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An uncertain SIR rumor spreading model

Hang Sun¹ , Yuhong Sheng^{1*} and Qing Cui²

*Correspondence:

shengyuhong1@sina.com

¹College of Mathematical and
System Sciences, Xinjiang
University, Urumqi, Xinjiang,
830046, China

Full list of author information is
available at the end of the article

Abstract

In this paper, an uncertain SIR (spreader, ignorant, stifler) rumor spreading model driven by one Liu process is formulated to investigate the influence of perturbation in the transmission mechanism of rumor spreading. The deduced process of the uncertain SIR rumor spreading model is presented. Then an existence and uniqueness theorem concerning the solution is proved. Moreover, the stability of uncertain SIR rumor spreading differential equation is proved. In addition, the influence of different parameters on rumor spreading is analyzed through numerical simulation. This paper also presents a paradox of stochastic SIR rumor spreading model.

Keywords: Uncertainty theory; Liu process; Rumor spreading; Existence and uniqueness; Stability

1 Introduction

Rumor is a statement that has no basis in fact, but is made up out of thin air or according to one's will and spread by some means. In other words, a rumor is a piece of information that is created out of nothing but spreads widely. With the development of the society and the internet, compared with the traditional interpersonal communication, rumor propagation has the characteristics of faster information transmission, wider influence, stronger force, and more concealment. Rumors have a great negative effect on our daily life. Rumors not only convey some inaccurate information to the society, but also cause bad social habits, crimes, and economic losses, as well as pose a great threat to the national and public security [1, 2]. At the same time, changes in rumor spreading are sudden and uncertain. How to remove rumors in a timely manner and how to effectively suppress rumors have become a crucial issue to the government and the society. Therefore, it is significant to study uncertain SIR rumor spreading model for rumor control and social stability.

Due to the similarity between rumor spreading process and propagating process of epidemic diseases, an epidemic model has been favored by most domestic and overseas scholars in the research on rumor spreading. Kermack and McKendrick [3] first proposed the classic SIR epidemic model in 1927. Subsequently, based on this research, many scholars used mathematical models to establish SI [4], SIS [4], SIRS [5], SEIRS [6], and other models, all of which are playing an increasingly important role in predicting and controlling infectious diseases. These models provide more ideas and insights for people to study the rumor propagation process. In 1964, based on theoretical analysis and practical obser-

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vation of infectious disease models, Daley and Kendall found that rumor and infectious disease spreads have many similarities on the surface, but their transmission systems have essential differences. Therefore, Daley and Kendall [7, 8] established the classical rumor spreading model for the first time and named it DK model. In the DK model, the general population is divided into three categories: those who don't know rumors (ignorant), those who know and spread rumors (spreader), and those who know but do not spread rumors (stiflers). Later, Maki [9] modified the DK model and established MK model in 1973. This model more truly reflected the rumor propagation process, because it indicated that direct contact between the disseminator and others was the way to spread rumors. Although DK and MK models have rigorous logic in establishing the model process, they ignore many factors that affect the rumor propagation process. Based on the DK, MK, and infectious disease models, more extensive studies have been conducted on rumor propagation from different perspectives, such as small-world networks [10, 11], complex networks [12, 13], forgetting mechanism [14], memory mechanism [15], retweeting mechanism [16], individual behavior [17], education [18], internal and external influences [19]. In 2016, Zhang et al. [20] used existing epidemic models to study the dynamics of information diffusion and applied them to complex networks, emphasizing the importance of information dissemination. In 2018, based on a nonlinear model of SIS in a complex network environment, Zhan et al. [21] studied the coupling dynamics of epidemic transmission and information diffusion.

The deterministic rumor spreading model has made great achievements in all aspects, but some scholars have found that in the process of rumor propagation, rumor spread form is largely influenced by the environment noise. Human contact is not fixed and can be disturbed by some random factors. Therefore, the use of deterministic rumor propagation model to describe the rumor propagation process will reduce the accuracy of rumor propagation results. There are a lot of studies on stochastic infectious disease model, but there are few studies on stochastic rumor propagation. Tornatore et al. [22] built a stochastic SIR model with or without distributed time delay and studied the stability of disease-free equilibrium of the model. Miao et al. [23] proposed a dynamic behavior of stochastic SIR infectious disease with horizontal and vertical characteristics by using stochastic differential equation theory and inequality technique. Ball et al. [24] considered the propagation of a Markovian SIR model over the configuration model network. In 2016, Chen [25] used Brownian movement to describe environmental noise to establish a random rumor propagation model with parametric perturbation. In 2020, Huo and Dong [26] analyzed the propagation mechanism of the rumor spreading model in media reports with white noise interference based on the classical rumor propagation model. However, it is unreasonable to use stochastic differential equations with Wiener process to study the uncertain factors in rumor propagation process in a short time. In this paper, we use another effective mathematical tool, uncertainty theory, to study the uncertainty factors in SIR rumor spreading model.

The uncertainty theory was established by Liu [27, 28] in 2007 and revised in 2009. Uncertainty theory is a new branch of mathematics based on normality, duality, subadditivity, and product axioms. It is used to describe and deal with the phenomenon of uncertainty [29]. In this theory, uncertain phenomena are characterized by uncertain variables, and uncertain variables are described by uncertain distributions. In 2008, Liu [30] proposed the uncertain process. An uncertain process is actually a series of uncertain variables with

time as the index, which describes the dynamic behavior of the uncertain phenomenon. Then, in 2009, Liu [28] proposed a special class of uncertain processes called Liu process, which is a stationary, independent increment process whose increments are normal uncertain variables [31]. Liu process is the uncertain counterpart of Wiener process. Uncertain differential equation is a kind of differential equation with Liu process, which was first proposed by Liu [30] in 2008. In 2010, Chen and Liu [32] proved the existence and uniqueness theorem of solutions for uncertain differential equations under Lipschitz and linear growth conditions. Moreover, some scholars presented some methods to solve special uncertain differential equations [32–34]. Based on the Lipschitz continuity of Liu process, Yao et al. [35] gives a sufficient condition for the stability of uncertain differential equations. In many cases, uncertain differential equations are difficult to solve to obtain analytic solutions. Yao and Chen [36] first put forward the concept of an α -path and Yao–Chen formula, which can transform uncertain differential equations into ordinary differential equations, and one can use Euler method to solve the uncertain differential equation. Based on the uncertain process in the uncertainty theory and the knowledge of an uncertain differential equation, we propose an uncertain SIR rumor spreading model.

In this paper, the uncertain differential equation and Liu process in the uncertainty theory is used to study the rumor spreading for the first time, and establish the uncertain SIR rumor spreading model. The rest of the paper is arranged as follows. In Sect. 2, the basic definitions and theorems of uncertain variable, uncertain process and uncertain differential equation in uncertainty theory are introduced. Section 3 establishes the uncertain SIR rumor spreading model with one Liu process, and gives the derivation process of the model. Section 4 proves the existence and uniqueness of the solutions of uncertain SIR rumor spreading differential equations. Section 5 proves that the uncertain SIR rumor spreading model is stable in measure under certain conditions. In Sect. 6, we give a numerical solution of the model. Section 7 introduces a paradox of stochastic rumor spreading model that shows it is unreasonable to use Wiener process to describe the uncertain disturbance in rumor propagation. In Sect. 8, a brief discussion is given.

2 Preliminaries

In this section, we review some definitions and useful results about uncertain variable, uncertain process, and uncertain differential equation.

2.1 Uncertain variable

Definition 1 (Liu [27]) Let \mathcal{L} be a σ -algebra on a nonempty set Γ . A set function $\mathcal{M} : \mathcal{L} \rightarrow [0, 1]$ is called an uncertain measure if it satisfies:

Axiom 1 (Normality axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ .

Axiom 2 (Duality axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ .

Axiom 3 (Subadditivity axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \dots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

In this case, the triple $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.

Beside, the product uncertain measure on the product σ -algebra \mathcal{L} was defined by Liu [28].

Axiom 4 (Product axiom) Let $(\Gamma_i, \mathcal{L}_i, \mathcal{M}_i)$ be uncertainty spaces for $i = 1, 2, \dots$, then the product uncertain measure \mathcal{M} is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{i=1}^{\infty} \Lambda_i\right\} = \bigwedge_{i=1}^{\infty} \mathcal{M}_i\{\Lambda_i\},$$

where Λ_i are arbitrarily chosen events from \mathcal{L}_i for $i = 1, 2, \dots$, respectively.

Definition 2 (Liu [27]) An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the real number set \Re , i.e., for any Borel set B of real numbers, the set $\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$ is an event.

Definition 3 (Liu [27]) The uncertainty distribution Φ of an uncertain variable ξ is defined by $\Phi(x) = \mathcal{M}\{\xi \leq x\}$ for any real number x .

Definition 4 (Liu [27]) An uncertain variable ξ is called normal if it has a normal uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \Re,$$

where e and σ are real numbers with $\sigma > 0$. In particular, if $e = 0$ and $\sigma = 1$, ξ is called standard normal.

If the inverse function Φ^{-1} exists and is unique for each $\alpha \in (0, 1)$, then it is called the inverse uncertainty distribution of ξ . In this case, the uncertainty distribution Φ is said to be regular.

Definition 5 (Liu [28]) The uncertain variables $\xi_1, \xi_2, \dots, \xi_m$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^m (\xi_i \in B_i)\right\} = \bigwedge_{i=1}^m \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets B_1, B_2, \dots, B_m of real numbers.

2.2 Uncertain process

Definition 6 (Liu [30]) Let T be an index set, and $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An uncertain process X_t is a measurable function from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for each $t \in T$ and any Borel set B of real numbers, the set $\{X_t \in B\} = \{\gamma \in \Gamma | X_t(\gamma) \in B\}$ is an event.

Definition 7 (Liu [31]) An uncertain process X_t is said to have an uncertainty distribution $\Phi_t(x)$ if at each time t , the uncertain variable X_t has the uncertainty distribution $\Phi_t(x)$.

Definition 8 (Liu [28]) An uncertain process C_t is said to be a Liu process if

- (i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous.

- (ii) C_t has stationary and independent increments.
- (iii) Every increment $C_{s+t} - C_s$ is a normal uncertain variable with an uncertainty distribution

$$\Phi_t(x) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3}t}\right)\right)^{-1}, \quad x \in \mathfrak{R}.$$

2.3 Uncertain differential equation

Definition 9 (Liu [30]) Let X_t be an uncertain process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as $\Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i|$. Then the time integral of X_t with respect to t is

$$\int_a^b X_t dt = \lim_{\Delta \rightarrow 0} \sum_{i=1}^k X_{t_i} (t_{i+1} - t_i)$$

provided that the limit exists almost surely and is finite.

Definition 10 (Liu [28]) Let X_t be an uncertain process and C_t be a Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as $\Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i|$. Then the Liu integral of X_t is defined by

$$\int_a^b X_t dC_t = \lim_{\Delta \rightarrow 0} \sum_{i=1}^k X_{t_i} (C_{t_{i+1}} - C_{t_i})$$

provided that the limit exists almost surely and is finite.

Definition 11 (Liu [30]) Suppose C_t is a Liu process, and f and g are two given functions. Then

$$dX_t = f(t, X_t) dt + g(t, X_t) dC_t \quad (1)$$

is called an uncertain differential equation. An uncertain process that satisfies equation (1) identically at each time t is called a solution of the uncertain differential equation.

Definition 12 (Yao and Chen [36]) The α -path ($0 < \alpha < 1$) of an uncertain differential equation $dX_t = f(t, X_t) dt + g(t, X_t) dC_t$ with initial value X_0 is a deterministic function X_t^α with respect to t that solves the corresponding ordinary differential equation

$$dX_t^\alpha = f(t, X_t^\alpha) dt + |g(t, X_t^\alpha)| \Phi^{-1}(\alpha) dt,$$

where $\Phi^{-1}(\alpha)$ is the inverse uncertainty distribution of standard normal uncertain variable, i.e.,

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}, \quad 0 < \alpha < 1.$$

Theorem 1 (Yao and Chen [36]) *Assume that $f(t, x)$ and $g(t, x)$ are continuous functions. Let X_t^α be an α -path of the uncertain differential equation*

$$dX_t^\alpha = f(t, X_t^\alpha) dt + |g(t, X_t^\alpha)| \Phi^{-1}(\alpha) dt, \quad t \in [0, s].$$

Then $\mathcal{M}\{X_s \leq X_s^\alpha\} = \alpha$, i.e., X_s has an inverse uncertainty distribution

$$\Phi_s^{-1}(\alpha) = X_s^\alpha, \quad 0 < \alpha < 1.$$

Definition 13 (Yao [37]) *Let X_t be an uncertain process. If for each $\alpha \in (0, 1)$, there exists a real function X_t^α such that*

$$\mathcal{M}\{X_t \leq X_t^\alpha, \forall t\} = \alpha, \quad \mathcal{M}\{X_t > X_t^\alpha, \forall t\} = 1 - \alpha.$$

then X_t is called a contour process. X_t^α is called the α -path of the uncertain process X_t .

Theorem 2 (Yao [37]) *An uncertain process X_t is a contour process if and only if for each $\alpha \in (0, 1)$, there exists a real function X_t^α such that*

$$\mathcal{M}\{X_t < X_t^\alpha, \forall t\} = \alpha, \quad \mathcal{M}\{X_t \geq X_t^\alpha, \forall t\} = 1 - \alpha.$$

Theorem 3 (Yao et al. [35]) *Let C_t be a Liu process. Then there exists an uncertain variable K such that for each γ , $K(\gamma)$ is a Lipschitz constant of the sample path $C_t(\gamma)$, and*

$$\mathcal{M}\{K \leq x\} \geq 2 \left(1 + \exp \left(-\frac{\pi x}{\sqrt{3}} \right) \right)^{-1} - 1.$$

Theorem 4 (Yao et al. [35]) *Let C_t be a Liu process. Then there exists an uncertain variable K such that for each γ , $K(\gamma)$ is a Lipschitz constant of the sample path $C_t(\gamma)$, and*

$$\lim_{x \rightarrow +\infty} \mathcal{M}\{K \leq x\} = 1.$$

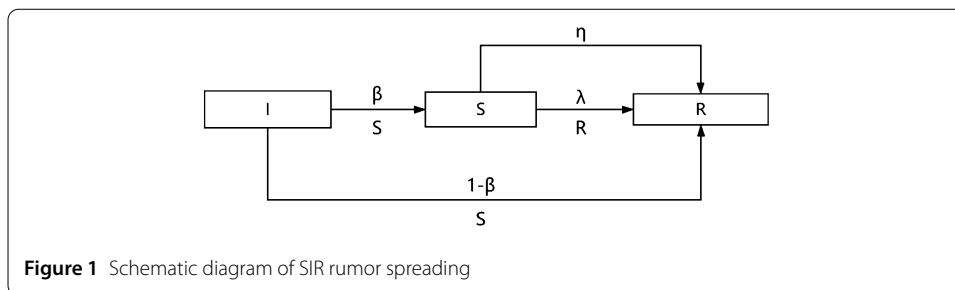
3 Model formulation

DK model is a classical rumor spreading model. On the basis of the DK model, the forgetting mechanism is added to establish the deterministic model of SIR rumor spreading. The ordinary differential equation SIR model is given as follows:

$$\begin{cases} \frac{dI_t}{dt} = -I_t S_t, \\ \frac{dS_t}{dt} = \beta I_t S_t - \lambda S_t R_t - \eta S_t, \\ \frac{dR_t}{dt} = \lambda S_t R_t + \eta S_t + (1 - \beta) I_t S_t. \end{cases} \quad (2)$$

In a closed environment, the total population is grouped into three categories: ignorants, spreaders, and stiflers, which are represented by I , S , and R , respectively:

- Rumor ignorant (I), individuals who know nothing about rumors;
- Rumor spreader (S), individuals who know rumors and will spread them;
- Rumor stifler (R), individuals who know rumors but don't spread them.



The population sizes of I , S , and R individuals at time t are denoted by I_t , S_t , and R_t , respectively. The parameters in the above model are expressed as follows:

- β , the spreading rate that an ignorant turns into a spreader after he comes into contact with the spreader;
- λ , the rate that the spreader converts into a stifler after contacting with the rumor stifler;
- η , the rate that a spreader becomes a stifler because he forgotten or lost interest in the rumor.

It is worth to mention that the deterministic SIR rumor spreading model and uncertain SIR rumor spreading model proposed in this manuscript are suitable to the following cases:

1. The total population size is considered as a constant, N . Suppose that there is no entry and exit in the total population.
2. Assume that the mixture of individuals in the population is sufficiently uniform.
3. Suppose that one spreader is in contact with another, and their status does not change.

The SIR rumor spreading process is shown in Fig. 1.

However, in real life, rumor spreading is affected by environmental noise. This paper, based on the ordinary differential equation (2), introduces the Liu process to describe the uncertain factors in the environment, the rate λ of a spreader turning into a stifler affected by uncertainties. In this part, we change the rate λ into an uncertain variable λ_t .

Let the time interval $[0, t]$ be divided uniformly into n subintervals such that $0 = t_0 < t_1 < t_2 < \dots < t_n = t$, and the interval length is $\Delta t = t_{i+1} - t_i$, $i = 0, 1, 2, \dots, n-1$. We assume that

$$\lambda_t = \lambda + \sigma \frac{\Delta C_t}{\Delta t},$$

where λ is a constant, C_t is a Liu process representing the uncertain disturbance, and σ is a constant denoting the intensity of C_t .

An increase in the number of spreader individuals, who are the original spreaders in the interval $(t_i, t_{i+1}]$, is equal to

$$(\beta I_{t_i} S_{t_i} - \lambda_{t_i} S_{t_i} R_{t_i} - \eta S_{t_i}) \Delta t = (\beta I_{t_i} S_{t_i} - \lambda S_{t_i} R_{t_i} - \eta S_{t_i}) \Delta t - \sigma S_{t_i} R_{t_i} \Delta C_t.$$

Thus

$$S_{t_{i+1}} - S_{t_i} = (\beta I_{t_i} S_{t_i} - \lambda S_{t_i} R_{t_i} - \eta S_{t_i}) \Delta t - \sigma S_{t_i} R_{t_i} \Delta C_t.$$

Then we have

$$S_t - S_0 = \sum_{i=0}^{n-1} (S_{t_{i+1}} - S_{t_i}) = \sum_{i=0}^{n-1} ((\beta I_{t_i} S_{t_i} - \lambda S_{t_i} R_{t_i} - \eta S_{t_i}) \Delta t - \sigma S_{t_i} R_{t_i} \Delta C_t).$$

Letting $\Delta t \rightarrow 0$, the above equation can be written as an uncertain integral equation,

$$S_t - S_0 = \int_0^t (\beta I_s S_s - \lambda S_s R_s - \eta S_s) ds - \int_0^t \sigma S_s R_s dC_s.$$

The number of stiflers in the interval $(t_i, t_{i+1}]$ is equal to

$$\begin{aligned} R_{t_{i+1}} - R_{t_i} &= (\lambda_{t_i} S_{t_i} R_{t_i} + \eta S_{t_i} + (1 - \beta) I_{t_i} S_{t_i}) \Delta t \\ &= (\lambda S_{t_i} R_{t_i} + \eta S_{t_i} + (1 - \beta) I_{t_i} S_{t_i}) \Delta t + \sigma S_{t_i} R_{t_i} \Delta C_t. \end{aligned}$$

Then we have

$$R_t - R_0 = \sum_{i=0}^{n-1} (R_{t_{i+1}} - R_{t_i}) = \sum_{i=0}^{n-1} ((\lambda S_{t_i} R_{t_i} + \eta S_{t_i} + (1 - \beta) I_{t_i} S_{t_i}) \Delta t + \sigma S_{t_i} R_{t_i} \Delta C_t).$$

Letting $\Delta t \rightarrow 0$, an uncertain integral equation can be obtained as

$$R_t - R_0 = \int_0^t (\lambda S_s R_s + \eta S_s + (1 - \beta) I_s S_s) ds + \int_0^t \sigma S_s R_s dC_s.$$

The number of ignorants in the interval $(t_i, t_{i+1}]$ is equal to

$$I_{t_{i+1}} - I_{t_i} = -I_{t_i} S_{t_i} \Delta t.$$

Then we have

$$I_t - I_0 = \sum_{i=0}^{n-1} (I_{t_{i+1}} - I_{t_i}) = \sum_{i=0}^{n-1} -I_{t_i} S_{t_i} \Delta t.$$

Letting $\Delta t \rightarrow 0$, an uncertain integral equation can be obtained as

$$I_t - I_0 = - \int_0^t I_s S_s ds.$$

We obtain a system of uncertain integral equations

$$\begin{cases} I_t = I_0 - \int_0^t I_s S_s ds, \\ S_t = S_0 + \int_0^t (\beta I_s S_s - \lambda S_s R_s - \eta S_s) ds - \int_0^t \sigma S_s R_s dC_s, \\ R_t = R_0 + \int_0^t (\lambda S_s R_s + \eta S_s + (1 - \beta) I_s S_s) ds + \int_0^t \sigma S_s R_s dC_s. \end{cases} \quad (3)$$

It is equal to a system of uncertain differential equations, called the uncertain SIR rumor spreading model, as follows:

$$\begin{cases} dI_t = -I_t S_t dt, \\ dS_t = (\beta I_t S_t - \lambda S_t R_t - \eta S_t) dt - \sigma S_t R_t dC_t, \\ dR_t = (\lambda S_t R_t + \eta S_t + (1 - \beta) I_t S_t) dt + \sigma S_t R_t dC_t, \end{cases} \quad (4)$$

where I_t, S_t, R_t are the numbers of ignorant, spreader, and stifler individuals, respectively, C_t is a Liu process representing the uncertain disturbers, σ is a constant denoting the intensity of C_t , and β, λ , and η are constants.

4 Existence and uniqueness of uncertain SIR rumor spreading model

Lemma 1 *If, in the uncertain SIR rumor spreading model (4), C_t is a Liu process, then we have*

$$\left| \int_0^t \sigma S_s(\gamma) R_s(\gamma) dC_s(\gamma) \right| \leq K(\gamma) \int_0^t |\sigma S_s(\gamma) R_s(\gamma)| ds, \quad (5)$$

where $K(\gamma)$ is the Lipschitz constant of the sample path $C_t(\gamma)$ for each $\gamma \in \Gamma$.

Proof According to model (3), we have

$$\begin{aligned} & \max \left\{ \left| - \int_0^t \sigma S_s(\gamma) R_s(\gamma) dC_s(\gamma) \right|, \left| \int_0^t \sigma S_s(\gamma) R_s(\gamma) dC_s(\gamma) \right| \right\} \\ &= \left| \int_0^t \sigma S_s(\gamma) R_s(\gamma) dC_s(\gamma) \right| \\ &= \left| \lim_{\Delta t \rightarrow 0} \sum_{i=1}^n \sigma S_{t_{i-1}}(\gamma) R_{t_{i-1}}(\gamma) (C_{t_i}(\gamma) - C_{t_{i-1}}(\gamma)) \right| \\ &\leq \lim_{\Delta t \rightarrow 0} \sum_{i=1}^n |\sigma S_{t_{i-1}}(\gamma) R_{t_{i-1}}(\gamma)| \cdot |C_{t_i}(\gamma) - C_{t_{i-1}}(\gamma)|. \end{aligned}$$

For every sample γ , it can be seen from the definition of Liu process that $C_t(\gamma)$ is Lipschitz continuous with respect to t . So there exists a finite number $K(\gamma)$, called Lipschitz constant, such that

$$|C_{t_i}(\gamma) - C_{t_{i-1}}(\gamma)| \leq K(\gamma) |t_i - t_{i-1}|, \quad i = 1, 2, \dots, n.$$

Thus

$$\begin{aligned} & \lim_{\Delta t \rightarrow 0} \sum_{i=1}^n |\sigma S_{t_{i-1}}(\gamma) R_{t_{i-1}}(\gamma)| \cdot |C_{t_i}(\gamma) - C_{t_{i-1}}(\gamma)| \\ &\leq K(\gamma) \lim_{\Delta t \rightarrow 0} \sum_{i=1}^n |\sigma S_{t_{i-1}}(\gamma) R_{t_{i-1}}(\gamma)| \cdot |t_i - t_{i-1}| \\ &\leq K(\gamma) \int_0^t |\sigma S_s(\gamma) R_s(\gamma)| ds. \end{aligned}$$

The lemma is proved. \square

Theorem 4.1 *If the coefficients of the uncertain SIR rumor spreading model (4) satisfy the Lipschitz condition*

$$\begin{aligned} & \max \left\{ \left| -(xy - \tilde{x}\tilde{y}) \right|, \left| \beta(xy - \tilde{x}\tilde{y}) - \lambda(yz - \tilde{y}\tilde{z}) - \eta(y - \tilde{y}) \right|, \right. \\ & \quad \left| \lambda(yz - \tilde{y}\tilde{z}) + \eta(y - \tilde{y}) + (1 - \beta)(xy - \tilde{x}\tilde{y}) \right| \Big\} \\ & \quad + \left| \sigma(yz - \tilde{y}\tilde{z}) \right| \\ & \leq L \max \left\{ |x - \tilde{x}|, |y - \tilde{y}|, |z - \tilde{z}| \right\}, \end{aligned}$$

for all $(x, y, z), (\tilde{x}, \tilde{y}, \tilde{z}) \in \mathbb{R}^3, t \geq 0$ and some constant L , then model (4) has a unique solution.

Proof To prove the existence of solution for uncertain SIR rumor spreading model (4), a successive approximation method is used to obtain the solution of the uncertain SIR rumor spreading model (4). For any sample γ , we define $(I_t^{(0)}(\gamma), S_t^{(0)}(\gamma), R_t^{(0)}(\gamma)) = (I_0, S_0, R_0)$, and set

$$\begin{aligned} I_t^{(n+1)}(\gamma) &= I_0 - \int_0^t I_s^{(n)}(\gamma) S_s^{(n)}(\gamma) \, ds, \\ S_t^{(n+1)}(\gamma) &= S_0 + \int_0^t \left(\beta I_s^{(n)}(\gamma) S_s^{(n)}(\gamma) - \lambda S_s^{(n)}(\gamma) R_s^{(n)}(\gamma) - \eta S_s^{(n)}(\gamma) \right) \, ds \\ &\quad - \int_0^t \sigma S_s^{(n)}(\gamma) R_s^{(n)}(\gamma) \, dC_s(\gamma), \\ R_t^{(n+1)}(\gamma) &= R_0 + \int_0^t \left(\lambda S_s^{(n)}(\gamma) R_s^{(n)}(\gamma) + \eta S_s^{(n)}(\gamma) + (1 - \beta) I_s^{(n)}(\gamma) S_s^{(n)}(\gamma) \right) \, ds \\ &\quad + \int_0^t \sigma S_s^{(n)}(\gamma) R_s^{(n)}(\gamma) \, dC_s(\gamma), \end{aligned}$$

and

$$D_t^{(n)}(\gamma) = \max_{0 \leq s \leq t} \left\{ \max \left\{ \left| I_s^{(n+1)}(\gamma) - I_s^{(n)}(\gamma) \right|, \left| S_s^{(n+1)}(\gamma) - S_s^{(n)}(\gamma) \right|, \left| R_s^{(n+1)}(\gamma) - R_s^{(n)}(\gamma) \right| \right\} \right\}$$

for $n = 0, 1, 2, \dots$. We assert that

$$\begin{aligned} D_t^{(n)}(\gamma) &\leq (1 + \max \{|I_0|, |S_0|, |R_0|\}) \\ &\quad \times \frac{L^{n+1}(1 + K(\gamma))^{n+1}}{(n+1)!} t^{n+1}, \quad n = 0, 1, 2, \dots, 0 \leq t \leq T, \end{aligned} \quad (6)$$

where T is a constant. Inequality (6) is proved by using the mathematical induction as follows. For $n = 0$, we have

$$\begin{aligned} D_t^{(0)}(\gamma) &= \max_{0 \leq v \leq t} \left\{ \max \left\{ \left| I_v^{(1)}(\gamma) - I_v^{(0)}(\gamma) \right|, \left| S_v^{(1)}(\gamma) - S_v^{(0)}(\gamma) \right|, \left| R_v^{(1)}(\gamma) - R_v^{(0)}(\gamma) \right| \right\} \right\} \\ &= \max_{0 \leq v \leq t} \left\{ \max \left\{ \left| - \int_0^v I_s^{(0)}(\gamma) S_s^{(0)}(\gamma) \, ds \right|, \right. \right. \\ &\quad \left. \left| \int_0^v \left(\beta I_s^{(0)}(\gamma) S_s^{(0)}(\gamma) - \lambda S_s^{(0)}(\gamma) R_s^{(0)}(\gamma) - \eta S_s^{(0)}(\gamma) \right) \, ds \right. \right. \end{aligned}$$

$$\begin{aligned}
& \left| - \int_0^\nu \sigma S_s^{(0)}(\gamma) R_s^{(0)}(\gamma) dC_s(\gamma) \right|, \\
& \left| \int_0^\nu (\lambda S_s^{(0)}(\gamma) R_s^{(0)}(\gamma) + \eta S_s^{(0)}(\gamma) + (1-\beta) I_s^{(0)}(\gamma) S_s^{(0)}(\gamma)) ds \right. \\
& \left. + \int_0^\nu \sigma S_s^{(0)}(\gamma) R_s^{(0)}(\gamma) dC_s(\gamma) \right| \Bigg\} \\
& \leq \max_{0 \leq \nu \leq t} \left\{ \max \left\{ \left| \int_0^\nu -I_s^{(0)}(\gamma) S_s^{(0)}(\gamma) ds \right|, \right. \right. \\
& \left. \left| \int_0^\nu (\beta I_s^{(0)}(\gamma) S_s^{(0)}(\gamma) - \lambda S_s^{(0)}(\gamma) R_s^{(0)}(\gamma) - \eta S_s^{(0)}(\gamma)) ds \right|, \right. \\
& \left. \left| \int_0^\nu (\lambda S_s^{(0)}(\gamma) R_s^{(0)}(\gamma) + \eta S_s^{(0)}(\gamma) + (1-\beta) I_s^{(0)}(\gamma) S_s^{(0)}(\gamma)) ds \right| \right. \\
& \left. + \left| \int_0^\nu \sigma S_s^{(0)}(\gamma) R_s^{(0)}(\gamma) dC_s(\gamma) \right| \right\} \\
& \leq \int_0^t \max \{ | -I_0 S_0 |, | \beta I_0 S_0 - \lambda S_0 R_0 - \eta S_0 |, | \lambda S_0 R_0 + \eta S_0 + (1-\beta) I_0 S_0 | \} ds \\
& \quad + \left| \int_0^t \sigma S_0 R_0 dC_s(\gamma) \right| \\
& \leq \int_0^t \max \{ | -I_0 S_0 |, | \beta I_0 S_0 - \lambda S_0 R_0 - \eta S_0 |, | \lambda S_0 R_0 + \eta S_0 + (1-\beta) I_0 S_0 | \} ds \\
& \quad + K(\gamma) \int_0^t | \sigma S_0 R_0 | ds \\
& \leq (1 + K(\gamma)) (\max \{ | I_0 S_0 |, | \beta I_0 S_0 - \lambda S_0 R_0 - \eta S_0 |, | \lambda S_0 R_0 + \eta S_0 + (1-\beta) I_0 S_0 | \} \\
& \quad + | \sigma S_0 R_0 |) t \\
& \leq (1 + K(\gamma)) L (1 + \max \{ | I_0 |, | S_0 |, | R_0 | \}) t,
\end{aligned}$$

which is satisfied if we take

$$L \geq \frac{\max \{ | I_0 S_0 |, | \beta I_0 S_0 - \lambda S_0 R_0 - \eta S_0 |, | \lambda S_0 R_0 + \eta S_0 + (1-\beta) I_0 S_0 | \} + | \sigma S_0 R_0 |}{1 + \max \{ | I_0 |, | S_0 |, | R_0 | \}}.$$

Assume inequality (6) holds for $n-1$. Then

$$\begin{aligned}
& D_t^{(n)}(\gamma) \\
& = \max_{0 \leq \nu \leq t} \left\{ \max \left\{ | I_\nu^{(n+1)}(\gamma) - I_\nu^{(n)}(\gamma) |, | S_\nu^{(n+1)}(\gamma) - S_\nu^{(n)}(\gamma) |, | R_\nu^{(n+1)}(\gamma) - R_\nu^{(n)}(\gamma) | \right\} \right. \\
& = \max_{0 \leq \nu \leq t} \left\{ \max \left\{ \left| - \int_0^\nu (I_s^{(n)}(\gamma) S_s^{(n)}(\gamma) - I_s^{(n-1)}(\gamma) S_s^{(n-1)}(\gamma)) ds \right|, \right. \right. \\
& \left. \left| \int_0^\nu \beta (I_s^{(n)}(\gamma) S_s^{(n)}(\gamma) - I_s^{(n-1)}(\gamma) S_s^{(n-1)}(\gamma)) - \lambda (S_s^{(n)}(\gamma) R_s^{(n)}(\gamma) - S_s^{(n-1)}(\gamma) R_s^{(n-1)}(\gamma)) \right. \right. \\
& \quad \left. \left. - \eta (S_s^{(n)}(\gamma) - S_s^{(n-1)}(\gamma)) ds - \int_0^\nu \sigma (S_s^{(n)}(\gamma) R_s^{(n)}(\gamma) - S_s^{(n-1)}(\gamma) R_s^{(n-1)}(\gamma)) dC_s(\gamma) \right|, \right. \\
& \left. \left| \int_0^\nu \lambda (S_s^{(n)}(\gamma) R_s^{(n)}(\gamma) - S_s^{(n-1)}(\gamma) R_s^{(n-1)}(\gamma)) + \eta (S_s^{(n)}(\gamma) - S_s^{(n-1)}(\gamma)) \right. \right. \\
& \quad \left. \left. \right| \right\}
\end{aligned}$$

$$\begin{aligned}
& + (1 - \beta)(I_s^{(n)}(\gamma)S_s^{(n)}(\gamma) - I_s^{(n-1)}(\gamma)S_s^{(n-1)}(\gamma)) \, ds \\
& + \int_0^\gamma \sigma(S_s^{(n)}(\gamma)R_s^{(n)}(\gamma) - S_s^{(n-1)}(\gamma)R_s^{(n-1)}(\gamma)) \, dC_s(\gamma) \Bigg\} \Bigg\} \\
\leq & \max \left\{ \left| \int_0^t - (I_s^{(n)}(\gamma)S_s^{(n)}(\gamma) - I_s^{(n-1)}(\gamma)S_s^{(n-1)}(\gamma)) \, ds \right|, \right. \\
& \left| \int_0^t \beta(I_s^{(n)}(\gamma)S_s^{(n)}(\gamma) - I_s^{(n-1)}(\gamma)S_s^{(n-1)}(\gamma)) \right. \\
& \left. - \lambda(S_s^{(n)}(\gamma)R_s^{(n)}(\gamma) - S_s^{(n-1)}(\gamma)R_s^{(n-1)}(\gamma)) - \eta(S_s^{(n)}(\gamma) - S_s^{(n-1)}(\gamma)) \, ds \right|, \\
& \left| \int_0^t \lambda(S_s^{(n)}(\gamma)R_s^{(n)}(\gamma) - S_s^{(n-1)}(\gamma)R_s^{(n-1)}(\gamma)) + \eta(S_s^{(n)}(\gamma) - S_s^{(n-1)}(\gamma)) \right. \\
& \left. + (1 - \beta)(I_s^{(n)}(\gamma)S_s^{(n)}(\gamma) - I_s^{(n-1)}(\gamma)S_s^{(n-1)}(\gamma)) \, ds \right| \Bigg\} \\
& + \max \left\{ \left| \int_0^t - \sigma(S_s^{(n)}(\gamma)R_s^{(n)}(\gamma) - S_s^{(n-1)}(\gamma)R_s^{(n-1)}(\gamma)) \, dC_s(\gamma) \right|, \right. \\
& \left| \int_0^t \sigma(S_s^{(n)}(\gamma)R_s^{(n)}(\gamma) - S_s^{(n-1)}(\gamma)R_s^{(n-1)}(\gamma)) \, dC_s(\gamma) \right| \Bigg\} \\
= & \max \left\{ \left| \int_0^t - (I_s^{(n)}(\gamma)S_s^{(n)}(\gamma) - I_s^{(n-1)}(\gamma)S_s^{(n-1)}(\gamma)) \, ds \right|, \right. \\
& \left| \int_0^t \beta(I_s^{(n)}(\gamma)S_s^{(n)}(\gamma) - I_s^{(n-1)}(\gamma)S_s^{(n-1)}(\gamma)) \right. \\
& \left. - \lambda(S_s^{(n)}(\gamma)R_s^{(n)}(\gamma) - S_s^{(n-1)}(\gamma)R_s^{(n-1)}(\gamma)) - \eta(S_s^{(n)}(\gamma) - S_s^{(n-1)}(\gamma)) \, ds \right|, \\
& \left| \int_0^t \lambda(S_s^{(n)}(\gamma)R_s^{(n)}(\gamma) - S_s^{(n-1)}(\gamma)R_s^{(n-1)}(\gamma)) + \eta(S_s^{(n)}(\gamma) - S_s^{(n-1)}(\gamma)) \right. \\
& \left. + (1 - \beta)(I_s^{(n)}(\gamma)S_s^{(n)}(\gamma) - I_s^{(n-1)}(\gamma)S_s^{(n-1)}(\gamma)) \, ds \right| \Bigg\} \\
& + \left| \int_0^t \sigma(S_s^{(n)}(\gamma)R_s^{(n)}(\gamma) - S_s^{(n-1)}(\gamma)R_s^{(n-1)}(\gamma)) \, dC_s(\gamma) \right| \\
\leq & \int_0^t \max \{ | - (I_s^{(n)}(\gamma)S_s^{(n)}(\gamma) - I_s^{(n-1)}(\gamma)S_s^{(n-1)}(\gamma)) |, \\
& | \beta(I_s^{(n)}(\gamma)S_s^{(n)}(\gamma) - I_s^{(n-1)}(\gamma)S_s^{(n-1)}(\gamma)) \\
& - \lambda(S_s^{(n)}(\gamma)R_s^{(n)}(\gamma) - S_s^{(n-1)}(\gamma)R_s^{(n-1)}(\gamma)) - \eta(S_s^{(n)}(\gamma) - S_s^{(n-1)}(\gamma)) |, \\
& | \lambda(S_s^{(n)}(\gamma)R_s^{(n)}(\gamma) - S_s^{(n-1)}(\gamma)R_s^{(n-1)}(\gamma)) \\
& + (1 - \beta)(I_s^{(n)}(\gamma)S_s^{(n)}(\gamma) - I_s^{(n-1)}(\gamma)S_s^{(n-1)}(\gamma)) \\
& + \eta(S_s^{(n)}(\gamma) - S_s^{(n-1)}(\gamma)) | \} \, ds \\
& + K(\gamma) \int_0^t | \sigma(S_s^{(n)}(\gamma)R_s^{(n)}(\gamma) - S_s^{(n-1)}(\gamma)R_s^{(n-1)}(\gamma)) | \, ds \\
\leq & L \int_0^t \max \{ | I_s^{(n)}(\gamma) - I_s^{(n-1)}(\gamma) |, | S_s^{(n)}(\gamma) - S_s^{(n-1)}(\gamma) |, | R_s^{(n)}(\gamma) - R_s^{(n-1)}(\gamma) | \} \, ds
\end{aligned}$$

$$\begin{aligned}
& + K(\gamma)L \int_0^t \max\{|I_s^{(n)}(\gamma) - I_s^{(n-1)}(\gamma)|, |S_s^{(n)}(\gamma) - S_s^{(n-1)}(\gamma)|, \\
& |R_s^{(n)}(\gamma) - R_s^{(n-1)}(\gamma)|\} ds \\
& \leq L(1 + K(\gamma)) \int_0^t \max\{|I_s^{(n)}(\gamma) - I_s^{(n-1)}(\gamma)|, |S_s^{(n)}(\gamma) - S_s^{(n-1)}(\gamma)|, \\
& |R_s^{(n)}(\gamma) - R_s^{(n-1)}(\gamma)|\} ds \\
& \leq L(1 + K(\gamma)) \int_0^t (1 + \max\{|I_0|, |S_0|, |R_0|\}) \frac{L^n(1 + K(\gamma))^n}{n!} s^n ds \\
& = (1 + \max\{|I_0|, |S_0|, |R_0|\}) \frac{L^{n+1}(1 + K(\gamma))^{n+1}}{(n+1)!} t^{n+1}.
\end{aligned}$$

Therefore, inequality (6) is proved. By using the Weierstrass' criterion, we obtain

$$\begin{aligned}
& \sum_{n=0}^{+\infty} (1 + \max\{|I_0|, |S_0|, |R_0|\}) \frac{L^{n+1}(1 + K(\gamma))^{n+1}}{(n+1)!} t^{n+1} \\
& \leq \sum_{n=0}^{+\infty} (1 + \max\{|I_0|, |S_0|, |R_0|\}) \frac{L^{n+1}(1 + K(\gamma))^{n+1}}{(n+1)!} T^{n+1} \\
& < +\infty, \quad \forall \gamma \in \Gamma.
\end{aligned}$$

Thus $(I_t^{(k)}(\gamma), S_t^{(k)}(\gamma), R_t^{(k)}(\gamma))$ converges uniformly in $t \in [0, T]$. For any $\gamma \in \Gamma$, $t \in [0, T]$, we obtain

$$I_t(\gamma) = \lim_{k \rightarrow +\infty} I_t^{(k)}(\gamma), \quad S_t(\gamma) = \lim_{k \rightarrow +\infty} S_t^{(k)}(\gamma), \quad R_t(\gamma) = \lim_{k \rightarrow +\infty} R_t^{(k)}(\gamma).$$

Therefore,

$$\begin{aligned}
I_t &= I_0 - \int_0^t I_s S_s ds, \\
S_t &= S_0 + \int_0^t (\beta I_s S_s - \lambda S_s R_s - \eta S_s) ds - \int_0^t \sigma S_s R_s dC_s, \\
R_t &= R_0 + \int_0^t (\lambda S_s R_s + \eta S_s + (1 - \beta) I_s S_s) ds + \int_0^t \sigma S_s R_s dC_s
\end{aligned}$$

is the solution of model (4) for any $t \geq 0$, where T is arbitrary.

Then we prove the uniqueness of the solution under the given conditions. Assume that $(I_t(\gamma), S_t(\gamma), R_t(\gamma))$ and $(I_t^*(\gamma), S_t^*(\gamma), R_t^*(\gamma))$ are two solutions of the uncertain SIR rumor spreading model (4) with the same initial value (I_0, S_0, R_0) . For each $\gamma \in \Gamma$, we obtain

$$\begin{aligned}
& \max\{|I_t(\gamma) - I_t^*(\gamma)|, |S_t(\gamma) - S_t^*(\gamma)|, |R_t(\gamma) - R_t^*(\gamma)|\} \\
& = \max\left\{\left|\int_0^t -(I_u(\gamma)S_u(\gamma) - I_u^*(\gamma)S_u^*(\gamma)) du\right|, \right. \\
& \quad \left|\int_0^t \beta(I_u(\gamma)S_u(\gamma) - I_u^*(\gamma)S_u^*(\gamma)) - \lambda(S_u(\gamma)R_u(\gamma) - S_u^*(\gamma)R_u^*(\gamma)) \right. \\
& \quad \left. - \eta(S_u(\gamma) - S_u^*(\gamma)) du\right|
\end{aligned}$$

$$\begin{aligned}
& + \left| \int_0^t -\sigma(S_u(\gamma)R_u(\gamma) - S_u^*(\gamma)R_u^*(\gamma)) dC_u(\gamma) \right|, \\
& \left| \int_0^t \lambda(S_u(\gamma)R_u(\gamma) - S_u^*(\gamma)R_u^*(\gamma)) + \eta(S_u(\gamma) - S_u^*(\gamma)) \right. \\
& + (1-\beta)(I_u(\gamma)S_u(\gamma) - I_u^*(\gamma)S_u^*(\gamma)) du \\
& + \left. \int_0^t \sigma(S_u(\gamma)R_u(\gamma) - S_u^*(\gamma)R_u^*(\gamma)) dC_u(\gamma) \right| \Big\} \\
& \leq \max \left\{ \left| \int_0^t (I_u(\gamma)S_u(\gamma) - I_u^*(\gamma)S_u^*(\gamma)) du \right|, \right. \\
& \left| \int_0^t \beta(I_u(\gamma)S_u(\gamma) - I_u^*(\gamma)S_u^*(\gamma)) - \lambda(S_u(\gamma)R_u(\gamma) - S_u^*(\gamma)R_u^*(\gamma)) \right. \\
& - \eta(S_u(\gamma) - S_u^*(\gamma)) du \Big|, \\
& \left| \int_0^t \lambda(S_u(\gamma)R_u(\gamma) - S_u^*(\gamma)R_u^*(\gamma)) + \eta(S_u(\gamma) - S_u^*(\gamma)) \right. \\
& + (1-\beta)(I_u(\gamma)S_u(\gamma) - I_u^*(\gamma)S_u^*(\gamma)) du \Big| \Big\} \\
& + \left| \int_0^t \sigma(S_u(\gamma)R_u(\gamma) - S_u^*(\gamma)R_u^*(\gamma)) dC_u(\gamma) \right| \\
& \leq \int_0^t \max \{ |I_u(\gamma)S_u(\gamma) - I_u^*(\gamma)S_u^*(\gamma)|, \\
& |\beta(I_u(\gamma)S_u(\gamma) - I_u^*(\gamma)S_u^*(\gamma)) - \lambda(S_u(\gamma)R_u(\gamma) - S_u^*(\gamma)R_u^*(\gamma)) \\
& - \eta(S_u(\gamma) - S_u^*(\gamma))|, \\
& |\lambda(S_u(\gamma)R_u(\gamma) - S_u^*(\gamma)R_u^*(\gamma)) + \eta(S_u(\gamma) - S_u^*(\gamma)) \\
& + (1-\beta)(I_u(\gamma)S_u(\gamma) - I_u^*(\gamma)S_u^*(\gamma))| \} du \\
& + K(\gamma) \int_0^t |\sigma(S_u(\gamma)R_u(\gamma) - S_u^*(\gamma)R_u^*(\gamma))| du \\
& \leq L(1+K(\gamma)) \int_0^t \max \{ |I_u(\gamma) - I_u^*(\gamma)|, |S_u(\gamma) - S_u^*(\gamma)|, |R_u(\gamma) - R_u^*(\gamma)| \} du.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \max \{ |I_t(\gamma) - I_t^*(\gamma)|, |S_t(\gamma) - S_t^*(\gamma)|, |R_t(\gamma) - R_t^*(\gamma)| \} \\
& \leq 0 \cdot \exp(L(1+K(\gamma))t) = 0
\end{aligned}$$

can be obtained by using the Grönwall's inequality. It is equivalent to $(I_t(\gamma), S_t(\gamma), R_t(\gamma)) = (I_t^*(\gamma), S_t^*(\gamma), R_t^*(\gamma))$ almost certainly. The uniqueness of the solution is verified. \square

5 Stability of uncertain SIR rumor spreading model

Lemma 2 Assume the uncertain SIR rumor spreading model (4) has a unique solution for each given initial value. Then it is stable in measure if the coefficients of the uncertain SIR

rumor spreading model (4) satisfy the strong Lipschitz condition

$$\begin{aligned} & \max \left\{ \left| -(xy - \tilde{x}\tilde{y}) \right|, \left| \beta(xy - \tilde{x}\tilde{y}) - \lambda(yz - \tilde{y}\tilde{z}) - \eta(y - \tilde{y}) \right|, \right. \\ & \quad \left| \lambda(yz - \tilde{y}\tilde{z}) + \eta(y - \tilde{y}) + (1 - \beta)(xy - \tilde{x}\tilde{y}) \right| \} \\ & \quad + \left| \sigma(yz - \tilde{y}\tilde{z}) \right| \\ & \leq L_t \max \{ |x - \tilde{x}|, |y - \tilde{y}|, |z - \tilde{z}| \} \end{aligned}$$

for any $(x, y, z), (\tilde{x}, \tilde{y}, \tilde{z}) \in \mathbb{R}^3$, $t \geq 0$, where L_t is some positive function satisfying

$$\int_0^{+\infty} L_t dt < +\infty.$$

Proof Let $(I_t(\gamma), S_t(\gamma), R_t(\gamma))$ and $(I_t^*(\gamma), S_t^*(\gamma), R_t^*(\gamma))$ be two solutions of the uncertain SIR rumor spreading model (4) with different initial values (I_0, S_0, R_0) and (I_0^*, S_0^*, R_0^*) , respectively. On the basis of Lipschitz continuous sample path $C_t(\gamma)$, we have

$$\begin{cases} I_t = I_0 - \int_0^t I_s S_s ds, \\ S_t = S_0 + \int_0^t (\beta I_s S_s - \lambda S_s R_s - \eta S_s) ds - \int_0^t \sigma S_s R_s dC_s, \\ R_t = R_0 + \int_0^t (\lambda S_s R_s + \eta S_s + (1 - \beta) I_s S_s) ds + \int_0^t \sigma S_s R_s dC_s \end{cases}$$

and

$$\begin{cases} I_t^* = I_0^* - \int_0^t I_s^* S_s^* ds, \\ S_t^* = S_0^* + \int_0^t (\beta I_s^* S_s^* - \lambda S_s^* R_s^* - \eta S_s^*) ds - \int_0^t \sigma S_s^* R_s^* dC_s, \\ R_t^* = R_0^* + \int_0^t (\lambda S_s^* R_s^* + \eta S_s^* + (1 - \beta) I_s^* S_s^*) ds + \int_0^t \sigma S_s^* R_s^* dC_s. \end{cases}$$

According to the strong Lipschitz condition, we have

$$\begin{aligned} & \max \{ |I_t(\gamma) - I_t^*(\gamma)|, |S_t(\gamma) - S_t^*(\gamma)|, |R_t(\gamma) - R_t^*(\gamma)| \} \\ & \leq \max \{ |I_0 - I_0^*|, |S_0 - S_0^*|, |R_0 - R_0^*| \} + \int_0^t \max \{ |-(I_s(\gamma) S_s(\gamma) - I_s^*(\gamma) S_s^*(\gamma))|, \\ & \quad |\beta(I_s(\gamma) S_s(\gamma) - I_s^*(\gamma) S_s^*(\gamma)) - \lambda(S_s(\gamma) R_s(\gamma) - S_s^*(\gamma) R_s^*(\gamma)) - \eta(S_s(\gamma) - S_s^*(\gamma))|, \\ & \quad |\lambda(S_s(\gamma) R_s(\gamma) - S_s^*(\gamma) R_s^*(\gamma)) + \eta(S_s(\gamma) - S_s^*(\gamma)) \\ & \quad + (1 - \beta)(I_s(\gamma) S_s(\gamma) - I_s^*(\gamma) S_s^*(\gamma))| \} ds \\ & \quad + \int_0^t \max \{ |-\sigma(S_s(\gamma) R_s(\gamma) - S_s^*(\gamma) R_s^*(\gamma))|, \\ & \quad |\sigma(S_s(\gamma) R_s(\gamma) - S_s^*(\gamma) R_s^*(\gamma))| \} dC_s(\gamma) \\ & \leq \max \{ |I_0 - I_0^*|, |S_0 - S_0^*|, |R_0 - R_0^*| \} \\ & \quad + \int_0^t L_s \max \{ |I_s(\gamma) - I_s^*(\gamma)|, |S_s(\gamma) - S_s^*(\gamma)|, |R_s(\gamma) - R_s^*(\gamma)| \} ds \\ & \quad + \int_0^t K(\gamma) L_s \max \{ |I_s(\gamma) - I_s^*(\gamma)|, |S_s(\gamma) - S_s^*(\gamma)|, |R_s(\gamma) - R_s^*(\gamma)| \} ds \end{aligned}$$

$$= \max\{|I_0 - I_0^*|, |S_0 - S_0^*|, |R_0 - R_0^*|\} \\ + (1 + K(\gamma)) \int_0^t L_s \max\{|I_s(\gamma) - I_s^*(\gamma)|, |S_s(\gamma) - S_s^*(\gamma)|, |R_s(\gamma) - R_s^*(\gamma)|\} ds,$$

where $K(\gamma)$ is the Lipschitz constant of $C_t(\gamma)$. By the Grönwall's inequality, we have

$$\max\{|I_t(\gamma) - I_t^*(\gamma)|, |S_t(\gamma) - S_t^*(\gamma)|, |R_t(\gamma) - R_t^*(\gamma)|\} \\ \leq \max\{|I_0 - I_0^*|, |S_0 - S_0^*|, |R_0 - R_0^*|\} \cdot \exp\left((1 + K(\gamma)) \int_0^t L_s ds\right) \\ \leq \max\{|I_0 - I_0^*|, |S_0 - S_0^*|, |R_0 - R_0^*|\} \cdot \exp\left((1 + K(\gamma)) \int_0^{+\infty} L_s ds\right)$$

for any $t \geq 0$. So we get

$$\sup_{t \geq 0} \max\{|I_t - I_t^*|, |S_t - S_t^*|, |R_t - R_t^*|\} \\ \leq \max\{|I_0 - I_0^*|, |S_0 - S_0^*|, |R_0 - R_0^*|\} \cdot \exp\left((1 + K(\gamma)) \int_0^{+\infty} L_s ds\right)$$

almost surely, where K is a nonnegative uncertain variable such that

$$\lim_{x \rightarrow \infty} \mathcal{M}\{\gamma \in \Gamma | K(\gamma) \leq x\} = 1$$

by Theorem 4. For any given $\epsilon > 0$, there exists a real number H such that $\mathcal{M}\{\gamma | K(\gamma) \leq H\} \geq 1 - \epsilon$. Let

$$\delta = \exp\left(-(1 + H) \int_0^{+\infty} L_s ds\right) \epsilon.$$

Then $\max\{|I_t(\gamma) - I_t^*(\gamma)|, |S_t(\gamma) - S_t^*(\gamma)|, |R_t(\gamma) - R_t^*(\gamma)|\} \leq \epsilon$ for any time t , provided that $\max\{|I_0 - I_0^*|, |S_0 - S_0^*|, |R_0 - R_0^*|\} \leq \delta$ and $K(\gamma) \leq H$. This means

$$\mathcal{M}\left\{\sup_{t \geq 0} \max\{|I_t - I_t^*|, |S_t - S_t^*|, |R_t - R_t^*|\} \leq \epsilon\right\} > 1 - \epsilon,$$

as long as $\max\{|I_0 - I_0^*|, |S_0 - S_0^*|, |R_0 - R_0^*|\} \leq \delta$. In other words,

$$\lim_{\max\{|I_0 - I_0^*|, |S_0 - S_0^*|, |R_0 - R_0^*|\} \rightarrow 0} \mathcal{M}\left\{\sup_{t \geq 0} \max\{|I_t - I_t^*|, |S_t - S_t^*|, |R_t - R_t^*|\} \leq \epsilon\right\} = 1$$

and the uncertain differential equation (4) is stable in measure. The theorem is proved. \square

6 Numerical algorithms

In this section, we use a numerical algorithm based on Euler method to solve the uncertain SIR rumor spreading model. For the uncertain SIR rumor spreading model (4), the most important thing is to obtain spectra of α -paths of I_t , S_t , and R_t .

Algorithm

- *Step 1.* Fix α on $(0, 1)$. Given the initial values I_0, S_0, R_0 and all parameters N, β, λ, η , and σ .
- *Step 2.* Solve the ordinary differential equation corresponding to the uncertain differential equation

$$dI_t^\alpha = -I_t^\alpha S_t^\alpha dt,$$

$$dR_t^\alpha = (\lambda S_t^\alpha R_t^\alpha + \eta S_t^\alpha + (1 - \beta)I_t^\alpha S_t^\alpha + |\sigma S_t^\alpha R_t^\alpha| \Phi^{-1}(\alpha)) dt,$$

then obtain the α -paths I_t^α and R_t^α . On the basis of $S_t^\alpha = N - I_t^{1-\alpha} - R_t^{1-\alpha}$, by using the recursion formula, we solve it as follows:

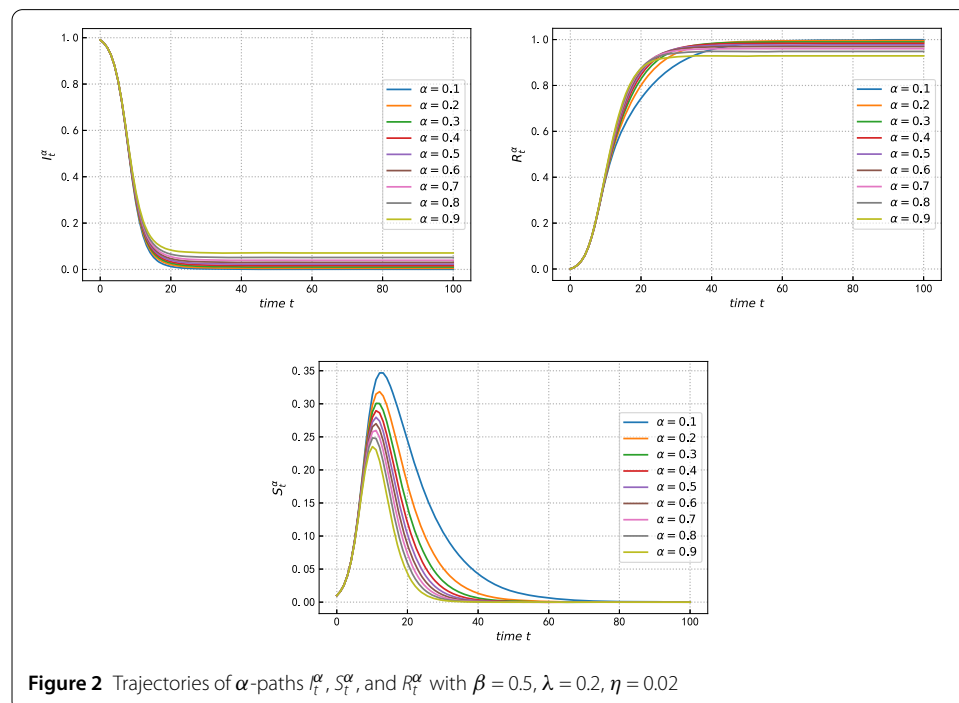
$$\begin{cases} I_{i+1}^\alpha = I_i^\alpha - (I_i^\alpha S_i^\alpha)h, \\ R_{i+1}^\alpha = R_i^\alpha + (\lambda S_i^\alpha R_i^\alpha + \eta S_i^\alpha + (1 - \beta)I_i^\alpha S_i^\alpha + |\sigma S_i^\alpha R_i^\alpha| \Phi^{-1}(\alpha))h, \\ S_{i+1}^\alpha = N - I_{i+1}^{1-\alpha} - R_{i+1}^{1-\alpha}, \end{cases}$$

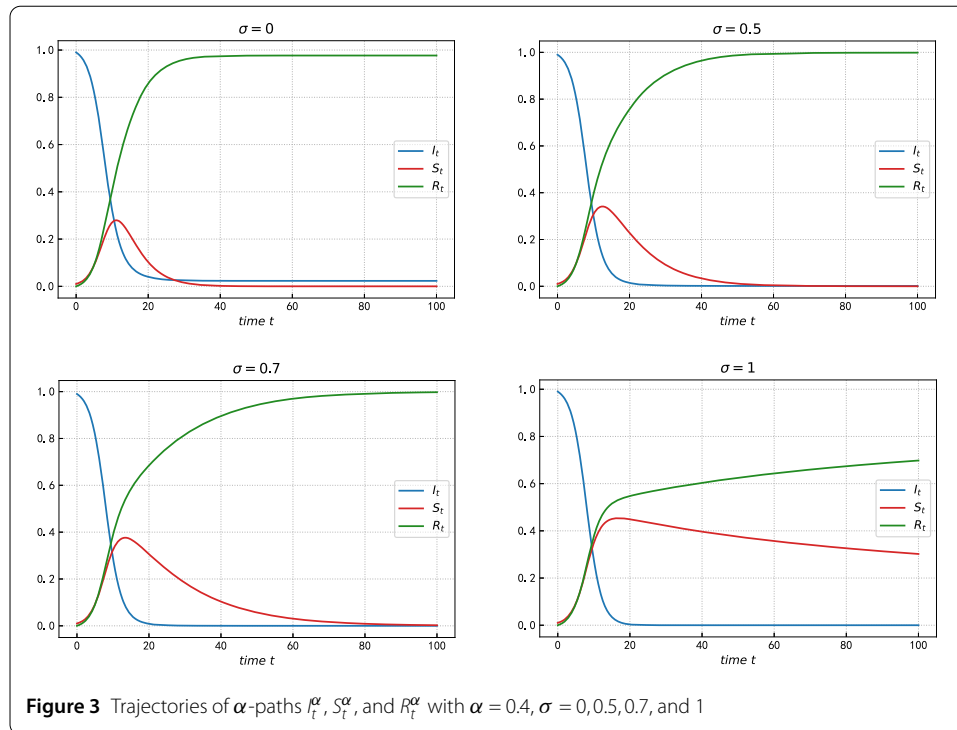
where h is the step length and $\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}$ is the inverse standard normal uncertainty distribution.

- *Step 3.* The α -paths I_t^α, S_t^α , and R_t^α are obtained.

Example 1 In the uncertain SIR rumor spreading model (4), take $N = 1, I_0 = 0.99, S_0 = 0.01, R_0 = 0$, and $\beta = 0.5, \lambda = 0.2, \eta = 0.02, \sigma = 0.1$. By the algorithm, for every α , we can obtain the corresponding α -paths S_t^α and R_t^α . Figure 2 shows the trajectories of I_t^α, S_t^α , and R_t^α when $\alpha = 0.1, 0.2, \dots, 0.9$.

From Fig. 2 we can see that no matter how α changes, the image will eventually stabilize. The number of spreaders goes up to its peak in a very short time and then down, eventually





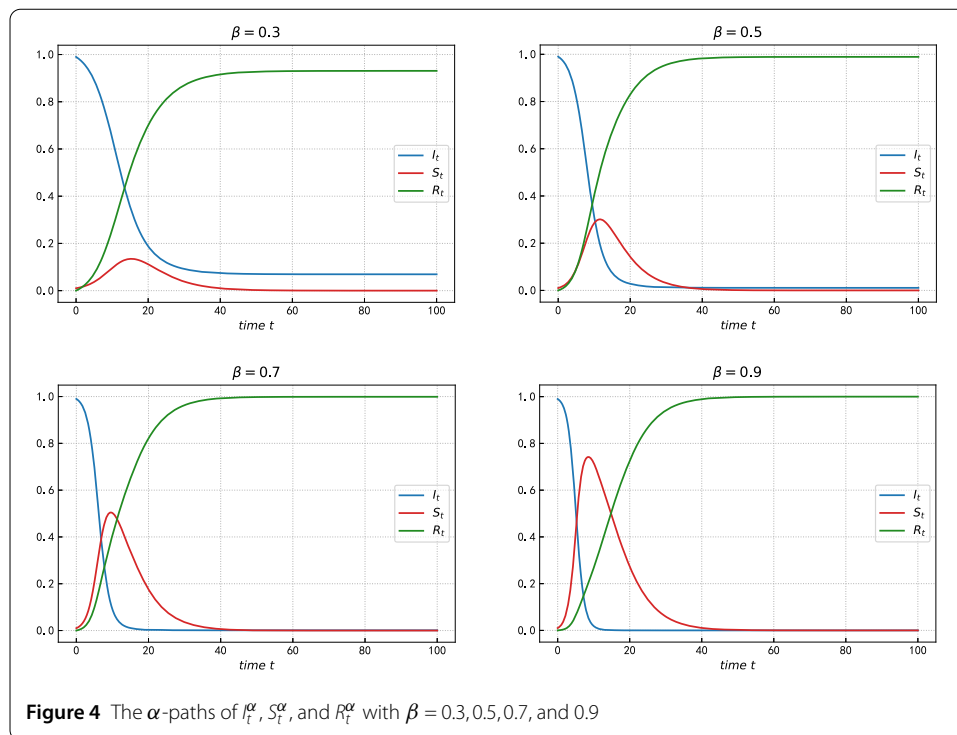
declining to 0, which suggests that the rumors will fade away. In addition, the uncertain SIR rumor spreading model (4) is equal to the deterministic rumor spreading model (2) when $\alpha = 0.5$. The range of the trajectories of I_t^α , S_t^α and R_t^α in the figure is between the 0.1-path and the 0.9-path of the uncertain differential equation, and the higher the value of α , the earlier the rumor dissipates.

Example 2 In this example, we study the effect of different σ on the uncertain SIR rumor spreading model (4), fixing $\alpha = 0.4$. The other parameters are the same as in Example 1. Figure 3 shows the trajectories of I_t^α , S_t^α , and R_t^α when $\sigma = 0, 0.5, 0.7$, and 1.

Figure 3 shows the paths of the I_t , S_t , and R_t with different σ when γ , λ , η , and α are fixed to 0.5, 0.2, 0.02, and 0.4. It can be clearly seen that when other parameters are fixed, the greater the value of σ , the larger the number of spreaders, and the longer the time for rumor to dissipate.

When $\sigma = 0$, the uncertain differential equation (4) is an ordinary differential equation (2). The peak of rumor spreading will be higher as the value of σ increases, which suggests that the bigger the value of σ , the more factors that affect λ , the fewer people change from spreaders to stiflers, so it takes longer for rumors to disappear.

Example 3 In this example, when other parameters are fixed, the influence of different values of β on rumor spreading is discussed. In UDE model (4), take $N = 1$, $I_0 = 0.99$, $S_0 = 0.01$, $R_0 = 0$, and $\lambda = 0.2$, $\eta = 0.02$, $\alpha = 0.4$, $\sigma = 0.2$. Base on the algorithm, Fig. 4 gives the α -paths of I_t^α , S_t^α , and R_t^α when $\beta = 0.3, 0.5, 0.7$, and 0.9. According to the image shown in Fig. 4, the change of β value has a great impact on the rumor spreading process. The smaller the value of β , the fewer the spreaders, the faster the rumor tends to stabilize. At the same time, when there is no one to spread rumors, a minority of the population will never know about the rumors when β is small. The result is in agreement with the actual



situation, the smaller the β , the less contact there is between ignorants and spreaders, thus the fewer people who spread rumors. In addition, for higher values of β , the proportion of spreaders will increase rapidly, the number of ignorants will decline quickly, and the spread of rumors will be wider. Therefore, the government can control the spread of rumors by reducing the rate β .

7 Paradox of stochastic SIR rumor spreading model

Let us consider a stochastic SIR rumor spreading model

$$\begin{cases} \frac{dI_t}{dt} = -I_t S_t, \\ \frac{dS_t}{dt} = \tilde{\beta} I_t S_t - \tilde{\lambda} S_t R_t - \tilde{\eta} S_t - \tilde{\sigma} S_t R_t \frac{dW_t}{dt}, \\ \frac{dR_t}{dt} = \tilde{\lambda} S_t R_t + \tilde{\eta} S_t + (1 - \tilde{\beta}) I_t S_t + \tilde{\sigma} S_t R_t \frac{dW_t}{dt}, \end{cases} \quad (7)$$

where I_t is the population size of ignorant individuals, S_t is the population size of spreaders, R_t is the population size of stifier individuals, $\tilde{\beta}$ is the rumor spread rate, $\tilde{\lambda}$ is the probability of not spreading rumors, $\tilde{\eta}$ is the forgetting rate, W_t is a standard Wiener process, $\tilde{\sigma}$ is a constant that represents the intensity of W_t , and time $t \geq 0$.

It has been widely accepted that Wiener process is used to describe the environmental noise in the rumor spreading process, but is this really justified? By deforming the model (7), one can obtain

$$\begin{cases} \frac{dI_t}{dt} = -I_t S_t, \\ \frac{dS_t}{dt} = \tilde{\beta} I_t S_t - \tilde{\eta} S_t - (\tilde{\lambda} + \tilde{\sigma} \frac{dW_t}{dt}) S_t R_t, \\ \frac{dR_t}{dt} = \tilde{\eta} S_t + (1 - \tilde{\beta}) I_t S_t + (\tilde{\lambda} + \tilde{\sigma} \frac{dW_t}{dt}) S_t R_t. \end{cases}$$

Let $\dot{\lambda} = \tilde{\lambda} + \tilde{\sigma} \frac{dW_t}{dt}$, which is called the effective rumor avoidance rate. It can be seen that the stochastic model (7) uses Wiener process to describe the noise influence of $\tilde{\lambda}$ in the process of rumor spreading, next we use $\frac{\Delta W_t}{\Delta t}$ instead of $\frac{dW_t}{dt}$. From $\Delta W_t \sim \mathcal{N}(0, \Delta t)$, we can get that

$$\frac{\Delta W_t}{\Delta t} \sim \mathcal{N}\left(0, \frac{1}{\Delta t}\right)$$

is a normal random variable, with expected value 0 and variance $1/\Delta t$. And from the properties of the normal distribution, we have

$$\dot{\lambda} \sim \mathcal{N}\left(\tilde{\lambda}, \frac{\tilde{\sigma}^2}{\Delta t}\right).$$

When $\Delta t \rightarrow 0$, we have

$$\begin{aligned} & \Pr\{\dot{\lambda} \in [0, 1]\} \\ &= \Pr\{0 \leq \dot{\lambda} \leq 1\} \\ &= \Pr\left\{-\frac{\tilde{\lambda}}{\tilde{\sigma}/\sqrt{\Delta t}} \leq \frac{\dot{\lambda} - \tilde{\lambda}}{\tilde{\sigma}/\sqrt{\Delta t}} \leq \frac{1 - \tilde{\lambda}}{\tilde{\sigma}/\sqrt{\Delta t}}\right\} \\ &= \Phi\left(\frac{1 - \tilde{\lambda}}{\tilde{\sigma}/\sqrt{\Delta t}}\right) - \Phi\left(-\frac{\tilde{\lambda}}{\tilde{\sigma}/\sqrt{\Delta t}}\right) \\ &\rightarrow 0, \end{aligned}$$

where $\tilde{\lambda} \in [0, 1]$, $\tilde{\sigma}$ is a constant, and $\Phi(\cdot)$ is the standard normal distribution function. It means that

$$\Pr\{\dot{\lambda} \in [0, 1]\} \approx 0, \quad \Pr\{\dot{\lambda} \notin [0, 1]\} \approx 1.$$

That is to say, the probability that the effective rumor avoidance rate $\dot{\lambda}$ is less than 0 or greater than 1 is almost 1. However, from the point of view of rumor spreading, $\dot{\lambda}$ is the effective avoidance rate of rumor spreader turning into rumor stifler. In the actual rumor spreading process, it should be bounded on $[0, 1]$. That is,

$$\dot{\lambda} = \frac{\text{Number of spreaders converted to rumor stiflers per unit time}}{\text{The total number of people who spread rumors}}.$$

Thus, it is incorrect to use Brownian motion to describe the uncertain disturbance in rumor propagation. Therefore, it is not reasonable to use stochastic SIR rumor spreading model (7) to describe the process of rumor spreading.

8 Conclusion

This paper mainly proposed an uncertain SIR rumor spreading model by uncertain differential equation and Liu process. It first deduced the process of establishing uncertain SIR rumor spreading model. Then the existence and uniqueness of solutions of uncertain differential equation SIR rumor spreading model were proved, and the model was proved to

be stable in measure when it satisfied certain conditions. Furthermore, we transformed the uncertain differential equation into an ordinary differential equation by using Yao–Chen formula. And through numerical simulation, several examples were given to compare the influence of different parameters on the uncertain SIR rumor spreading model. The results showed that in daily life, we can predict the time when rumor spreading reaches its peak and rumor dissipates through uncertain SIR rumor spreading model. Finally, a paradox of stochastic SIR rumor spreading model was introduced.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

HS and YHS conceived and designed the study. HS established the uncertain model, theoretical analysis, and wrote the original draft. YHS provided modeling ideas and analysis methods. QC provided the code for the numerical simulation. All authors read and approved the final manuscript.

Author details

¹College of Mathematical and System Sciences, Xinjiang University, Urumqi, Xinjiang, 830046, China. ²College of Information Science and Engineering, Xinjiang University, Urumqi, Xinjiang, 830046, China.

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