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Derivative Feedback Control for a Class of Uncertain Linear Systems Subject to Actuator Saturation

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Abstract
This manuscript considers a class of linear systems with time-invariant uncertainties, where only the derivative of the state vector is considered for feedback. In this scenario, the proposed strategy uses auxiliary dynamics, whose state is accessible for feedback, to control the original plant. It is proposed a design procedure by means of linear matrix inequalities, adding an auxiliary dynamics and subject to actuator saturation. If the conditions are feasible, they assure that the equilibrium point of the closed-loop system is locally asymptotically stable, for all initial conditions in an ellipsoidal region, which is within a given region defined for the plant and the new dynamics. Although the proposed design includes an auxiliary dynamics, it ensures the stability and decay rate proprieties for the original plant. Simulations examples illustrate the effectiveness of the proposed approach.

Keywords Derivative feedback · Uncertain linear system · Actuator saturation · Linear matrix inequalities

1 Introduction

Derivative feedback may be useful in the control of real plants in which the state-derivative signals are easier to obtain than the state signal. For instance, in the active automotive suspension system (Reithmeier and Leitmann 2003; da Silva et al. 2011, 2013; Assunção et al. 2007; Yazici and Sever 2017b), in the mass–spring system with damping (Moreira et al. 2010), in the vibration control of an offshore steel jacket platform (Yazici and Sever 2017a), in the design of controllers for mechanical systems and vibration damping systems (Abdelaziz and Valášek 2004, 2005; Cardim et al. 2007; da Silva et al. 2012; Rossi et al. 2018), in the control of vibrations in cable-suspended bridges (Duan et al. 2005), in the optimization problem for a four-wheel-drive front-wheel-steerable vehicle (Fallah et al. 2013) and in the attitude control system of refueling spacecraft in orbit (Abdelaziz 2017).

The saturation of the actuators is present in large part of the practical applications due to operational restrictions in the equipments. In Hu et al. (2002), the domain of attraction of the equilibrium point for a linear system saturated with feedback of the state vector is estimated using a Lyapunov function candidate of the quadratic type. The composite quadratic Lyapunov function is used in Hu and Lin (2003) for

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continuous linear systems with saturation, where the authors show, in that case, the convex hull of a set of invariant ellipsoids is also invariant.

To the best of the authors knowledge, there are not available in the literature papers which consider state-derivative feedback and actuator saturation.

In view of the aforementioned scenario, we propose a robust control structure for a class of uncertain linear system in which only the state derivative is available for the feedback. The main idea is to use an auxiliary dynamic, whose state is available for the feedback, in the control of the original plant. For the proposed scheme, we use linear matrix inequalities (LMIs) to design the gain from the control structure. To deal with the presence of saturation in the control signal, we define an operating region for the control signal where the saturation function is described by a convex combination. Then, as a result of the design procedure, it finds an invariant ellipsoidal set, and for all initial condition in this set, the origin of the state space is an equilibrium point locally asymptotically stable.

The rest of the manuscript is organized as follows: In Sect. 2, we present an alternative representation of the saturation of the control vector, we define the region of operation for the auxiliary dynamics, in which the saturation representation is valid, and present the LMIs that guarantee local stability for the equilibrium point of the closed-loop system. In Sect. 3, we present the relation between auxiliary dynamics and plant dynamics. In Sect. 4, we present the examples, where we use the MATLAB software, and the LMILab (Gahinet et al. 1994), interfaced by YALMIP (Lofberg 2004), to solve the design conditions and perform the simulations. Section 5 draws the conclusions.

Throughout this manuscript, $\Re$ represents the set of real numbers, Z_+ represents the set of positive integers, $\Re^n$ and $\Re^{n\times m}$ denote the set of vectors $n \times 1$ with real elements and the set of matrices $n \times m$ with real elements, respectively. The set $\mathbb{I}_r = \{1, 2, \ldots, r\}, \quad r \in Z_+$. We denote the convex combination of vectors $\mathbf{w}_i$, $\forall i \in \mathbb{I}_r$, by co $= \{\mathbf{w}_1, \ldots, \mathbf{w}_r\}$. The block diagonal matrix formed by the matrices $\mathbf{M}_1, \ldots, \mathbf{M}_r$ is indicated by diag$\{\mathbf{M}_1, \ldots, \mathbf{M}_r\}$, $\mathbf{M}_{(l)}$ represents lth row (element) of a matrix (vector) $\mathbf{M}$. $\|\mathbf{M}\|$ represents the Euclidean norm, $\lambda_{\min}(\mathbf{P})$ is the minimum eigenvalue of $\mathbf{P}$, and $\lambda_{\max}(\mathbf{P})$ is the maximum eigenvalue of $\mathbf{P}$. $\mathbf{M} > 0$ ($\mathbf{M} < 0$, $\mathbf{M} \geq 0$ and $\mathbf{M} \leq 0$) means that the matrix $\mathbf{M}$ is positive definite (negative definite, positive semi-definite and negative semi-definite, respectively).

2 Derivative Control with Actuator Saturation

Consider a linear system with time-invariant uncertainties, subject to actuator saturation, given by:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\alpha)\mathbf{x}(t) + \mathbf{B}\mathbf{sat}(\mathbf{u}(t)), \tag{1}$$

where $\mathbf{A}(\alpha) \in \Re^{n\times n}$ and $\mathbf{B} \in \Re^{n\times m}$ are the matrices that represent the dynamics of the uncertain system, $\mathbf{x}(t) \in \Re^n$ the state vector and $\mathbf{u}(t) \in \Re^m$ the control vector. The matrix $\mathbf{A}(\alpha)$ is represented by the convex combination of known matrices described by $\mathbf{A}(\alpha) = \sum_{i=1}^{r_1} \alpha_i \mathbf{A}_i$ (Boyd et al. 1994), with the uncertain but constant vector $\alpha \in \mathscr{P}$,

$$\mathscr{P} = \left\{ \alpha = [\alpha_1 \ldots \alpha_{r_1}]^T : \sum_{i=1}^{r_1} \alpha_i = 1,\ \alpha_i \geq 0 \right\}, \tag{2}$$

where $r_1 \in Z_+$.

The saturation of the control signal is given by

$$\begin{aligned}&\mathbf{sat}(\mathbf{u}(t)) = [\mathrm{sat}(u_1(t)) \quad \ldots \quad \mathrm{sat}(u_m(t))]^T \in \Re^m,\\ &\mathrm{sat}(u_l(t)) = \mathrm{sgn}(u_l(t)) \min\{\rho_l, |u_l(t)|\},\end{aligned}$$

with $\mathbf{u}(t) = [u_1(t) \quad \ldots \quad u_m(t)]^T$ and $\rho_l > 0$ for all $l \in \mathbb{I}_m$, are known constants (Hu and Lin 2003; Hu et al. 2002; Alves et al. 2016). The function $\mathrm{sat}(u_l(t))$ is displayed in Fig. 1.

Given the uncertain linear system (1), consider that $\mathbf{A}(\alpha)$, for all $\alpha \in \mathscr{P}$, is a full rank matrix.

Then, we have

$$\begin{aligned}&\dot{\mathbf{x}}(t) = \mathbf{A}(\alpha)\mathbf{x}(t) + \mathbf{B}\mathbf{sat}(\mathbf{u}(t))\\ &\quad \Rightarrow \mathbf{A}(\alpha)\mathbf{x}(t) = \dot{\mathbf{x}}(t) - \mathbf{B}\mathbf{sat}(\mathbf{u}(t)).\end{aligned} \tag{3}$$

Once again, since the derivative of the state vector and the saturation of the control vector are available, we have that $\mathbf{A}(\alpha)\mathbf{x}(t)$ is also available for feedback. Now, from (3) it is possible to define a new state vector of the system (1) (Moreira 2015) taking:

$$\hat{\mathbf{x}}(t) = \mathbf{A}(\alpha)\mathbf{x}(t). \tag{4}$$

Therefore, from (3) and (4), $\hat{\mathbf{x}}(t)$ is available for feedback. Also, taking the time-derivative from (4) it follows that

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}(\alpha)\dot{\mathbf{x}}(t). \tag{5}$$

Premultiplying (1) by $\mathbf{A}(\alpha)$, and considering (4) and (5),

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}(\alpha)\hat{\mathbf{x}}(t) + \mathbf{A}(\alpha)\mathbf{B}\mathbf{sat}(\mathbf{u}(t)). \tag{6}$$

Remark 1 For the state-derivative feedback studied in this paper, the assumption that the matrix $\mathbf{A}(\alpha)$ is a full rank matrix for all α defined in (2) is a necessary condition for the stabilizability of the plant given in (1) (Abdelaziz and Valášek 2004; Assunção et al. 2007; Moreira et al. 2010; da Silva et al. 2011, 2012).

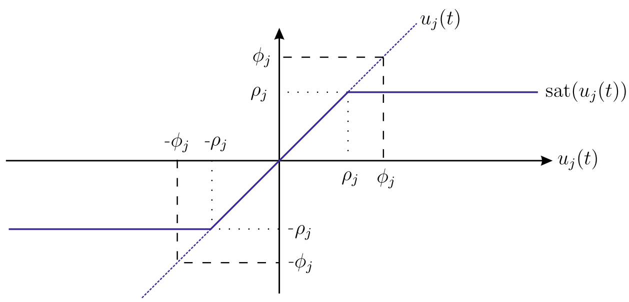

Fig. 1 Representation of $\mathrm{sat}(u_j(t))$ as a function of $u_j(t)$ and its operation region

2.1 A Convex Description for the Control Signal Saturation

In Fig. 1, we present the graphical representation of the saturation of one control signal $u_j(t)$ and the assumed limits $|u_j(t)| \leq \phi_j$.

In order to obtain a convex combination that describes the saturation in the control signal, we consider that there exist $\phi_j > 0$ for all $j \in \mathbb{I}_m$, such that $-\phi_j \leq u_j \leq \phi_j$. Then, an alternative representation for the saturation of $\mathbf{u}(t)$ is given by

$$\begin{aligned} &\mathbf{sat}(\mathbf{u}(t)) = \left[\mathrm{sat}(u_1(t)) \ldots \mathrm{sat}(u_m(t)) \right]^T \in \Re^m, \\ &\mathrm{sat}(u_j(t)) = u_j(t)\varphi_j(t) \end{aligned} \tag{7}$$

with

$$\varphi_j(t) = \begin{cases} 1, & \text{if } |u_j(t))| \leq \rho_j \\ \rho_j/|u_j(t)|, & \text{if } |u_j(t)| > \rho_j. \end{cases} \tag{8}$$

Therefore, $\max\{\varphi_j(t)\} = 1$ and $\min\{\varphi_j(t)\} = \rho_j/\phi_j$.

Thus, we can represent the function $\varphi_j(t)$ by the following convex combination:

$$\varphi_j(t) = \tau_{1_{(j)}}(t)v_{1_{(j)}} + \tau_{2_{(j)}}(t)v_{2_{(j)}} \tag{9}$$

with $v_{1_{(j)}} = 1$, $v_{2_{(j)}} = \rho_j/\phi_j$. Observe that $\tau_{1_{(j)}}(t) + \tau_{2_{(j)}}(t) = 1$ and $\tau_{1_{(j)}}(t) \geq 0$, $\tau_{2_{(j)}}(t) \geq 0$, for all $j \in \mathbb{I}_m$.

Lemma 1 *Consider that there exist constants $\phi_j > 0$, $j \in \mathbb{I}_m$, such that $-\phi_j \leq u_j \leq \phi_j$ for all $j \in \mathbb{I}_m$. Define $\mathbf{Z}_s \in \Re^{m\times m}$, $s \in \mathbb{I}_{2^m}$, the diagonal matrices whose elements (j, j) are all the possible combinations of $v_{1_{(j)}} = 1$ and $v_{2_{(j)}} = \rho_j/\phi_j$, for all $j \in \mathbb{I}_m$. Then, from* (7), (8) and (9)*, the condition below holds:*

$$\boldsymbol{sat}(\mathbf{u}(t)) = \mathbf{Z}(\lambda)\mathbf{u}(t), \quad \textit{with} \;\; \mathbf{Z}(\lambda) = \sum_{s=1}^{2^m} \lambda_s \mathbf{Z}_s, \tag{10}$$

with the uncertain vector $\lambda \in \mathscr{L}$,

$$\mathscr{L} = \left\{ \lambda = [\lambda_1 \ldots \lambda_{2^m}]^T : \sum_{s=1}^{2^m} \lambda_s = 1,\ \lambda_s \geq 0 \right\},$$

where $2^m \in Z_+$.

Proof For $m = 1$, note that (10) holds, from (7), (8) and (9), for $Z_1 = v_{1_{(1)}} = 1$, $Z_2 = v_{2_{(1)}} = \rho_1/\phi_1$, $\lambda_1 = \tau_{1_{(1)}}(t)$ and $\lambda_2 = \tau_{2_{(1)}}(t)$.

Now, for $m = 2$, then $\mathbf{u}(t) = [u_1(t) \quad u_2(t)]^T$, $-\phi_1 \leq u_1(t) \leq \phi_1$, $-\phi_2 \leq u_2(t) \leq \phi_2$, where ϕ_1 and ϕ_2 are known, and from (7), (8) and (9),

$$\begin{aligned} \mathbf{sat}(\mathbf{u}(t)) &= \begin{bmatrix} \mathrm{sat}(u_1(t)) \\ \mathrm{sat}(u_2(t)) \end{bmatrix} \\ &= \begin{bmatrix} (\tau_{1_{(1)}}(t)v_{1_{(1)}} + \tau_{2_{(1)}}(t)v_{2_{(1)}})u_1(t) \\ (\tau_{1_{(2)}}(t)v_{1_{(2)}} + \tau_{2_{(2)}}(t)v_{2_{(2)}})u_2(t) \end{bmatrix} \\ &= \begin{bmatrix} \left(\tau_{1_{(2)}}(t) + \tau_{2_{(2)}}(t)\right)\left(\tau_{1_{(1)}}(t)v_{1_{(1)}}\right)u_1(t) \\ \left(\tau_{1_{(1)}}(t) + \tau_{2_{(1)}}(t)\right)\left(\tau_{1_{(2)}}(t)v_{1_{(2)}}\right)u_2(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} \left(\tau_{1_{(2)}}(t) + \tau_{2_{(2)}}(t)\right)\left(\tau_{2_{(1)}}(t)v_{2_{(1)}}\right)u_1(t) \\ \left(\tau_{1_{(1)}}(t) + \tau_{2_{(1)}}(t)\right)\left(\tau_{2_{(2)}}(t)v_{2_{(2)}}\right)u_2(t) \end{bmatrix} \\ &= \tau_{1_{(1)}}(t)\tau_{1_{(2)}}(t)\begin{bmatrix} v_{1_{(1)}} & 0 \\ 0 & v_{1_{(2)}} \end{bmatrix}\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \\ &\quad + \tau_{1_{(2)}}(t)\tau_{2_{(1)}}(t)\begin{bmatrix} v_{2_{(1)}} & 0 \\ 0 & v_{1_{(2)}} \end{bmatrix}\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \\ &\quad + \tau_{1_{(1)}}(t)\tau_{2_{(2)}}(t)\begin{bmatrix} v_{1_{(1)}} & 0 \\ 0 & v_{2_{(2)}} \end{bmatrix}\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \\ &\quad + \tau_{2_{(2)}}(t)\tau_{2_{(1)}}(t)\begin{bmatrix} v_{2_{(1)}} & 0 \\ 0 & v_{2_{(2)}} \end{bmatrix}\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \\ &= [\lambda_1(t)\mathbf{Z}_1 + \lambda_2(t)\mathbf{Z}_2 + \lambda_3(t)\mathbf{Z}_3 + \lambda_4(t)\mathbf{Z}_4]\,\mathbf{u}(t). \end{aligned} \tag{11}$$

From (10), (11) and Table 1, note that for $s = 1$: $j = 1 \Rightarrow l_1 = 1$, $j = 2 \Rightarrow l_2 = 1$ and $\lambda_1(t) = \tau_{l_1(1)}(t)\tau_{l_2(1)}(t) = \tau_{1(1)}(t)\tau_{1(2)}(t)$; $s = 2$: $j = 1 \Rightarrow l_1 = 2$, $j = 2 \Rightarrow l_2 = 1$ and $\lambda_2(t) = \tau_{l_1(1)}(t)\ \tau_{l_2(2)}(t) = \tau_{2(1)}(t)\tau_{1(2)}(t)$; $s = 3$: $j = 1 \Rightarrow l_1 = 1$, $j = 2 \Rightarrow l_2 = 2$ and $\lambda_3(t) = \tau_{l_1(1)}(t)\tau_{l_2(2)}(t) = \tau_{1(1)}(t)\ \tau_{2(2)}(t)$; $s = 4$: $j = 1 \Rightarrow l_1 = 2$, $j = 2 \Rightarrow l_2 = 2$ and $\lambda_4(t) = \tau_{l_1(1)}(t)\ \tau_{l_2(2)}(t) = \tau_{2(1)}(t)\tau_{2(2)}(t)$, Table 1 presents all combinations of the these elements. Therefore, we can verify (10) with $v_{1_{(1)}} = v_{1_{(2)}} = 1$, $v_{2_{(1)}} = \rho_1/\phi_1$, $v_{2_{(2)}} = \rho_2/\phi_2$, and from (9), $\sum_{s=1}^{4} \lambda_s = \tau_{1(1)}(t)\tau_{1(2)}(t) + \tau_{2(1)}(t)\tau_{1(2)}(t) + \tau_{1(1)}(t)\tau_{2(2)}(t) + \tau_{2(1)}(t)\tau_{2(2)}(t) = (\tau_{1(1)}(t) + \tau_{2(1)}(t))(\tau_{1(2)}(t) + \tau_{2(2)}(t)) = 1$, in which

$$[\mathbf{Z}_1|\mathbf{Z}_2|\mathbf{Z}_3|\mathbf{Z}_4] = \left[\begin{array}{cc|cc|cc|cc} v_{1_{(1)}} & 0 & v_{1_{(1)}} & 0 & v_{2_{(1)}} & 0 & v_{2_{(1)}} & 0 \\ 0 & v_{1_{(2)}} & 0 & v_{2_{(2)}} & 0 & v_{1_{(2)}} & 0 & v_{2_{(2)}} \end{array}\right].$$

Table 1 Combinations for variable changes in the description of saturation by a convex combination for $m = 2$

s	$l_2\|\tau_{l_2}(2)$	$l_1\|\tau_{l_1}(1)$
1	1	1
2	1	2
3	2	1
4	2	2

Table 2 Combinations for variable changes in the description of saturation by a convex combination

s	$l_m\|\tau_{l_m}(m)$	$l_{m-1}\|\tau_{l_{m-1}}(m-1)$	$\cdots$	$l_2\|\tau_{l_2(2)}$	$l_1\|\tau_{l_1(1)}$
1	1	1	$\cdots$	1	1
2	1	1	$\cdots$	1	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^m	2	2	$\cdots$	2	2

Thus, following the same idea, for all $m \geq 1$, $j \in \mathbb{I}_m$ and $s \in \mathbb{I}_{2^m}$ one obtains (10), defining $s = 1 + \sum_{j=1}^{m} 2^{(j-1)}(l_j - 1)$, $\lambda_s = (\prod_{j=1}^{m} \tau_{l_j}(j)(t)) = \tau_{l_1}(1)(t)\tau_{l_2}(2)(t)\cdots\tau_{l_m}(m)(t)$, $l_j \in \mathbb{I}_2$ and the combinations presented in Table 2.

Observe that Table 2 is created for changing variables, the order of the factors $\tau_{l_j}(j)$ being determined by the value of j, where $j = 1$ is the rightmost column and $j = m$ the leftmost column at right of s index. □

Replacing (10) in (6), we obtain

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}(\alpha)\hat{\mathbf{x}}(t) + \mathbf{A}(\alpha)\mathbf{B}\mathbf{Z}(\lambda)\mathbf{u}(t), \tag{12}$$

where $\lambda = [\lambda_1 \ \ldots \ \lambda_{2^m}]^T$, $\lambda_s \geq 0$, $\sum_{s=1}^{2^m} \lambda_s = 1$ for $s \in \mathbb{I}_{2^m}$ and $\alpha \in \mathscr{P}$ defined in (2). Hence, it follows from (12) that

$$\dot{\hat{\mathbf{x}}}(t) = \sum_{s=1}^{2^m} \lambda_s \sum_{i=1}^{r_1} \alpha_i \mathbf{A}_i \hat{\mathbf{x}}(t) + \sum_{i=1}^{r_1} \alpha_i \mathbf{A}_i \mathbf{B} \sum_{s=1}^{2^m} \lambda_s \mathbf{Z}_s \mathbf{u}(t). \tag{13}$$

In order to rewrite the dynamics in (13) in a single simplex, we perform the following change of variable: $\mu_k = \lambda_s \alpha_i$ with $k = r_1(s-1) + (i-1) + 1$ for all $i \in \mathbb{I}_{r_1}$ and $s \in \mathbb{I}_{2^m}$ in (13). Following this procedure, we get

$$\begin{aligned}
\dot{\hat{\mathbf{x}}}(t) &= \sum_{s=1}^{2^m} \lambda_s \sum_{i=1}^{r_1} \alpha_i \mathbf{A}_i \hat{\mathbf{x}}(t) + \sum_{i=1}^{r_1} \alpha_i \mathbf{A}_i \mathbf{B} \sum_{s=1}^{2^m} \lambda_s \mathbf{Z}_s \mathbf{u}(t) \\
&= \lambda_1 \left\{ \sum_{i=1}^{r_1} \alpha_i \mathbf{A}_i \hat{\mathbf{x}}(t) + \sum_{i=1}^{r_1} \alpha_i \mathbf{A}_i \mathbf{B}\mathbf{Z}_1 \mathbf{u}(t) \right\} \\
&\quad + \cdots + \lambda_{2^m} \left\{ \sum_{i=1}^{r_1} \alpha_i \mathbf{A}_i \hat{\mathbf{x}}(t) + \sum_{i=1}^{r_1} \alpha_i \mathbf{A}_i \mathbf{B}\mathbf{Z}_{2^m} \mathbf{u}(t) \right\} \\
&= \lambda_1 \left\{ \alpha_1 \left\{ \mathbf{A}_1 \hat{\mathbf{x}}(t) + \mathbf{A}_1 \mathbf{B}\mathbf{Z}_1 \mathbf{u}(t) \right\} \right\} \\
&\quad + \cdots + \lambda_1 \left\{ \alpha_{r_1} \left\{ \mathbf{A}_{r_1} \hat{\mathbf{x}}(t) + \mathbf{A}_{r_1} \mathbf{B}\mathbf{Z}_1 \mathbf{u}(t) \right\} \right\} \\
&\quad + \cdots + \lambda_{2^m} \left\{ \alpha_1 \left\{ \mathbf{A}_1 \hat{\mathbf{x}}(t) + \mathbf{A}_1 \mathbf{B}\mathbf{Z}_{2^m} \mathbf{u}(t) \right\} \right\} \\
&\quad + \cdots + \lambda_{2^m} \left\{ \alpha_{r_1} \left\{ \mathbf{A}_{r_1} \hat{\mathbf{x}}(t) + \mathbf{A}_{r_1} \mathbf{B}\mathbf{Z}_{2^m} \mathbf{u}(t) \right\} \right\} \\
&= \mu_1 \left\{ \mathbf{A}_1 \hat{\mathbf{x}}(t) + \mathbf{A}_1 \mathbf{B}\mathbf{Z}_1 \mathbf{u}(t) \right\} \\
&\quad + \cdots + \mu_{r_1} \left\{ \mathbf{A}_{r_1} \hat{\mathbf{x}}(t) + \mathbf{A}_{r_1} \mathbf{B}\mathbf{Z}_1 \mathbf{u}(t) \right\} \\
&\quad + \mu_{r_1+1} \left\{ \mathbf{A}_1 \hat{\mathbf{x}}(t) + \mathbf{A}_1 \mathbf{B}\mathbf{Z}_2 \mathbf{u}(t) \right\} \\
&\quad + \cdots + \mu_{2r_1} \left\{ \mathbf{A}_{r_1} \hat{\mathbf{x}}(t) + \mathbf{A}_{r_1} \mathbf{B}\mathbf{Z}_2 \mathbf{u}(t) \right\} \\
&\quad + \cdots + \mu_{(2^m-1)r_1+1} \left\{ \mathbf{A}_1 \hat{\mathbf{x}}(t) + \mathbf{A}_1 \mathbf{B}\mathbf{Z}_{2^m} \mathbf{u}(t) \right\} \\
&\quad + \cdots + \mu_{2^m r_1} \left\{ \mathbf{A}_{r_1} \hat{\mathbf{x}}(t) + \mathbf{A}_{r_1} \mathbf{B}\mathbf{Z}_{2^m} \mathbf{u}(t) \right\},
\end{aligned} \tag{14}$$

with $k \in \mathbb{I}_r$, $r = 2^m r_1$ the number of vertices of the polytopic representation, $\sum_{k=1}^{r} \mu_k = 1$ and $\mu_k \geq 0$.

In (14), we take

$$\begin{aligned}
\hat{\mathbf{A}}_{fr_1+k} &= \mathbf{A}_k, \quad \text{for all} \quad k \in \{1, \ldots, r_1\} \quad \text{and} \quad \text{for each} \\
&\quad f \in \left\{0, 1, \ldots, 2^m - 1\right\}, \\
\hat{\mathbf{Z}}_k &= \mathbf{Z}_s \quad \text{for} \quad k = (s-1)r_1 + 1, \cdots, sr_1 \quad \text{for each} \\
&\quad s \in \left\{1, \ldots, 2^m\right\}, \\
\hat{\mathbf{B}}_k &= \hat{\mathbf{A}}_k \mathbf{B} \hat{\mathbf{Z}}_k \quad \text{for all} \quad k \in \{1, \ldots, r\}.
\end{aligned} \tag{15}$$

Then, defining $\mu = [\mu_1 \ \ldots \ \mu_r]^T$, we can rewrite (14) as

$$\dot{\hat{\mathbf{x}}}(t) = \sum_{k=1}^{r} \mu_k \left\{ \hat{\mathbf{A}}_k \hat{\mathbf{x}}(t) + \hat{\mathbf{B}}_k \mathbf{u}(t) \right\} = \hat{\mathbf{A}}(\mu)\hat{\mathbf{x}}(t) + \hat{\mathbf{B}}(\mu)\mathbf{u}(t). \tag{16}$$

2.2 Robust Control of the Auxiliary Dynamic with Actuator Saturation

Consider the uncertain linear system subject to actuator saturation (1). Using the auxiliar dynamics (16), we propose the control scheme presented in Fig. 2. Note that the control law dynamics is given by $\mathbf{u}_N(t) = \dot{\mathbf{u}}(t) \in \Re^m$ presented in (17):

$$\mathbf{u}_N(t) = -\mathbf{K}\mathbf{x}_N(t), \ \text{with} \ \mathbf{x}_N(t) = \left[\hat{\mathbf{x}}(t)^T \ \mathbf{u}(t)^T \right]^T. \tag{17}$$

From (16) and (17), we have the following system:

$$\begin{cases} \dot{\hat{\mathbf{x}}}(t) = \hat{\mathbf{A}}(\mu)\hat{\mathbf{x}}(t) + \hat{\mathbf{B}}(\mu)\mathbf{u}(t) \\ \dot{\mathbf{u}} = \mathbf{u}_N(t) = -\mathbf{K}\mathbf{x}_N(t), \end{cases} \tag{18}$$

which can also be represented by

$$\dot{\mathbf{x}}_N(t) = \begin{bmatrix} \hat{\mathbf{A}}(\mu) & \hat{\mathbf{B}}(\mu) \\ \mathbf{0}_{m\times n} & \mathbf{0}_{m\times m} \end{bmatrix} \mathbf{x}_N(t) + \begin{bmatrix} \mathbf{0}_{n\times m} \\ \mathbf{I}_{m\times m} \end{bmatrix} \mathbf{u}_N(t)$$

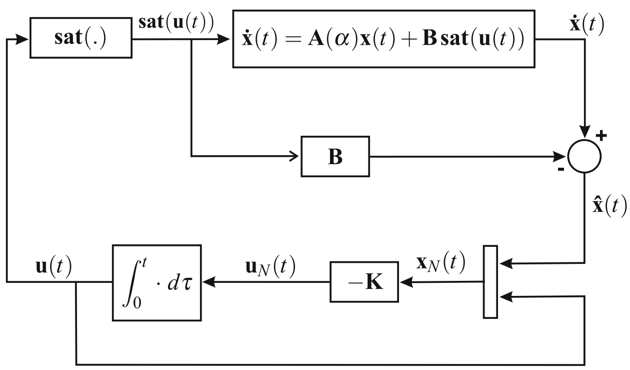

Fig. 2 Schematic of the proposed control law (17) for the uncertain linear system subject to actuator saturation (1)

$$= \sum_{k=1}^{r} \mu_k \mathbf{A}_{N_k}\mathbf{x}_N(t) + \sum_{k=1}^{r} \mu_k \mathbf{B}_N \mathbf{u}_N(t), \tag{19}$$

with $\mathbf{x}_N(t) = \left[\hat{\mathbf{x}}(t)^T \;\; \mathbf{u}(t)^T\right]^T$, $\mathbf{B}_N = \left[\mathbf{0}_{n\times m}{}^T \;\; \mathbf{I}_{m\times m}\right]^T$, $\mathbf{A}_{N_k} = \begin{bmatrix} \hat{\mathbf{A}}_k & \hat{\mathbf{B}}_k \\ \mathbf{0}_{m\times n} & \mathbf{0}_{m\times m} \end{bmatrix}$, where $\hat{\mathbf{A}}_k$ and $\hat{\mathbf{B}}_k$ were defined in (15), or equivalently (Barmish 1983)

$$\dot{\mathbf{x}}_N(t) = \mathbf{A}_N(\mu)(t)\mathbf{x}_N(t) + \mathbf{B}_N\mathbf{u}_N(t). \tag{20}$$

It is interesting to note that the control signal $\mathbf{u}(t)$ composes the state vector $\mathbf{x}_N(t)$. Therefore the restriction $|u_j(t)| \leq \phi_j$, used to obtain (19), can be viewed as an operation region for the auxiliary dynamics. Thus, consider matrices $\mathbf{N} = \left[\mathbf{0}_{m\times n} \;\; \mathbf{I}_{m\times m}\right] \in \Re^{m\times(n+m)}$, $\mathbf{P} \in \Re^{(n+m)\times(n+m)}$ with $\mathbf{P} = \mathbf{P}^T > \mathbf{0}$, vectors $\boldsymbol{\phi} = [\phi_1 \; \ldots \; \phi_m]^T \in \Re^m$, $\boldsymbol{\rho} = [\rho_1 \; \ldots \; \rho_m]^T \in \Re^m$, $\mathbb{I}_m = \{1, \cdots, m\}$, $\rho_j > 0$, $\phi_j \geq \rho_j$, for all $j \in \mathbb{I}_m$.

Let $\mathscr{X}$ and $\mathscr{E}(\mathbf{P}, \delta)$ be the following sets:

$$\mathscr{X} \triangleq \left\{\mathbf{x}_\mathbf{N}(t) \in \Re^{n+m} : \left|\mathbf{N}_{(h)}\mathbf{x}_N(t)\right| \leq \phi_h, \;\; h \in \mathbb{I}_m\right\}, \tag{21}$$

with $\mathbf{N} = \left[\mathbf{0}_{m\times n} \;\; \mathbf{I}_{m\times m}\right]$,

$$\mathscr{E}(\mathbf{P}, \delta) \triangleq \left\{\mathbf{x}_N(t) \in \Re^{n+m} : \mathbf{x}_N(t)^T\mathbf{P}\mathbf{x}_N(t) \leq \delta\right\}, \tag{22}$$

where $\mathbf{N}$ and $\boldsymbol{\phi}$ are known, $\mathbf{N}_{(h)}$ represents the hth row of the matrix $\mathbf{N}$, ϕ_h, the hth element of the vector $\boldsymbol{\phi}$ and a Lyapunov function candidate $V(\mathbf{x}_N(t)) = \mathbf{x}_N(t)^T\mathbf{P}\mathbf{x}_N(t)$.

Theorem 1 *Consider a linear system subject to actuator saturation described by (20) with control law given in (17) and an operation region with $\mathbf{x}_N(t) \in \mathscr{X}$, $t \geq 0$ given in (21) where $\mathbf{N} = \left[\mathbf{0}_{m\times n} \;\; \mathbf{I}_{m\times m}\right]$, $\boldsymbol{\phi} > 0 \in \Re^m$, $\boldsymbol{\rho} > 0 \in \Re^m$, are known. Assume that there exist a symmetric positive definite matrix $\mathbf{X} \in \Re^{(n+m)\times(n+m)}$, a matrix $\mathbf{M} \in \Re^{m\times(m+n)}$ and a scalar $\beta > 0$, such that*

$$\mathbf{A}_{N_k}\mathbf{X} + \mathbf{X}\mathbf{A}_{N_k}{}^T - \mathbf{B}_N\mathbf{M} - \mathbf{M}^T\mathbf{B}_N^T + 2\beta\mathbf{X} < 0, \tag{23}$$

$$\begin{bmatrix} \phi_h{}^2 & \mathbf{N}_{(h)}\mathbf{X} \\ \mathbf{X}\mathbf{N}_{(h)}{}^T & \mathbf{X} \end{bmatrix} \geq 0, \tag{24}$$

for all $h \in \mathbb{I}_m$, $k \in \mathbb{I}_r$. Then, the control law (17), $\mathbf{u}_N(t) = -\mathbf{K}\mathbf{x}_N(t)$, with $\mathbf{K} = \mathbf{M}\mathbf{X}^{-1}$, makes the origin of the state space of the system (20) locally asymptotically stable with decay rate equal to or greater than β for all $\mathbf{x}_N(0) \in \mathscr{E}(\mathbf{P}, 1)$ given in (22), where $\mathbf{P} = \mathbf{X}^{-1}$.

Proof Consider a Lyapunov function candidate $V(\mathbf{x}_N(t)) = \mathbf{x}_N(t)^T\mathbf{P}\,\mathbf{x}_N(t)$, with $0 < \mathbf{P} = \mathbf{P}^T \in \Re^{(n+m)\times(n+m)}$. Assume that $\mathbf{X} = \mathbf{P}^{-1}$, $\mathbf{M} = \mathbf{K}\mathbf{X}$ and suppose that the LMIs described in (23) and (24) , for all $k \in \mathbb{I}_r$ and $h \in \mathbb{I}_m$, hold.

Then, from (23) and $\mathbf{M} = \mathbf{K}\mathbf{X}$ we have

$$\begin{aligned} 0 &> \mathbf{A}_{N_k}\mathbf{X} + \mathbf{X}\mathbf{A}_{N_k}{}^T - \mathbf{B}_N\mathbf{M} - \mathbf{M}^T\mathbf{B}_N{}^T + 2\beta\mathbf{X} \\ &= \mathbf{A}_{N_k}\mathbf{X} + \mathbf{X}\mathbf{A}_{N_k}{}^T - \mathbf{B}_N\mathbf{K}\mathbf{X} - \mathbf{X}\mathbf{K}^T\mathbf{B}_N{}^T + 2\beta\mathbf{X}. \end{aligned} \tag{25}$$

Premultiplying and postmultiplying (25) by $\mathbf{X}^{-1} = \mathbf{P} = \mathbf{P}^T > 0$, it follows that

$$\mathbf{P}\mathbf{A}_{N_k} + \mathbf{A}_{N_k}{}^T\mathbf{P} - \mathbf{P}\mathbf{B}_N\mathbf{K} - \mathbf{K}^T\mathbf{B}_N{}^T\mathbf{P} + 2\beta\mathbf{P} < 0. \tag{26}$$

Assume $\mathbf{x}_N(t) \neq 0$ and premultiplying and postmultiplying (26) by $\mathbf{x}_N(t)^T$ and $\mathbf{x}_N(t)$, respectively, we obtain

$$\begin{aligned} 0 &< \mathbf{x}_N(t)^T\left[\mathbf{P}\mathbf{A}_{N_k} + \mathbf{A}_{N_k}{}^T\mathbf{P} - \mathbf{P}\mathbf{B}_N\mathbf{K} - \mathbf{K}^T\mathbf{B}_N{}^T\mathbf{P}\right]\mathbf{x}_N(t) \\ &\quad +2\beta\mathbf{x}_N(t)^T\mathbf{P}\mathbf{x}_N(t). \end{aligned} \tag{27}$$

Now, multiplying (27) by μ_k, where $\mu_k \geq 0$, $\sum_{k=1}^r \mu_k = 1$ for $k \in \mathbb{I}_r$, and taking the sum from $k = 1$ to r, considering (20), we have

$$\begin{aligned} 0 &> \sum_{k=1}^{r} \mu_k \mathbf{x}_N(t)^T\left[\mathbf{P}\mathbf{A}_{N_k} + \mathbf{A}_{N_k}{}^T\mathbf{P} - \mathbf{P}\mathbf{B}_N\mathbf{K}\right]\mathbf{x}_N(t) \\ &\quad + \sum_{k=1}^{r} \mu_k \mathbf{x}_N(t)^T\left[-\mathbf{K}^T\mathbf{B}_N{}^T\mathbf{P} + 2\beta\mathbf{P}\right]\mathbf{x}_N(t) \\ &= \mathbf{x}_N(t)^T\left[\mathbf{P}\mathbf{A}_N(\mu) + \mathbf{A}_N(\mu)^T\mathbf{P} - \mathbf{P}\mathbf{B}_N\mathbf{K}\right]\mathbf{x}_N(t) \\ &\quad +\mathbf{x}_N(t)^T\left[-\mathbf{K}^T\mathbf{B}_N{}^T\mathbf{P} + 2\beta\mathbf{P}\right]\mathbf{x}_N(t) \\ &= \mathbf{x}_N(t)^T\mathbf{P}\dot{\mathbf{x}}_N(t) + \dot{\mathbf{x}}_N(t)^T\mathbf{P}\mathbf{x}_N(t) + 2\beta\dot{\mathbf{x}}_N(t)^T\mathbf{P}\mathbf{x}_N(t) \\ &= \dot{V}(\mathbf{x}_N(t)) + 2\beta V(\mathbf{x}_N(t)). \end{aligned} \tag{28}$$

Premultiplying and postmultiplying (24) by diag $\{1, \mathbf{P}\}$, with $\mathbf{P} = \mathbf{X}^{-1}$, it follows that

$$\begin{bmatrix} \phi_h{}^2 & \mathbf{N}_{(h)} \\ \mathbf{N}_{(h)}{}^T & \mathbf{P} \end{bmatrix} \geq 0. \tag{29}$$

Applying the Schur complement in (29), observe that

$$\mathbf{P} - \mathbf{N}_{(h)}{}^T \phi_h{}^{-2} \mathbf{N}_{(h)} \geq 0. \tag{30}$$

Premultiplying (30) by $\mathbf{x}_N(t)^T$ and postmultiplying by $\mathbf{x}_N(t) \neq 0$, respectively, note that

$$\mathbf{x}_N(t)^T \mathbf{P} \mathbf{x}_N(t) \geq \mathbf{x}_N(t)^T \mathbf{N}_{(h)}{}^T \phi_h{}^{-2} \mathbf{N}_{(h)} \mathbf{x}_N(t). \tag{31}$$

If $\mathbf{x}_N(t) \in \mathscr{E}(\mathbf{P}, 1)$ given in (22), then $\mathbf{x}_N(t)^T \mathbf{P} \mathbf{x}_N(t) \leq 1$, and from (31),

$$\phi_h{}^2 \geq \mathbf{x}_N(t)^T \mathbf{N}_{(h)}{}^T \mathbf{N}_{(h)} \mathbf{x}_N(t) = \left|\mathbf{N}_{(h)} \mathbf{x}_N(t)\right|^2,$$
$$\left|\mathbf{N}_{(h)} \mathbf{x}_N(t)\right| \leq \phi_h \quad \forall \quad h \in \mathbb{I}_m. \tag{32}$$

Note that for $\mathbf{N} = \begin{bmatrix} \mathbf{0}_{m \times n} & \mathbf{I}_{m \times m} \end{bmatrix}$,

$$\mathbf{N}\mathbf{x}_N(t) = \begin{bmatrix} \mathbf{0}_{m \times n} & \mathbf{I}_{m \times m} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}}(t) \\ \mathbf{u}(t) \end{bmatrix} = \mathbf{u}(t).$$

Hence, $\left|\mathbf{N}_{(h)} \mathbf{x}_N(t)\right| = |\mathbf{u}_h(t)| \leq \phi_h$ for all $h \in \mathbb{I}_m$. These conditions allow the use of the results from Lemma 1, for representing the plant (6) by (16) and the plant with auxiliary dynamics by (20).

Then, if (24) is feasible, for all $\mathbf{x}_N(t) \in \mathscr{E}(\mathbf{P}, 1)$ we have $\mathscr{E}(\mathbf{P}, 1) \subset \mathscr{X}$. In view of (28), for $\mathbf{x}_N(t) \neq 0$, $\dot{V}(\mathbf{x}_N(t)) < 0$, from (22) if $\mathbf{x}_N(0) \in \mathscr{E}(\mathbf{P}, 1)$ then, as long as (24) hold for all $h \in \mathbb{I}_m$, $\mathbf{x}_N(t) \in \mathscr{E}(\mathbf{P}, 1)$ for all $t \geq 0$.

Therefore, for $\mathbf{x}_N(t) \neq 0$, $\dot{V}(\mathbf{x}_N(t)) < -2\beta V(\mathbf{x}_N(t))$, and the closed-loop system (20) and (17) is locally asymptotically stable with decay rate equal to or greater than β (Boyd et al. 1994). □

The sufficient conditions presented in Theorem 1 guarantee the local stability with restriction in the decay rate for the controlled system (17) and (20). The existence of matrices $\mathbf{X}$ and $\mathbf{M}$ satisfying (23) and (24) is a sufficient condition for the proposed controller design. The search for these matrices can be done computationally. In this work, we use the MATLAB software and the LMILab (Gahinet et al. 1994), interfaced by YALMIP (Lofberg 2004) in the search of such matrices.

Theorem 1 guarantees the local asymptotic stability, with decay rate β, of the uncertain linear system (20) in closed loop, with the control law (17), for any initial condition $\mathbf{x}_N(0) \in \mathscr{E}(\mathbf{P}, 1)$. In the same way as in Alves et al. (2016), given the initial conditions polytope, the plant state vector $\mathbf{x}(0) \in \mathscr{X}_0$, where $\mathscr{X}_0 = \text{co}\left\{\mathbf{x}_{0_1}, \ldots, \mathbf{x}_{0_q}\right\}$, $\mathbf{x}_{0_e} \in \Re^n$ for all $e \in \mathbb{I}_q$, $x(0) = \sum_{e=1}^{q} \eta_e \mathbf{x}_{0_e}$, $\eta_e \geq 0$ and $\sum_{e=1}^{q} \eta_e = 1$, which is the convex hull of known vectors $\mathbf{x}_{0_1}, \ldots, \mathbf{x}_{0_q}$. From $\mathscr{X}_0$, we define the sets of initial conditions of interest for the state vector $\mathbf{x}(t)$ and for $\mathbf{x}_N(t) = [(\mathbf{A}(\alpha)\mathbf{x}(t))^t \;\; \mathbf{u}(t)^T]^T$, respectively, as $\hat{w}\mathscr{X}_0$ and $\hat{w}\mathscr{W} \in \mathscr{E}(\mathbf{P}, 1)$, with $\hat{w} > 0$, $\mathbf{u}(0) = \mathbf{0}_{m \times 1}$, where $\mathscr{W} = \text{co}\left\{\begin{bmatrix} \mathbf{A}(\alpha)\mathbf{x}_{0_1} \\ \mathbf{0}_{m \times 1} \end{bmatrix}, \ldots, \begin{bmatrix} \mathbf{A}(\alpha)\mathbf{x}_{0_q} \\ \mathbf{0}_{m \times 1} \end{bmatrix}\right\}$ is a convex hull of vectors $\begin{bmatrix} \mathbf{A}(\alpha)\mathbf{x}_{0_1} \\ \mathbf{0}_{m \times 1} \end{bmatrix}, \ldots, \begin{bmatrix} \mathbf{A}(\alpha)\mathbf{x}_{0_q} \\ \mathbf{0}_{m \times 1} \end{bmatrix}$, with $\sum_{e=1}^{q} \eta_e \begin{bmatrix} \mathbf{A}(\alpha)\mathbf{x}_{0_e} \\ \mathbf{0}_{m \times 1} \end{bmatrix} \in \mathscr{W}$, $\sum_{e=1}^{q} \eta_e = 1$, $\eta_e \geq 0$ for all $e \in \mathbb{I}_q$, with the uncertain but constant vector $\alpha \in \mathscr{P}$ (2).

Lemma 2 *The condition $\hat{w}\mathscr{W} \subset \mathscr{E}(\mathbf{P}, 1)$ holds if*

$$\begin{bmatrix} \hat{w}^{-2} & \begin{bmatrix} \mathbf{A}_i \mathbf{x}_{0_e} \\ \mathbf{0}_{m \times 1} \end{bmatrix}^T \\ \begin{bmatrix} \mathbf{A}_i \mathbf{x}_{0_e} \\ \mathbf{0}_{m \times 1} \end{bmatrix} & \mathbf{X} \end{bmatrix} > 0, \tag{33}$$

for all $e \in \mathbb{I}_q$ and $i \in \mathbb{I}_{r_1}$, where $\hat{w}$ is a positive constant, with $\sum_{e=1}^{q} \eta_e = 1$, $\eta_e \geq 0$. Thus, $\hat{w}$ can be used as a variable to obtain a less conservative condition for the attraction domain in the search for the largest ellipsoid $\mathscr{E}(\mathbf{P}, 1)$ (Alves et al. 2016; Hu et al. 2002; Cao and Lin 2003).

Proof Let us define an initial condition $\mathbf{x}_N(0) = \begin{bmatrix} \hat{\mathbf{x}}(0) \\ \mathbf{0}_{m \times 1} \end{bmatrix} = \begin{bmatrix} \mathbf{A}(\alpha)\mathbf{x}_0(\eta) \\ \mathbf{0}_{m \times 1} \end{bmatrix}$ and multiplying (33) by α_i and η_e, with $\sum_{i=1}^{r_1} \alpha_i = 1$, $\sum_{e=1}^{q} \eta_e = 1$, $\alpha_i \geq 0$ and $\eta_e \geq 0$ we have

$$\begin{bmatrix} \hat{w}^{-2} & \begin{bmatrix} \sum_{i=1}^{r_1} \alpha_i \mathbf{A}_i \sum_{e=1}^{q} \eta_e \mathbf{x}_{0_e} \\ \mathbf{0}_{m \times 1} \end{bmatrix}^T \\ \begin{bmatrix} \sum_{i=1}^{r_1} \alpha_i \mathbf{A}_i \sum_{e=1}^{q} \eta_e \mathbf{x}_{0_e} \\ \mathbf{0}_{m \times 1} \end{bmatrix} & \mathbf{X} \end{bmatrix} > 0.$$
$$\begin{bmatrix} \hat{w}^{-2} & \begin{bmatrix} \mathbf{A}(\alpha)\mathbf{x}_0(\eta) \\ \mathbf{0}_{m \times 1} \end{bmatrix}^T \\ \begin{bmatrix} \mathbf{A}(\alpha)\mathbf{x}_0(\eta) \\ \mathbf{0}_{m \times 1} \end{bmatrix} & \mathbf{X} \end{bmatrix} > 0. \tag{34}$$

Applying the Schur complement (Boyd et al. 1994) to (34) with $\mathbf{P} = \mathbf{X}^{-1}$

$$\hat{w}^{-2} - \begin{bmatrix} \mathbf{A}(\alpha)\mathbf{x}_0(\eta) \\ \mathbf{0}_{m \times 1} \end{bmatrix}^T \mathbf{P} \begin{bmatrix} \mathbf{A}(\alpha)\mathbf{x}_0(\eta) \\ \mathbf{0}_{m \times 1} \end{bmatrix} > 0, \tag{35}$$

multiplying (35) by $-\hat{w}^2$

$$\hat{w} \begin{bmatrix} \mathbf{A}(\alpha)\mathbf{x}_0(\eta) \\ \mathbf{0}_{m \times 1} \end{bmatrix}^T \mathbf{P} \hat{w} \begin{bmatrix} \mathbf{A}(\alpha)\mathbf{x}_0(\eta) \\ \mathbf{0}_{m \times 1} \end{bmatrix} < 1. \tag{36}$$

Therefore, from (36) and (22) it follows that $\hat{w}\mathbf{x}_N(0) \in \mathscr{E}(\mathbf{P}, 1)$. □

Note that in (33), $\hat{w}$ is a scaling factor of the set $\mathscr{W} = \text{co}\left\{\begin{bmatrix} \mathbf{A}(\alpha)\mathbf{x}_{0_1} \\ \mathbf{0}_{m \times 1} \end{bmatrix}, \ldots, \begin{bmatrix} \mathbf{A}(\alpha)\mathbf{x}_{0_q} \\ \mathbf{0}_{m \times 1} \end{bmatrix}\right\}$ (Boyd et al. 1994; Hu et al. 2002; Cao and Lin 2003; Alves et al. 2016). Hence, $\hat{w}$

can be used as a variable to be maximized ($\hat{w}^{-2}$ minimization) in order to obtain a less conservative estimate of the set of initial conditions $\mathscr{E}(\mathbf{P}, 1)$.

3 Relation Between the Auxiliary Dynamics and the Plant Dynamics

In Theorem 1, we guarantee that the system (20) with the control law (17) is locally asymptotically stable with decay rate equal to or greater than β for all $\mathbf{x}_N(0) \in \mathscr{E}(\mathbf{P}, 1)$ given in (22). We need to ensure that there is a decay rate for the system (1) for the state vector $\mathbf{x}(t)$. For this, we will use the following result that can be found in Boyd et al. (1994).

The decay rate of the system (20) (or largest Lyapunov exponent) is defined to be the largest β such that

$$\lim_{t\to\infty} e^{\beta t} \|\mathbf{x}_N(t)\| = 0 \text{ holds for all trajectories } \mathbf{x}_N(t).$$

Consider a Lyapunov function candidate $V(\mathbf{x}_N(t))$ with $V(\mathbf{x}_N(t)) = \mathbf{x}_N(t)^T \mathbf{P}\, \mathbf{x}_N(t)$. If

$$\dot{V}(\mathbf{x}_N(t)) \leq -2\beta V(\mathbf{x}_N(t)) \tag{37}$$

is satisfied, for all $\mathbf{x}_N(t) \neq 0$, we have (Slotine et al. 1991)

$$V(\mathbf{x}_N(t)) \leq V(\mathbf{x}_N(0))e^{-2\beta t}, \quad \forall \quad t > 0 \quad \text{and furthermore,}$$

$$\|\mathbf{x}_N(t)\| \leq e^{-\beta t}\sqrt{\lambda_{\max}(\mathbf{P})/\lambda_{\min}(\mathbf{P})}\, \|\mathbf{x}_N(0)\|, \tag{38}$$

for every trajectory $\mathbf{x}_N(t)$. This fact guarantees that the system (20) has a decay rate greater than or equal to β (Boyd et al. 1994).

In fact, $\mathbf{x}_N(t) = \begin{bmatrix} \hat{\mathbf{x}}(t)^T & \mathbf{u}(t)^T \end{bmatrix}^T$ and

$$\|\mathbf{x}_N(t)\|^2 = \mathbf{x}_N(t)^T \mathbf{x}_N(t) = \hat{\mathbf{x}}(t)^T \hat{\mathbf{x}}(t) + \mathbf{u}(t)^T \mathbf{u}(t). \tag{39}$$

Then, from (38) it follows that

$$\|\mathbf{x}_N(t)\|^2 \leq e^{-2\beta t}(\lambda_{\max}(\mathbf{P})/\lambda_{\min}(\mathbf{P}))\, \|\mathbf{x}_N(0)\|^2. \tag{40}$$

Now, from (39), (40) and considering $\mathbf{u}(0) = \mathbf{0}$ (note that from Fig. 2 one can implement the control system with $\mathbf{u}(0) = \mathbf{0}$) we have

$$\begin{aligned}\hat{\mathbf{x}}(t)^T \hat{\mathbf{x}}(t) &\leq e^{-2\beta t}(\lambda_{\max}(\mathbf{P})/\lambda_{\min}(\mathbf{P}))\, \|\mathbf{x}_N(0)\|^2 - \mathbf{u}(t)^T \mathbf{u}(t)\\ &\leq e^{-2\beta t}(\lambda_{\max}(\mathbf{P})/\lambda_{\min}(\mathbf{P}))\, \|\mathbf{x}_N(0)\|^2\\ &= e^{-2\beta t}(\lambda_{\max}(\mathbf{P})/\lambda_{\min}(\mathbf{P}))\, \left\|\hat{\mathbf{x}}(0)\right\|^2.\end{aligned} \tag{41}$$

Thus,

$$\left\|\hat{\mathbf{x}}(t)\right\| \leq e^{-\beta t}\sqrt{\lambda_{\max}(\mathbf{P})/\lambda_{\min}(\mathbf{P})}\, \left\|\hat{\mathbf{x}}(0)\right\|. \tag{42}$$

It is assumed that $\mathbf{A}(\alpha)$ is non-singular; therefore, $\mathbf{A}(\alpha)^T\mathbf{A}(\alpha)$ is symmetric definite positive, for all $\alpha \in \mathscr{P}$ (2), then,

$$\begin{aligned}\lambda_{\min}(\mathbf{A}(\alpha)^T\mathbf{A}(\alpha))\, \|\mathbf{x}(t)\|^2 &\leq \mathbf{x}(t)^T\mathbf{A}(\alpha)^T\mathbf{A}(\alpha)\mathbf{x}(t)\\ &= \hat{\mathbf{x}}(t)^T\hat{\mathbf{x}}(t)\\ &\leq \lambda_{\max}(\mathbf{A}(\alpha)^T\mathbf{A}(\alpha))\, \|\mathbf{x}(t)\|^2.\end{aligned} \tag{43}$$

From (43), we obtain

$$\begin{aligned}\lambda_{\min}(\mathbf{A}(\alpha)^T\mathbf{A}(\alpha))\, \|\mathbf{x}(0)\|^2 &\leq \mathbf{x}(0)^T\mathbf{A}(\alpha)^T\mathbf{A}(\alpha)\mathbf{x}(0)\\ &\leq \lambda_{\max}(\mathbf{A}(\alpha)^T\mathbf{A}(\alpha))\, \|\mathbf{x}(0)\|^2.\end{aligned} \tag{44}$$

Note that considering $\kappa(\mathbf{P}) = (\lambda_{\max}(\mathbf{P})/\lambda_{\min}(\mathbf{P})), \kappa(\mathbf{A}) = (\lambda_{\max}(\mathbf{A}(\alpha)^T\mathbf{A}(\alpha))/(\lambda_{\min}(\mathbf{A}(\alpha)^T\mathbf{A}(\alpha))$, from (42), (43) and (44) it follows that

$$\begin{aligned}\lambda_{\min}(\mathbf{A}(\alpha)^T\mathbf{A}(\alpha))\, \|\mathbf{x}(t)\|^2 &\leq \hat{\mathbf{x}}(t)^T\hat{\mathbf{x}}(t)\\ &\leq e^{-2\beta t}\kappa(\mathbf{P})\, \|\mathbf{A}(\alpha)\mathbf{x}(0)\|^2\\ &\leq e^{-2\beta t}\kappa(\mathbf{P})\lambda_{\max}(\mathbf{A}(\alpha)^T\mathbf{A}(\alpha))\\ &\quad \|\mathbf{x}(0)\|^2.\end{aligned} \tag{45}$$

Hence, from (45),

$$\|\mathbf{x}(t)\| \leq e^{-\beta t}\sqrt{\kappa(\mathbf{P})}\sqrt{\kappa(\mathbf{A})}\, \|\mathbf{x}(0)\|. \tag{46}$$

Then, $\|\mathbf{x}(t)\|$ is bounded for $t \geq 0$ and for every trajectory $\mathbf{x}(t)$, we guarantee that there is a decay rate greater than or equal to β.

It is important to note that $\mathbf{x}_N(t)^T = [\hat{\mathbf{x}}(t)^T \ \ \mathbf{u}(t)^T]$; thus, once $\mathbf{x}_N(t) \to 0$ for $t \to \infty$, it follows that $\hat{\mathbf{x}}(t) \to 0$ for $t \to \infty$. Therefore, because $\hat{\mathbf{x}}(t) = \mathbf{A}(\alpha)\mathbf{x}(t)$ and $\mathbf{A}(\alpha)$ is a full rank matrix for all $\alpha \in \mathscr{P}$ given in (2), if $\hat{\mathbf{x}}(t) \to 0$, then $\mathbf{x}(t) \to 0$.

4 Examples

In this section, we provide simulation examples to illustrate the effectiveness of our approach. In Sect. 4.1, we consider an academic example. In Sect. 4.2, we will examine the vibration damping system (Abdelaziz 2012) considering damper failures and actuator saturation.

4.1 Example 1

Consider the uncertain linear system (1) with polytopic vertices:

$$\mathbf{A}(\alpha)=\begin{bmatrix}\check{a} & -100\\ 10 & \check{b}\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}10\\ 10\end{bmatrix} \tag{47}$$

with $-1 \le \check{a} \le a$ and $b \le \check{b} \le 110$. The vertices of the polytope related to $\mathbf{A}(\alpha)$ are given by:

$$[\mathbf{A}_1|\mathbf{A}_2|\mathbf{A}_3|\mathbf{A}_4]=\left[\begin{array}{cc|cc|cc|cc}-1 & -100 & a & -100 & -1 & -100 & a & -100\\ 10 & 110 & 10 & 110 & 10 & b & 10 & b\end{array}\right]. \tag{48}$$

In this case, solving the conditions from Theorem 1 and Lemma 2,

$$A_{N_k}=\begin{bmatrix}\hat{A}_k & \hat{A}_k B Z_1\\ 0_{1\times 2} & 0_{1\times 1}\end{bmatrix}\ k=1,\ldots,4,$$

$$A_{N_k}=\begin{bmatrix}\hat{A}_k & \hat{A}_k B Z_2\\ 0_{1\times 2} & 0_{1\times 1}\end{bmatrix}\ k=5,\ldots,8,$$

with $B_N=\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$, $x_N(t)=\begin{bmatrix}\hat{x}(t)^T & u(t)^T\end{bmatrix}^T$, $Z_1=1$, $Z_2=\frac{\rho}{\phi}$, $\hat{A}_5=\hat{A}_1=A_1$, $\hat{A}_6=\hat{A}_2=A_2$, $\hat{A}_7=\hat{A}_3=A_3$ and $\hat{A}_8=\hat{A}_4=A_4$, the decay rate specification $\beta=2.44$, $\boldsymbol{\phi}=1.2$, $\boldsymbol{\rho}=1$, $(a,b)=(60,50)$ and $\mathbf{N}=\begin{bmatrix}\mathbf{0}_{1\times 2} & \mathbf{I}_{1\times 1}\end{bmatrix}$. In Lemma 2, we minimize $\bar{w}$ with $\bar{w}=\hat{w}^{-2}$ and consider the initial conditions: $-0.1 \le x_1(0) \le 0.1$ and $-0.01 \le x_2(0) \le 0.01$. Therefore, the initial conditions polytope has four vertices.

In order for system (1) to be in the proper form for the use of Theorem 1 and Lemma 2, we obtain $\hat{w}=0.3350$,

$$\mathbf{K}=10^8\begin{bmatrix}-1.7658\\ 1.6874\\ 9.9911\end{bmatrix}^T,\ \mathbf{P}=\begin{bmatrix}0.1798 & -0.0704 & -0.3505\\ -0.0704 & 0.0921 & 0.3350\\ -0.3505 & 0.3350 & 1.9834\end{bmatrix}. \tag{49}$$

It is possible to reduce the norm of the gain matrix K, adding new LMIs in Theorem 1 (Alves et al. 2016; Assunção et al. 2007). However, observe that from Fig. 4 the control input $u(t)$ is bounded, due to the specification of the actuator saturation.

For the simulations, we consider the initial condition $\mathbf{x}(0)=\hat{w}\mathbf{x}_{0_4}=[0.0335\quad -0.0034]^T \in \hat{w}\mathscr{X}_0$, $\mathbf{x}_N(0)=[(\mathbf{A}_4\mathbf{x}_{0_4})^T\quad 0]^T=[2.3452\ \ 0.1675\ \ 0]^T$. Then, as discussed before, $V(\mathbf{x}_N(0))=\mathbf{x}_N(0)^T\mathbf{P}\mathbf{x}_N(0)\le 1$ and $\mathbf{x}_N(0)\in\mathscr{E}(\mathbf{P},1)$.

In Figs. 3, 4 and 5, the curves represent the simulations of the system (1), (47), (48), with control law (17) and (49).

Note that $\mathbf{x}_{N_3}(t)=\mathbf{u}(t)$ (17), once we have no restraint for the control signal. In this case, the system is locally asymptotically stable with decay rate greater than or equal to $\beta=2.44$.

4.2 Example 2

This example is based on the vibration absorption system shown in Fig. 6 and given in Abdelaziz (2012). The dynamic equation of the system can be described by a state space form, with state vector $\mathbf{x}(t)=[x_1(t)\ x_2(t)\ \dot{x}_1(t)\ \dot{x}_2(t)]^T$, considering actuator saturation and the possibility of dampers failures, as follows:

$$\dot{\mathbf{x}}(t)=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -k_1c_1 & -k_2c_2 & -f_1b_1c_1 & -f_2b_2c_2\\ -k_1c_2 & -k_2c_1 & -f_1b_1c_2 & -f_2b_2c_1\end{bmatrix}\mathbf{x}(t)$$
$$+\begin{bmatrix}0 & 0\\ 0 & 0\\ c_1 & c_2\\ c_2 & c_1\end{bmatrix}\mathbf{sat}(\mathbf{u}(t)) \tag{50}$$

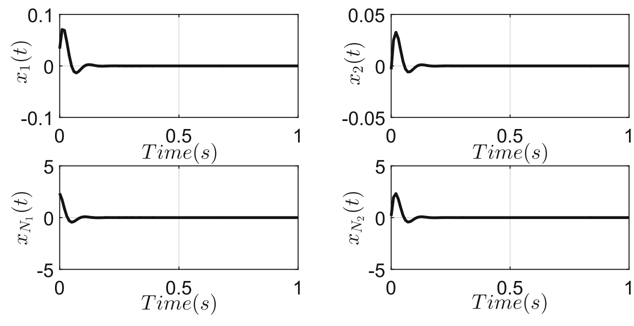

Fig. 3 State variables and auxiliary state variables from the simulation of the system (1), (47), (48), with control law (17) and (49)

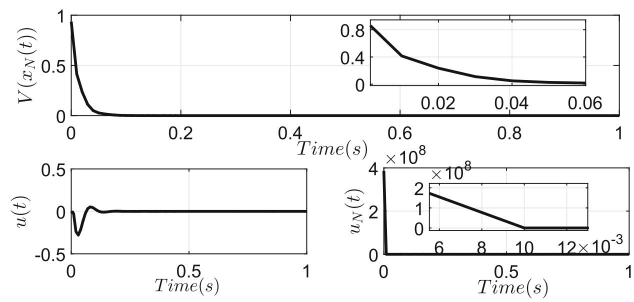

Fig. 4 Lyapunov function $V(\mathbf{x}_N(t))$, control input $u(t)=x_{N_3}$ and signal $\mathbf{u}_N(t)$ from the simulation of the system (1), (47), (48), with control law (17) and (49)

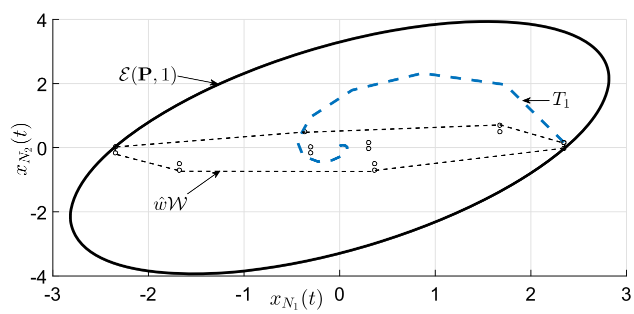

Fig. 5 The ellipsoidal region for $\mathbf{x}_N(t)$ with $\mathbf{u}(0)=0$ for $\mathbf{x}_N(0)=[(\hat{\mathbf{x}}(0))^T\ \ \mathbf{u}(0)^T]^T$, the set of initial condition of interest $\hat{w}\mathscr{W}$ and the 16 initial conditions in $\hat{w}\mathscr{W}$ represented by $\circ$. The curve T_1 is the trajectory for the initial condition $\mathbf{x}(0)=\hat{w}\mathbf{x}_{0_4}=[0.0335\quad -0.0034]^T$ and $\mathbf{x}_N(0)=[(\mathbf{A}_4\hat{w}\mathbf{x}_{0_4})^T\quad 0]^T=[2.3452\ \ 0.1675\ \ 0]^T$

where $c_1 = \frac{1}{m} + \frac{L^2}{I}$, $c_2 = \frac{1}{m} - \frac{L^2}{I}$, m and I represent the mass and inertia, k_1 and k_2 are the constants of the springs, b_1 and b_2 are the constants of the dampers, x_1 and x_2 are the mass displacement of both sides, φ is the angle of inclination of the mass with the horizontal, $2L$ is the distance between two support points and u_1 and u_2 are the control inputs. We consider the following system parameters: $m = 10$ kg, $I = 1$ Kg m^2, $L = 1$ m, $k_1 = 500$ N/m, $k_2 = 600$ N/m, $b_1 = 10$ N s/m and $b_2 = 15$ N s/m. We represent the dampers failures using the uncertain parameters f_1 and f_2 as follows: $f_1 = 0$ and $f_2 = 0$ mean that the dampers have a total failure, if $f_1 = 1$ and $f_2 = 1$ it means that there are no failures in the dampers, and $0 < f_1 < 1$ and $0 < f_2 < 1$ mean that there are partial failures in the dampers. Then, a damping failure can be represented as a parametric uncertainty, obtaining an uncertain matrix $\mathbf{A}(\alpha)$ and the constant matrix $\mathbf{B}$, necessaries for this approach. For the solution, in Theorem 1 we consider the decay rate $\beta = 4.9152$, $\boldsymbol{\phi} = [100.2 \;\; 100.2]^T$, $\boldsymbol{\rho} = [100 \;\; 100]^T$ and $\mathbf{N} = \left[\mathbf{0}_{2\times 4} \; \mathbf{I}_{2\times 2}\right]$. In Lemma 2, we minimize $\bar{w}$ with $\bar{w} = \hat{w}^{-2}$ and consider the initial conditions: $-0.01 \leq x_1(0), x_2(0) \leq 0.01$ and $-0.02 \leq x_3(0), x_4(0) \leq 0.02$. Therefore, the initial conditions polytope has sixteen vertices. Let us assume failures $(f_1, f_2) = (0.6, 0.2)$ on the dampers. In order for system (1) to be in the proper form for the use of Theorem 1, we have the following vertices:

$$[\mathbf{A}_1|\mathbf{A}_2] = \left[\begin{array}{cccc|cccc} 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ -550 & 540 & -7.7 & 13.5 & -550 & 540 & -11 & 12.825 \\ 450 & -660 & 6.3 & -16.5 & 450 & -660 & 9 & -15.675 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$[\mathbf{A}_3|\mathbf{A}_4] = \left[\begin{array}{cccc|cccc} 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ -550 & 540 & -7.7 & 12.825 & -550 & 540 & -11 & 13.5 \\ 450 & -660 & 6.3 & -15.675 & 450 & -660 & 9 & -16.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$\hat{\mathbf{A}}_5 = \hat{\mathbf{A}}_9 = \hat{\mathbf{A}}_{13} = \hat{\mathbf{A}}_1 = \mathbf{A}_1, \hat{\mathbf{A}}_6 = \hat{\mathbf{A}}_{10} = \hat{\mathbf{A}}_{14} = \hat{\mathbf{A}}_2 = \mathbf{A}_2,$$
$$\hat{\mathbf{A}}_7 = \hat{\mathbf{A}}_{11} = \hat{\mathbf{A}}_{15} = \hat{\mathbf{A}}_3 = \mathbf{A}_3, \hat{\mathbf{A}}_8 = \hat{\mathbf{A}}_{12} = \hat{\mathbf{A}}_{16} = \hat{\mathbf{A}}_4 = \mathbf{A}_4,$$
$$[\mathbf{Z}_1|\mathbf{Z}_2|\mathbf{Z}_3] = \left[\begin{array}{cc|cc|cc} 1 & 0 & 0.9980 & 0 & 1.0000 & 0 \\ 0 & 1 & 0 & 1.0000 & 0 & 0.9980 \end{array}\right],$$
$$[\mathbf{Z}_4|\mathbf{B}_N{}^T] = \left[\begin{array}{cc|cccccc} 0.9980 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0.9980 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right],$$
$$\hat{\mathbf{Z}}_1 = \hat{\mathbf{Z}}_2 = \hat{\mathbf{Z}}_3 = \hat{\mathbf{Z}}_4 = \mathbf{Z}_1, \; \hat{\mathbf{Z}}_{13} = \hat{\mathbf{Z}}_{14} = \hat{\mathbf{Z}}_{15} = \hat{\mathbf{Z}}_{16} = \mathbf{Z}_4,$$
$$\hat{\mathbf{Z}}_5 = \hat{\mathbf{Z}}_6 = \hat{\mathbf{Z}}_7 = \hat{\mathbf{Z}}_8 = \mathbf{Z}_2, \; \hat{\mathbf{Z}}_9 = \hat{\mathbf{Z}}_{10} = \hat{\mathbf{Z}}_{11} = \hat{\mathbf{Z}}_{12} = \mathbf{Z}_3. \tag{51}$$

In this case, solving the conditions from Theorem 1 and Lemma 2, we obtain $\hat{w} = 15.6438$

$$\mathbf{K} = 10^7 \begin{bmatrix} 1.0010 & -0.8542 & 0.0003 & 0.0003 & 0.5082 & -0.4996 \\ -0.8301 & 0.9987 & 0.0009 & 0.0009 & -0.4995 & 0.4984 \end{bmatrix},$$

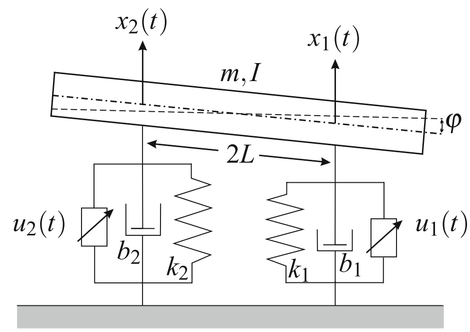

Fig. 6 Vibration absorber system (Abdelaziz 2012)

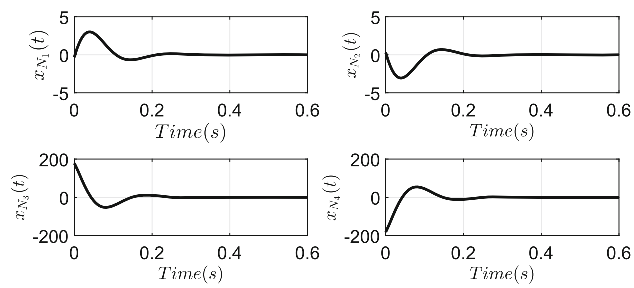

Fig. 7 Auxiliary state variables from the simulation of the system (1), (50), (51), with control law (17) and (52)

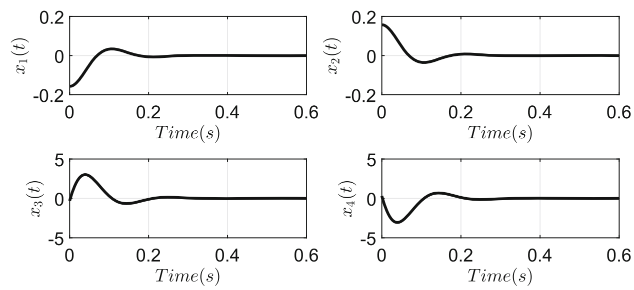

Fig. 8 State variables from the simulation of the system (1), (50), (51), with control law (17) and (52)

$$\mathbf{P} = \begin{bmatrix} 0.8215 & 0.2956 & 0.0008 & 0.0009 & 0.1427 & -0.1184 \\ 0.2956 & 0.7671 & 0.0017 & 0.0018 & -0.1218 & 0.1424 \\ 0.0008 & 0.0017 & 0.0007 & 0.0007 & 0.0000 & 0.0001 \\ 0.0009 & 0.0018 & 0.0007 & 0.0007 & 0.0000 & 0.0001 \\ 0.1427 & -0.1218 & 0.0000 & 0.0000 & 0.0725 & -0.0712 \\ -0.1184 & 0.1424 & 0.0001 & 0.0001 & -0.0712 & 0.0711 \end{bmatrix}. \tag{52}$$

It is possible to reduce the norm of the gain matrix K, adding new LMIs in Theorem 1 (Alves et al. 2016; Assunção et al. 2007). However, observe that from Fig. 9 the control input $u(t)$ is bounded, due to the specification of the actuator saturation.

For the simulations, we consider the initial condition $\mathbf{x}(0) = \hat{w}\mathbf{x}_{0_{14}} = [-0.1564 \;\; 0.1564 \;\; -0.3129 \;\; 0.3129]^T$, $\mathbf{x}_N(0) = [(\mathbf{A}_4\mathbf{x}_{0_{14}})^T \;\; 0 \;\; 0]^T = [-0.3129 \;\; 0.3129 \;\; 177.3376 \;\; -180.5915 \;\; 0 \;\; 0]^T$. Then, as discussed before, $V(\mathbf{x}_N(0)) = \mathbf{x}_N(0)^T\mathbf{P}\mathbf{x}_N(0) \leq 1$ and $\mathbf{x}_N(0) \in \mathscr{E}(\mathbf{P}, 1)$.

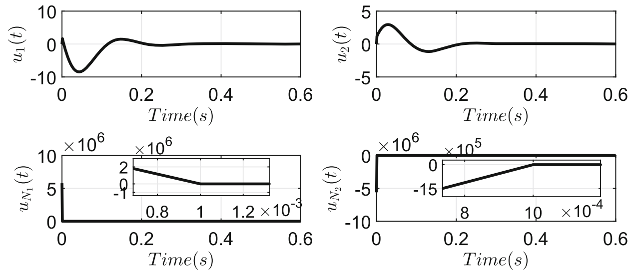

Fig. 9 Control input $\mathbf{u}(t) = [x_{N_5}\ x_{N_6}]^T$ and signal $\mathbf{u}_N(t)$ from the simulation of the system (1), (50), (51), with control law (17) and (52)

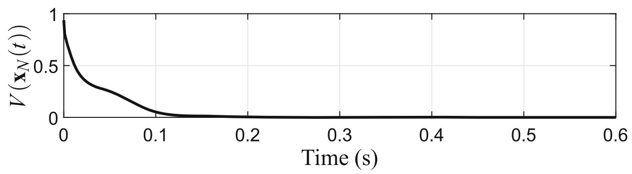

Fig. 10 Lyapunov function $V(\mathbf{x}_N(t)) = \mathbf{x}_N(t)^T\mathbf{P}\mathbf{x}_N(t)$ from the simulation of the system (1), (50), (51), with control law (17) and (52)

In Figs. 7, 8, 9 and 10, the curves represent the simulations of the system (1), (50), (51), with control law (17) and (52).

Note that $x_{N_5}(t) = u_1(t), x_{N_6}(t) = u_2(t)$ in (17), once we have no restraint for the control signal. In this case, the system is locally asymptotically stable with decay rate greater than or equal to $\beta = 4.9152$.

5 Conclusions

The derivative feedback is adequate to practical implementations where the derivative of the system state vector is easier to measure than the state vector of the system. In real implementations, usually the control signal is subject to actuator saturation. This manuscript investigated the derivative control design for time-invariant linear systems with polytopic uncertainties and subject to actuator saturation. We proposed control structures and design procedures based in an auxiliary dynamic, for plants given in (1) and (2), where $\mathbf{A}(\alpha)$ must be a full rank matrix and $\mathbf{B}$ a constant matrix, which allowed us to use a convex description for the actuator saturation. It is important to note that this description for the saturation is valid in an operating region of the auxiliary dynamics and it is guaranteed for all initial condition in an ellipsoidal set that the state remains in this operation region for all $t \geq 0$. Using the proposed strategy, it is possible to guarantee adequate decay rate for the closed-loop system and the effectiveness of the approach was shown by means of examples.

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