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# $L_2$-Gain Analysis and Design of Uncertain Switched Saturated Systems

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**Abstract**
This paper studies the problem on $L_2$-gain analysis and anti-windup design of uncertain discrete-time switched saturated systems by the multiple Lyapunov functions approach. Firstly, we obtain a sufficient condition of tolerable disturbances, under which the state trajectory starting from the origin will remain inside a bounded set. Then, the upper bound of the restricted $L_2$-gain is obtained. Furthermore, the anti-windup compensators and the switched rule, aiming to determine the maximum disturbance tolerance capability and the minimum upper bound of the restricted $L_2$-gain, are presented by solving a constraints optimization problem. Finally, we give a numerical example to show the effectiveness of the proposed method.

**Keywords** $L_2$-Gain · Anti-windup · Switched systems · Actuator saturation · Multiple Lyapunov function · Tolerable disturbances

## 1 Introduction

The switched systems that are an important class of a hybrid system have attracted much attention in recent years (Liberzon and Morse 1999; Lin and Antsaklis 2009; Sun et al. 2002; Varaiya 1993; Zhao and Spong 2001; Sun et al. 2008). Generally, switched systems consist of a set of subsystems that interact with a logical or decision-making process. It is generally known that the stability is of most importance in analysis and design of switched systems. For studying the stability (Zhao and Dimirovski 2004; Cheng 2004; Pettersson 2003; Branicky 1998; Zhai 2001) and synthesis problem for switched systems (Cheng et al. 2003; Hespanha and Morse 1999; Zhai et al. 2001; Zhao and Hill 2008; Lin and Antsaklis 2006; Xie et al. 2004), many approaches have been introduced. Thereinto, the common Lyapunov function is used to check the stability property under arbitrary switchings (Cheng 2004). Although this property is a desirable property, the most switched systems do not possess a common Lyapunov function. Yet, the switched system still is stable under certain switching laws by using other methods. Among them, the multiple Lyapunov functions method (Pettersson 2003; Branicky 1998), the single Lyapunov function method (Zhai 2001; Cheng et al. 2003) and the average dwell-time technique (Hespanha and Morse 1999; Zhai et al. 2001) are effective tools for choosing switching laws.

On the other hand, the practice system is often subject to exogenous disturbances. The $L_2$ analysis has also an important influence for systems with disturbance, which can provide a kind of measure of the certain extent of the influence of disturbance (Zhai et al. 2001; Zhao and Hill 2008). All the results mentioned above study continuous-time switched systems. However, from a practical point of view, studying discrete-time switched systems with disturbances is meaningful. The exponential stability and $L_2$ induced gain performance were investigated for a class of discrete-time switched systems by multiple functions (Lin and Antsaklis 2006). Using the switched Lyapunov function method, Xie et al. (2004) studied the $L_2$-gain analysis and control synthesis of uncertain discrete-time switched systems.

In addition, the actuator saturation appears in almost all practical control systems owing to physical constraints. The input saturation can degrade system performance and even make system unstable. Thereby, the study of saturated systems has received much attention (Silva et al. 2008; Zheng and Wu 2008; Fang et al. 2004; da Silva et al. 2008; Wada et al. 2004). There are many methods developed to deal with saturation (Tarbouriech et al. 2002; Zhang et al. 2008; Hu

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et al. 2002; Gomes da Silva and Tarbouriech 2006). However, the anti-windup method is better for handling input saturation. This method is to firstly design a linear controller that meets the performance requirement of the closed-loop system without considering actuator saturation and then to design an anti-windup compensator to reduce the effects of the actuator saturation (Gomes da Silva and Tarbouriech 2006). So the anti-windup compensators can maintain the performance of the closed-loop system in the absence of saturation, while minimizing the degradation of the performance for the closed-loop system subject to actuator saturation. For switched saturated systems, studying its property becomes more difficult. It is because the saturation and switching signals interact with each other. The existing results are relatively few (Lu and Lin 2008, 2010; Zhang et al. 2011; Lu et al. 2009; Benzaouiaa et al. 2010). And then for the study of the $L_2$-gain and anti-windup design problem of uncertain switched discrete systems on the strength of the multiple Lyapunov functions method, to the best of our ability, there are nearly no results in the existing literature. That is our motivation.

Compared with the existing results for switched saturated systems (Zhang et al. 2012; Lu and Lin 2008, 2010; Zhang et al. 2011; Lu et al. 2009; Benzaouiaa et al. 2010), there are two features of our results. First of all, the L2-gain analysis and anti-windup design problem are simultaneously addressed for discrete-time uncertain switched systems with saturating actuator, while most existing works considered only the problem of stability; second, the multiple Lyapunov functions method is used to study the disturbance tolerance/rejection problem for discrete-time uncertain switched systems with actuator saturation for the first time and no solvability of the problem for subsystem is required, while in the existing literature, the problem has been investigated by using the switched Lyapunov function method which requires the solvability for each subsystem.

Based on the multiple Lyapunov function approach, the $L_2$-gain analysis and anti-windup compensators design are studied for a class of uncertain discrete-time switched saturated systems in this paper. Firstly, we obtain a sufficient condition of disturbance tolerance under which the state trajectory starting from the origin will remain inside a bounded set. Then, we analyzed the restricted $L_2$-gain. Furthermore, in order to obtain the maximal disturbance tolerance capacity and the minimum upper bound of the restricted $L_2$-gain, the problem of designing the anti-windup compensators and the switched rule is formulated and solved as a constraints optimization problem.

## 2 Problem Statement and Preliminaries

The following discrete-time switched systems with actuator saturation are considered:

$$\begin{aligned} x(k+1) &= (A_\sigma + \Delta A_\sigma)x(k) + (B_\sigma + \Delta B_\sigma)\mathrm{sat}(u(k)) \\ &\quad + E_\sigma w(k), \\ y(k) &= C_{\sigma 1} x(k), \\ z(k) &= C_{\sigma 2} x(k), \end{aligned} \tag{1}$$

where $k \in Z^+$, $x(k) \in R^n$ is the state vector, $u(k) \in R^m$ is the control input vector, $y(k) \in R^p$ is the measured output vector, $z(k) \in R^l$ is the controlled output, and $w(k) \in R^q$ is the external disturbance input. $\sigma(k)$ is a switching signal which takes its values in the finite set $I_N = \{1, \cdots, N\}$; $\sigma(k) = i$ means that the $i$th subsystem is active. $A_i$, $B_i$, $E_i$, $C_{i1}$ and $C_{i2}$ are real constant matrices of appropriate dimensions. $\Delta A_i$, $\Delta B_i$ are unknown matrices with time-varying parameter uncertainties in the system matrices and having the following form

$$[\Delta A_i,\ \Delta B_i] = T_i \Gamma(k) [F_{1i},\ F_{2i}], \quad \forall i \in I_N,$$

where $T_i$, $F_{1i}$ and $F_{2i}$ are given constant matrices with proper dimensions which characterize the structure of uncertainties and $\Gamma(k)$ is an unknown time-varying matrix function satisfying

$$\Gamma^{\mathrm{T}}(k)\Gamma(k) \leq I.$$

Due to the presence of actuator saturation, the $L_2$-gain may not be well defined when the external disturbances are sufficiently large, because a sufficiently large external disturbance may drive the system state or output unbounded under any control input (Fang et al. 2004; Lu et al. 2009). Therefore, we assume that

$$W_\beta^2 := \left\{ w : R_+ \to R^q,\ \sum_{k=0}^{\infty} w^{\mathrm{T}}(k)w(k) \leq \beta \right\}, \tag{2}$$

where $\beta$ is some positive number that is aimed at representing disturbance tolerance capability of system. sat : $R^m \to R^m$ is the vector-valued standard saturation function defined as

$$\mathrm{sat}(u) = \left[\mathrm{sat}(u^1),\ \cdots,\ \mathrm{sat}(u^m)\right]^{\mathrm{T}}, \tag{3}$$

$$\mathrm{sat}(u^j) = \mathrm{sign}(u^j)\min\left\{1,\ \left|u^j\right|\right\}, \quad \forall j \in Q_m = \{1, \ldots, m\}. \tag{4}$$

Notice that here we have slightly abused the notation by using "sat(·)" to stand for both scalar- and vector-valued saturation functions. It is generally known that it is without loss of generality to assume unity saturation level. The non-unity saturation level can always be transformed into unity saturation level by scaling the matrix $B_i$ and $u$ (Hu et al. 2002).

For system (1), suppose that a set of $n_c$-order dynamic output feedback controllers are of the form

$$x_c(k+1) = A_{ci}x_c(k) + B_{ci}u_c(k),$$
$$v_c(k) = C_{ci}x_c(k) + D_{ci}u_c(k), \quad \forall i \in I_N, \tag{5}$$

where $x_c(k) \in R^{n_c}$, $u_c(k) = y(k)$ and $v_c(k) = u(k)$ are the vector of state, input and controller output, respectively. In this paper, we focus on $L_2$-gain analysis and anti-windup gains design, so we assume that the dynamic compensators have been designed for system (1) without actuator saturation, as commonly adopted in the literature (see, for example, Gomes da Silva and Tarbouriech 2006).

For the sake of weakening the undesirable effects of the windup caused by actuator saturation, a typical anti-windup compensator includes adding to the controller dynamics a "correction" term of the form $E_{ci}(\mathrm{sat}(v_c(k)) - v_c(k))$. Then, the modified controller structure has the form

$$x_c(k+1) = A_{ci}x_c(k) + B_{ci}u_c(k) + E_{ci}(\mathrm{sat}(v_c(k)) - v_c(k)),$$
$$v_c(k) = C_{ci}x_c(k) + D_{ci}u_c(k), \quad \forall i \in I_N. \tag{6}$$

Clearly, through adding such the correction terms, the dynamic controllers (6) go on operating in the linear domain without actuator saturation, which does not affect the system's performance. Then the controller state of the system with input saturation can be revised by using the anti-windup compensators which restore the system nominal performance as much as possible.

Then, when we adopt the above controllers and anti-windup tactic, the closed-loop system will be written as

$$\begin{aligned} x(k+1) &= (A_i + \Delta A_i)x(k) \\ &\quad + (B_i + \Delta B_i)\mathrm{sat}(v_c(k)) + E_i w(k), \\ y(k) &= C_{i1}x(k), \\ z(k) &= C_{i2}x(k), \\ x_c(k+1) &= A_{ci}x_c(k) + B_{ci}C_{i1}x(k) \\ &\quad + E_{ci}(\mathrm{sat}(v_c(k)) - v_c(k)), \\ v_c(k) &= C_{ci}x_c(k) + D_{ci}C_{i1}x(k), \quad \forall i \in I_N. \end{aligned} \tag{7}$$

Now, define a new state vector

$$\zeta(k) = \begin{bmatrix} x(k) \\ x_c(k) \end{bmatrix} \in R^{n+n_c} \tag{8}$$

and the matrices

$$\tilde{A}_i = \begin{bmatrix} A_i + B_i D_{ci} C_{i1} & B_i C_{ci} \\ B_{ci} C_{i1} & A_{ci} \end{bmatrix}, \ \tilde{B}_i = \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \ G = \begin{bmatrix} 0 \\ I_{n_c} \end{bmatrix},$$

$$K_i = \begin{bmatrix} D_{ci}C_{i1} & C_{ci} \end{bmatrix}, \ \tilde{E}_i = \begin{bmatrix} E_i \\ 0 \end{bmatrix}, \ \tilde{C}_{i2} = \begin{bmatrix} C_{i2} & 0 \end{bmatrix}.$$

$$\tilde{T}_i = \begin{bmatrix} T_i \\ 0 \end{bmatrix}, \ \tilde{F}_i = \begin{bmatrix} F_{1i} + F_{2i}D_{ci}C_{i1} & F_{2i}C_{ci} \end{bmatrix}.$$

Therefore, in combination with (7) and (8), the closed-loop system can be rewritten as

$$\begin{aligned} \zeta(k+1) &= (\tilde{A}_i + \tilde{T}_i\Gamma(k)\tilde{F}_i)\zeta(k) \\ &\quad - (\tilde{B}_i + GE_{ci} + \tilde{T}_i\Gamma(k)F_{2i})\psi(v_c) + \tilde{E}_i w(k), \\ z(k) &= \tilde{C}_{i2}\zeta(k), \quad \forall i \in I_N, \end{aligned} \tag{9}$$

where $v_c = K_i\zeta(k)$, $\psi(v_c) = v_c - \mathrm{sat}(v_c)$.

In this paper, we design the switched law and the anti-windup compensation gains via multiple Lyapunov such that the largest disturbance tolerance level of system (9) is obtained at the beginning and then the minimized upper bound of the restricted $L_2$-gain is achieved.

**Definition 1** (Fang et al. 2004; Lu et al. 2009) Given $\gamma > 0$. System (9) is said to have a restricted $L_2$-gain less than $\gamma$, if there exists a switching signal $\sigma(k)$ such that the following condition is satisfied under the zero initial condition,

$$\sum_{k=0}^{\infty} z^{\mathrm{T}}(k)z(k) < \gamma^2 \sum_{k=0}^{\infty} w^{\mathrm{T}}(k)w(k),$$

for all nonzero $w(k) \in W_\beta^2$.

To develop the main results, we need the following lemma.

For a positive definite matrix $P \in R^{(n+n_c)\times(n+n_c)}$ and a scalar $\rho > 0$, an ellipsoid $\Omega(P, \rho)$ is defined as

$$\Omega(P, \rho) = \left\{ \zeta \in R^{n+n_c} : \zeta^{\mathrm{T}} P \zeta \le \rho \right\}.$$

Consider matrices $K_i$, $H_i \in R^{m\times(n+n_c)}$ and define the following polyhedral set:

$$L(K_i, H_i) = \left\{ \zeta \in R^{n+n_c} : \left| (K_i^j - H_i^j)\zeta \right| \le 1, \ i \in I_N, \ j \in Q_m \right\},$$

where $K_i^j$, $H_i^j$ are the $j$th row of matrices $K_i$ and $H_i$, respectively.

**Lemma 1** (Gomes da Silva and Tarbouriech 2006) *Consider the function $\psi(v_c)$ defined above. If $\zeta \in L(K_i, H_i)$, then the relation*

$$\psi^{\mathrm{T}}(K_i\zeta)J_i[\psi(K_i\zeta) - H_i\zeta] \le 0, \quad \forall i \in I_N, \tag{10}$$

*holds for any matrix $J_i \in R^{m\times m}$ diagonal and positive definite.*

## 3 Disturbance Tolerance

In this section, we derive a sufficient condition under the given anti-windup gain matrices $E_{ci}$ via the multiple Lyapunov function method, which guarantees that the state trajectory of the system (9) starting from the origin will remain inside a bounded set for any disturbance satisfying (2). The approach obtaining the largest disturbance tolerance level by designing the switched law and the anti-windup compensation gains will be stated in Sect. 5.

**Theorem 1** *Suppose there exist positive definite matrices $P_i$, matrices $H_i$ and diagonal positive definite matrices $J_i$ and a set of scalars $\beta_{ir} \geq 0$ and $\lambda_i > 0$ such that*

$$\begin{bmatrix} -P_i + \sum_{r=1,\, r\neq i}^{N} \beta_{ir}(P_r - P_i) & H_i^{\mathrm{T}} J_i & 0 & \tilde{A}_i^{\mathrm{T}} P_i & \tilde{F}_i^{\mathrm{T}} \\ * & -2J_i & 0 & -(\tilde{B}_i + GE_{ci})^{\mathrm{T}} P_i & -F_{2i}^{\mathrm{T}} \\ * & * & -I & \tilde{E}_i^{\mathrm{T}} P_i & 0 \\ * & * & * & -P_i + \lambda_i P_i \tilde{T}_i \tilde{T}_i^{\mathrm{T}} P_i & 0 \\ * & * & * & * & -\lambda_i I \end{bmatrix} < 0, \quad \forall i \in I_N, \tag{11}$$

*and*

$$\Omega(P_i,\ \beta) \cap \Phi_i \subset L(K_i,\ H_i), \quad \forall i \in I_N. \tag{12}$$

*Then under the switched law*

$$\sigma = \arg\min\{\zeta^{\mathrm{T}}(k) P_i \zeta(k),\ i \in I_N\}, \tag{13}$$

*where $\Phi_i = \{\zeta(k) \in R^{n+n_c} : \zeta^{\mathrm{T}}(k)(P_r - P_i)\zeta(k) \geq 0,\ \forall r \in I_N,\ r \neq i\}$; any trajectory of system (9) starting from the origin will remain inside the region $\cup_{i=1}^{N}(\Omega(P_i,\ \beta) \cap \Phi_i \subset L(K_i,\ H_i))$ for every $w \in W_\beta^2$.*

***Proof*** By condition (12), if $\forall\zeta \in \Omega(P_i,\ \beta) \cap \Phi_i$, then $\zeta \in L(K_i,\ H_i)$. Therefore, in view of Lemma 1, for $\forall\zeta \in \Omega(P_i,\ \beta) \cap \Phi_i$ it follows that $\psi(K_i\zeta(k)) = K_i\zeta(k) - \mathrm{sat}(K_i\zeta(k))$ satisfies the sector condition (10).

In view of the switching law (13), for $\forall\zeta(k) \in \Omega(P_i,\ \beta) \cap \Phi_i \subset L(K_i,\ H_i)$, the $i$th subsystem is active.

Then, we choose the following quadratic Lyapunov function candidate for system (9) as

$$V(\zeta(k)) = V_{\sigma(k)}(\zeta(k)) = \zeta^{\mathrm{T}}(k) P_{\sigma(k)} \zeta(k). \tag{14}$$

We split the proof into two parts.

**Case 1** When $\sigma(k+1) = \sigma(k) = i$, for $\forall\zeta(k) \in \Omega(P_i,\ \beta) \cap \Phi_i \subset L(K_i,\ H_i)$, the difference of $V(\zeta(k))$ along the solution of the closed-loop switched system (9) is

$$\begin{aligned} \Delta V(\zeta(k)) &= \zeta^{\mathrm{T}}(k+1) P_i \zeta(k+1) - \zeta^{\mathrm{T}}(k) P_i \zeta(k) \\ &= [(\tilde{A}_i + \tilde{T}_i\Gamma(k)\tilde{F}_i)\zeta(k) - (\tilde{B}_i + GE_{ci} \\ &\quad + \tilde{T}_i\Gamma(k)F_{2i})\psi(K_i\zeta(k)) + \tilde{E}_i w(k)]^{\mathrm{T}} \\ &\quad \times P_i[(\tilde{A}_i + \tilde{T}_i\Gamma(k)\tilde{F}_i)\zeta(k) - (\tilde{B}_i + GE_{ci} \\ &\quad + \tilde{T}_i\Gamma(k)F_{2i})\psi(K_i\zeta(k)) \\ &\quad + \tilde{E}_i w(k)] - \zeta^{\mathrm{T}}(k) P_i \zeta(k). \end{aligned} \tag{15}$$

Therefore, by using Lemma 1 and condition (12), we have

$$\begin{aligned} &\Delta V(\zeta(k)) \\ &\quad \leq [(\tilde{A}_i + \tilde{T}_i\Gamma(k)\tilde{F}_i)\zeta(k) - (\tilde{B}_i + GE_{ci} \\ &\qquad + \tilde{T}_i\Gamma(k)F_{2i})\psi(K_i\zeta(k)) + \tilde{E}_i w(k)]^{\mathrm{T}} \\ &\qquad \times P_r[(\tilde{A}_i + \tilde{T}_i\Gamma(k)\tilde{F}_i)\zeta(k) - (\tilde{B}_i + GE_{ci} \\ &\qquad + \tilde{T}_i\Gamma(k)F_{2i})\psi(K_i\zeta(k)) + \tilde{E}_i w(k)] - \zeta^{\mathrm{T}}(k)(P_i) \\ &\qquad \times \zeta(k) - 2\psi^{\mathrm{T}}(K_i\zeta(k))J_i[\psi(K_i\zeta(k)) - H_i\zeta(k)]. \end{aligned}$$

**Case 2** $\sigma(k) = i,\ \sigma(k+1) = r$ and $i \neq r$, for $\forall\zeta(k) \in \Omega(P_i,\ \beta) \cap \Phi_i \subset L(K_i,\ H_i)$. Then using the switching law (13) gives

$$\begin{aligned} \Delta V(\zeta(k)) &= \zeta^{\mathrm{T}}(k+1) P_r \zeta(k+1) - \zeta^{\mathrm{T}}(k) P_i \zeta(k) \\ &\leq \zeta^{\mathrm{T}}(k+1) P_i \zeta(k+1) - \zeta^{\mathrm{T}}(k) P_i \zeta(k). \end{aligned}$$

In view of Cases 1 and 2, we get

$$\begin{aligned} \Delta V(\zeta(k)) &\leq [(\tilde{A}_i + \tilde{T}_i\Gamma(k)\tilde{F}_i)\zeta(k) - (\tilde{B}_i + GE_{ci} \\ &\quad + \tilde{T}_i\Gamma(k)F_{2i})\psi(K_i\zeta(k)) + \tilde{E}_i w(k)]^{\mathrm{T}} \\ &\quad \times P_r[(\tilde{A}_i + \tilde{T}_i\Gamma(k)\tilde{F}_i)\zeta(k) - (\tilde{B}_i + GE_{ci} \end{aligned}$$

$$+\tilde{T}_i\Gamma(k)F_{2i})\psi(K_i\zeta(k)) + \tilde{E}_i w(k)]$$
$$-\zeta^{\mathrm{T}}(k)(P_i)\zeta(k) - 2\psi^{\mathrm{T}}(K_i\zeta(k))J_i$$
$$[\psi(K_i\zeta(k)) - H_i\zeta(k)],$$

Then, from Schur's complements, (11) is equivalent to

$$\begin{bmatrix} (\tilde{A}_i+\tilde{T}_i\Gamma(k)\tilde{F}_i)^{\mathrm{T}}P_i(\tilde{A}_i \\ \quad +\tilde{T}_i\Gamma(k)\tilde{F}_i)-P_i \\ \quad +\sum_{r=1,\ r\neq i}^{N}\beta_{ir}(P_r-P_i) & \begin{array}{l}-(\tilde{A}_i+\tilde{T}_i\Gamma(k)\tilde{F}_i)^{\mathrm{T}}\\ \times P_i(\tilde{B}_i+GE_{ci}\\ +\tilde{T}_i\Gamma(k)F_{2i})\\ +H_i^{\mathrm{T}}J_i\end{array} & \begin{array}{l}(\tilde{A}_i+\tilde{T}_i\Gamma(k)\tilde{F}_i)^{\mathrm{T}}\\ \times P_i\tilde{E}_i\end{array} \\ * & \begin{array}{l}(\tilde{B}_i+GE_{ci}\\ +\tilde{T}_i\Gamma(k)F_{2i})^{\mathrm{T}}\\ \times P_i(\tilde{B}_i+GE_{ci}\\ +\tilde{T}_i\Gamma(k)F_{2i})-2J_i\end{array} & \begin{array}{l}-(\tilde{B}_i+GE_{ci}\\ +\tilde{T}_i\Gamma(k)F_{2i})^{\mathrm{T}}P_i\tilde{E}_i\end{array} \\ * & * & \tilde{E}_i^{\mathrm{T}}P_i\tilde{E}_i-I \end{bmatrix} < 0. \tag{16}$$

Multiplying (17) from the left by $[x^{\mathrm{T}}\ \psi^{\mathrm{T}}\ w^{\mathrm{T}}]$ and from the right by $[x^{\mathrm{T}}\ \psi^{\mathrm{T}}\ w^{\mathrm{T}}]^{\mathrm{T}}$, we have

$$\Delta V(k) = V(k+1) - V(k) < w^{\mathrm{T}}(k)w(k) - \sum_{r=1,\ r\neq i}^{N} \beta_{ir}\zeta^{\mathrm{T}}(k)(P_r - P_i)\zeta(k). \tag{17}$$

Again by the switching law (13), we obtain

$$\sum_{r=1,\ r\neq i}^{N} \beta_{ir}\zeta^{\mathrm{T}}(k)(P_r - P_i)\zeta(k) \geq 0,$$

which in turn gives

$$\Delta V(k) = V(k+1) - V(k) < w^{\mathrm{T}}(k)w(k). \tag{18}$$

Then, when we consider $V(k)$ as the overall Lyapunov function of system (9), it follows that

$$\Delta V(k) = V(k+1) - V(k) < w^{\mathrm{T}}(k)w(k), \quad \forall \zeta(k) \in \cup_{i=1}^{N}(\Omega(P_i,\ \beta) \cap \Phi_i). \tag{19}$$

Therefore, it follows

$$\sum_{t=0}^{k} \Delta V(t) < \sum_{t=0}^{k} w^{\mathrm{T}}(t)w(t),$$

which indicates

$$V(k+1) < V(0) + \sum_{n=0}^{k} w^{\mathrm{T}}(n)w(n), \quad \forall k \geq 0.$$

Due to $x(0) = 0$ and $\sum_{k=0}^{\infty} w^{\mathrm{T}}(k)w(k) \leq \beta$, we can obtain

$$V(k+1) < \beta, \tag{20}$$

which implies that the state trajectory of the system (9) starting from the origin will always remain inside the region $\cup_{i=1}^{N}(\Omega(P_i,\ \beta) \cap \Phi_i)$ for all times. Thus, this completes the proof. □

In view of the above-established result, we easily know that the disturbance tolerance capability is estimated firstly before we analyze the restricted $L_2$-gain for the closed-loop system (9). Clearly, constant $\beta$ provides a kind of measure of the system's disturbance tolerance capability. Thus, the largest disturbance tolerance level $\beta^*$ is able to be determined by solving the following optimization problem,

$$\begin{aligned} &\sup_{P_i,\ H_i,\ J_i,\ \beta_{ir}} \quad \beta \\ &\text{s.t. } (a) \text{ inequality } (11), \quad \forall i \in I_N, \\ &\quad (b)\ \Omega(P_i,\ \beta) \cap \Phi_i \subset L(K_i,\ H_i), \quad \forall i \in I_N. \end{aligned} \tag{21}$$

Then, pre- and post-multiplying both sides of inequality (11) by *block-diagonal* $\{P_i^{-1},\ J_i^{-1},\ I,\ P_i^{-1},\ I\}$ and letting $P_i^{-1} = X_i,\ H_i P_i^{-1} = M_i,\ J_i^{-1} = S_i$, it follows that

$$\begin{bmatrix} -X_i - \sum_{r=1,\, r\neq i}^{N}\beta_{ir}X_i & M_i^{\mathrm{T}} & 0 & X_i\tilde{A}_i^{\mathrm{T}} & X_i\tilde{F}_i^{\mathrm{T}} & X_i & X_i & X_i \\ * & -2S_i & 0 & -S_i(\tilde{B}_i + GE_{ci})^{\mathrm{T}} & -S_iF_{2i}^{\mathrm{T}} & 0 & 0 & 0 \\ * & * & -I & \tilde{E}_i^{\mathrm{T}} & 0 & 0 & 0 & 0 \\ * & * & * & -X_i + \lambda_i\tilde{T}_i\tilde{T}_i^{\mathrm{T}} & 0 & 0 & 0 & 0 \\ * & * & * & * & -\lambda_i I & 0 & 0 & 0 \\ * & * & * & * & * & -\beta_{i1}^{-1}X_i & 0 & 0 \\ * & * & * & * & * & * & \ddots & 0 \\ * & * & * & * & * & * & * & -\beta_{iN}^{-1}X_N \end{bmatrix} < 0. \tag{22}$$

By using a similar method as in Zhang et al. (2011), the condition (b) is guaranteed by

$$P_i - \sum_{r=1,\, r\neq i}^{N}\delta_{ir}(P_r - P_i) - \beta(K_i^j - H_i^j)^{\mathrm{T}}(K_i^j - H_i^j) \geq 0, \tag{23}$$

where $K_i^j$, $H_i^j$ are the $j$th row of matrices $K_i$ and $H_i$, respectively, and $\delta_{ir} > 0$.

Then from Schur's complements, (23) is equivalent to

$$\begin{bmatrix} P_i - \sum_{r=1,\, r\neq i}^{N}\delta_{ir}(P_r - P_i) & (K_i^j - H_i^j)^{\mathrm{T}} \\ * & \mu \end{bmatrix} \geq 0, \tag{24}$$

where $\mu = \beta^{-1}$.
Thus, pre- and post-multiplying both sides of inequality (24) by *block-diagonal* $\{P_i^{-1}, \ I\}$, we also have

$$\begin{bmatrix} X_i + \sum_{r=1,\, r\neq i}^{N}\delta_{ir}X_i & X_iK_i^{jT} - M_i^{jT} & X_i & X_i & X_i \\ * & \mu & 0 & 0 & 0 \\ * & * & \delta_{i1}^{-1}X_1 & 0 & 0 \\ * & * & * & \ddots & 0 \\ * & * & * & * & \delta_{iN}^{-1}X_N \end{bmatrix} \geq 0, \tag{25}$$

where $M_i^j$ denotes the $j$th row of $M_i$.

As a result, the optimization problem (21) can be formulated as

$$\inf_{X_i,\, M_i,\, S_i,\, \beta_{ir},\, \delta_{ir}} \mu$$
$$\text{s.t. } (a) \text{ inequality (22)}, \quad \forall i \in I_N, \tag{26}$$
$$(b) \text{ inequality (25)}, \quad \forall i \in I_N, \quad \forall j \in Q_m.$$

## 4 $L_2$-Gain Analysis

The $L_2$-gain which can measure the disturbance rejection capability is one of the important performance indexes for control systems. However, due to the presence of actuator saturation, the disturbance rejection capability of the system with actuator saturation is measured by means of the restricted $L_2$-gain over a set of tolerable disturbances. Thus, we study the restricted $L_2$-gain problem for system (9) via the multiple Lyapunov function method in this section. Similarly, we suppose that the anti-windup compensation gains $E_{ci}$ are given beforehand.

**Theorem 2** *Consider switched systems (9). For given positive scalar $\beta \in (0, \ \beta^*]$ and constant $\gamma$, suppose there exist positive definite matrices $P_i$, matrices $H_i$, and diagonal positive definite matrices $J_i$ and a set of scalars $\beta_{ir} \geq 0$ such that*

$$\begin{bmatrix} -P_i + \sum_{r=1,\, r\neq i}^{N} \beta_{ir}(P_r - P_i) & H_i^{\mathrm{T}} J_i & 0 & \tilde{A}_i^{\mathrm{T}} P_i & \tilde{F}_i^{\mathrm{T}} & \tilde{C}_{i2}^{\mathrm{T}} \\ * & -2J_i & 0 & -(\tilde{B}_i + G E_{ci})^{\mathrm{T}} P_i & -F_{2i}^{\mathrm{T}} & 0 \\ * & * & -I & \tilde{E}_i^{\mathrm{T}} P_i & 0 & 0 \\ * & * & * & -P_i + \lambda_i P_i \tilde{T}_i \tilde{T}_i^{\mathrm{T}} P_i & 0 & 0 \\ * & * & * & * & -\lambda_i I & 0 \\ * & * & * & * & * & -\gamma^2 I \end{bmatrix} < 0, \tag{27}$$

$\forall i \in I_N$,

*and*

$$\Omega(P_i,\ \beta) \cap \Phi_i \subset L(K_i,\ H_i), \quad \forall i \in I_N. \tag{28}$$

*Then, under the switching law*

$$\sigma = \arg\min \left\{ \zeta^{\mathrm{T}} P_i \zeta,\ i \in I_N \right\}, \tag{29}$$

*the restricted $L_2$-gain from $w$ to $z$ over $W_\beta^2$ is less than $\gamma$.*

***Proof*** Using the similar method as for proving Theorem 1, we choose the same the multiple Lyapunov function candidate for system (10) as

$$V(\zeta(k)) = V_{\sigma(k)}(\zeta(k)) = \zeta^{\mathrm{T}}(k) P_{\sigma(k)} \zeta(k). \tag{30}$$

We still split the proof into two parts.

**Case 1** $\sigma(k+1) = \sigma(k) = i$, for $\forall \zeta(k) \in \Omega(P_i,\ \beta) \cap \Phi_i \subset L(K_i,\ H_i)$. Then, computing the variation of $V(\zeta(k))$ along the trajectory of the switched system (9), we have

$$\begin{aligned} \Delta V(\zeta(k)) &= \zeta^{\mathrm{T}}(k+1) P_i \zeta(k+1) - \zeta^{\mathrm{T}}(k) P_i \zeta(k) \\ &= [(\tilde{A}_i + \tilde{T}_i \Gamma(k) \tilde{F}_i)\zeta(k) - (\tilde{B}_i + G E_{ci} \\ &\quad + \tilde{T}_i \Gamma(k) F_{2i})\psi(K_i \zeta(k)) + \tilde{E}_i w(k)]^{\mathrm{T}} \\ &\quad \times P_i[(\tilde{A}_i + \tilde{T}_i \Gamma(k) \tilde{F}_i)\zeta(k) - (\tilde{B}_i + G E_{ci} \\ &\quad + \tilde{T}_i \Gamma(k) F_{2i})\psi(K_i \zeta(k)) \\ &\quad + \tilde{E}_i w(k)] - \zeta^{\mathrm{T}}(k) P_i \zeta(k). \end{aligned}$$

Then, in view of Lemma 1 and Condition (29), it follows that

$$\begin{aligned} \Delta V(\zeta(k)) \leq\ & [(\tilde{A}_i + \tilde{T}_i \Gamma(k) \tilde{F}_i)\zeta(k) - (\tilde{B}_i + G E_{ci} \\ & + \tilde{T}_i \Gamma(k) F_{2i})\psi(K_i \zeta(k)) + \tilde{E}_i w(k)]^{\mathrm{T}} \\ & \times P_i[(\tilde{A}_i + \tilde{T}_i \Gamma(k) \tilde{F}_i)\zeta(k) - (\tilde{B}_i + G E_{ci} \\ & + \tilde{T}_i \Gamma(k) F_{2i})\psi(K_i \zeta(k)) + \tilde{E}_i w(k)] - \zeta^{\mathrm{T}}(k) P_i \zeta(k) \\ & - 2\psi^{\mathrm{T}}(K_i \zeta) J_i [\psi(K_i \zeta) - H_i \zeta]. \end{aligned}$$

**Case 2** $\sigma(k) = i,\ \ \sigma(k+1) = r$ and $i \neq r$, for $\forall \zeta(k) \in \Omega(P_i,\ \beta) \cap \Phi_i \subset L(K_i,\ H_i)$. Then applying the switching law (30), we obtain

$$\begin{aligned} \Delta V(\zeta(k)) &= \zeta^{\mathrm{T}}(k+1) P_r \zeta(k+1) - \zeta^{\mathrm{T}}(k) P_i \zeta(k) \\ &\leq \zeta^{\mathrm{T}}(k+1) P_i \zeta(k+1) - \zeta^{\mathrm{T}}(k) P_i \zeta(k). \end{aligned}$$

From Cases 1 and 2, we have

$$\begin{aligned} \Delta V(\zeta(k)) \leq\ & [(\tilde{A}_i + \tilde{T}_i \Gamma(k) \tilde{F}_i)\zeta(k) - (\tilde{B}_i + G E_{ci} \\ & + \tilde{T}_i \Gamma(k) F_{2i})\psi(K_i \zeta(k)) + \tilde{E}_i w(k)]^{\mathrm{T}} \\ & \times P_i[(\tilde{A}_i + \tilde{T}_i \Gamma(k) \tilde{F}_i)\zeta(k) - (\tilde{B}_i + G E_{ci} \\ & + \tilde{T}_i \Gamma(k) F_{2i})\psi(K_i \zeta(k)) + \tilde{E}_i w(k)] - \zeta^{\mathrm{T}}(k) P_i \zeta(k) \\ & - 2\psi^{\mathrm{T}}(K_i \zeta) J_i [\psi(K_i \zeta) - H_i \zeta]. \end{aligned}$$

Then, in view of Schur's complements, (27) is equivalent to

$$\begin{bmatrix} \begin{array}{l} (\tilde{A}_i + \tilde{T}_i \Gamma(k) \tilde{F}_i)^{\mathrm{T}} \\ \times P_i(\tilde{A}_i + \tilde{T}_i \Gamma(k) \tilde{F}_i) \\ -P_i + \gamma^{-2} \tilde{C}_{i2}^{\mathrm{T}} \tilde{C}_{i2} \\ + \sum_{r=1,\, r\neq i}^{N} \beta_{ir}(P_r - P_i) \end{array} & \begin{array}{l} -(\tilde{A}_i + \tilde{T}_i \Gamma(k) \tilde{F}_i)^{\mathrm{T}} \\ \times P_i(\tilde{B}_i + G E_{ci} \\ + \tilde{T}_i \Gamma(k) F_{2i}) + H_i^{\mathrm{T}} J_i \end{array} & \begin{array}{l} (\tilde{A}_i + \tilde{T}_i \Gamma(k) \tilde{F}_i)^{\mathrm{T}} \\ \times P_i \tilde{E}_i \end{array} \\ * & \begin{array}{l} (\tilde{B}_i + G E_{ci} + \tilde{T}_i \Gamma(k) F_{2i})^{\mathrm{T}} \\ \times P_i(\tilde{B}_i + G E_{ci} \\ + \tilde{T}_i \Gamma(k) F_{2i}) - 2J_i \end{array} & \begin{array}{l} -(\tilde{B}_i + G E_{ci} \\ + \tilde{T}_i \Gamma(k) F_{2i})^{\mathrm{T}} P_i \tilde{E}_i \end{array} \\ * & * & \tilde{E}_i^{\mathrm{T}} P_i \tilde{E}_i - I \end{bmatrix} < 0. \tag{31}$$

Multiplying (31) from the left by $[\zeta^{\mathrm{T}}\ \psi^{\mathrm{T}}\ w^{\mathrm{T}}]$ and from the right by $[\zeta^{\mathrm{T}}\ \psi^{\mathrm{T}}\ w^{\mathrm{T}}]^{\mathrm{T}}$, we obtain

$$\begin{aligned}\Delta V(k) = V(k+1) - V(k) < w^{\mathrm{T}}(k)w(k) - \gamma^{-2}z^{\mathrm{T}}(k)z(k)\\ - \sum_{r=1,\ r\neq i}^{N} \beta_{ir}\zeta^{\mathrm{T}}(k)(P_r - P_i)\zeta(k).\end{aligned} \tag{32}$$

Again from the switching law (30), we have

$$\sum_{r=1,\ r\neq i}^{N} \beta_{ir}\zeta^{\mathrm{T}}(k)(P_r - P_i)\zeta(k) \geq 0,$$

which implies that

$$\Delta V(k) = V(k+1)-V(k) < w^{\mathrm{T}}(k)w(k)-\gamma^{-2}z^{\mathrm{T}}(k)z(k). \tag{33}$$

Then, considering $V(k)$ as the overall Lyapunov function of system (9), we obtain

$$\begin{aligned}\Delta V(k) = V(k+1) - V(k) < w^{\mathrm{T}}(k)w(k) - \gamma^{-2}z^{\mathrm{T}}(k)z(k),\\ \forall\zeta(k) \in \cup_{i=1}^{N}(\Omega(P_i,\ \beta)\cap\Phi_i).\end{aligned} \tag{34}$$

Therefore,

$$\sum_{k=0}^{\infty} \Delta V(k) < \sum_{k=0}^{\infty} w^{\mathrm{T}}(k)w(k) - \gamma^{-2}\sum_{k=0}^{\infty} z^{\mathrm{T}}(k)z(k). \tag{35}$$

Then,

$$V(\infty) < V(0)+\sum_{k=0}^{\infty} w^{\mathrm{T}}(k)w(k)-\gamma^{-2}\sum_{k=0}^{\infty} z^{\mathrm{T}}(k)z(k). \tag{36}$$

Due to $V(0) = 0$ and $V(\infty) \geq 0$, we obtain

$$\sum_{k=0}^{\infty} z^{\mathrm{T}}(k)z(k) < \gamma^2\sum_{k=0}^{\infty} w^{\mathrm{T}}(k)w(k), \tag{37}$$

which implies that system (9) has its restricted $L_2$-gain from $w$ to $z$ over $W_{\beta}^2$ less than $\gamma$. Thus, the proof is complete. □

In order to minimize the upper bound of the restricted $L_2$-gain of system (9), the optimization problem can be solved for given $\beta \in (0,\ \beta^*]$ as follows:

$$\begin{aligned}&\inf_{P_i,\ H_i,\ J_i,;\beta_{ir}} \gamma^2\\ &\text{s.t. } (a) \text{ inequality (27)},\quad \forall i \in I_N,\\ &\quad (b)\ \Omega(P_i,\ \beta)\cap\Phi_i \subset L(K_i,\ H_i),\quad \forall i \in I_N.\end{aligned} \tag{38}$$

Applying a similar method as used in changing (21) into (26), we can convert the optimization problem (38) into a constraints optimization problem. Therefore, constraint (a) in (38) is equivalent to

$$\begin{bmatrix} -X_i - \sum_{r=1,\ r\neq i}^{N}\beta_{ir}X_i & M_i^{\mathrm{T}} & 0 & X_i\tilde{A}_i^{\mathrm{T}} & X_i\tilde{F}_i^{\mathrm{T}} & X_i\tilde{C}_{i2}^{\mathrm{T}} & X_i & X_i & X_i\\ * & -2S_i & 0 & -S_i(\tilde{B}_i+GE_{ci})^{\mathrm{T}} & -S_iF_{2i}^{\mathrm{T}} & 0 & 0 & 0 & 0\\ * & * & -I & \tilde{E}_i^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0\\ * & * & * & -X_i+\lambda_i\tilde{T}_i\tilde{T}_i^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0\\ * & * & * & * & -\lambda_iI & 0 & 0 & 0 & 0\\ * & * & * & * & * & -\theta I & 0 & 0 & 0\\ * & * & * & * & * & * & -\beta_{i1}^{-1}X_1 & 0 & 0\\ * & * & * & * & * & * & * & \ddots & 0\\ * & * & * & * & * & * & * & * & -\beta_{iN}^{-1}X_N \end{bmatrix} < 0, \tag{39}$$

where $\theta = \gamma^2$ and constraint (b) in (38) is guaranteed by

$$\begin{bmatrix} X_i+\sum_{r=1,\ r\neq i}^{N}\delta_{ir}X_i & X_iK_i^{jT}-M_i^{jT} & X_i & X_i & X_i\\ * & \mu & 0 & 0 & 0\\ * & * & \delta_{i1}^{-1}X_1 & 0 & 0\\ * & * & * & \ddots & 0\\ * & * & * & * & \delta_{iN}^{-1}X_N \end{bmatrix} \geq 0. \tag{40}$$

Then, the optimization problem (38) can be formulated as

$$\begin{aligned}&\inf_{X_i,\ M_i,\ S_i,\ \beta_{ir},\ \delta_{ir}} \theta\\ &\text{s.t. } (a) \text{ inequality (39)},\quad \forall i \in I_N,\\ &\quad (b) \text{ inequality (40)},\quad \forall i \in I_N,\quad \forall j \in Q_m.\end{aligned} \tag{41}$$

## 5 Anti-windup Synthesis

In fact, anti-windup compensation gains can be designed in order to further improve the closed-loop system (9) performance. Thus, the optimum solutions in Sect. 3 and 4 can be obtained by anti-windup compensation gains design.

Let $N_i = E_{ci}S_i$. Then, (22) and (39) are, respectively, equivalent to

$$\begin{bmatrix} -X_i - \sum_{r=1,\, r\neq i}^{N} \beta_{ir}X_i & M_i^{\mathrm{T}} & 0 & X_i\tilde{A}_i^{\mathrm{T}} & X_i\tilde{F}_i^{\mathrm{T}} & X_i & X_i & X_i \\ * & -2S_i & 0 & -S_i\tilde{B}_i^{\mathrm{T}} - N_i^{\mathrm{T}}G^{\mathrm{T}} & -S_iF_{2i}^{\mathrm{T}} & 0 & 0 & 0 \\ * & * & -I & \tilde{E}_i^{\mathrm{T}} & 0 & 0 & 0 & 0 \\ * & * & * & -X_i + \lambda_i\tilde{T}_i\tilde{T}_i^{\mathrm{T}} & 0 & 0 & 0 & 0 \\ * & * & * & * & -\lambda_i I & 0 & 0 & 0 \\ * & * & * & * & * & -\beta_{i1}^{-1}X_i & 0 & 0 \\ * & * & * & * & * & * & \ddots & 0 \\ * & * & * & * & * & * & * & -\beta_{iN}^{-1}X_N \end{bmatrix} < 0 \tag{42}$$

and

$$\begin{bmatrix} -X_i - \sum_{r=1,\, r\neq i}^{N} \beta_{ir}X_i & M_i^{\mathrm{T}} & 0 & X_i\tilde{A}_i^{\mathrm{T}} & X_i\tilde{F}_i^{\mathrm{T}} & X_i\tilde{C}_{i2}^{\mathrm{T}} & X_i & X_i & X_i \\ * & -2S_i & 0 & -S_i\tilde{B}_i^{\mathrm{T}} - N_i^{\mathrm{T}}G^{\mathrm{T}} & -S_iF_{2i}^{\mathrm{T}} & 0 & 0 & 0 & 0 \\ * & * & -I & \tilde{E}_i^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -X_i + \lambda_i\tilde{T}_i\tilde{T}_i^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\lambda_i I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\theta I & 0 & 0 & 0 \\ * & * & * & * & * & * & -\beta_{i1}^{-1}X_1 & 0 & 0 \\ * & * & * & * & * & * & * & \ddots & 0 \\ * & * & * & * & * & * & * & * & -\beta_{iN}^{-1}X_N \end{bmatrix} < 0. \tag{43}$$

Therefore, the optimization problem which aims to obtain the largest disturbance tolerance level $\beta^*$ is formalized as follows:

$$\begin{aligned} &\inf_{X_i,\, M_i,\, N_i,\, S_i,\, \beta_{ir},\, \delta_{ir}} \mu \\ &\text{s.t. } (a) \text{ inequality (42)}, \quad \forall i \in I_N, \\ &\quad (b) \text{ inequality (25)}, \quad \forall i \in I_N, \quad \forall j \in Q_m, \end{aligned} \tag{44}$$

and then, when any $\beta \in (0,\ \beta^*]$ is given, the minimum upper bound of the restricted $L_2$-gain will be obtained by solving the following optimization problem,

$$\begin{aligned} &\inf_{X_i,\, M_i,\, N_i,\, S_i,\, \beta_{ir},\, \delta_{ir}} \theta \\ &\text{s.t. } (a) \text{ inequality (43)}, \quad \forall i \in I_N, \\ &\quad (b) \text{ inequality (40)}, \quad \forall i \in I_N, \quad \forall j \in Q_m. \end{aligned} \tag{45}$$

When these optimization problems (44) and (45) are solved, we can compute the anti-windup compensation gains $E_{ci} = N_iS_i^{-1}$.

## 6 An Illustrative Example

In order to illustrate the effectiveness of the proposed method, we give the following example in the section.

$$\begin{aligned} x(k+1) &= (A_i + \Delta A_i)x(k) + (B_i + \Delta B_i) \\ &\quad \mathrm{sat}(v_c(k)) + E_iw(k), \\ y(k) &= C_{i1}x(k), \\ z(k) &= C_{i2}x(k), \end{aligned} \tag{46}$$

and the dynamic output feedback controllers with the anti-windup terms are given as

$$x_c(k+1) = A_{ci}x_c(k) + B_{ci}C_{i1}x(k) + E_{ci}(\mathrm{sat}(v_c(k)) - v_c(k)),$$
$$v_c(k) = C_{ci}x_c(k) + D_{ci}C_ix(k), \quad (47)$$

where $\sigma(k) \in I_2 = \{1,\ 2\}$,

$$A_1 = \begin{bmatrix} 1.25 & 0 \\ 0 & 0 \end{bmatrix},\quad A_2 = \begin{bmatrix} 0.339 & 0 \\ 0 & 1.487 \end{bmatrix},\quad B_1 = \begin{bmatrix} 0 \\ 0.75 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} 0 \\ -1.3 \end{bmatrix}, E_1 = \begin{bmatrix} 0.3 & 0.02 \\ 0.44 & 0.04 \end{bmatrix},\quad E_2 = \begin{bmatrix} 0.6 & 0.35 \\ 0.55 & 0.1 \end{bmatrix},$$
$$C_{11} = \begin{bmatrix} 0.345 \\ 0.69 \end{bmatrix}^{\mathrm{T}},\quad C_{21} = \begin{bmatrix} 0.17 \\ -0.3 \end{bmatrix}^{\mathrm{T}}, C_{12} = \begin{bmatrix} 0.058 \\ 0.030 \end{bmatrix}^{\mathrm{T}},$$
$$C_{22} = \begin{bmatrix} -0.019 \\ 0.017 \end{bmatrix}^{\mathrm{T}},\quad A_{c1} = \begin{bmatrix} 0.1133 & 0 \\ 0.0138 & -0.1143 \end{bmatrix},$$
$$A_{c2} = \begin{bmatrix} -0.0515 & 0 \\ 0.0043 & -0.0309 \end{bmatrix},\quad B_{c1} = \begin{bmatrix} -0.0209 \\ -0.0904 \end{bmatrix},$$
$$B_{c2} = \begin{bmatrix} -0.0525 \\ 0.0286 \end{bmatrix}, C_{c1} = \begin{bmatrix} 2.3191 \\ -0.4768 \end{bmatrix}^{\mathrm{T}},\quad C_{c2} = \begin{bmatrix} -2.9468 \\ -1.5688 \end{bmatrix}^{\mathrm{T}},$$
$$D_{c1} = -0.5437,\quad D_{c2} = -1.5199.$$
$$x(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix},\quad x_c(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The uncertain term $[\Delta A_i,\ \Delta B_i] = T_i\Gamma(k)[F_{1i},\ F_{2i}]$ with

$$T_1 = T_2 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix},\quad F_{11} = F_{12} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}^{\mathrm{T}},\quad F_{21} = 0.1,$$
$$F_{22} = 0.2,\ \Gamma(\mathrm{k}) = \sin(\mathrm{k}).$$

Firstly, we design the set of anti-windup compensation gains by using the proposed method in Sect. 5 such that the capability of disturbance tolerance of systems (46)–(47) is maximized via the multiple Lyapunov function method. Thus, solving the optimization problem (44), we obtain the optimal solutions as follows:

$$\mu^* = 0.0546,\ \beta^* = \mu^{*-1} = 18.3150,\ S_1 = 95.2784,$$
$$S_2 = 52.1235,$$
$$X_1 = \begin{bmatrix} 35.4219 & -4.3051 & -3.4539 & -5.6234 \\ * & 91.2354 & -5.2536 & 2.2782 \\ * & * & 8.5237 & 0.0857 \\ * & * & * & 32.5473 \end{bmatrix},$$
$$X_2 = \begin{bmatrix} 38.6719 & -3.9653 & -3.1277 & -4.9839 \\ * & 89.6518 & -6.0253 & 1.8567 \\ * & * & 9.3615 & 0.0975 \\ * & * & * & 42.3566 \end{bmatrix},$$
$$N_1 = \begin{bmatrix} -6.8532 \\ 3.9136 \end{bmatrix}, N_2 = \begin{bmatrix} 6.8735 \\ -1.7584 \end{bmatrix},$$
$$E_{c1} = N_1S_1^{-1} = \begin{bmatrix} -0.0719 \\ 0.0411 \end{bmatrix},\ E_{c2} = N_2S_2^{-1} = \begin{bmatrix} 0.1319 \\ -0.0337 \end{bmatrix}.$$

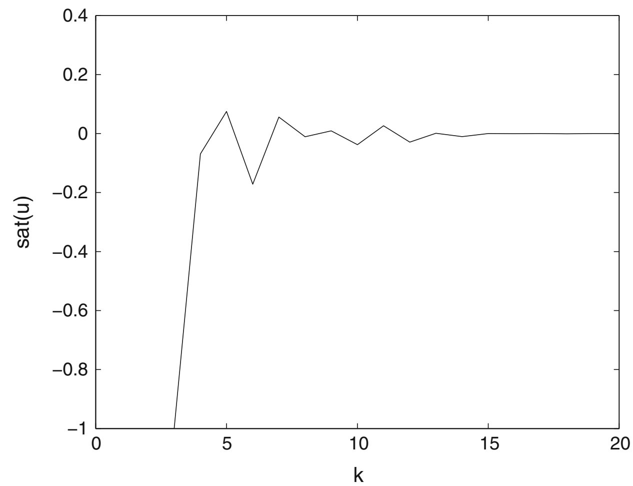


**Fig. 1** Input signal of systems (46)–(47)

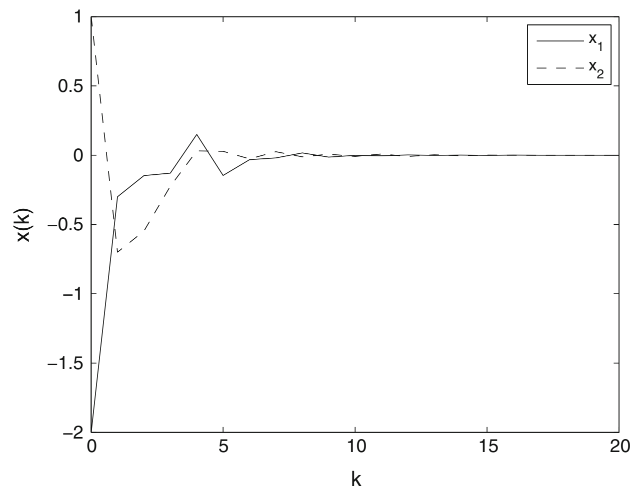


**Fig. 2** State response of systems (46)–(47)

The input signal is shown in Fig. 1. The state response and the controller state response are shown in Figs. 2 and 3, respectively.

In addition, if we let $E_{c1} = E_{c2} = 0$, the obtained optimal solution is $\beta^* = 2.8937$, which implies the disturbance tolerance capacity of the system expanded under the effect of the anti-windup compensators.

Finally, for any given $\beta \in (0,\ \beta^*]$, we can obtain the minimum upper bound of the restricted $L_2$-gain of the switched systems (46)–(47) by solving optimization problem (45). Figure 4 shows the relation of the restricted $L_2$-gain $\gamma$ and different values $\beta \in (0,\ \beta^*]$ of the corresponding system.

On the other hand, we apply the method in Benzaouiaa et al. (2010) to the considered system and find that all the optimization problems have no solutions, which is because the problem of disturbance tolerance/rejection is required to be solvable for every subsystem in Benzaouiaa et al. (2010). However, it is easy to verify that in this example, the problem

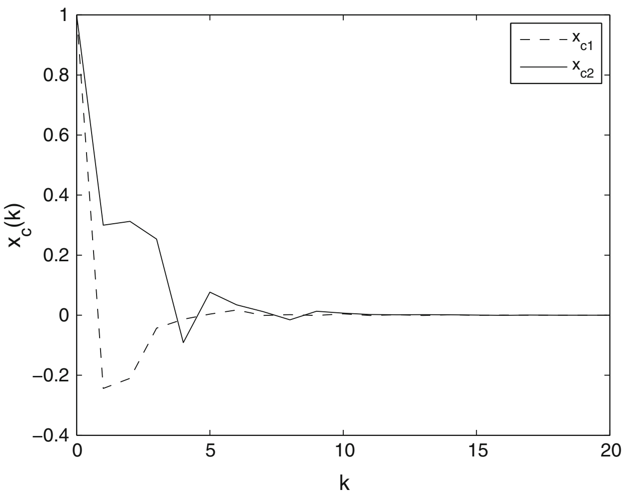


**Fig. 3** Controller state response of systems (46)–(47)

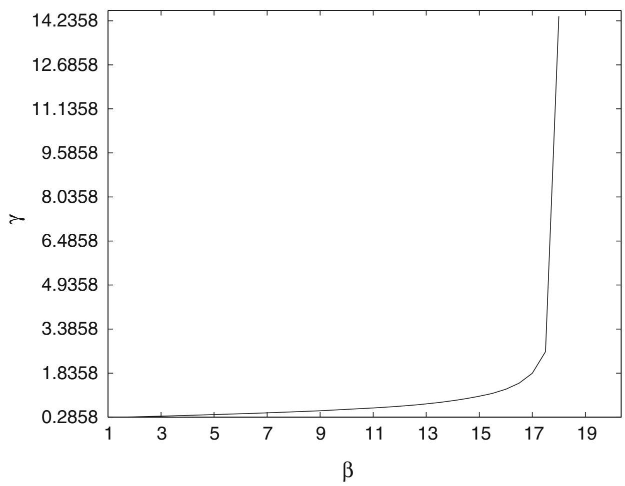


**Fig. 4** Restricted $L_2$-gain of the switched systems (46)–(47) for any $\beta \in (0, \ \beta^*]$

of disturbance tolerance/rejection for each subsystem is not solvable.

## 7 Conclusions

The problem of $L_2$-gain analysis and anti-windup design has been investigated for a class of discrete-time uncertain switched systems subject to actuator saturation. We derive some sufficient conditions of disturbance tolerance and restricted $L_2$-gain by using the multiple Lyapunov function method. Furthermore, we propose a method of designing the anti-windup compensators of the considered system such that the disturbance tolerance capacity is maximized and the upper bound of the restricted $L_2$-gain over the set of tolerable disturbances is minimized, respectively.

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