observe that the matrix is non-singular.

Let us suppose there is an eigenvalue  $\lambda = 0$  and its corresponding eigenvector  $v \neq 0$ , then

$$Gv = 0 \quad (20)$$

therefore

$$\sum_j g_{ij} v_j = 0 \quad (21)$$

$$g_{ii} v_i + \sum_{i \neq j} g_{ij} v_j = 0 \quad (22)$$

$$\beta_i v_i + \sum_{i \neq j} \rho_{ij} v_i - \sum_{i \neq j} \rho_{ij} v_j = 0 \quad (23)$$

$$\beta_i v_i + \sum_{i \neq j} \rho_{ij} (v_i - v_j) = 0 \quad (24)$$

since the graph is connected, the only way this is true, is if  $v_i = v_j = 0$  which contradicts the supposition. Therefore, the matrix is not singular. This property is maintained even after a Kron's reduction (Dorfler and Bullo 2013).

Other proofs of this lemma can be obtained from graph theory (Bollobas 1998) or by using the properties of weakly chained diagonal dominant matrices (Azimzadeh and Forsyth 2016).

## References

- Azimzadeh, P., & Forsyth, P. (2016). Weakly chained matrices, policy iteration, and impulse control. *SIAM Journal on Numerical Analysis*, 54(3), 1341–1364. <https://doi.org/10.1137/15M1043431>.
- Barabanov, N., Ortega, R., Grino, R., & Polyak, B. (2016). On existence and stability of equilibria of linear time-invariant systems with constant power loads. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 63(1), 114–121. <https://doi.org/10.1109/TCSI.2015.2497559>.
- Bollobas, B. (1998). *Modern graph theory*, 10 (1st ed., Vol. 1). Philadelphia: Springer.
- Dorfler, F., & Bullo, F. (2013). Kron reduction of graphs with applications to electrical networks. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 60(1), 150–163. <https://doi.org/10.1109/TCSI.2012.2215780>.
- Doria-Cerezo, A., Olm, J. M., di Bernardo, M., & Nuno, E. (2016). Modelling and control for bounded synchronization in multi-terminal VSC-HVDC transmission networks. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 63(6), 916–925. <https://doi.org/10.1109/TCSI.2016.2537938>.
- Dragicevic, T., Lu, X., Vasquez, J. C., & Guerrero, J. M. (2016). Dc microgrids part i: A review of control strategies and stabilization technique. *IEEE Transactions on Power Electronics*, 31(7), 4876.
- Garces, A. (2016). Matlab central file exchange. <http://www.mathworks.com/matlabcentral/profile/authors/3009175-alejandro-garces>.
- Garces, A. (2017). Uniqueness of the power flow solutions in low voltage direct current grids. *Electric Power Systems Research*, 151, 149–153. <https://doi.org/10.1016/j.epsr.2017.05.031>.
- Garces, A. (2018). On the convergence of Newton's method in power flow studies for DC microgrids. *IEEE Transactions on Power Systems*, 33(5), 5770–5777. <https://doi.org/10.1109/TPWRS.2018.2820430>.
- Garces A., & Gutierrez, A. (2018). On the stability of DC microgrids with two constant power devices. In *2018 IEEE green technologies conference (GreenTech)* (pp. 33–37). <https://doi.org/10.1109/GreenTech.2018.00015>.
- Garces, A., & Montoya, D. (2019). A potential function for the power flow in dc microgrids: An analysis of the uniqueness and existence of the solution and convergence of the algorithms. *Journal of Control, Automation and Electrical Systems*, 2, 1–8. <https://doi.org/10.1007/s40313-019-00489-4>.
- Herrera, L., Zhang, W., & Wang, J. (2017). Stability analysis and controller design of DC microgrids with constant power loads. *IEEE Transactions on Smart Grid*, 8(2), 881–888. <https://doi.org/10.1109/TSG.2015.2457909>.
- Inam, W., Belk, J. A., Turitsyn, K., & Perreault, D. J. (2016). Stability, control, and power flow in ad hoc DC microgrids. In *2016 IEEE 17th workshop on control and modeling for power electronics (COMPEL)* (pp. 1–8). <https://doi.org/10.1109/COMPEL.2016.7556704>.
- Kabalan, M., Singh, P., & Niebur, D. (2017). Large signal Lyapunov-based stability studies in microgrids: A review. *IEEE Transactions on Smart Grid*, 8(5), 2287–2295. <https://doi.org/10.1109/TSG.2016.2521652>.
- Kron, G. (1942). *Tensors for circuits*. Mineola: Dover Publications.
- Li, Z., & Shahidehpour, M. (2018). Small-signal modeling and stability analysis of hybrid AC/DC microgrids. *IEEE Transactions on Smart Grid*. <https://doi.org/10.1109/TSG.2017.2788042>.
- Liu, J., Zhang, W., & Rizzoni, G. (2018). Robust stability analysis of DC microgrids with constant power loads. *IEEE Transactions on Power Systems*, 33(1), 851–860. <https://doi.org/10.1109/TPWRS.2017.2697765>.



- Majumder, R. (2013). Some aspects of stability in microgrids. *IEEE Transactions on Power Systems*, 28(3), 3243–3252. <https://doi.org/10.1109/TPWRS.2012.2234146>.
- Malik, O. P. (2013). Evolution of power systems into smarter networks. *Journal of Control, Automation and Electrical Systems*, 24(1), 139–147. <https://doi.org/10.1007/s40313-013-0005-6>.
- Meng, L., Shafiee, Q., Trecate, G. F., Karimi, H., Fulwani, D., Lu, X., et al. (2017). Review on control of DC microgrids and multiple microgrid clusters. *IEEE Journal of Emerging and Selected Topics in Power Electronics*, 5(3), 928–948.
- Montoya, O. D. (2018). Numerical approximation of the maximum power consumption in DC-mgs with CPLS via an sdp model. *IEEE Transactions on Circuits and Systems II: Express Briefs*, <https://doi.org/10.1109/TCSII.2018.2866447>.
- Nesterov, Y., & Nemirovskii, A. (1994). *Interior point polynomial algorithms in convex programming*, 10 (1st ed., Vol. 1). Philadelphia: SIAM Studies in Applied Mathematics.
- Radwan, A. A. A., & Mohamed, Y. A. I. (2012). Linear active stabilization of converter-dominated DC microgrids. *IEEE Transactions on Smart Grid*, 3(1), 203–216. <https://doi.org/10.1109/TSG.2011.2162430>.
- Saad, Y. (2003). *Iterative methods for sparse linear systems* (1st ed.). Philadelphia: Society for Industrial and Applied Mathematics.
- Sanchez, S., Ortega, R., Grino, R., Bergna, G., & Molinas, M. (2014). Conditions for existence of equilibria of systems with constant power loads. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 61(7), 2204–2211. <https://doi.org/10.1109/TCSI.2013.2295953>.
- Sanchez, S., Garcés, A., Bergna-Díaz, G., & Tedeschi, E. (2019). Dynamics and stability of meshed multiterminal hvdc networks. *IEEE Transactions on Power Systems*, 34(3), 1824–1833. <https://doi.org/10.1109/TPWRS.2018.2889516>.
- Su, M., Liu, Z., Sun, Y., Han, H., & Hou, X. (2018). Stability analysis and stabilization methods of DC microgrid with multiple parallel-connected DC–DC converters loaded by CPLS. *IEEE Transactions on Smart Grid*, 9(1), 132–142. <https://doi.org/10.1109/TSG.2016.2546551>.

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.