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Some trapezoid and midpoint type inequalities via fractional (p,q)-calculus

Pheak Neang[1], Kamsing Nonlaopon[1*], Jessada Tariboon[2], Sotiris K. Ntouyas[3,4] and Praveen Agarwal[5,6]

*Correspondence: nkamsi@kku.ac.th
[1] Department of Mathematics, Khon Kaen University, 40002 Khon Kaen, Thailand
Full list of author information is available at the end of the article

Abstract

Fractional calculus is the field of mathematical analysis that investigates and applies integrals and derivatives of arbitrary order. Fractional q-calculus has been investigated and applied in a variety of research subjects including the fractional q-trapezoid and q-midpoint type inequalities. Fractional (p,q)-calculus on finite intervals, particularly the fractional (p,q)-integral inequalities, has been studied. In this paper, we study two identities for continuous functions in the form of fractional (p,q)-integral on finite intervals. Then, the obtained results are used to derive some fractional (p,q)-trapezoid and (p,q)-midpoint type inequalities.

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1 Introduction

The ordinary calculus of Newton and Leibniz is well known to be investigated extensively and intensively to produce a large number of related formulas and properties as well as applications in a variety of fields ranging from natural sciences to social sciences. In the early eighteenth century, the well-known mathematician Leonhard Euler (1707–1783) established quantum calculus or q-calculus, which is the study of calculus without limits, in the way of Newton's work for infinite series. Later, F. H. Jackson initiated a study of q-calculus in a symmetrical manner in 1910 and introduced q-derivative and q-integral in [1], see [2] for more details.

Many physical and mathematical problems have led to the necessity of studying q-calculus; for instance, Fock [3] studied the symmetry of hydrogen atoms using the q-difference equation. In addition, in modern mathematical analysis, q-calculus has lots of applications such as combinatorics, orthogonal polynomials, basic hypergeometric functions, number theory, quantum theory, mechanics, and theory of relativity, see also [4–24] and the references cited therein. The book by Kac and Cheung [25] covers the basic theoretical concepts of q-calculus.

As one of the major driving forces behind the modern approach of real analysis, inequalities have played a vital role in almost all branches of mathematics along with other fields of science. In 2015, Noor et al. [26] established q-analogue of classical integral identity to obtain q-trapezoid type inequalities for convex functions. Moreover, in 2016, Necmettin, Mehmet, and İmdat [27] proved the correctness of left part of q-Hermite–Hadamard and gave some q-midpoint type integral inequalities through q-differentiable convex function and q-differentiable quasi-convex functions. With these results, many researchers have extended some important topics of q-calculus together with applications in many fields, such as q-integral inequalities, see [28–37] for more details.

Since the exploration has been continued to generalize the existing results through creative thoughts and novel techniques of fractional calculus, in 2015, Tariboon, Ntouyas, and Agarwal [38] proposed a new q-shifting operator ${}_a\Phi_q(m) = qm + (1-q)a$ for studying new concepts of fractional q-calculus. In 2016, Sudsutad, Ntouyas, and Tariboon [39] studied some fractional q-integral inequalities. In 2020, Kunt and Aljasem [40] proved Riemann–Liouville fractional q-trapezoid and q-midpoint type inequalities for convex functions. Furthermore, in 2021, Neang et al. [41] introduced fractional (p,q)-calculus on finite intervals and proved some well-known integral inequalities.

In 2018, as one of the most attractive areas, Kunt et al. [42] proved (p,q)-Hermite–Hadamard inequalities and gave some (p,q)-midpoint type integral inequalities via (p,q)-differentiable convex and (p,q)-differentiable quasi-convex functions. In 2019, Latif et al. [43] proved some (p,q)-trapezoid integral inequalities for convex and quasi-convex functions. Based on these results, many authors have generalized and developed (p,q)-calculus, which is used efficiently in many fields, and some results on the study of (p,q)-calculus can be found in [44–71].

Motivated by some of the above studies and applications, in this paper, we study two identities for continuous functions in the form of fractional (p,q)-integral on finite intervals. Then, the obtained results are used to derive some fractional (p,q)-trapezoid and (p,q)-midpoint type inequalities.

2 Preliminaries

In this section, we recall some well-known facts on fractional (p,q)-calculus, which can be found in [10, 11, 38, 53, 55]. Throughout this paper, let $[a,b] \subset \mathbb{R}$ be an interval with $a < b$, and $0 < q < p \leq 1$ be constants,

$$[k]_{p,q} = \left\{ \frac{p^k - q^k}{p-q}, \quad k \in \mathbb{N}, \right. \tag{2.1}$$

$$[k]_{p,q}! = \begin{cases} [k]_{p,q}[k-1]_{p,q}\cdots[1]_{p,q} = \prod_{i=1}^{k} \frac{p^i - q^i}{p-q}, & k \in \mathbb{N}, \\ 1, & k = 0. \end{cases}$$

Property 2.1 ([38]) Let ${}_a\Phi_q(m) = qm + (1-q)a$. For any $m,n \in \mathbb{R}$ and for all positive integers j, k, we have

(i) ${}_a\Phi_q^k(m) = {}_a\Phi_{q^k}(m)$;
(ii) ${}_a\Phi_q^j({}_a\Phi_q^k(m)) = {}_a\Phi_q^k({}_a\Phi_q^j(m)) = {}_a\Phi_q^{j+k}(m)$;
(iii) ${}_a\Phi_q(a) = a$;
(iv) ${}_a\Phi_q^k(m) - a = q^k(m-a)$;
(v) $m - {}_a\Phi_q^k(m) = (1-q^k)(m-a)$;

(vi) ${}_a\Phi_q^k(m) = m\,{}_{a/m}\Phi_q^k(1)$ for $m \neq 0$;
(vii) ${}_a\Phi_q(m) - {}_a\Phi_q^k(n) = q(m - {}_a\Phi_q^{k-1}(n))$.

Property 2.2 ([38]) For any $\gamma, n, m \in \mathbb{R}$ with $n \neq a$ and $k \in \mathbb{N} \cup \{0\}$, we have
(i) $(n-m)_a^{(k)} = (n-a)^k (\frac{m-a}{n-a}; q)_k$;
(ii) $(n-m)^{(\gamma)}{}_a = (n-a)^\gamma \prod_{i=0}^{\infty} \frac{1-\frac{m-a}{n-a}q^i}{1-\frac{m-a}{n-a}q^{\gamma+i}} = (n-a)^\gamma \frac{(\frac{m-a}{n-a};q)_\infty}{(\frac{m-a}{n-a}q^\gamma;q)_\infty}$;
(iii) $(n - {}_a\Phi_q^k(n))_a^\gamma = (n-a)^\gamma \frac{(q^k;q)_\infty}{(q^{\gamma+k};q)_\infty}$.

For $m, n \in \mathbb{R}$, the (p,q)-analogue of the power function ${}_a(m-n)_{p,q}^k$ with $k \in \mathbb{N} \cup \{0\}$ is defined follows:

$$ {}_a(m-n)_{p,q}^{(0)} := 1, \qquad {}_a(m-n)_{p,q}^{(k)} := \prod_{i=0}^{k-1} \bigl({}_a\Phi_p^i(m) - {}_a\Phi_q^i(n)\bigr), \tag{2.2} $$

$$ {}_a(m-n)_{p,q}^{(k)} = (m-a)^k \prod_{i=0}^{k-1} p^i \left(1 - \left(\frac{n-a}{m-a}\right)\left(\frac{q}{p}\right)^i\right). \tag{2.3} $$

More generally, if $\alpha \in \mathbb{R}$, then

$$ {}_a(m-n)_{p,q}^{(\alpha)} = (m-a)^\alpha \prod_{i=0}^{\infty} \frac{p^i}{p^{\alpha+i}} \frac{1-(\frac{n-a}{m-a})(\frac{q}{p})^i}{1-(\frac{n-a}{m-a})(\frac{q}{p})^{\alpha+i}}, \tag{2.4} $$

or

$$ {}_a(m-n)_{p,q}^{(\alpha)} = (m-a)^\alpha p^{\binom{\alpha}{2}} \prod_{i=0}^{\infty} \frac{1-(\frac{n-a}{m-a})(\frac{q}{p})^i}{1-(\frac{n-a}{m-a})(\frac{q}{p})^{\alpha+i}}. \tag{2.5} $$

Property 2.3 ([41]) For $\alpha > 0$, the following formulas hold:
(i) ${}_a\Phi_{q/p}^k(m) - a = (\frac{q}{p})^k (m-a)$;
(ii) ${}_a(m - {}_a\Phi_{q/p}^k(m))_{p,q}^{(\alpha)} = (m-a)^\alpha \prod_{i=0}^{\infty} \frac{p^i}{p^{\alpha+i}} \frac{1-(\frac{q}{p})^k(\frac{q}{p})^i}{1-(\frac{q}{p})^k(\frac{q}{p})^{(\alpha+i)}} = (m-a)^\alpha (1-(\frac{q}{p})^k)_{p,q}^{(\alpha)}$.

Definition 2.1 ([72]) If $f : [a,b] \to \mathbb{R}$ is a continuous function, then the (p,q)-derivative of f on $[a, \frac{1}{p}(b-a)+a]$ at x is defined by

$$ {}_aD_{p,q}f(x) = \frac{f(px+(1-p)a) - f(qx+(1-q)a)}{(p-q)(x-a)}, \quad x \neq a, \tag{2.6} $$
$$ {}_aD_{p,q}f(a) = \lim_{x\to a} {}_aD_{p,q}f(x). $$

Obviously, a function f is (p,q)-differentiable on $[a, \frac{1}{p}(b-a)+a]$ if ${}_aD_{p,q}f(x)$ exists for all $x \in [a, \frac{1}{p}(b-a)+a]$. In Definition 2.1, if $a = 0$, then ${}_0D_{p,q}f = D_{p,q}f$, where $D_{p,q}f$ is defined by

$$ D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}, \quad x \neq 0. \tag{2.7} $$

Furthermore, if $p = 1$ in (2.7), then it reduces to $D_q f$, which is q-derivative of the function f, see [25, 73] for more details.

Definition 2.2 ([72]) If $f:[a,b]\to\mathbb{R}$ is a continuous function, then the (p,q)-integral is defined by

$$\int_a^x f(t)\,{}_a d_{p,q}t=(p-q)(x-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\tag{2.8}$$

for $x\in[a,\frac{1}{p}(b-a)+a]$. If $a=0$ and $p=1$ in (2.8), then we have the classical q-integral, see [25] for more details.

Theorem 2.1 ([72]) *The following formulas hold for* $t\in[a,b]$:
(i) ${}_aD_{p,q}\int_a^t f(s)\,{}_a d_{p,q}s=f(t)$;
(ii) $\int_a^b {}_aD_{p,q}f(s)\,{}_a d_{p,q}s=f(t)-f(a)$;
(iii) $\int_c^t {}_aD_{p,q}f(s)\,{}_a d_{p,q}s=f(t)-f(c)$ for $c\in(a,t)$.

Theorem 2.2 ([72]) *If* $f,g:[a,b]\to\mathbb{R}$ *are continuous functions and* $\lambda\in\mathbb{R}$, *then the following formulas hold*:
(i) $\int_a^t[f(s)+g(s)]\,{}_a d_{p,q}s=\int_a^t f(s)\,{}_a d_{p,q}s+\int_a^t g(s)\,{}_a d_{p,q}s$;
(ii) $\int_a^t \lambda f(s)\,{}_a d_{p,q}s=\lambda\int_a^t f(s)\,{}_a d_{p,q}s$;
(iii) $\int_a^t f(ps+(1-p)a)\,{}_aD_{p,q}g(s)\,{}_a d_{p,q}s=(fg)(s)|_a^t-\int_a^t g(qs+(1-q)a)\,{}_aD_{p,q}(f(s))\,{}_a d_{p,q}s$.

For $t\in\mathbb{R}\setminus\{0,-1,-2,\dots\}$, the (p,q)-gamma function is defined by

$$\Gamma_{p,q}(t)=\frac{(p-q)_{p,q}^{(t-1)}}{(p-q)^{t-1}},\tag{2.9}$$

and an equivalent definition of (2.9) is given in [56] as

$$\Gamma_{p,q}(t)=p^{\frac{t(t-1)}{2}}\int_0^\infty x^{t-1}E_{p,q}^{-qx}\,d_{p,q}x,\tag{2.10}$$

where

$$E_{p,q}^{-qx}=\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}}{[n]_{p,q}}(-qx)^n.$$

Obviously, $\Gamma_{p,q}(t+1)=[t]_{p,q}\Gamma_{p,q}(t)$. For $s,t>0$, the definition of the (p,q)-beta function is defined by

$$B_{p,q}(s,t)=\int_0^1 u^{s-1}{}_0\big(1-{}_0\Phi_q(u)\big)_{p,q}^{(t-1)}\,{}_0d_{p,q}u,\tag{2.11}$$

and (2.11) can also be written as

$$B_{p,q}(s,t)=p^{(t-1)(2s+t-2)/2}\frac{\Gamma_{p,q}(s)\Gamma_{p,q}(t)}{\Gamma_{p,q}(s+t)},\tag{2.12}$$

see [74, 75] for more details.

Definition 2.3 ([41]) Let f be a function defined on $[a,b]$, and let $\alpha > 0$. The Riemann–Liouville fractional (p,q)-integral is defined by

$$\begin{aligned}
&\left({}_aI^{\alpha}_{p,q}f\right)(t)\\
&= \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_a^t {}_a\left(t-{}_a\Phi_q(s)\right)^{(\alpha-1)}_{p,q} f\left(\frac{s}{p^{\alpha-1}}+\left(1-\frac{1}{p^{\alpha-1}}\right)a\right)\,{}_ad_{p,q}s\\
&= \frac{(p-q)(t-a)}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\,{}_a\left(t-{}_a\Phi^{n+1}_{q/p}(t)\right)^{(\alpha-1)}_{p,q} f\left(\frac{q^n}{p^{\alpha+n}}t+\left(1-\frac{q^n}{p^{\alpha+n}}\right)a\right) \qquad (2.13)
\end{aligned}$$

for $t \in [a, p^{\alpha}(b-a)+a]$.

Theorem 2.3 ([41]) *If $f : [a,b] \to \mathbb{R}$ is a convex differentiable function and $\alpha > 0$, then we have*

$$\begin{aligned}
f\left(\frac{([\alpha+1]_{p,q}-p^{\alpha})a+p^{\alpha}b}{[\alpha+1]_{p,q}}\right) &\le \frac{\Gamma_{p,q}(\alpha+1)}{p^{\alpha^2}(b-a)^{\alpha}}\left({}_aI^{\alpha}_{p,q}f(s)\right)\left(p^{\alpha}b+\left(1-p^{\alpha}\right)a\right)\\
&\le \frac{([\alpha+1]_{p,q}-p^{\alpha})f(a)+p^{\alpha}f(b)}{[\alpha+1]_{p,q}}. \qquad (2.14)
\end{aligned}$$

3 Main results

In this section, we give two identities for continuous functions in the form of fractional Riemann–Liouville (p,q)-integral type which will be used to prove the fractional Riemann–Liouville (p,q)-trapezoid and (p,q)-midpoint type inequalities.

Lemma 3.1 *Let $f : [a,b] \to \mathbb{R}$ be a continuous function and $\alpha > 0$. If ${}_aD_{p,q}f$ is (p,q)-integrable on $(a, \frac{1}{p}(b-a)+a)$, then the following equality holds*:

$$\begin{aligned}
&\frac{\Gamma_{p,q}(\alpha+1)}{p^{\alpha^2}(b-a)^{\alpha}}\left({}_aI^{\alpha}_{p,q}f\right)\left(p^{\alpha}b+\left(1-p^{\alpha}\right)a\right)-\frac{([\alpha+1]_{p,q}-p^{\alpha})f(a)+p^{\alpha}f(b)}{[\alpha+1]_{p,q}}\\
&= \frac{(b-a)}{[\alpha+1]_{p,q}}\int_0^1\left([\alpha+1]_{p,q}\left(1-\Phi_q(t)\right)^{\alpha}_{p,q}-p^{\alpha}\right)\,{}_aD_{p,q}f\left((1-t)a+tb\right)\,{}_0d_{p,q}t. \qquad (3.1)
\end{aligned}$$

Proof By simple computation and using Definition 2.3, we have

$$\begin{aligned}
A_1 &= \frac{b-a}{p^{\binom{\alpha}{2}}}\int_0^1\left(1-{}_0\Phi_q(t)\right)^{(\alpha)}_{p,q}\,{}_aD_{p,q}f\left((1-t)a+tb\right)\,{}_0d_{p,q}t\\
&= \frac{b-a}{p^{\binom{\alpha}{2}}}\int_0^1\left(1-{}_0\Phi_q(t)\right)^{(\alpha)}_{p,q}\frac{f((1-pt)a+ptb)-f((1-qt)a+qtb)}{(p-q)(b-a)t}\,{}_0d_{p,q}t\\
&= \frac{1}{p^{\binom{\alpha}{2}}(p-q)}\int_0^1\left(1-{}_0\Phi_q(t)\right)^{(\alpha)}_{p,q}\frac{f((1-pt)a+ptb)}{t}\,{}_0d_{p,q}t\\
&\quad - \frac{1}{p^{\binom{\alpha}{2}}(p-q)}\int_0^1\left(1-{}_0\Phi_q(t)\right)^{(\alpha)}_{p,q}\frac{f((1-qt)a+qtb)}{t}\,{}_0d_{p,q}t\\
&= \frac{1}{p^{\binom{\alpha}{2}}}\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(1-{}_0\Phi^{n+1}_{q/p}(1)\right)^{(\alpha)}_{p,q}\frac{f((1-{}_0\Phi^n_{q/p}(1))a+{}_0\Phi^n_{q/p}(1)b)}{\frac{q^n}{p^{n+1}}}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{p^{\binom{\alpha}{2}}}\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(1-{}_0\Phi_{q/p}^{n+1}(1)\right)_{p,q}^{(\alpha)}\frac{f((1-{}_0\Phi_{q/p}^{n+1}(1))a+{}_0\Phi_{q/p}^{n+1}(1)b)}{\frac{q^n}{p^{n+1}}}\\
&=\sum_{n=0}^{\infty}\frac{((\frac{q}{p})^{n+1};\frac{q}{p})_\infty}{((\frac{q}{p})^{\alpha+n+1};\frac{q}{p})_\infty}f\left(\left(1-\left(\frac{q}{p}\right)^n\right)a+\left(\frac{q}{p}\right)^n b\right)\\
&\quad-\sum_{n=0}^{\infty}\frac{((\frac{q}{p})^{n+1};\frac{q}{p})_\infty}{((\frac{q}{p})^{\alpha+n+1};\frac{q}{p})_\infty}f\left(\left(1-\left(\frac{q}{p}\right)^{n+1}\right)a+\left(\frac{q}{p}\right)^{n+1} b\right)\\
&=\sum_{n=0}^{\infty}\left(1-\left(\frac{q}{p}\right)^{\alpha+n}\right)\frac{((\frac{q}{p})^{n+1};\frac{q}{p})_\infty}{((\frac{q}{p})^{\alpha+n};\frac{q}{p})_\infty}f\left(\left(1-\left(\frac{q}{p}\right)^n\right)a+\left(\frac{q}{p}\right)^n b\right)\\
&\quad-\sum_{n=0}^{\infty}\left(1-\left(\frac{q}{p}\right)^{n+1}\right)\frac{((\frac{q}{p})^{n+2};\frac{q}{p})_\infty}{((\frac{q}{p})^{\alpha+n+1};\frac{q}{p})_\infty}f\left(\left(1-\left(\frac{q}{p}\right)^{n+1}\right)a+\left(\frac{q}{p}\right)^{n+1} b\right)\\
&=\sum_{n=0}^{\infty}\frac{((\frac{q}{p})^{n+1};\frac{q}{p})_\infty}{((\frac{q}{p})^{\alpha+n};\frac{q}{p})_\infty}f\left(\left(1-\left(\frac{q}{p}\right)^n\right)a+\left(\frac{q}{p}\right)^n b\right)\\
&\quad-\sum_{n=0}^{\infty}\frac{((\frac{q}{p})^{n+2};\frac{q}{p})_\infty}{((\frac{q}{p})^{\alpha+n+1};\frac{q}{p})_\infty}f\left(\left(1-\left(\frac{q}{p}\right)^{n+1}\right)a+\left(\frac{q}{p}\right)^{n+1} b\right)\\
&\quad-\left[\sum_{n=0}^{\infty}\left(\frac{q}{p}\right)^{\alpha+n}\frac{((\frac{q}{p})^{n+1};\frac{q}{p})_\infty}{((\frac{q}{p})^{\alpha+n};\frac{q}{p})_\infty}f\left(\left(1-\left(\frac{q}{p}\right)^n\right)a+\left(\frac{q}{p}\right)^n b\right)\right.\\
&\quad\left.-\sum_{n=0}^{\infty}\left(\frac{q}{p}\right)^{n+1}\frac{((\frac{q}{p})^{n+2};\frac{q}{p})_\infty}{((\frac{q}{p})^{\alpha+n+1};\frac{q}{p})_\infty}f\left(\left(1-\left(\frac{q}{p}\right)^{n+1}\right)a+\left(\frac{q}{p}\right)^{n+1} b\right)\right]\\
&=\frac{((\frac{q}{p})^1;\frac{q}{p})_\infty}{((\frac{q}{p})^\alpha;\frac{q}{p})_\infty}f(b)-f(a)-\left[\sum_{n=0}^{\infty}\left(\frac{q}{p}\right)^{\alpha+n}\frac{((\frac{q}{p})^{n+1};\frac{q}{p})_\infty}{((\frac{q}{p})^{\alpha+n};\frac{q}{p})_\infty}f\left(\left(1-\left(\frac{q}{p}\right)^n\right)a+\left(\frac{q}{p}\right)^n b\right)\right.\\
&\quad\left.-\sum_{n=1}^{\infty}\left(\frac{q}{p}\right)^{n}\frac{((\frac{q}{p})^{n+1};\frac{q}{p})_\infty}{((\frac{q}{p})^{\alpha+n};\frac{q}{p})_\infty}f\left(\left(1-\left(\frac{q}{p}\right)^{n}\right)a+\left(\frac{q}{p}\right)^{n} b\right)\right]\\
&=\frac{((\frac{q}{p})^1;\frac{q}{p})_\infty}{((\frac{q}{p})^\alpha;\frac{q}{p})_\infty}f(b)-f(a)-\left[\sum_{n=0}^{\infty}\left(\frac{q}{p}\right)^{\alpha+n}\frac{((\frac{q}{p})^{n+1};\frac{q}{p})_\infty}{((\frac{q}{p})^{\alpha+n};\frac{q}{p})_\infty}f\left(\left(1-\left(\frac{q}{p}\right)^n\right)a+\left(\frac{q}{p}\right)^n b\right)\right.\\
&\quad\left.-\sum_{n=0}^{\infty}\left(\frac{q}{p}\right)^{n}\frac{((\frac{q}{p})^{n+1};\frac{q}{p})_\infty}{((\frac{q}{p})^{\alpha+n};\frac{q}{p})_\infty}f\left(\left(1-\left(\frac{q}{p}\right)^{n}\right)a+\left(\frac{q}{p}\right)^{n} b\right)+\frac{((\frac{q}{p})^1;\frac{q}{p})_\infty}{((\frac{q}{p})^\alpha;\frac{q}{p})_\infty}f(b)\right]\\
&=-f(a)+\left(1-\left(\frac{q}{p}\right)^{\alpha}\right)\sum_{n=0}^{\infty}\left(\frac{q}{p}\right)^{n}\frac{((\frac{q}{p})^{n+1};\frac{q}{p})_\infty}{((\frac{q}{p})^{\alpha+n};\frac{q}{p})_\infty}f\left(\left(1-\left(\frac{q}{p}\right)^{n}\right)a+\left(\frac{q}{p}\right)^{n} b\right)\\
&=-f(a)+\frac{[\alpha]_{p,q}(p-q)}{p^\alpha}\sum_{n=0}^{\infty}\left(\frac{q}{p}\right)^{n}\frac{((\frac{q}{p})^{n+1};\frac{q}{p})_\infty}{((\frac{q}{p})^{\alpha+n};\frac{q}{p})_\infty}f\left(\left(1-\left(\frac{q}{p}\right)^{n}\right)a+\left(\frac{q}{p}\right)^{n} b\right)\\
&=-f(a)+\frac{[\alpha]_{p,q}\Gamma_{p,q}(\alpha)}{p^{\alpha^2}(b-a)^\alpha}\frac{(p-q)p^\alpha(b-a)}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}p^{\alpha(\alpha-1)}(b-a)^{\alpha-1}p^{\binom{\alpha-1}{2}}\\
&\quad\times\frac{((\frac{q}{p})^{n+1};\frac{q}{p})_\infty}{((\frac{q}{p})^{(\alpha-1)+(n+1)};\frac{q}{p})_\infty}f\left(\left(1-\left(\frac{q}{p}\right)^{n}\right)a+\left(\frac{q}{p}\right)^{n} b\right)\\
&=-f(a)+\frac{\Gamma_{p,q}(\alpha+1)}{p^{\alpha^2}(b-a)^\alpha}\left[\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_a^{{}_a\Phi_{p^\alpha}(b)}{}_a\left({}_a\Phi_{p^\alpha}(b)-{}_a\Phi_q(t)\right)_{p,q}^{(\alpha-1)}\right.
\end{aligned}
$$

$$
\begin{aligned}
&\times f\left(\frac{t}{p^{\alpha-1}}+\left(1-\frac{1}{p^{\alpha-1}}\right)a\right){}_a d_{p,q}t\Bigg]\\
&= -f(a)+\frac{\Gamma_{p,q}(\alpha+1)}{p^{\alpha^2}(b-a)^{\alpha}}\left({}_aI^{\alpha}_{p,q}f\right)\left(p^{\alpha}b+\left(1-p^{\alpha}\right)a\right),
\end{aligned} \tag{3.2}
$$

and

$$
\begin{aligned}
A_2 &= \frac{p^{\alpha}(b-a)}{[\alpha+1]_{p,q}}\int_0^1 {}_aD_{p,q}f\left((1-t)a+tb\right){}_0d_{p,q}t\\
&= \frac{p^{\alpha}(b-a)}{[\alpha+1]_{p,q}}\int_0^1 \frac{f((1-pt)a+ptb)-f((1-qt)a+qtb)}{(p-q)(b-a)t}\,{}_0d_{p,q}t\\
&= \Bigg[\frac{p^{\alpha}}{(p-q)[\alpha+1]_{p,q}}\int_0^1\frac{f((1-pt)a+ptb)}{t}\,{}_0d_{p,q}t\\
&\quad -\frac{p^{\alpha}}{(p-q)[\alpha+1]_{p,q}}\int_0^1\frac{f((1-qt)a+qtb)}{t}\,{}_0d_{p,q}t\Bigg]\\
&= \frac{p^{\alpha}}{[\alpha+1]_{p,q}}\Bigg[\sum_{n=0}^{\infty}f\left(\left(1-\left(\frac{q}{p}\right)^{n}\right)a+\left(\frac{q}{p}\right)^{n}b\right)\\
&\quad -\sum_{n=0}^{\infty}f\left(\left(1-\left(\frac{q}{p}\right)^{n+1}\right)a+\left(\frac{q}{p}\right)^{n+1}b\right)\Bigg]\\
&= \frac{p^{\alpha}f(b)-p^{\alpha}f(a)}{[\alpha+1]_{p,q}}.
\end{aligned} \tag{3.3}
$$

From (3.2) and (3.3), we obtain

$$
\begin{aligned}
&\frac{(b-a)}{[\alpha+1]_{p,q}}\int_0^1\left(\frac{[\alpha+1]_{p,q}}{p^{\binom{\alpha}{2}}}\left(1-{}_0\Phi_q(s)\right)^{(\alpha)}_{p,q}-p^{\alpha}\right){}_aD_{p,q}f\left((1-t)a+tb\right){}_0d_{p,q}t\\
&= A_1-A_2\\
&= \frac{\Gamma_{p,q}(\alpha+1)}{p^{\alpha^2}(b-a)^{\alpha}}\left({}_aI^{\alpha}_{p,q}f\right)\left(p^{\alpha}b+\left(1-p^{\alpha}\right)a\right)-\frac{([\alpha+1]_{p,q}-p^{\alpha})f(a)+p^{\alpha}f(b)}{[\alpha+1]_{p,q}}.
\end{aligned} \tag{3.4}
$$

Thus the proof is completed. □

Remark 3.1 If $\alpha = 1$, then (3.1) reduces to Lemma 3.2 in [43] as

$$
\begin{aligned}
&\frac{1}{p(b-a)}\int_a^{pb+(1-p)a}f(x)\,{}_ad_{p,q}x-\frac{pf(a)+qf(a)}{p+q}\\
&= \frac{q(b-a)}{p+q}\int_0^1\left(1-(p+q)t\right){}_aD_{p,q}f\left(tb+(1-t)a\right){}_ad_{p,q}t.
\end{aligned} \tag{3.5}
$$

If $p = 1$, then (3.1) reduces to Lemma 5.2 in [40] as

$$
\begin{aligned}
&\frac{\Gamma_q(\alpha+1)}{(b-a)^{\alpha}}\left({}_aI^{\alpha}_{q}f\right)(b)-\frac{([\alpha+1]_q-1)f(a)+f(b)}{[\alpha+1]_q}\\
&= \frac{(b-a)}{[\alpha+1]_q}\int_0^1\left([\alpha+1]_q\left(1-\Phi_q(t)\right)^{(\alpha)}_q-1\right){}_aD_qf\left((1-t)a+tb\right){}_0d_qt.
\end{aligned} \tag{3.6}
$$

Moreover, if $q \to 1$ and $\alpha = 1$, then (3.6) reduces to

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{b-a}{2}\int_0^1 (1-2t)f'\bigl(ta+(1-t)b\bigr)\,dt, \tag{3.7}$$

which can be found in [76].

Theorem 3.1 *Let $f:[a,b]\to\mathbb{R}$ be a continuous function, $\alpha>0$, and ${}_aD_{p,q}f$ be (p,q)-integrable on $(a,\frac{1}{p}(b-a)+a)$. If $|{}_aD_{p,q}f|$ is convex on*

$$\left(a,\frac{1}{p}(b-a)+a\right),$$

then the following Riemann–Liouville fractional (p,q)-trapezoid type inequality holds:

$$\begin{aligned}&\left|\frac{\Gamma_{p,q}(\alpha+1)}{p^{\alpha^2}(b-a)^\alpha}\bigl({}_aI^\alpha_{p,q}f\bigr)\bigl(p^\alpha b+\bigl(1-p^\alpha\bigr)a\bigr)-\frac{([\alpha+1]_{p,q}-p^\alpha)f(a)+p^\alpha f(b)}{[\alpha+1]_{p,q}}\right|\\&\quad\le\frac{(b-a)}{[\alpha+1]_{p,q}}\bigl(\bigl|{}_aD_{p,q}f(a)\bigr|B_1+\bigl|{}_aD_{p,q}f(b)\bigr|B_2\bigr),\end{aligned}\tag{3.8}$$

where

$$B_1=\int_0^1\left|\frac{[\alpha+1]_{p,q}}{p^{\binom{\alpha}{2}}}\bigl(1-{}_0\Phi_q(t)\bigr)^{(\alpha)}_{p,q}-p^\alpha\right|(1-t)\,{}_0d_{p,q}t$$

and

$$B_2=\int_0^1\left|\frac{[\alpha+1]_{p,q}}{p^{\binom{\alpha}{2}}}\bigl(1-{}_0\Phi_q(t)\bigr)^{(\alpha)}_{p,q}-p^\alpha\right|t\,{}_0d_{p,q}t.$$

Proof Using Lemma 3.1 and the convexity of $|{}_aD_{p,q}f|$, we have

$$\begin{aligned}&\left|\frac{\Gamma_{p,q}(\alpha+1)}{p^{\alpha^2}(b-a)^\alpha}\bigl({}_aI^\alpha_{p,q}f\bigr)\bigl(p^\alpha b+\bigl(1-p^\alpha\bigr)a\bigr)-\frac{([\alpha+1]_{p,q}-p^\alpha)f(a)+p^\alpha f(b)}{[\alpha+1]_{p,q}}\right|\\&\quad\le\frac{(b-a)}{[\alpha+1]_{p,q}}\int_0^1\left|\frac{[\alpha+1]_{p,q}}{p^{\binom{\alpha}{2}}}\bigl(1-{}_0\Phi_q(t)\bigr)^{(\alpha)}_{p,q}-p^\alpha\right|\bigl|{}_aD_{p,q}f\bigl((1-t)a+tb\bigr)\bigr|\,{}_0d_{p,q}t\\&\quad\le\frac{(b-a)}{[\alpha+1]_{p,q}}\int_0^1\left|\frac{[\alpha+1]_{p,q}}{p^{\binom{\alpha}{2}}}\bigl(1-{}_0\Phi_q(t)\bigr)^{(\alpha)}_{p,q}-p^\alpha\right|\\&\qquad\times\bigl[\bigl|{}_aD_{p,q}f(a)\bigr|(1-t)+\bigl|{}_aD_{p,q}f(b)\bigr|t\bigr]\,{}_0d_{p,q}t\\&\quad\le\frac{(b-a)}{[\alpha+1]_{p,q}}\left[\bigl|{}_aD_{p,q}f(a)\bigr|\int_0^1\left|\frac{[\alpha+1]_{p,q}}{p^{\binom{\alpha}{2}}}\bigl(1-{}_0\Phi_q(t)\bigr)^{(\alpha)}_{p,q}-p^\alpha\right|(1-t)\right]{}_0d_{p,q}t\\&\qquad+\frac{(b-a)}{[\alpha+1]_{p,q}}\left[\bigl|{}_aD_{p,q}f(b)\bigr|\int_0^1\left|\frac{[\alpha+1]_{p,q}}{p^{\binom{\alpha}{2}}}\bigl(1-{}_0\Phi_q(t)\bigr)^{(\alpha)}_{p,q}-p^\alpha\right|t\right]{}_0d_{p,q}t.\end{aligned}$$

This completes the proof. □

Remark 3.2 If $p = 1$, then (3.8) reduces to

$$\begin{aligned}
&\left|\frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha}\left({}_aI_q^\alpha f\right)(b) - \frac{([\alpha+1]_q - 1)f(a) + f(b)}{[\alpha+1]_q}\right| \\
&\quad \le \frac{(b-a)}{[\alpha+1]_q}\left(\left|{}_aD_qf(a)\right|\delta_1 + \left|{}_aD_qf(b)\right|\delta_2\right),
\end{aligned} \tag{3.9}$$

where

$$\delta_1 = \int_0^1 \left|[\alpha+1]_q\left(1 - {}_0\Phi_q(t)\right)_q^{(\alpha)} - 1\right|(1-t)\,{}_0d_qt$$

and

$$\delta_2 = \int_0^1 \left|[\alpha+1]_q\left(1 - {}_0\Phi_q(t)\right)_q^{(\alpha)} - 1\right|t\,{}_0d_qt,$$

which appeared in [40].

Theorem 3.2 *Let $f : [a,b] \to \mathbb{R}$ be a continuous function, $\alpha > 0$, and ${}_aD_{p,q}f$ be (p,q)-integrable on $(a, \frac{1}{p}(b-a)+a)$. If $|{}_aD_{p,q}f|^r$ is convex on $(a, \frac{1}{p}(b-a)+a)$ for $r \ge 0$, then the following Riemann–Liouville fractional (p,q)-trapezoid type inequality holds*:

$$\begin{aligned}
&\left|\frac{\Gamma_{p,q}(\alpha+1)}{p^{\alpha^2}(b-a)^\alpha}\left({}_aI_{p,q}^\alpha f\right)\left(p^\alpha b + \left(1-p^\alpha\right)a\right) - \frac{([\alpha+1]_{p,q} - p^\alpha)f(a) + p^\alpha f(b)}{[\alpha+1]_{p,q}}\right| \\
&\quad \le \frac{(b-a)}{[\alpha+1]_{p,q}}B_3^{1-1/r}\left(\left|{}_aD_{p,q}f(a)\right|^r B_1 + \left|{}_aD_{p,q}f(b)\right|^r B_2\right)^{1/r},
\end{aligned} \tag{3.10}$$

where B_1 and B_2 are given in Theorem 3.1 and

$$B_3 = \int_0^1 \left|\frac{[\alpha+1]_{p,q}}{p^{\binom{\alpha}{2}}}\left(1 - {}_0\Phi_q(t)\right)_{p,q}^{(\alpha)} - p^\alpha\right|{}_0d_{p,q}t.$$

Proof Using Lemma 3.1, the convexity of $|{}_aD_{p,q}f|^r$, and the power mean inequality, we have

$$\begin{aligned}
&\left|\frac{\Gamma_{p,q}(\alpha+1)}{p^{\alpha^2}(b-a)^\alpha}\left({}_aI_{p,q}^\alpha f\right)\left(p^\alpha b + \left(1-p^\alpha\right)a\right) - \frac{([\alpha+1]_{p,q} - p^\alpha)f(a) + p^\alpha f(b)}{[\alpha+1]_{p,q}}\right| \\
&\quad \le \frac{(b-a)}{[\alpha+1]_{p,q}}\int_0^1 \left|\frac{[\alpha+1]_{p,q}}{p^{\binom{\alpha}{2}}}\left(1 - {}_0\Phi_q(t)\right)_{p,q}^{(\alpha)} - p^\alpha\right|\left|{}_aD_{p,q}f\left((1-t)a + tb\right)\right|{}_0d_{p,q}t \\
&\quad \le \frac{(b-a)}{[\alpha+1]_{p,q}}\left(\int_0^1 \left|\frac{[\alpha+1]_{p,q}}{p^{\binom{\alpha}{2}}}\left(1 - {}_0\Phi_q(t)\right)_{p,q}^{(\alpha)} - p^\alpha\right|{}_0d_{p,q}t\right)^{1-1/r} \\
&\qquad \times \left(\int_0^1 \left|\frac{[\alpha+1]_{p,q}}{p^{\binom{\alpha}{2}}}\left(1 - {}_0\Phi_q(t)\right)_{p,q}^{(\alpha)} - p^\alpha\right|\left|{}_aD_{p,q}f\left((1-t)a + tb\right)\right|^r{}_0d_{p,q}t\right)^{1/r} \\
&\quad \le \frac{(b-a)}{[\alpha+1]_{p,q}}\left(\int_0^1 \left|\frac{[\alpha+1]_{p,q}}{p^{\binom{\alpha}{2}}}\left(1 - {}_0\Phi_q(t)\right)_{p,q}^{(\alpha)} - p^\alpha\right|{}_0d_{p,q}t\right)^{1-1/r} \\
&\qquad \times \left(\int_0^1 \left|\frac{[\alpha+1]_{p,q}}{p^{\binom{\alpha}{2}}}\left(1 - {}_0\Phi_q(t)\right)_{p,q}^{(\alpha)} - p^\alpha\right|\left[\left|{}_aD_{p,q}f(a)\right|^r(1-t)\right.\right.
\end{aligned}$$

$$
\begin{aligned}
&+ \left|{}_aD_{p,q}f(b)\right|^r t\big]\, {}_0d_{p,q}t\Bigg)^{1/r} \\
&\le \frac{(b-a)}{[\alpha+1]_{p,q}}\left(\int_0^1\left|\frac{[\alpha+1]_{p,q}}{p^{\binom{\alpha}{2}}}\big(1-{}_0\Phi_q(t)\big)_{p,q}^{(\alpha)}-p^{\alpha}\right|{}_0d_{p,q}t\right)^{1-1/r} \\
&\times\Bigg[\left|{}_aD_{p,q}f(a)\right|^r\int_0^1\left|\frac{[\alpha+1]_{p,q}}{p^{\binom{\alpha}{2}}}\big(1-{}_0\Phi_q(t)\big)_{p,q}^{(\alpha)}-p^{\alpha}\right|(1-t)\,{}_0d_{p,q}t \\
&+ \left|{}_aD_{p,q}f(b)\right|^r\int_0^1\left|\frac{[\alpha+1]_{p,q}}{p^{\binom{\alpha}{2}}}\big(1-{}_0\Phi_q(t)\big)_{p,q}^{(\alpha)}-p^{\alpha}\right|(1-t)\,{}_0d_{p,q}t\Bigg]^{1/r}.
\end{aligned}
$$

Therefore, the proof is completed. □

Remark 3.3 If $\alpha = 1$, then (3.10) reduces to

$$
\begin{aligned}
&\left|\frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)\,{}_ad_{p,q}x - \frac{pf(a)+qf(a)}{p+q}\right| \\
&= \frac{q(b-a)}{p+q}\left[\frac{2(p+q-1)}{(p+q)^2}\right]^{1-1/r}\left[\lambda_1(p,q)\left|{}_aD_{p,q}f(b)\right|^r + \lambda_2(p,q)\left|{}_aD_{p,q}f(a)\right|^r\right]^{1/r}, \quad (3.11)
\end{aligned}
$$

where

$$
\lambda_1(p,q) = \frac{q[(p^3-2+2p)+(2p^2+2)q+pq^2pq^2]+2p^2-2p}{(p+q)^3(p^2+pq+q^2)}
$$

and

$$
\begin{aligned}
\lambda_2(p,q) = \frac{1}{(p+q)^3(p^2+pq+q^2)}&\big\{q\big[\big(5p^3-4p^2-2p+2\big)+\big(6p^2-4p-2\big)q \\
&+(5p-2)q^2+2q^3\big]+\big(2p^4-2p^3-2p^3-2p^2+2p\big)\big\},
\end{aligned}
$$

which appeared in [43].

Moreover, if $p = 1$, then (3.10) reduces to

$$
\begin{aligned}
&\left|\frac{\Gamma_q(\alpha+1)}{(b-a)^{\alpha}}\big({}_aI_q^{\alpha}f\big)(b) - \frac{([\alpha+1]_q-1)f(a)+f(b)}{[\alpha+1]_q}\right| \\
&\le \frac{(b-a)}{[\alpha+1]_q}M_3^{1-1/r}\big(\left|{}_aD_qf(a)\right|^r M_1 + \left|{}_aD_qf(b)\right|^r M_2\big)^{1/r}, \quad (3.12)
\end{aligned}
$$

where δ_1 and δ_2 are given in Remark 3.2 and

$$
\delta_3 = \int_0^1\left|[\alpha+1]_q\big(1-{}_0\Phi_q(t)\big)_q^{(\alpha)}-1\right|{}_0d_qt,
$$

which appeared in [40].

Theorem 3.3 *Let $f:[a,b]\to\mathbb{R}$ be a continuous function, $\alpha>0$ and ${}_aD_{p,q}f$ be (p,q)-integrable on $(a,\frac{1}{p}(b-a)+a)$. If $|{}_aD_{p,q}f|^r$ is convex on $[a,\frac{1}{p}(b-a)+a]$ for $r>1$ and*

$1/r+1/p=1$, *then the following Riemann–Liouville fractional* (p,q)*-trapezoid type inequality holds*:

$$\left|\frac{\Gamma_{p,q}(\alpha+1)}{p^{\alpha^2}(b-a)^{\alpha}}\left({}_aI^{\alpha}_{p,q}f\right)\left(p^{\alpha}b+\left(1-p^{\alpha}\right)a\right)-\frac{([\alpha+1]_{p,q}-p^{\alpha})f(a)+p^{\alpha}f(b)}{[\alpha+1]_{p,q}}\right| \leq \frac{(b-a)}{[\alpha+1]_{p,q}}B_4^{1/s}\left(\frac{(p+q-1)|{}_aD_{p,q}f(a)|^r+|{}_aD_{p,q}f(b)|^r}{p+q}\right)^{1/r}, \tag{3.13}$$

where

$$B_4=\int_0^1\left|\frac{[\alpha+1]_{p,q}}{p^{\binom{\alpha}{2}}}\left(1-{}_0\Phi_q(t)\right)^{(\alpha)}_{p,q}-p^{\alpha}\right|^s {}_0d_{p,q}t.$$

Proof Using Lemma 3.1, the convexity of $|{}_aD_{p,q}f|^r$, and Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{\Gamma_{p,q}(\alpha+1)}{p^{\alpha^2}(b-a)^{\alpha}}\left({}_aI^{\alpha}_{p,q}f\right)\left(p^{\alpha}b+\left(1-p^{\alpha}\right)a\right)-\frac{([\alpha+1]_{p,q}-p^{\alpha})f(a)+p^{\alpha}f(b)}{[\alpha+1]_{p,q}}\right| \\
&\quad\leq \frac{(b-a)}{[\alpha+1]_{p,q}}\int_0^1\left|\frac{[\alpha+1]_{p,q}}{p^{\binom{\alpha}{2}}}\left(1-{}_0\Phi_q(t)\right)^{(\alpha)}_{p,q}-p^{\alpha}\right|\left|{}_aD_{p,q}f\left((1-t)a+tb\right)\right| {}_0d_{p,q}t \\
&\quad\leq \frac{(b-a)}{[\alpha+1]_{p,q}}\left(\int_0^1\left|\frac{[\alpha+1]_{p,q}}{p^{\binom{\alpha}{2}}}\left(1-{}_0\Phi_q(t)\right)^{(\alpha)}_{p,q}-p^{\alpha}\right|^s {}_0d_{p,q}t\right)^{1/s} \\
&\qquad\times\left(\int_0^1\left|{}_aD_{p,q}f\left((1-t)a+tb\right)\right|^r {}_0d_{p,q}t\right)^{1/r} \\
&\quad\leq \frac{(b-a)}{[\alpha+1]_{p,q}}\left(\int_0^1\left|\frac{[\alpha+1]_{p,q}}{p^{\binom{\alpha}{2}}}\left(1-{}_0\Phi_q(t)\right)^{(\alpha)}_{p,q}-p^{\alpha}\right|^s {}_0d_{p,q}t\right)^{1/s} \\
&\qquad\times\left(\int_0^1\left[\left|{}_aD_{p,q}f(a)\right|^r(1-t)+\left|{}_aD_{p,q}f(b)\right|^r t\right] {}_0d_{p,q}t\right)^{1/r} \\
&\quad\leq \frac{(b-a)}{[\alpha+1]_{p,q}}\left(\int_0^1\left|\frac{[\alpha+1]_{p,q}}{p^{\binom{\alpha}{2}}}\left(1-{}_0\Phi_q(t)\right)^{(\alpha)}_{p,q}-p^{\alpha}\right|^s {}_0d_{p,q}t\right)^{1/s} \\
&\qquad\times\left(\frac{(p+q-1)|{}_aD_{p,q}f(a)|^r+|{}_aD_{p,q}f(b)|^r}{p+q}\right)^{1/r}.
\end{aligned}$$

This completes the proof. □

Remark 3.4 If $\alpha=1$, then (3.13) reduces to

$$\left|\frac{1}{p(b-a)}\int_a^{pb+(1-p)a}f(x)\,{}_ad_{p,q}x-\frac{pf(a)+qf(a)}{p+q}\right| = \frac{q(b-a)}{p+q}[\lambda_3]^{1/s}\left(\frac{|{}_aD_{p,q}f(b)|^r+(p+q-1)|{}_aD_{p,q}f(a)|^r}{p+q}\right)^{1/r}, \tag{3.14}$$

where

$$\lambda_3=\int_0^1\left|1-(p+q)t\right|^s {}_0d_{p,q}t,$$

which appeared in [43].

Moreover, if $p = 1$, then (3.13) reduces to

$$\left|\frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha}\left({}_aI_q^\alpha f\right)(b) - \frac{([\alpha+1]_q -)f(a) + f(b)}{[\alpha+1]_q}\right| \\ \leq \frac{(b-a)}{[\alpha+1]_q}\delta_4^{1/s}\left(\frac{q|{}_aD_qf(a)|^r + |{}_aD_qf(b)|^r}{1+q}\right)^{1/r},$$

where

$$\delta_4 = \int_0^1 \left|[\alpha+1]_q\left(1 - {}_0\Phi_q(t)\right)_q^{(\alpha)} - 1\right|^s {}_0d_qt, \tag{3.15}$$

which appeared in [40].

Now we will prove the following lemma to obtain the Riemann–Liouville fractional (p,q)-midpoint type inequalities.

Lemma 3.2 *Let $f : [a,b] \to \mathbb{R}$ be a continuous function and $\alpha > 0$. If ${}_aD_{p,q}f$ is (p,q)-integrable on $(a, \frac{1}{p}(b-a)+a)$, then the following equality holds*:

$$f\left(\frac{([\alpha+1]_{p,q} - p^\alpha)a + p^\alpha b}{[\alpha+1]_{p,q}}\right) - \frac{\Gamma_{p,q}(\alpha+1)}{p^{\alpha^2}(b-a)^\alpha}\left({}_aI_{p,q}^\alpha f\right)\left(p^\alpha b + \left(1-p^\alpha\right)a\right) \\ = (b-a)\left[\int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}}\left(1 - \frac{(1-{}_0\Phi_q(t))_{p,q}^{(\alpha)}}{p^{\binom{\alpha}{2}}}\right){}_aD_{p,q}f\left((1-t)a+tb\right){}_0d_{p,q}t \right. \\ \left. + \int_{\frac{p^\alpha}{[\alpha+1]_{p,q}}}^1 -\frac{(1-{}_0\Phi_q(t))_{p,q}^{(\alpha)}}{p^{\binom{\alpha}{2}}}{}_aD_{p,q}f\left((1-t)a+tb\right){}_0d_{p,q}t\right]. \tag{3.16}$$

Proof By direct computation and using Definitions 2.1 and 2.2, we have

$$A_3 = \int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}} {}_aD_{p,q}f\left((1-t)a+tb\right){}_0d_{p,q}t \\ = \int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}} \frac{f((1-pt)a+ptb) - f((1-qt)a+qtb)}{(p-q)(b-a)t}{}_0d_{p,q}t \\ = \frac{1}{(p-q)(b-a)}\int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}}\frac{f((1-pt)a+ptb)}{t}{}_0d_{p,q}t \\ - \frac{1}{(p-q)(b-a)}\int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}}\frac{f((1-qt)a+qtb)}{t}{}_0d_{p,q}t \\ = \frac{p^\alpha}{(b-a)[\alpha+1]_{p,q}}\sum_{n=0}^\infty \frac{q^n}{p^{n+1}}\frac{f((1-\frac{q^np^\alpha}{p^n[\alpha+1]_{p,q}})a + \frac{q^np^\alpha}{p^n[\alpha+1]_{p,q}}b)}{\frac{q^np^\alpha}{p^{n+1}[\alpha+1]_{p,q}}} \\ - \frac{p^\alpha}{(b-a)[\alpha+1]_{p,q}}\sum_{n=0}^\infty \frac{q^n}{p^{n+1}}\frac{f((1-\frac{q^{n+1}p^\alpha}{p^{n+1}[\alpha+1]_{p,q}})a + \frac{q^{n+1}p^\alpha}{p^{n+1}[\alpha+1]_{p,q}}b)}{\frac{q^np^\alpha}{p^{n+1}[\alpha+1]_{p,q}}}$$

$$
\begin{aligned}
&= \frac{1}{(b-a)}\Bigg[\sum_{n=0}^{\infty} f\bigg(\bigg(1-\frac{q^n p^\alpha}{p^n[\alpha+1]_{p,q}}\bigg)a+\frac{q^n p^\alpha}{p^n[\alpha+1]_{p,q}}b\bigg) \\
&\quad -\sum_{n=0}^{\infty} f\bigg(\bigg(1-\frac{q^{n+1} p^\alpha}{p^{n+1}[\alpha+1]_{p,q}}\bigg)a+\frac{q^{n+1} p^\alpha}{p^{n+1}[\alpha+1]_{p,q}}b\bigg)\Bigg] \\
&= \frac{1}{(b-a)}\bigg[f\bigg(\frac{([\alpha+1]_{p,q}-p^\alpha)a+p^\alpha b}{[\alpha+1]_{p,q}}\bigg)-f(a)\bigg].
\end{aligned} \tag{3.17}
$$

On the other hand, in Lemma 3.1, the following integral was given:

$$
\begin{aligned}
A_1 &= \frac{b-a}{p^{\binom{\alpha}{2}}}\int_0^1 \big(1-{}_0\Phi_q(t)\big)_{p,q}^{(\alpha)}\, {}_aD_{p,q}f\big((1-t)a+tb\big)\, {}_0d_{p,q}t \\
&= -f(a)+\frac{\Gamma_{p,q}(\alpha+1)}{p^{\alpha^2}(b-a)^\alpha}\big({}_aI_{p,q}^\alpha f\big)\big(p^\alpha b+\big(1-p^\alpha\big)a\big).
\end{aligned} \tag{3.18}
$$

Consequently, from (3.17) and (3.18), we have

$$
\begin{aligned}
&A_3+A_1 \\
&= (b-a)\Bigg[\int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}}\Bigg(1-\frac{(1-{}_0\Phi_q(t))_{p,q}^{(\alpha)}}{p^{\binom{\alpha}{2}}}\Bigg)\, {}_aD_{p,q}f\big((1-t)a+tb\big)\, {}_0d_{p,q}t \\
&\quad + \int_{\frac{p^\alpha}{[\alpha+1]_{p,q}}}^1 -\frac{(1-{}_0\Phi_q(t))_{p,q}^{(\alpha)}}{p^{\binom{\alpha}{2}}}\, {}_aD_{p,q}f\big((1-t)a+tb\big)\, {}_0d_{p,q}t\Bigg] \\
&= (b-a)\Bigg[\int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}} {}_aD_{p,q}f\big((1-t)a+tb\big)\, {}_0d_{p,q}t \\
&\quad - \int_0^1 \frac{(1-{}_0\Phi_q(t))_{p,q}^{(\alpha)}}{p^{\binom{\alpha}{2}}}\, {}_aD_{p,q}f\big((1-t)a+tb\big)\, {}_0d_{p,q}t\Bigg] \\
&= f\bigg(\frac{([\alpha+1]_{p,q}-p^\alpha)a+p^\alpha b}{[\alpha+1]_{p,q}}\bigg)-\frac{\Gamma_{p,q}(\alpha+1)}{p^{\alpha^2}(b-a)^\alpha}\big({}_aI_{p,q}^\alpha f\big)\big(p^\alpha b+\big(1-p^\alpha\big)a\big).
\end{aligned}
$$

Therefore, the proof is completed. □

Remark 3.5 If $\alpha = 1$, then (3.16) reduces to

$$
\begin{aligned}
&\bigg|f\bigg(\frac{qa+pb}{p+q}\bigg)-\frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)\, {}_ad_{p,q}x\bigg| \\
&= q(b-a)\bigg[\int_0^{\frac{p}{p+q}} t\, {}_aD_{p,q}f\big((1-t)a+tb\big)\, {}_0d_{p,q}t \\
&\quad + \int_{\frac{p}{p+q}}^1 \bigg(t-\frac{1}{q}\bigg)\, {}_aD_{p,q}f\big((1-t)a+tb\big)\, {}_0d_{p,q}t\bigg],
\end{aligned} \tag{3.19}
$$

which appeared in [42].

Moreover, if $p = 1$, then (3.16) reduces to

$$f\left(\frac{([\alpha+1]_q - 1)a + b}{[\alpha+1]_q}\right) - \frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha}\left({}_aI_q^\alpha f\right)(b)$$
$$= (b-a)\left[\int_0^{\frac{1}{[\alpha+1]_q}} \left(1 - {}_0\Phi_q(t)\right)_q^{(\alpha)} {}_aD_q f\left((1-t)a + tb\right) {}_0d_q t\right.$$
$$\left. - \int_{\frac{1}{[\alpha+1]_q}}^1 \left(1 - {}_0\Phi_q(t)\right)_q^{(\alpha)} {}_aD_q f\left((1-t)a + tb\right) {}_0d_q t, \right] \tag{3.20}$$

which appeared in [40].

Theorem 3.4 *Let $f : [a,b] \to \mathbb{R}$ be a continuous function, $\alpha > 0$, and ${}_aD_{p,q}f$ be (p,q)-integrable on $(a, \frac{1}{p}(b-a)+a)$. If $|{}_aD_{p,q}f|$ is convex on $(a, \frac{1}{p}(b-a)+a)$, then the following Riemann–Liouville fractional (p,q)-midpoint type inequality holds*:

$$\left|f\left(\frac{([\alpha+1]_{p,q} - p^\alpha)a + p^\alpha b}{[\alpha+1]_{p,q}}\right) - \frac{\Gamma_{p,q}(\alpha+1)}{p^{\alpha^2}(b-a)^\alpha}\left({}_aI_{p,q}^\alpha f\right)\left(p^\alpha b + \left(1-p^\alpha\right)a\right)\right|$$
$$\leq (b-a)\left[B_5\left|{}_aD_{p,q}f(a)\right| + B_6\left|{}_aD_{p,q}f(b)\right| + B_7\left|{}_aD_{p,q}f(a)\right| + B_8\left|{}_aD_{p,q}f(b)\right|\right], \tag{3.21}$$

where

$$B_5 = \left[\int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}} \left|1 - \frac{(1 - {}_0\Phi_q(t))_{p,q}^{(\alpha)}}{p^{\binom{\alpha}{2}}}\right| (1-t)\, {}_0d_{p,q}t\right],$$
$$B_6 = \left[\int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}} \left|1 - \frac{(1 - {}_0\Phi_q(t))_{p,q}^{(\alpha)}}{p^{\binom{\alpha}{2}}}\right| t\, {}_0d_{p,q}t\right],$$
$$B_7 = \left[\int_{\frac{p^\alpha}{[\alpha+1]_{p,q}}}^1 \left|- \frac{(1 - {}_0\Phi_q(t))_{p,q}^{(\alpha)}}{p^{\binom{\alpha}{2}}}\right| (1-t)\, {}_0d_{p,q}t\right],$$
$$B_8 = \left[\int_{\frac{p^\alpha}{[\alpha+1]_{p,q}}}^1 \left|- \frac{(1 - {}_0\Phi_q(t))_{p,q}^{(\alpha)}}{p^{\binom{\alpha}{2}}}\right| t\, {}_0d_{p,q}t\right].$$

Proof Using Lemma 3.2 and the convexity of $|{}_aD_{p,q}f|$, we have

$$\left|f\left(\frac{([\alpha+1]_{p,q} - p^\alpha)a + p^\alpha b}{[\alpha+1]_{p,q}}\right) - \frac{\Gamma_{p,q}(\alpha+1)}{p^{\alpha^2}(b-a)^\alpha}\left({}_aI_{p,q}^\alpha f\right)\left(p^\alpha b + \left(1-p^\alpha\right)a\right)\right|$$
$$\leq (b-a)\left[\int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}} \left|1 - \frac{(1 - {}_0\Phi_q(t))_{p,q}^{(\alpha)}}{p^{\binom{\alpha}{2}}}\right| \left|{}_aD_{p,q}f\left((1-t)a+tb\right)\right| {}_0d_{p,q}t\right.$$
$$\left. + \int_{\frac{p^\alpha}{[\alpha+1]_{p,q}}}^1 \left|- \frac{(1 - {}_0\Phi_q(t))_{p,q}^{(\alpha)}}{p^{\binom{\alpha}{2}}}\right| \left|{}_aD_{p,q}f\left((1-t)a+tb\right)\right| {}_0d_{p,q}t\right]$$
$$\leq (b-a)\left[\int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}} \left|1 - \frac{(1 - {}_0\Phi_q(t))_{p,q}^{(\alpha)}}{p^{\binom{\alpha}{2}}}\right| \left[\left|{}_aD_{p,q}f(a)\right|(1-t) + \left|{}_aD_{p,q}f(b)\right|t\right] {}_0d_{p,q}t\right.$$
$$\left. + \int_{\frac{p^\alpha}{[\alpha+1]_{p,q}}}^1 \left|- \frac{(1 - {}_0\Phi_q(t))_{p,q}^{(\alpha)}}{p^{\binom{\alpha}{2}}}\right| \left[\left|{}_aD_{p,q}f(a)\right|(1-t) + \left|{}_aD_{p,q}f(b)\right|t\right] {}_0d_{p,q}t\right]$$

$$
\begin{aligned}
&\leq (b-a)\Bigg[\int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}} \left|1-\frac{(1-{}_0\Phi_q(t))_{p,q}^{(\alpha)}}{p^{\binom{\alpha}{2}}}\right| \left|{}_aD_{p,q}f(a)\right|(1-t)\,{}_0d_{p,q}t \\
&\quad + \int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}} \left|1-\frac{(1-{}_0\Phi_q(t))_{p,q}^{(\alpha)}}{p^{\binom{\alpha}{2}}}\right| \left|{}_aD_{p,q}f(b)\right| t\,{}_0d_{p,q}t\Bigg] \\
&\quad + (b-a)\Bigg[\int_{\frac{p^\alpha}{[\alpha+1]_{p,q}}}^1 \left|-\frac{(1-{}_0\Phi_q(t))_{p,q}^{(\alpha)}}{p^{\binom{\alpha}{2}}}\right| \left|{}_aD_{p,q}f(a)\right|(1-t)\,{}_0d_{p,q}t \\
&\quad + \int_{\frac{p^\alpha}{[\alpha+1]_{p,q}}}^1 \left|-\frac{(1-{}_0\Phi_q(t))_{p,q}^{(\alpha)}}{p^{\binom{\alpha}{2}}}\right| \left|{}_aD_{p,q}f(b)\right| t\,{}_0d_{p,q}t\Bigg].
\end{aligned}
$$

This completes the proof. □

Remark 3.6 If $\alpha = 1$, then (3.21) reduces to

$$
\begin{aligned}
&\left|f\left(\frac{qa+pb}{p+q}\right) - \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)\,{}_ad_{p,q}x\right| \\
&\quad \leq q(b-a)\left[\lambda_4(p,q)\left|{}_aD_{p,q}f(a)\right| + \lambda_5(p,q)\left|{}_aD_{p,q}f(b)\right|\right. \\
&\quad \left. + \lambda_6(p,q)\left|{}_aD_{p,q}f(a)\right| + \lambda_7(p,q)\left|{}_aD_{p,q}f(b)\right|\right],
\end{aligned}
\tag{3.22}
$$

where

$$
\begin{aligned}
\lambda_4(p,q) &= \frac{p^3}{(p+q)^3(p^2+pq+q^2)}, & \lambda_5(p,q) &= \frac{p^2(p^2+pq+q^2)-p^3}{(p+q)^3(p^2+pq+q^2)}, \\
\lambda_6(p,q) &= \frac{2p^3}{(p+q)^3(p^2+pq+q^2)}, & \lambda_7(p,q) &= \frac{p^4+p^3q+p^2q^2-2p^3}{(p+q)^3(p^2+pq+q^2)},
\end{aligned}
$$

which appeared in [42].

Moreover, if $p = 1$, then (3.21) reduces to

$$
\begin{aligned}
&\left|f\left(\frac{([\alpha+1]_q-1)a+b}{[\alpha+1]_q}\right) - \frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha}\left({}_aI_q^\alpha f\right)(b)\right| \\
&\quad \leq (b-a)\left[\delta_5\left|{}_aD_qf(a)\right| + \delta_6\left|{}_aD_qf(b)\right| + \delta_7\left|{}_aD_qf(a)\right| + \delta_8\left|{}_aD_qf(b)\right|\right],
\end{aligned}
\tag{3.23}
$$

where

$$
\begin{aligned}
\delta_5 &= \left[\int_0^{\frac{1}{[\alpha+1]_q}} \left|1-\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right|(1-t)\,{}_0d_qt\right], \\
\delta_6 &= \left[\int_0^{\frac{1}{[\alpha+1]_q}} \left|1-\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right| t\,{}_0d_qt\right], \\
\delta_7 &= \left[\int_{\frac{1}{[\alpha+1]_q}}^1 \left|-\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right|(1-t)\,{}_0d_qt\right], \\
\delta_8 &= \left[\int_{\frac{1}{[\alpha+1]_q}}^1 \left|-\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right| t\,{}_0d_qt\right],
\end{aligned}
$$

which appeared in [40].

Theorem 3.5 *Let $f:[a,b]\to\mathbb{R}$ be a continuous function, $\alpha>0$ and ${}_aD_{p,q}f$ be (p,q)-integrable on $(a,\frac{1}{p}(b-a)+a)$. If $|{}_aD_{p,q}f|^r$ is convex on $(a,\frac{1}{p}(b-a)+a)$ for $r\geq 0$, then the following Riemann–Liouville fractional (p,q)-midpoint type inequality holds*:

$$\begin{aligned}
&\left|f\left(\frac{([\alpha+1]_{p,q}-p^\alpha)a+p^\alpha b}{[\alpha+1]_{p,q}}\right)-\frac{\Gamma_{p,q}(\alpha+1)}{p^{\alpha^2}(b-a)^\alpha}\left({}_aI^\alpha_{p,q}f\right)\left(p^\alpha b+\left(1-p^\alpha\right)a\right)\right|\\
&\quad\leq(b-a)\left[B_9^{1-1/r}\left(B_5\left|{}_aD_{p,q}f(a)\right|^r+B_6\left|{}_aD_{p,q}f(b)\right|^r\right)^{1/r}\right.\\
&\qquad\left.+B_{10}^{1-1/r}\left(B_7\left|{}_aD_{p,q}f(a)\right|^r+B_8\left|{}_aD_{p,q}f(b)\right|^r\right)^{1/r}\right],
\end{aligned}\tag{3.24}$$

where B_5, B_6, B_7, and B_8 are given in Theorem 3.4 and

$$B_9=\int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}}\left|1-\frac{(1-{}_0\Phi_q(t))^{(\alpha)}_{p,q}}{p^{\binom{\alpha}{2}}}\right|{}_0d_{p,q}t$$

and

$$B_{10}=\int_{\frac{p^\alpha}{[\alpha+1]_{p,q}}}^1\left|-\frac{(1-{}_0\Phi_q(t))^{(\alpha)}_{p,q}}{p^{\binom{\alpha}{2}}}\right|{}_0d_{p,q}t.$$

Proof Using Lemma 3.2, the power mean inequality and the convexity of $|{}_aD_{p,q}f|^r$, we have

$$\begin{aligned}
&\left|f\left(\frac{([\alpha+1]_{p,q}-p^\alpha)a+p^\alpha b}{[\alpha+1]_{p,q}}\right)-\frac{\Gamma_{p,q}(\alpha+1)}{p^{\alpha^2}(b-a)^\alpha}\left({}_aI^\alpha_{p,q}f\right)\left(p^\alpha b+\left(1-p^\alpha\right)a\right)\right|\\
&\quad\leq(b-a)\left[\int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}}\left|1-\frac{(1-{}_0\Phi_q(t))^{(\alpha)}_{p,q}}{p^{\binom{\alpha}{2}}}\right|\left|{}_aD_{p,q}f\left((1-t)a+tb\right)\right|{}_0d_{p,q}t\right.\\
&\qquad\left.+\int_{\frac{p^\alpha}{[\alpha+1]_{p,q}}}^1\left|-\frac{(1-{}_0\Phi_q(t))^{(\alpha)}_{p,q}}{p^{\binom{\alpha}{2}}}\right|\left|{}_aD_{p,q}f\left((1-t)a+tb\right)\right|{}_0d_{p,q}t\right]\\
&\quad\leq(b-a)\left[\left(\int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}}\left|1-\frac{(1-{}_0\Phi_q(t))^{(\alpha)}_{p,q}}{p^{\binom{\alpha}{2}}}\right|{}_0d_{p,q}t\right)^{1-1/r}\right.\\
&\qquad\times\left(\int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}}\left|1-\frac{(1-{}_0\Phi_q(t))^{(\alpha)}_{p,q}}{p^{\binom{\alpha}{2}}}\right|\left|{}_aD_{p,q}f\left((1-t)a+tb\right)\right|^r{}_0d_{p,q}t\right)^{1/r}\\
&\qquad+\left(\int_{\frac{p^\alpha}{[\alpha+1]_{p,q}}}^1\left|-\frac{(1-{}_0\Phi_q(t))^{(\alpha)}_{p,q}}{p^{\binom{\alpha}{2}}}\right|{}_0d_{p,q}t\right)^{1-1/r}\\
&\qquad\left.\times\left(\int_{\frac{p^\alpha}{[\alpha+1]_{p,q}}}^1\left|-\frac{(1-{}_0\Phi_q(t))^{(\alpha)}_{p,q}}{p^{\binom{\alpha}{2}}}\right|\left|{}_aD_{p,q}f\left((1-t)a+tb\right)\right|^r{}_0d_{p,q}t\right)^{1/r}\right]\\
&\quad\leq(b-a)\left[\left(\int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}}\left|1-\frac{(1-{}_0\Phi_q(t))^{(\alpha)}_{p,q}}{p^{\binom{\alpha}{2}}}\right|{}_0d_{p,q}t\right)^{1-1/r}\right.\\
&\qquad\times\left(\int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}}\left|1-\frac{(1-{}_0\Phi_q(t))^{(\alpha)}_{p,q}}{p^{\binom{\alpha}{2}}}\right|\left[\left|{}_aD_{p,q}f(a)\right|^r(1-t)+\left|{}_aD_{p,q}f(b)\right|^rt\right]{}_0d_{p,q}t\right)^{1/r}\\
&\qquad+\left(\int_{\frac{p^\alpha}{[\alpha+1]_{p,q}}}^1\left|-\frac{(1-{}_0\Phi_q(t))^{(\alpha)}_{p,q}}{p^{\binom{\alpha}{2}}}\right|{}_0d_{p,q}t\right)^{1-1/r}
\end{aligned}$$

$$\times\left(\int_{\frac{p^\alpha}{[\alpha+1]_{p,q}}}^{1}\left|-\frac{(1-{}_0\Phi_q(t))_{p,q}^{(\alpha)}}{p^{\binom{\alpha}{2}}}\right|\left[\left|{}_aD_{p,q}f(a)\right|^r(1-t)+\left|{}_aD_{p,q}f(b)\right|^r t\right]{}_0d_{p,q}t\right)^{1/r}\Bigg]$$

$$\le(b-a)\Bigg[\left(\int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}}\left|1-\frac{(1-{}_0\Phi_q(t))_{p,q}^{(\alpha)}}{p^{\binom{\alpha}{2}}}\right|{}_0d_{p,q}t\right)^{1-1/r}$$

$$\times\left(\left|{}_aD_{p,q}f(a)\right|^r\int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}}\left|1-\frac{(1-{}_0\Phi_q(t))_{p,q}^{(\alpha)}}{p^{\binom{\alpha}{2}}}\right|(1-t)\,{}_0d_{p,q}t\right.$$

$$\left.+\left|{}_aD_{p,q}f(b)\right|^r\int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}}\left|1-\frac{(1-{}_0\Phi_q(t))_{p,q}^{(\alpha)}}{p^{\binom{\alpha}{2}}}\right|t\,{}_0d_{p,q}t\right)^{1/r}\Bigg]$$

$$+(b-a)\Bigg[\left(\int_{\frac{p^\alpha}{[\alpha+1]_{p,q}}}^{1}\left|-\frac{(1-{}_0\Phi_q(t))_{p,q}^{(\alpha)}}{p^{\binom{\alpha}{2}}}\right|{}_0d_{p,q}t\right)^{1-1/r}$$

$$\times\left(\left|{}_aD_{p,q}f(a)\right|^r\int_{\frac{p^\alpha}{[\alpha+1]_{p,q}}}^{1}\left|-\frac{(1-{}_0\Phi_q(t))_{p,q}^{(\alpha)}}{p^{\binom{\alpha}{2}}}\right|(1-t)\,{}_0d_{p,q}t\right.$$

$$\left.+\left|{}_aD_{p,q}f(b)\right|^r\int_{\frac{p^\alpha}{[\alpha+1]_{p,q}}}^{1}\left|-\frac{(1-{}_0\Phi_q(t))_{p,q}^{(\alpha)}}{p^{\binom{\alpha}{2}}}\right|t\,{}_0d_{p,q}t\right)^{1/r}\Bigg].$$

This completes the proof. □

Remark 3.7 If $\alpha=1$, then (3.24) reduces to

$$\begin{aligned}&\left|f\left(\frac{qa+pb}{p+q}\right)-\frac{1}{p(b-a)}\int_a^{pb+(1-p)a}f(x)\,{}_ad_{p,q}x\right|\\&\quad\le q(b-a)\left(\frac{p^2}{(p+q)^3}\right)^{1-1/r}\left[\left(\lambda_4(p,q)\left|{}_aD_{p,q}f(a)\right|^r+\lambda_5(p,q)\left|{}_aD_{p,q}f(b)\right|r\right)^{1/r}\right.\\&\qquad\left.+\left(\lambda_6(p,q)\left|{}_aD_{p,q}f(a)\right|^r+\lambda_7(p,q)\left|{}_aD_{p,q}f(b)\right|^r\right)^{1/r}\right],\end{aligned}\tag{3.25}$$

where $\lambda_4(p,q)$, $\lambda_5(p,q)$, $\lambda_6(p,q)$, and $\lambda_7(p,q)$ are given in Remark (3.6), which appeared in [42].

Moreover, if $p=1$, then (3.24) reduces to

$$\begin{aligned}&\left|f\left(\frac{([\alpha+1]_q-1)a+b}{[\alpha+1]_q}\right)-\frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha}\left({}_aI_q^\alpha f\right)(b)\right|\\&\quad\le(b-a)\left[\delta_9^{1-1/r}\left(\delta_5\left|{}_aD_qf(a)\right|^r+\delta_6\left|{}_aD_qf(b)\right|^r\right)^{1/r}\right.\\&\qquad\left.+\delta_{10}^{1-1/r}\left(\delta_7\left|{}_aD_qf(a)\right|^r+\delta_8\left|{}_aD_qf(b)\right|^r\right)^{1/r}\right],\end{aligned}\tag{3.26}$$

where δ_5, δ_6, δ_7, and δ_8 are given in Remark (3.6) and

$$\delta_9=\int_0^{\frac{1}{[\alpha+1]_q}}\left|1-\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right|{}_0d_qt,$$

$$\delta_{10}=\int_{\frac{1}{[\alpha+1]_q}}^{1}\left|-\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right|{}_0d_qt,$$

which appeared in [40].

Theorem 3.6 *Let $f:[a,b]\to\mathbb{R}$ be a continuous function, $\alpha>0$, and ${}_aD_{p,q}f$ be (p,q)-integrable on $(a,\frac{1}{p}(b-a)+a)$. If $|{}_aD_{p,q}f|^r$ is convex on $[a,\frac{1}{p}(b-a)+a]$ for $r>1$ and $1/r+1/s=1$, then the following Riemann–Liouville fractional (p,q)-midpoint type inequality holds*:

$$\begin{aligned}
&\left|f\left(\frac{([\alpha+1]_{p,q}-p^\alpha)a+p^\alpha b}{[\alpha+1]_{p,q}}\right)-\frac{\Gamma_{p,q}(\alpha+1)}{p^{\alpha^2}(b-a)^\alpha}\left({}_aI^\alpha_{p,q}f\right)\left(p^\alpha b+\left(1-p^\alpha\right)a\right)\right|\\
&\quad\leq(b-a)\left[(B_{11})^{1/s}\left(\left|{}_aD_{p,q}f(a)\right|^r\left(\frac{p^\alpha(p+q)[\alpha+1]_{p,q}-p^\alpha}{(p+q)([\alpha+1]_{p,q})^2}\right)\right.\right.\\
&\quad\quad\left.\left.+\left|{}_aD_{p,q}f(b)\right|^r\left(\frac{p^\alpha}{(p+q)([\alpha+1]_{p,q})^2}\right)\right)^{1/r}\right]\\
&\quad\quad+(b-a)\left[(B_{12})^{1/s}\left(\left|{}_aD_{p,q}f(a)\right|^r\left(\frac{p+q-1}{p+q}-\frac{p^\alpha(p+q)[\alpha+1]_{p,q}-p^{2\alpha}}{(p+q)([\alpha+1]_{p,q})^2}\right)\right.\right.\\
&\quad\quad\left.\left.+\left|{}_aD_{p,q}f(b)\right|^r\left(\frac{1}{p+q}-\frac{p^{2\alpha}}{(p+q)([\alpha+1]_{p,q})^2}\right)\right)^{1/r}\right],
\end{aligned}\tag{3.27}$$

where

$$B_{11}=\int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}}\left|1-\frac{(1-{}_0\Phi_q(t))^{(\alpha)}_{p,q}}{p^{\binom{\alpha}{2}}}\right|^s{}_0d_{p,q}t$$

and

$$B_{12}=\int_{\frac{p^\alpha}{[\alpha+1]_{p,q}}}^1\left|-\frac{(1-{}_0\Phi_q(t))^{(\alpha)}_{p,q}}{p^{\binom{\alpha}{2}}}\right|^s{}_0d_{p,q}t.$$

Proof Applying Lemma 3.2, Hölder's inequality, and the convexity of $|{}_aD_{p,q}f|^r$, we have

$$\begin{aligned}
&\left|f\left(\frac{([\alpha+1]_{p,q}-p^\alpha)a+p^\alpha b}{[\alpha+1]_{p,q}}\right)-\frac{\Gamma_{p,q}(\alpha+1)}{p^{\alpha^2}(b-a)^\alpha}\left({}_aI^\alpha_{p,q}f\right)\left(p^\alpha b+\left(1-p^\alpha\right)a\right)\right|\\
&\quad\leq(b-a)\left[\int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}}\left|1-\frac{(1-{}_0\Phi_q(t))^{(\alpha)}_{p,q}}{p^{\binom{\alpha}{2}}}\right|\left|{}_aD_{p,q}f\left((1-t)a+tb\right)\right|{}_0d_{p,q}t\right.\\
&\quad\quad\left.+\int_{\frac{p^\alpha}{[\alpha+1]_{p,q}}}^1\left|-\frac{(1-{}_0\Phi_q(t))^{(\alpha)}_{p,q}}{p^{\binom{\alpha}{2}}}\right|\left|{}_aD_{p,q}f\left((1-t)a+tb\right)\right|{}_0d_{p,q}t\right]\\
&\quad\leq(b-a)\left[\left(\int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}}\left|1-\frac{(1-{}_0\Phi_q(t))^{(\alpha)}_{p,q}}{p^{\binom{\alpha}{2}}}\right|^p{}_0d_{p,q}t\right)^{1/p}\right.\\
&\quad\quad\times\left(\int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}}\left|{}_aD_{p,q}f\left((1-t)a+tb\right)\right|^r{}_0d_{p,q}t\right)^{1/r}\\
&\quad\quad+\left(\int_{\frac{p^\alpha}{[\alpha+1]_{p,q}}}^1\left|-\frac{(1-{}_0\Phi_q(t))^{(\alpha)}_{p,q}}{p^{\binom{\alpha}{2}}}\right|^p{}_0d_{p,q}t\right)^{1/p}\\
&\quad\quad\left.\times\left(\int_{\frac{p^\alpha}{[\alpha+1]_{p,q}}}^1\left|{}_aD_{p,q}f\left((1-t)a+tb\right)\right|^r{}_0d_{p,q}t\right)^{1/r}\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq (b-a)\Bigg[\Bigg(\int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}}\bigg|1-\frac{(1-{}_0\Phi_q(t))_{p,q}^{(\alpha)}}{p^{\binom{\alpha}{2}}}\bigg|^p\,{}_0d_{p,q}t\Bigg)^{1/p}\\
&\quad\times\Bigg(\big|{}_aD_{p,q}f(a)\big|^r\int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}}(1-t)\,{}_0d_{p,q}t+\big|{}_aD_{p,q}f(b)\big|^r\int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}}t\,{}_0d_{p,q}t\Bigg)^{1/r}\Bigg]\\
&\quad+(b-a)\Bigg[\Bigg(\int_{\frac{p^\alpha}{[\alpha+1]_{p,q}}}^1\bigg|-\frac{(1-{}_0\Phi_q(t))_{p,q}^{(\alpha)}}{p^{\binom{\alpha}{2}}}\bigg|^p\,{}_0d_{p,q}t\Bigg)^{1/p}\\
&\quad\times\Bigg(\big|{}_aD_{p,q}f(a)\big|^r\int_{\frac{p^\alpha}{[\alpha+1]_{p,q}}}^1(1-t)\,{}_0d_{p,q}t\\
&\quad+\big|{}_aD_{p,q}f(b)\big|^r\int_{\frac{p^\alpha}{[\alpha+1]_{p,q}}}^1 t\,{}_0d_{p,q}t\Bigg)^{1/r}\Bigg]\\
&\leq (b-a)\Bigg[\Bigg(\int_0^{\frac{p^\alpha}{[\alpha+1]_{p,q}}}\bigg|1-\frac{(1-{}_0\Phi_q(t))_{p,q}^{(\alpha)}}{p^{\binom{\alpha}{2}}}\bigg|^p\,{}_0d_{p,q}t\Bigg)^{1/p}\\
&\quad\times\Bigg(\big|{}_aD_{p,q}f(a)\big|^r\bigg(\frac{p^\alpha(p+q)[\alpha+1]_{p,q}-p^\alpha}{(p+q)([\alpha+1]_{p,q})^2}\bigg)\\
&\quad+\big|{}_aD_{p,q}f(b)\big|^r\bigg(\frac{p^\alpha}{(p+q)([\alpha+1]_{p,q})^2}\bigg)\Bigg)^{1/r}\Bigg]\\
&\quad+(b-a)\Bigg[\Bigg(\int_{\frac{p^\alpha}{[\alpha+1]_{p,q}}}^1\bigg|-\frac{(1-{}_0\Phi_q(t))_{p,q}^{(\alpha)}}{p^{\binom{\alpha}{2}}}\bigg|^p\,{}_0d_{p,q}t\Bigg)^{1/p}\\
&\quad\times\Bigg(\big|{}_aD_{p,q}f(a)\big|^r\bigg(\frac{p+q-1}{p+q}-\frac{p^\alpha(p+q)[\alpha+1]_{p,q}-p^{2\alpha}}{(p+q)([\alpha+1]_{p,q})^2}\bigg)\\
&\quad+\big|{}_aD_{p,q}f(b)\big|^r\bigg(\frac{1}{p+q}-\frac{p^{2\alpha}}{(p+q)([\alpha+1]_{p,q})^2}\bigg)\Bigg)^{1/r}\Bigg].
\end{aligned}
$$

This completes the proof. □

Remark 3.8 If $\alpha=1$, then (3.27) reduces to

$$
\begin{aligned}
&\bigg|f\bigg(\frac{qa+pb}{p+q}\bigg)-\frac{1}{p(b-a)}\int_a^{pb+(1-p)a}f(x)\,{}_ad_{p,q}x\bigg|\\
&\leq q(b-a)\Bigg[\Bigg(\bigg(\frac{p}{p+q}\bigg)^{s+1}\bigg(\frac{p-q}{p^{s+1}-q^{s+1}}\bigg)\Bigg)^{1/s}\Bigg(\big|{}_aD_{p,q}f(a)\big|^r\bigg(\frac{p^3+2p^2q+pq^2-p^2}{(p+q)^3}\bigg)\\
&\quad+\big|{}_aD_{p,q}f(b)\big|^r\bigg(\frac{p^2}{(p+q)^3}\bigg)\Bigg)^{1/r}\\
&\quad+\Bigg(\int_{\frac{p}{p+q}}^1\bigg(\frac{1}{q}-t\bigg)^s\,{}_0d_{p,q}t\Bigg)^{1/s}\Bigg(\big|{}_aD_{p,q}f(b)\big|^r\bigg(\frac{2pq+q^2}{(p+q)^3}\bigg)\\
&\quad+\big|{}_aD_{p,q}f(a)\big|^r\bigg(\frac{p^2q+2pq^2-2pq-q^2+q^3}{(p+q)^3}\bigg)\Bigg)^{1/r}\Bigg],
\end{aligned}
$$

which appeared in [42].

Moreover, if $p = 1$, then (3.27) reduces to

$$\begin{aligned}
&\left|f\left(\frac{([\alpha+1]_q-1)a+b}{[\alpha+1]_q}\right)-\frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha}\left({}_aI_q^\alpha f\right)(b)\right| \\
&\quad\le (b-a)\left[(\delta_{11})^{1/s}\left(\left|{}_aD_qf(a)\right|^r\left(\frac{(1+q)[\alpha+1]_q-1}{(1+q)([\alpha+1]_q)^2}\right)\right.\right. \\
&\qquad\left.\left.+\left|{}_aD_qf(b)\right|^r\left(\frac{1}{(1+q)([\alpha+1]_q)^2}\right)\right)^{1/r}\right] \\
&\qquad+(b-a)\left[(\delta_{12})^{1/s}\left(\left|{}_aD_qf(a)\right|^r\left(\frac{q}{1+q}-\frac{(1+q)[\alpha+1]_q-1}{(1+q)([\alpha+1]_q)^2}\right)\right.\right. \\
&\qquad\left.\left.+\left|{}_aD_qf(b)\right|^r\left(\frac{1}{1+q}-\frac{1}{(1+q)([\alpha+1]_q)^2}\right)\right)^{1/r}\right],
\end{aligned}$$

where

$$\delta_{11}=\int_0^{\frac{1}{[\alpha+1]_q}}\left|1-\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right|^s\,{}_0d_qt$$

and

$$\delta_{12}=\int_{\frac{1}{[\alpha+1]_q}}^1\left|-\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right|^s\,{}_0d_qt,$$

which appeared in [40].

4 Conclusions

In this work, we studied two identities for continuous functions in the form of fractional Riemann–Liouville (p,q)-integral. Based on these two identities, some fractional Riemann–Liouville (p,q)-trapezoid and (p,q)-midpoint type inequalities are given. From this idea, as well as the techniques of this paper, we hope that it will inspire interested readers working in this field.

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Author details
[1] Department of Mathematics, Khon Kaen University, 40002 Khon Kaen, Thailand. [2] Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, 10800 Bangkok, Thailand. [3] Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece. [4] Nonlinear Analysis and Applied Mathematics (NAAM)—Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, 21588 Jeddah, Saudi Arabia. [5] International Center for Basic and Applied Sciences, 302029 Jaipur, India. [6] Department of Mathematics, Anand International College of Engineering, 302029 Jaipur, India.

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