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Anti-periodic behavior for quaternion-valued delayed cellular neural networks

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Abstract

In this manuscript, quaternion-valued delayed cellular neural networks are studied. Applying the continuation theorem of coincidence degree theory, inequality techniques and a Lyapunov function approach, a new sufficient condition that guarantees the existence and exponential stability of anti-periodic solutions for quaternion-valued delayed cellular neural networks is presented. The obtained results supplement some earlier publications that deal with the anti-periodic solutions of quaternion-valued neural networks with distributed delay or impulse or state-dependent delay or inertial term. Computer simulations are displayed to check the derived analytical results.

Keywords: Quaternion-valued delayed cellular neural networks; Anti-periodic solution; Exponential stability; Time delay

1 Introduction

It is well known that cellular neural networks have widely been applied in many areas such as optimization, associative memories, image processing, psychophysics, and adaptive pattern recognition [1–3]. Time delay is unavoidable in neural networks and it often makes the networks lose their stability and even destroy the periodic behavior of networks [4–6]. Therefore it is necessary for us to investigate the dynamics of cellular neural networks with delays. In recent years, many excellent works on stability, periodic solution, almost periodic solution, anti-periodic solution, pseudo almost periodic solution and synchronization of cellular neural networks with delays have been reported. For example, Wang et al. [7] investigated the global stability of periodic solution of cellular neural networks; Li and Wang [8] studied the almost periodic solutions of delayed cellular neural networks; Aouiti et al. [9] analyzed the exponential stability of piecewise pseudo almost periodic solution for neutral-type neural networks; Li and Xiang [10] dealt with the anti-periodic solution of Cohen–Grossberg neural networks; Wang [11] handled the finite-time synchronization of fuzzy delayed cellular neural networks. For more related studies, one can see [12–19].

Complex-valued neural networks (CVNNs), which can be regarded as an extension of real-valued neural networks (RVNNs), play an important role in characterizing the signal and information of neural networks. In particular, CVNNs have potential application in various waves such as light wave, sonic wave, electron wave and so on. In addition, quaternion-valued neural networks (QVNNs), which were proposed by Hamilton [20], are extension form of RVNNs and CVNNs. The skew of a quaternion is denoted by $\mathcal{Q} := \{h = h_0 + ih_1 + jh_2 + kh_3\}$, where $h_0, h_1, h_2, h_3 \in R$ and i, j, k satisfy the following rules:

$$ij = -ji = k, \qquad jk = -kj = i, \qquad ki = -ik = j, \qquad i^2 = j^2 = k^2 = ijk = -1.$$

$\forall h \in \mathcal{Q}$, we denote the conjugate of h as follows:

$$h^* = h = h_0 - ih_1^* - jh_2^* - kh_3^*.$$

The norm of h is given by

$$\|h\| = \sqrt{hh^*} = \sqrt{(h_0)^2 + \left(h_1^*\right)^2 + \left(h_2^*\right)^2 + \left(h_3^*\right)^2}.$$

QVNNs have been widely applied in spatial rotation, color night vision, image impression of three dimension geometrical affine transformation, etc. [21–23]. Recently, some scholars dealt with the dynamical behavior of QVNNs. For example, Tu et al. [23] studied the stability issue of QVNNs with discrete and distributed delays, Qi et al. [24] discussed the exponential input-to-state stability of QVNNs. Liu and Jian [25] analyzed the global dissipativity of QVNNs with delays. For more related publications, one can see [22, 26–28].

The signal transmission of neural networks usually displays anti-periodic phenomenon. Some researchers argued that anti-periodic solutions can effectively depict the dynamical behavior of nonlinear differential equations [29–31]. In particular, the anti-periodic solution of neural networks plays an important role in designing and controlling neural networks. Also, the research results on anti-periodic solution of neural networks can be applied to automatic control, artificial intelligence, disease diagnosis and many engineering technologies. Therefore it is important for us to discuss the anti-periodic phenomenon of neural networks. At present, some research results on anti-periodic solution of neural networks have been available. We refer the reader to [32–39].

Nowadays the anti-periodic solution of QVCNNs can be widely applied in robotics, attitude control of satellites, artificial intelligence, ensemble control, image processing, disease diagnosis in medicine and so on [21–26]. Thus the research on anti-periodic solution of QVCNNs has become a topic of focus in today's society. Here we would like to point out that the report on anti-periodic solution of QVNNs is very rare [10, 40–42]. In order to make up for this deficiency, in the present manuscript, we will consider the anti-periodic solution of a class of quaternion-valued cellular neural networks (QVCNNs).

In 2018, Li and Qin [43] studied the following QVCNNs:

$$\dot{u}_a(t) = -\gamma_a u_a(t) + \sum_{b=1}^{m} \alpha_{ab}(t) g_b\big(u_b(t)\big) + \sum_{b=1}^{m} \beta_{ab}(t) g_b\big(u_b\big(t - \varrho_{ab}(t)\big)\big) + R_a(t), \tag{1.1}$$

where $a = 1, 2, \dots, m$, $u_a(t) \in \mathcal{Q}$ represents the state of the ath unit, γ_a denotes the rate with which the ath unit will reset its potential to the resting state when disconnected from the network and external inputs, $\alpha_{ab}(t) \in \mathcal{Q}$ stands for the strength of the bth unit on the ath unit, $\varrho_{ab}(t)$ denotes the transmission delay along the axon of the bth unit on the ath unit. $\beta_{ab}(t) \in \mathcal{Q}$ stands for the strength of the bth unit on the ath unit at time $t - \varrho_{ab}(t)$, $R_a(t) \in \mathcal{Q}$ stands for the external input on the ath unit, $g : \mathcal{Q} \to \mathcal{Q}$ is an activation function of signal transmission. In details, one can see [43]. With the aid of the continuation theorem of coincidence degree theory, inequality techniques and Lyapunov function, Li and Qin [43] established the sufficient conditions to ensure the existence of periodic solutions and the global exponential stability of periodic solutions for model (1.1). Their work can be thought of as an important complement to the earlier publications. Notice that the rate γ_a will change with the environment, thus the model (1.1) can be modified as follows:

$$\dot{u}_a(t) = -\gamma_a(t)u_a(t) + \sum_{b=1}^{m} \alpha_{ab}(t)g_b\bigl(u_b(t)\bigr) + \sum_{b=1}^{m} \beta_{ab}(t)g_b\bigl(u_b\bigl(t - \varrho_{ab}(t)\bigr)\bigr) + R_a(t). \tag{1.2}$$

The key object of this manuscript is to focus on the existence of anti-periodic solutions and the global exponential stability of anti-periodic solutions for model (1.2). Up to now, few researchers have discussed the anti-periodic solutions of QVCNNs.

In order to obtain the key results of this manuscript, we make some preparations. Firstly, we give the following assumptions for model (1.2):

($\mathcal{A}1$) For $a, b = 1, 2, \ldots, m$, $\gamma_a \in C(R, R^+), \varrho_{ab} \in BC(R, R), g_b \in C(\mathcal{Q}, \mathcal{Q}), \alpha_{ab}, \beta_{ab}, R_a \in C(R, \mathcal{Q})$ and $\exists$ a positive constant $\varpi > 0$ such that $\forall t \in R, u \in \mathcal{Q}$,

$$\begin{aligned} &\gamma_a\left(t + \frac{\varpi}{2}\right) = \gamma_a(t), \qquad \alpha_{ab}\left(t + \frac{\varpi}{2}\right)g_b(u) = -\alpha_{ab}(t)g_b(-u), \\ &\beta_{ab}\left(t + \frac{\varpi}{2}\right)g_b(u) = -\beta_{ab}(t)g_b(-u), \qquad R_a\left(t + \frac{\varpi}{2}\right) = -R_a(t). \end{aligned}$$

($\mathcal{A}2$) For $b = 1, 2, \ldots, m$, $\exists$ a positive constant L_b such that $\forall x, y \in \mathcal{Q}$, $\|g_b(x) - g_b(y)\| \leq L_b\|x - y\|$.

($\mathcal{A}3$) For $b = 1, 2, \ldots, m, x \in \mathcal{Q}$, $\exists$ positive constants G_a such that $\|g_a(x)\| \leq G_a$.

Let

$$\begin{aligned} &\gamma_a^- = \inf_{t \in [0,\varpi]} \bigl\{\gamma_a(t)\bigr\}, \qquad \alpha_{ab}^+ = \sup_{t \in [0,\varpi]} \bigl\{\bigl\|\alpha_{ab}(t)\bigr\|\bigr\}, \qquad \beta_{ab}^+ = \sup_{t \in [0,\varpi]} \bigl\{\bigl\|\beta_{ab}(t)\bigr\|\bigr\}, \\ &\varrho^+ = \max_{1 \leq a,b \leq m} \Bigl\{\sup_{t \in [0,\varpi]} \varrho_{ab}(t)\Bigr\}, \qquad R_a^+ = \sup_{t \in [0,\varpi]} \bigl\|R_a(t)\bigr\|. \end{aligned}$$

We give the initial conditions of (1.2) as follows:

$$u_a(s) = \vartheta_a(s), \quad s \in \bigl[-\varrho^+, 0\bigr], \tag{1.3}$$

where $\vartheta_a \in C([-\varrho^+, 0], R)$.

The rest of the manuscript is planned as follows. In Sect. 2 we present some preliminary results. In Sect. 3, applying coincidence degree theory, we investigate the existence of anti-periodic solutions of (1.2). In Sect. 4, constructing Lyapunov functional, we establish a new sufficient condition to ensure the global exponential stability of anti-periodic solutions of model (1.2). In Sect. 5, numerical simulations are implemented. The conclusion is given finally in Sect. 6.

Remark 1.1 In physics, there are many anti-periodic phenomena such as the anti-periodic wave, anti-periodic vibration, and anti-periodic wavelet. For neural networks, the signal transmission of the neurons displays anti-periodic behavior.

2 Preliminaries

In this section, we give a definition and three lemmas that are needed in proving the key results of this manuscript.

Definition 2.1 ([40]) Assume that $u = (u_1, u_2, \dots, u_m)^T$ and $\bar{u} = (\bar{u}_1, \bar{u}_2, \dots, \bar{u}_m)^T$ are two arbitrary solutions of model (1.2) with the initial values $\phi = (\phi_1, \phi_2, \dots, \phi_m)^T$ and $\bar{\phi} = (\bar{\phi}_1, \bar{\phi}_2, \dots, \bar{\phi}_m)^T$, respectively. If $\exists$ two positive constants ϵ and $\mathcal{M}$ which satisfy

$$\left\| u(t) - \bar{u}(t) \right\|_{\mathcal{Q}} \leq \mathcal{M} \|\phi - \bar{\phi}\|_0 e^{-\epsilon t}, \quad t > 0,$$

where

$$\left\| u(t) - \bar{u}(t) \right\|_{\mathcal{Q}} = \left[\sum_{a=1}^{m} \left\| u_a(t) - \bar{u}_a(t) \right\|^2 \right]^{\frac{1}{2}}$$

and

$$\|\phi - \bar{\phi}\|_0 = \left[\sum_{a=1}^{m} \left\| \sup_{s \in [-\varrho^+, 0]} \vartheta_a(s) - \bar{\vartheta}_a(s) \right\|^2 \right]^{\frac{1}{2}},$$

then every solution of model (1.2) is said to be globally exponentially stable.

Lemma 2.1 ([44]) *Assume that $v \in C^1$ and $v(0) = v(\varpi)$, then*

$$\|v - \bar{v}\|_{L_2} \leq \frac{\varpi}{2\pi} \|\dot{v}\|_{L_2},$$

where

$$\|v\|_{L_2} = \left(\int_0^{\varpi} \left| v(t) \right|^2 dt \right)^{\frac{1}{2}}, \qquad \bar{v} = \frac{1}{\varpi} \int_0^{\varpi} v(t)\, dt.$$

Lemma 2.2 ([40]) *$\forall p, q \in \mathcal{Q}$, one has $p^* q + q^* p \leq p^* p + q^* q$.*

Lemma 2.3 ([44]) *Assume that $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, $\mathcal{L} : \mathrm{Dom}\, \mathcal{L} \subset \mathcal{X} \to \mathcal{Y}$ is linear and $\mathcal{N} : \mathcal{X} \to \mathcal{Y}$ is continuous. If $\mathcal{L}$ is one-to-one and $\mathcal{K} := \mathcal{L}^{-1}\mathcal{N}$ is compact. Furthermore, suppose that $\exists$ a bounded and open subset $\Omega \subset \mathcal{X}$ with $0 \in \Omega$ such that $\mathcal{L}v = \lambda \mathcal{N} v$ has no solutions in $\partial\Omega \cap \mathrm{Dom}\, \mathcal{L}$, $\forall \lambda \in (0,1)$. Then the equation $\mathcal{L}v = \lambda \mathcal{N} v$ has at least one solution in $\bar{\Omega}$.*

3 Existence of anti-periodic solutions

In view of $(\mathcal{A}2)$, one knows that the solution of system (1.2) with the initial condition (1.3) exists.

Theorem 3.1 *Assume that* $(\mathcal{A}1)$–$(\mathcal{A}3)$ *hold and assume that* $(\mathcal{A}4)$ $2\pi > \gamma_a^+\varpi$ *is satisfied, then model* (1.2) *has at least one* $\frac{\varpi}{2}$*-anti-periodic solution that remains in*

$$\mathcal{X}_0 = \left\{u \in \mathcal{X} \,|\, \|u\|_{\mathcal{X}} \le \sum_{a=1}^{m} \frac{\varpi[\sum_{b=1}^{m}((\alpha_{ab}^+ + \beta_{ab}^+)G_b) + R_a^+]}{1 - \frac{\gamma_a^+\varpi}{2\pi}} + 1\right\},$$

where $a = 1, 2, \ldots, m$.

Proof Let

$$\mathcal{X} = \left\{u \,|\, u = (u_1, u_2, \ldots, u_m)^T \in C(R, \mathcal{Q}), u\left(t + \frac{\varpi}{2}\right) = -u(t), \forall t \in R\right\},$$

$$\|u\|_{\mathcal{X}} = \sum_{a=1}^{m} |u_a|_0,$$

where

$$|u_a|_0 = \sup_{t\in[0,\varpi]} \sqrt{u_a(t)u_a^*(t)}, \quad a = 1, 2, \ldots, m.$$

Obviously, $\mathcal{X}$ is a Banach space under the norm $\|.\|_{\mathcal{X}}$. Define a linear operator $\mathcal{L} : \operatorname{Dom}\mathcal{L} \subset \mathcal{X} \to \mathcal{X}$ by $\mathcal{L}u = \dot{u}$, where $\operatorname{Dom}\mathcal{L} = \{u | u \in \mathcal{X}, \dot{u} \in \mathcal{X}\}$ and a continuous operator $\mathcal{N} : \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{N}(u)(t) = \bigl(\Psi_1(u,t), \Psi_2(u,t), \ldots, \Psi_m(u,t)\bigr)^T, \tag{3.1}$$

where

$$\Psi_j(u,t) = -\gamma_j(t)u_j(t) + \sum_{b=1}^{m} \alpha_{jb}(t)g_b\bigl(u_b(t)\bigr) + \sum_{b=1}^{m} \beta_{jb}(t)g_b\bigl(u_b\bigl(t - \varrho_{jb}(t)\bigr)\bigr) + R_j(t), \tag{3.2}$$

where $j = 1, 2, \ldots, m$. We can easily obtain

$$\operatorname{Ker}\mathcal{L} = \{0\}, \qquad \operatorname{Im}\mathcal{L} = \left\{v \in \mathcal{X}, \int_0^{2\varpi} v(t)\,dt = 0\right\} = \mathcal{X}.$$

Then $\mathcal{L} : \operatorname{Dom}\mathcal{L} \to \mathcal{X}$ is one-to-one. Let $\mathcal{K} = \mathcal{L}^{-1}\mathcal{N}$. Then $\mathcal{K}$ is compact.

Assume that $u \in \mathcal{X}$ is an arbitrary solution of the equation $\mathcal{L}u = \lambda\mathcal{N}u$, where $\lambda \in (0,1)$, then one has

$$\dot{u}_a(t) = \lambda\left[-\gamma_a(t)u_a(t) + \sum_{b=1}^{m} \alpha_{ab}(t)g_b\bigl(u_b(t)\bigr) + \sum_{b=1}^{m} \beta_{ab}(t)g_b\bigl(u_b\bigl(t - \varrho_{ab}(t)\bigr)\bigr) + R_a(t)\right], \tag{3.3}$$

where $a = 1, 2, \ldots, m$. It follows from (3.3) that, for $a = 1, 2, \ldots, m$,

$$\dot{u}_a(t)u_a^*(t) = \lambda u_a^*(t)\Biggl[-\gamma_a(t)u_a(t) + \sum_{b=1}^{m} \alpha_{ab}(t)g_b\bigl(u_b(t)\bigr)$$

$$+\sum_{b=1}^{m}\beta_{ab}(t)g_b\big(u_b\big(t-\varrho_{ab}(t)\big)\big)+R_a(t)\Bigg], \tag{3.4}$$

which leads to

$$\begin{aligned}\int_0^{\varpi}\|\dot{u}_a(t)\|^2\,dt &\le \gamma_a^+\int_0^{\varpi}\|u_a(t)\dot{u}_a(t)\|\,dt+\Bigg[\sum_{b=1}^{m}\big(\big(\alpha_{ab}^++\beta_{ab}^+\big)G_b\big)+R_a^+\Bigg]\int_0^{\varpi}\|\dot{u}_a(t)\|\,dt\\ &\le \gamma_a^+\bigg(\int_0^{\varpi}\|u_a(t)\|^2\,dt\bigg)^{\frac{1}{2}}\bigg(\int_0^{\varpi}\|\dot{u}_a(t)\|^2\,dt\bigg)^{\frac{1}{2}}\\ &\quad+\sqrt{\varpi}\Bigg[\sum_{b=1}^{m}\big(\big(\alpha_{ab}^++\beta_{ab}^+\big)G_b\big)+R_a^+\Bigg]\bigg(\int_0^{\varpi}\|\dot{u}_a(t)\|^2\,dt\bigg)^{\frac{1}{2}},\end{aligned} \tag{3.5}$$

where $a=1,2,\dots,m$. Hence

$$\bigg(\int_0^{\varpi}\|\dot{u}_a(t)\|^2\,dt\bigg)^{\frac{1}{2}}\le\gamma_a^+\bigg(\int_0^{\varpi}\|u_a(t)\|^2\,dt\bigg)^{\frac{1}{2}}+\sqrt{\varpi}\Bigg[\sum_{b=1}^{m}\big(\big(\alpha_{ab}^++\beta_{ab}^+\big)G_b\big)+R_a^+\Bigg]. \tag{3.6}$$

Notice that $u_a(t)\in C^1$ is $\frac{\varpi}{2}$-anti-periodic, in view of Lemma 2.1, one has

$$\bigg(\int_0^{\varpi}\|u_a(t)\|^2\,dt\bigg)^{\frac{1}{2}}\le\frac{\varpi}{2\pi}\bigg(\int_0^{\varpi}\|\dot{u}_a(t)\|^2\,dt\bigg)^{\frac{1}{2}}. \tag{3.7}$$

In view of (3.6) and (3.7), one gets

$$\begin{aligned}&\bigg(\int_0^{\varpi}\|\dot{u}_a(t)\|^2\,dt\bigg)^{\frac{1}{2}}\\ &\quad\le\frac{\gamma_a^+\varpi}{2\pi}\bigg(\int_0^{\varpi}\|\dot{u}_a(t)\|^2\,dt\bigg)^{\frac{1}{2}}+\sqrt{\varpi}\Bigg[\sum_{b=1}^{m}\big(\big(\alpha_{ab}^++\beta_{ab}^+\big)G_b\big)+R_a^+\Bigg].\end{aligned} \tag{3.8}$$

Then

$$\bigg(\int_0^{\varpi}\|\dot{u}_a(t)\|^2\,dt\bigg)^{\frac{1}{2}}\le\frac{\sqrt{\varpi}[\sum_{b=1}^{m}((\alpha_{ab}^++\beta_{ab}^+)G_b)+R_a^+]}{1-\frac{\gamma_a^+\varpi}{2\pi}}. \tag{3.9}$$

We assume that

$$u_a(t)=u_a^R(t)+iu_a^I(t)+ju_a^J(t)+ku_a^K(t)$$

and

$$\begin{aligned}&u_a^R\bigg(t+\frac{\varpi}{2}\bigg)=u_a^R(t), &&u_a^I\bigg(t+\frac{\varpi}{2}\bigg)=u_a^I(t),\\ &u_a^J\bigg(t+\frac{\varpi}{2}\bigg)=u_a^J(t), &&u_a^K\bigg(t+\frac{\varpi}{2}\bigg)=u_a^K(t),\end{aligned}$$

where $u_a^R, u_a^I, u_a^J, u_a^K \in C(R,R), a = 1,2,\dots,m$. Notice that $u_a^R, u_a^I, u_a^J, u_a^K$ are $\frac{\varpi}{2}$-anti-periodic real-valued functions, then $\exists\ \vartheta_a^R, \vartheta_a^I, \vartheta_a^J, \vartheta_a^K \in [0,\varpi]$ such that

$$u_a^R(\vartheta_a^R) = u_a^I(\vartheta_a^I) = u_a^J(\vartheta_a^J) = u_a^K(\vartheta_a^K) = 0, \quad a = 1,2,\dots,m.$$

Then one has

$$\begin{cases} |u_a^R|_\infty = \sup_{t\in[0,\varpi]} |u_a^R(t)| = \sup_{t\in[0,\varpi]} |u_a^R(\vartheta_a^R) + \int_{\vartheta_a^R}^t \dot{u}_a^R(s)\,ds| \\ \qquad \le \sqrt{\varpi}(\int_0^\varpi \|\dot{u}_a^R(t)\|^2\,dt)^{\frac{1}{2}}, \\ |u_a^I|_\infty = \sup_{t\in[0,\varpi]} |u_a^I(t)| = \sup_{t\in[0,\varpi]} |u_a^I(\vartheta_a^I) + \int_{\vartheta_a^I}^t \dot{u}_a^I(s)\,ds| \\ \qquad \le \sqrt{\varpi}(\int_0^\varpi \|\dot{u}_a^I(t)\|^2\,dt)^{\frac{1}{2}}, \\ |u_a^J|_\infty = \sup_{t\in[0,\varpi]} |u_a^J(t)| = \sup_{t\in[0,\varpi]} |u_a^J(\vartheta_a^J) + \int_{\vartheta_a^J}^t \dot{u}_a^J(s)\,ds| \\ \qquad \le \sqrt{\varpi}(\int_0^\varpi \|\dot{u}_a^J(t)\|^2\,dt)^{\frac{1}{2}}, \\ |u_a^K|_\infty = \sup_{t\in[0,\varpi]} |u_a^K(t)| = \sup_{t\in[0,\varpi]} |u_a^K(\vartheta_a^K) + \int_{\vartheta_a^K}^t \dot{u}_a^K(s)\,ds| \\ \qquad \le \sqrt{\varpi}(\int_0^\varpi \|\dot{u}_a^K(t)\|^2\,dt)^{\frac{1}{2}}, \end{cases} \tag{3.10}$$

where $a = 1,2,\dots,m$. Then

$$\begin{cases} |u_a^R|_\infty^2 \le \varpi \int_0^\varpi \|\dot{u}_a^R(t)\|^2\,dt, \\ |u_a^I|_\infty^2 \le \varpi \int_0^\varpi \|\dot{u}_a^I(t)\|^2\,dt, \\ |u_a^J|_\infty^2 \le \varpi \int_0^\varpi \|\dot{u}_a^J(t)\|^2\,dt, \\ |u_a^K|_\infty^2 \le \varpi \int_0^\varpi \|\dot{u}_a^K(t)\|^2\,dt, \end{cases} \tag{3.11}$$

where $a = 1,2,\dots,m$. According to (3.9) and (3.11), one gets

$$\begin{aligned} \|u\|_{\mathcal{X}} &\le \sum_{a=1}^m \sqrt{\left|u_a^R\right|_\infty^2 + \left|u_a^I\right|_\infty^2 + \left|u_a^J\right|_\infty^2 + \left|u_a^K\right|_\infty^2} \\ &\le \sum_{a=1}^m \sqrt{\varpi}\left(\int_0^\varpi \left\|\dot{u}_a(t)\right\|^2 dt\right)^{\frac{1}{2}} \\ &\le \sum_{a=1}^m \frac{\varpi\,[\sum_{b=1}^m((\alpha_{ab}^+ + \beta_{ab}^+)G_b) + R_a^+]}{1 - \frac{\gamma_a^+\varpi}{2\pi}} := \mathcal{Q}. \end{aligned} \tag{3.12}$$

Let $\Omega = \{u \in \mathcal{X} \,|\, \|u\|_{\mathcal{X}} < \mathcal{Q}+1\}$, then $\Omega \subset \mathcal{X}$ with $0 \in \Omega$ such that $\mathcal{L}u = \lambda\mathcal{N}u$ has no solution in $\partial\Omega \cap \operatorname{Dom}\mathcal{L}$, $\forall\lambda \in (0,1)$. It follows from Lemma 2.3 that model (1.2) has at least $\frac{\varpi}{2}$-anti-periodic solution in $\mathcal{X}_0$. This ends the proof. □

4 Exponential stability of anti-periodic solution

In this section, we discuss the global exponential stability of anti-periodic solution for model (1.2) by applying inequality theory and constructing an appropriate Lyapunov function.

Theorem 4.1 *Let* $(\mathcal{A}1)$–$(\mathcal{A}4)$ *be satisfied. Assume that*

$(\mathcal{A}5)$ $\varrho_{ab} \in C^1(R,R^+)$ *and* $\theta = \max_{1\le a,b\le m}\{\sup_{t\in[0,\varpi]} \dot{\varrho}_{ab}(t)\} < 1$.

($\mathcal{A}6$) $\exists$ *a constant* $\epsilon > 0$ *such that*

$$\delta = \max_{1\le a\le m}\left\{\left(2\epsilon - 2\gamma_a^- + 2\right) + \sum_{b=1}^{m}\left(\alpha_{ba}^+\right)^2 (L_b)^2 + \sum_{b=1}^{m}\left(\beta^+\right)^2 (L_b)^2 \frac{e^{2\epsilon\varrho^+}}{1-\theta}\right\} < 0,$$

then system (1.2) *possesses a unique* $\frac{\varpi}{2}$*-anti-periodic solution, which is globally exponentially stable.*

Proof In view of Theorem 3.1, one knows that model (1.2) possesses a $\frac{\varpi}{2}$-anti-periodic solution $\bar{u}(t) = (\bar{u}_1(t), \bar{u}_2(t), \dots, \bar{u}_m(t))^T$ with the initial value $\bar{\vartheta}(t) = (\bar{\vartheta}_1(t), \bar{\vartheta}_2(t), \dots, \bar{\vartheta}_m(t))^T$. Let $u(t) = (u_1(t), u_2(t), \dots, u_m(t))^T$ be an arbitrary solution of model (1.2) with the initial value $\psi(t) = (\psi_1(t), \psi_2(t), \dots, \psi_m(t))^T$. Let $v_i(t) = u_i(t) - \bar{u}_i(t)$ $(i = 1, 2, \dots, m)$. Then one has

$$\begin{aligned}\dot{v}_a(t) = {}& -\gamma_a(t)v_a(t) + \sum_{b=1}^{m}\left[\alpha_{ab}(t)\left(g_b\left(u_b(t)\right) - g_b\left(\bar{u}_b(t)\right)\right)\right] \\ & + \sum_{b=1}^{m}\left[\beta_{ab}(t)\left(g_b\left(u_b\left(t-\varrho_{ab}(t)\right)\right) - g_b\left(\bar{u}_b\left(t-\varrho_{ab}(t)\right)\right)\right)\right],\end{aligned} \tag{4.1}$$

where $a = 1, 2, \dots, m$. Define the following Lyapunov function:

$$\mathcal{V}(t) = \sum_{a=1}^{m} e^{2\epsilon t} v_a^*(t)v_a(t) + \sum_{a=1}^{m}\sum_{b=1}^{m}\left(\beta^+\right)^2 (L_b)^2 \frac{e^{2\epsilon\varrho^+}}{1-\theta}\int_{t-\varrho_{ab}(t)}^{t} e^{2\epsilon s} v_b^*(t)v_b(t)\,ds. \tag{4.2}$$

According to (4.1) and Lemma 2.2, one gets

$$\begin{aligned}
&\mathcal{D}^+\left|\mathcal{V}(t)\right| \\
&\quad\le \sum_{a=1}^{m} 2\epsilon e^{2\epsilon t} v_a^*(t)v_a(t) + \sum_{a=1}^{m} e^{2\epsilon t}\left|\dot{v}_a^*(t)v_a(t) + v_a^*(t)\dot{v}_a(t)\right| \\
&\qquad + e^{2\epsilon t}\sum_{a=1}^{m}\sum_{b=1}^{m}\left(\beta^+\right)^2 (L_b)^2 \frac{e^{2\epsilon\varrho^+}}{1-\theta} v_b^*(t)v_b(t) + e^{2\epsilon t} \\
&\qquad \times \sum_{a=1}^{m}\sum_{b=1}^{m}\left(\beta^+\right)^2 (L_b)^2 \frac{e^{2\epsilon\varrho^+}}{1-\theta}\left(1 - \dot{\varrho}_{ab}(t)\right)e^{-2\epsilon\varrho_{ab}(t)} v_b^*\left(t-\varrho_{ab}(t)\right)v_b\left(t-\varrho_{ab}(t)\right) \\
&\quad= \sum_{a=1}^{m} 2\epsilon e^{2\epsilon t} v_a^*(t)v_a(t) + \sum_{a=1}^{m} e^{2\epsilon t}\left\{-\gamma_a(t)v_a^*(t) + \sum_{b=1}^{m}\left|\alpha_{ab}(t)\left(g_b\left(u_b(t)\right) - g_b\left(\bar{u}_b(t)\right)\right)\right|^*\right. \\
&\qquad \left. + \sum_{b=1}^{m}\left|\beta_{ab}(t)\left(g_b\left(u_b^*\left(t-\varrho_{ab}(t)\right)\right) - g_b\left(\bar{u}_b^*\left(t-\varrho_{ab}(t)\right)\right)\right)\right|^*\right\} v_a(t) \\
&\qquad + \sum_{a=1}^{m} e^{2\epsilon t}\left\{-\gamma_a(t)v_a(t) + \sum_{b=1}^{m}\left|\alpha_{ab}(t)\left[g_b\left(u_b(t)\right) - g_b\left(\bar{u}_b(t)\right)\right]\right|\right. \\
&\qquad \left. + \sum_{b=1}^{m}\left|\beta_{ab}(t)\left[g_b\left(u_b\left(t-\varrho_{ab}(t)\right)\right) - g_b\left(\bar{u}_b\left(t-\varrho_{ab}(t)\right)\right)\right]\right|\right\} v_a^*(t) \\
&\qquad + e^{2\epsilon t}\sum_{a=1}^{m}\sum_{b=1}^{m}\left(\beta^+\right)^2 (L_b)^2 \frac{e^{2\epsilon\varrho^+}}{1-\theta} v_b^*(t)v_b(t) - e^{2\epsilon t}
\end{aligned}$$

$$\times \sum_{a=1}^{m}\sum_{b=1}^{m}\left(\beta^{+}\right)^{2}(L_b)^{2}\frac{e^{2\epsilon\varrho^{+}}}{1-\theta}\left(1-\dot{\varrho}_{ab}(t)\right)e^{-2\epsilon\varrho_{ab}(t)}v_b^{*}\left(t-\varrho_{ab}(t)\right)v_b\left(t-\varrho_{ab}(t)\right)$$

$$\le \sum_{a=1}^{m}\left(2\epsilon-2\gamma_a^{-}\right)e^{2\epsilon t}v_a^{*}(t)v_a(t)+\sum_{a=1}^{m}e^{2\epsilon t}\left\{\sum_{b=1}^{m}\left|\left[\left(\alpha_{ab}(t)\left(g_b\left(u_b(t)\right)-g_b\left(\bar{u}_b(t)\right)\right)\right)^{*}v_a(t)\right.\right.\right.$$

$$\left.\left.+\left(\alpha_{ab}(t)\left(g_b\left(u_b(t)\right)-g_b\left(\bar{u}_b(t)\right)\right)\right)v_a^{*}(t)\right]\right|$$

$$+\sum_{b=1}^{m}\left|\left(\beta_{ab}(t)\left[\left(g_b\left(u_b^{*}\left(t-\varrho_{ab}(t)\right)\right)-g_b\left(\bar{u}_b^{*}\left(t-\varrho_{ab}(t)\right)\right)\right)\right)^{*}v_a(t)\right.$$

$$\left.\left.+v_a^{*}(t)\left(g_b\left(u_b^{*}\left(t-\varrho_{ab}(t)\right)\right)-g_b\left(\bar{u}_b^{*}\left(t-\varrho_{ab}(t)\right)\right)\right)\right)\right]\Big|\Bigg\}$$

$$+e^{2\epsilon t}\sum_{a=1}^{m}\sum_{b=1}^{m}\left(\beta^{+}\right)^{2}(L_b)^{2}\frac{e^{2\epsilon\varrho^{+}}}{1-\theta}v_b^{*}(t)v_b(t)-e^{2\epsilon t}$$

$$\times \sum_{a=1}^{m}\sum_{b=1}^{m}\left(\beta^{+}\right)^{2}(L_b)^{2}\frac{e^{2\epsilon\varrho^{+}}}{1-\theta}\left(1-\dot{\varrho}_{ab}(t)\right)e^{-2\epsilon\varrho_{ab}(t)}v_b^{*}\left(t-\varrho_{ab}(t)\right)v_b\left(t-\varrho_{ab}(t)\right)$$

$$\le \sum_{a=1}^{m}\left(2\epsilon-2\gamma_a^{-}\right)e^{2\epsilon t}v_a^{*}(t)v_a(t)+\sum_{a=1}^{m}e^{2\epsilon t}\left\{\sum_{b=1}^{m}\left[\left(\alpha_{ab}(t)\left(g_b\left(u_b(t)\right)-g_b\left(\bar{u}_b(t)\right)\right)\right)^{*}\right.\right.$$

$$\left.\times\left(g_b\left(u_b(t)\right)-g_b\left(\bar{u}_b(t)\right)\right)\right)+v_a^{*}(t)v_a(t)\Big]$$

$$+\sum_{b=1}^{m}\left(\beta_{ab}(t)\left[\left(g_b\left(u_b^{*}\left(t-\varrho_{ab}(t)\right)\right)-g_b\left(\bar{u}_b^{*}\left(t-\varrho_{ab}(t)\right)\right)\right)\right)^{*}$$

$$\left.\times\left(g_b\left(u_b^{*}\left(t-\varrho_{ab}(t)\right)\right)-g_b\left(\bar{u}_b^{*}\left(t-\varrho_{ab}(t)\right)\right)\right)\right)+v_a^{*}(t)v_a(t)\Big]\Bigg\}$$

$$+e^{2\epsilon t}\sum_{a=1}^{m}\sum_{b=1}^{m}\left(\beta^{+}\right)^{2}(L_b)^{2}\frac{e^{2\epsilon\varrho^{+}}}{1-\theta}v_b^{*}(t)v_b(t)-e^{2\epsilon t}$$

$$\times \sum_{a=1}^{m}\sum_{b=1}^{m}\left(\beta^{+}\right)^{2}(L_b)^{2}\frac{e^{2\epsilon\varrho^{+}}}{1-\theta}\left(1-\dot{\varrho}_{ab}(t)\right)e^{-2\epsilon\varrho_{ab}(t)}v_b^{*}\left(t-\varrho_{ab}(t)\right)v_b\left(t-\varrho_{ab}(t)\right)$$

$$\le \sum_{a=1}^{m}\epsilon e^{2\epsilon t}\left[\left(2\epsilon-2\gamma_a^{-}+2\right)v_a^{*}(t)v_a(t)+\sum_{b=1}^{m}\left(\alpha_{ab}^{+}\right)^{2}(L_b)^{2}v_b^{*}(t)v_b(t)\right.$$

$$\left.+\sum_{b=1}^{m}\left(\beta_{ab}^{+}\right)^{2}(L_b)^{2}v_b^{*}\left(t-\varrho_{ab}(t)\right)v_b\left(t-\varrho_{ab}(t)\right)\right]$$

$$+e^{2\epsilon t}\sum_{a=1}^{m}\sum_{b=1}^{m}\left(\beta^{+}\right)^{2}(L_b)^{2}\frac{e^{2\epsilon\varrho^{+}}}{1-\theta}v_b^{*}(t)v_b(t)-e^{2\epsilon t}$$

$$\times \sum_{a=1}^{m}\sum_{b=1}^{m}\left(\beta^{+}\right)^{2}(L_b)^{2}\frac{e^{2\epsilon\varrho^{+}}}{1-\theta}\left(1-\dot{\varrho}_{ab}(t)\right)e^{-2\epsilon\varrho_{ab}(t)}v_b^{*}\left(t-\varrho_{ab}(t)\right)v_b\left(t-\varrho_{ab}(t)\right)$$

$$\le \sum_{a=1}^{m}\epsilon e^{2\epsilon t}\left[\left(2\epsilon-2\gamma_a^{-}+2\right)v_a^{*}(t)v_a(t)+\sum_{b=1}^{m}\left(\alpha_{ab}^{+}\right)^{2}(L_b)^{2}v_b^{*}(t)v_b(t)\right.$$

$$
\begin{aligned}
&\quad + \sum_{b=1}^{m} \big(\beta_{ab}^{+}\big)^2 (L_b)^2 \nu_b^*\big(t-\varrho_{ab}(t)\big)\nu_b\big(t-\varrho_{ab}(t)\big)\Bigg] \\
&\quad + e^{2\epsilon t}\sum_{a=1}^{m}\sum_{b=1}^{m}\big(\beta^{+}\big)^2 (L_b)^2 \frac{e^{2\epsilon\varrho^{+}}}{1-\theta}\nu_b^*(t)\nu_b(t) - e^{2\epsilon t} \\
&\quad \times \sum_{a=1}^{m}\sum_{b=1}^{m}\big(\beta^{+}\big)^2 (L_b)^2 \frac{e^{2\epsilon\varrho^{+}}}{1-\theta}\big(1-\dot{\varrho}_{ab}(t)\big)e^{-2\epsilon\varrho_{ab}(t)}\nu_b^*\big(t-\varrho_{ab}(t)\big)\nu_b\big(t-\varrho_{ab}(t)\big) \\
&\le \sum_{a=1}^{m}\epsilon e^{2\epsilon t}\Bigg[\big(2\epsilon-2\gamma_a^{-}+2\big)\nu_a^*(t)\nu_a(t) + \sum_{b=1}^{m}\big(\alpha_{ab}^{+}\big)^2(L_b)^2\nu_b^*(t)\nu_b(t)\Bigg] \\
&\quad + e^{2\epsilon t}\sum_{a=1}^{m}\sum_{b=1}^{m}\big(\beta^{+}\big)^2 (L_b)^2 \frac{e^{2\epsilon\varrho^{+}}}{1-\theta}\nu_b^*(t)\nu_b(t) \\
&= \sum_{a=1}^{m}\epsilon e^{2\epsilon t}\Bigg[\big(2\epsilon-2\gamma_a^{-}+2\big)\nu_a^*(t)\nu_a(t) + \sum_{b=1}^{m}\big(\alpha_{ba}^{+}\big)^2(L_b)^2\nu_a^*(t)\nu_a(t)\Bigg] \\
&\quad + e^{2\epsilon t}\sum_{a=1}^{m}\sum_{b=1}^{m}\big(\beta^{+}\big)^2 (L_b)^2 \frac{e^{2\epsilon\varrho^{+}}}{1-\theta}\nu_a^*(t)\nu_a(t) \\
&= \sum_{a=1}^{m}\epsilon e^{2\epsilon t}\Bigg[\big(2\epsilon-2\gamma_a^{-}+2\big) + \sum_{b=1}^{m}\big(\alpha_{ba}^{+}\big)^2(L_b)^2 + \sum_{b=1}^{m}\big(\beta^{+}\big)^2(L_b)^2\frac{e^{2\epsilon\varrho^{+}}}{1-\theta}\Bigg]\nu_a^*(t)\nu_a(t) \\
&\le \delta e^{2\epsilon t}\sum_{a=1}^{m}\nu_a^*(t)\nu_a(t) \le 0,
\end{aligned}
\tag{4.3}
$$

which leads to $\mathcal{V}(t) \le \mathcal{V}(0)$, $\forall t \ge 0$. Then

$$
\mathcal{V}(t) \ge e^{2\epsilon t}\sum_{a=1}^{m}\nu_a^*(t)\nu_a(t) = \big(e^{\epsilon t}\big\|u(t)-\bar{u}(t)\big\|_{\mathcal{Q}}\big)^2 \tag{4.4}
$$

and

$$
\begin{aligned}
\mathcal{V}(0) &\le \sum_{a=1}^{m}\Big(\sup_{s\in[-\varrho^{+},0]}\big\|\nu_a(s)\big\|^2\Big) + \sum_{a=1}^{m}\sum_{b=1}^{m}\big(\beta^{+}\big)^2(L_b)^2\frac{e^{2\epsilon\varrho^{+}}}{1-\theta}\varrho^{+}\sup_{s\in[-\varrho^{+},0]}\big\|\nu_a(s)\big\|^2 \\
&\le \mathcal{K}\big\|\psi_a(s)-\bar{\vartheta}(s)\big\|_0^2,
\end{aligned}
\tag{4.5}
$$

where

$$
\mathcal{K} = \sum_{a=1}^{m}\Bigg\{1+\sum_{b=1}^{m}\big(\beta^{+}\big)^2(L_b)^2\frac{e^{2\epsilon\varrho^{+}}}{1-\theta}\varrho^{+}\Bigg\} > 1.
$$

Then

$$
\big(e^{\epsilon t}\big\|u(t)-\bar{u}(t)\big\|_{\mathcal{Q}}\big)^2 \le \mathcal{K}\big\|\phi-\bar{\phi}(s)\big\|_0^2. \tag{4.6}
$$

Thus

$$
\big\|u(t)-\bar{u}(t)\big\|_{\mathcal{Q}} \le \mathcal{U}\big\|\phi-\bar{\phi}(s)\big\|_0^2 e^{-\epsilon t}, \quad \forall t > 0, \tag{4.7}
$$

where $\mathcal{U} = \sqrt{\mathcal{K}}$. In view of Definition 2.1, one knows that model (1.2) has a unique $\frac{\varpi}{2}$-anti-periodic solution that is global exponentially stable. This ends the proof. □

Remark 4.1 In [29–39], the authors have studied the anti-periodic solution of different type RVNNs. They do not investigate the anti-periodic solution of QVNNs. The analysis method to investigate the anti-periodic solution of QVNNs is different from that for RVNNs. All the results in [29–39] cannot be transferred to model (1.2) to establish the conditions that guarantee the existence and globally exponential stability of the anti-periodic solutions. In [45], the authors have investigated the anti-periodic solutions of quaternion-valued neural networks with multiple time-varying delays and the product of multiple activation functions. The analysis method is more complex and all the results in [45] cannot be applied to model (1.2) to obtain the results of this manuscript. This manifests that the results of this manuscript are essentially innovative.

Remark 4.2 In [29, 37, 46–67], the authors dealt with stability of delayed neural networks or other delayed models, but all the scholars in [29, 37, 46–67] did not investigate the stability of anti-periodic solution of QVNNs. Moreover, to establish the sufficient condition to ensure the exponentially stability of involved delayed models, how to construct a suitable Lyapunov function is a challenging work. In this work, we successfully construct an appropriate Lyapunov function to establish the sufficient condition to ensure the exponentially stability of considered delayed QVNNs. So we think that this work has some novelties.

Remark 4.3 In [43], the authors discussed the periodic solution of quaternion-valued cellular neural networks with time-varying delays. This article did not consider the anti-periodic solution that this article involved. In [40], the authors investigated the anti-periodic solution of inertial delayed quaternion-valued high-order Hopfield neural networks. The neural networks involved the state-dependent delays which is different from the neural network with time-varying delays in this article.

Remark 4.4 In this paper, we have skillfully applied some suitable inequalities to establish our main results except a series of mathematical analysis when we deal with the anti-periodic solution by applying coincidence degree theory. Also the choice of Lyapunov function has some novelties.

5 Computer simulations

In previous section, we have found that under some appropriate parameter conditions, the quaternion-valued delayed cellular neural networks have a unique anti-periodic solution that is global exponentially stable. To check the correctness of the theoretical predictions, we give the following neural networks:

$$\begin{cases} \dot{u}_1(t) = -\gamma_1(t)u_1(t) + \sum_{b=1}^{2} \alpha_{1b}(t)g_b(u_b(t)) \\ \qquad + \sum_{b=1}^{2} \beta_{1b}(t)g_b(u_b(t-\varrho_{1b}(t))) + R_1(t), \\ \dot{u}_2(t) = -\gamma_2(t)u_2(t) + \sum_{b=1}^{2} \alpha_{2b}(t)g_b(u_b(t)) \\ \qquad + \sum_{b=1}^{2} \beta_{2b}(t)g_b(u_b(t-\varrho_{2b}(t))) + R_2(t), \end{cases} \tag{5.1}$$

where $g_b(u_b) = 0.3\cos u_b^R + i0.3\cos u_a^I + j0.3\cos u_a^J + k0.3\cos u_a^K$ $(b = 1,2)$ and

$$\begin{cases} \gamma_1(t) = 4 - 0.5\sin 6t, \\ \gamma_2(t) = 3 - 0.2\sin 6t, \\ \varrho_{11}(t) = 0.5\sin 6t - 0.3, \\ \varrho_{12}(t) = 0.4\cos 6t - 0.2, \\ \varrho_{21}(t) = 0.3\cos 6t - 0.1, \\ \varrho_{22}(t) = 0.5\cos 6t - 0.3, \\ R_1(t) = 1 + i\sin 6t + j\cos 6t + k\sin 6t, \\ R_2(t) = 2 + i\cos 6t + j\sin 6t + k\cos 6t, \\ \alpha_{11}(t) = 0.5\cos 6t + 0.5i\sin 6t + 0.3j\cos 6t + 0.3k\sin 6t, \\ \alpha_{12}(t) = 0.2\cos 6t + 0.2i\sin 6t + 0.3j\cos 6t + 0.3k\sin 6t, \\ \alpha_{21}(t) = 0.3\sin 6t + 0.3i\cos 6t + 0.2j\sin 6t + 0.2k\cos 6t, \\ \alpha_{22}(t) = 0.1\sin 6t + 0.1i\cos 6t + 0.4j\sin 6t + 0.4k\cos 6t, \\ \beta_{11}(t) = 0.3\sin 6t + 0.3i\cos 6t + 0.2j\sin 6t + 0.2k\cos 6t, \\ \beta_{12}(t) = 0.1\sin 6t + 0.1i\cos 6t + 0.4j\sin 6t + 0.4k\cos 6t, \\ \beta_{21}(t) = 0.5\sin 6t + 0.5i\cos 6t + 0.3j\cos 6t + 0.3k\sin 6t, \\ \beta_{22}(t) = 0.2\sin 6t + 0.2i\cos 6t + 0.2j\sin 6t + 0.2k\cos 6t. \end{cases} \tag{5.2}$$

Then $L_1 = L_2 = 0.3, G_1 = G_2 = 0.36, \gamma_1^- = 3.5, \gamma_2^- = 2.8, \gamma_1^+ = 4.5, \gamma_2^+ = 3.2, \varrho^+ = 0.2, \theta = 0.3, \alpha_{11}^+ = 0.5831, \alpha_{12}^+ = 0.3606, \alpha_{21}^+ = 0.3606, \alpha_{22}^+ = 0.4123, \beta_{11}^+ = 0.3606, \beta_{12}^+ = 0.4123, \beta_{21}^+ = 0.5831, \beta_{22}^+ = 0.2828$. Let $\epsilon = 0.02$, then one has

$$\delta = \max_{1\le a\le 2}\left\{ \left(2\epsilon - 2\gamma_a^- + 2\right) + \sum_{b=1}^{2}\left(\alpha_{ba}^+\right)^2 (L_b)^2 + \sum_{b=1}^{2}\left(\beta^+\right)^2 (L_b)^2 \frac{e^{2\epsilon\varrho^+}}{1-\theta} \right\} = -0.56 < 0.$$

Then one can easily check that all the required assumptions of Theorem 3.1 and Theorem 4.1 hold true. Thus one knows that model (5.1) possesses at least one $\frac{\pi}{3}$-periodic solution. Moreover this periodic solution is exponentially stable. The fact can be shown in Figs. 1–4. The numerical results show that under some suitable conditions, the states u_1, u_2 of the two units will exponentially converge to stability. It plays an important role in designing and optimizing neural networks.

6 Conclusions

During the past decades, the anti-periodic solution of neural networks has been widely studied [68]. But many works on the anti-periodic solution of neural networks mainly focus on the CVNNs and RVNNs. In this present manuscript, we mainly handle the anti-periodic solution of a class of QVNNs. With the aid of inequality techniques, coincidence degree theory and constructing a suitable Lyapunov function, we discuss the existence and exponential stability of anti-periodic solutions of the involved QVNNs. The derived results are helpful in designing and optimizing neural networks. For example, we can adjust the parameters and time delays to meet the requirement of the established neural network models to obtain our desired anti-periodic phenomenon. Then it can be applied

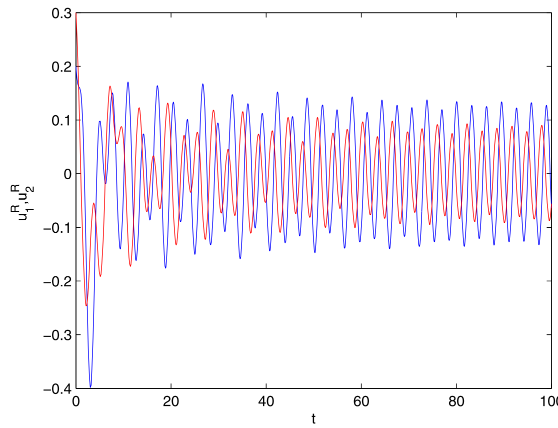

Figure 1 The anti-periodic solution of model (5.1): t–u_1^R and t–u_2^R. The blue line stands for u_1^R and the red line stands for u_2^R

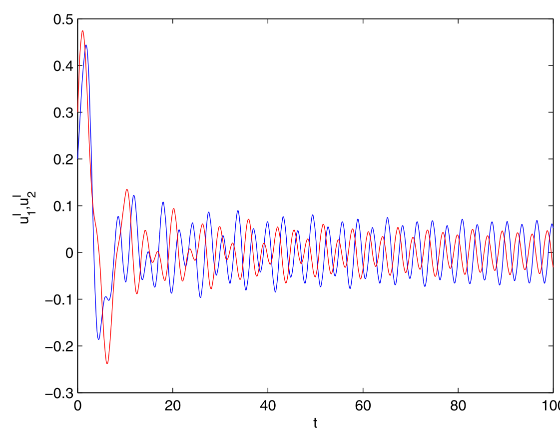

Figure 2 The anti-periodic solution of model (5.1): t–u_1^I and t–u_2^I. The blue line stands for u_1^I and the red line stands for u_2^I

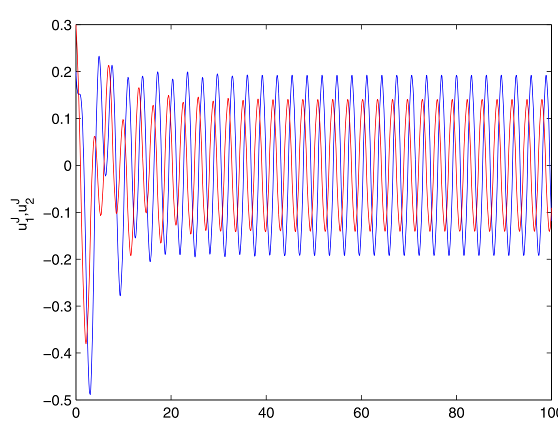

Figure 3 The anti-periodic solution of model (5.1): t–u_1^J and t–u_2^J. The blue line stands for u_1^J and the red line stands for u_2^J

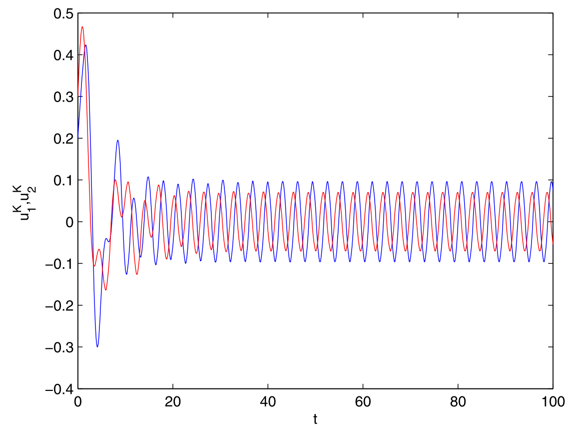

Figure 4 The anti-periodic solution of model (5.1): t–u_1^K and t–u_2^K. The blue line stands for u_1^K and the red line stands for u_2^K

in disease diagnosis for medical science, artificial intelligence, etc. The research method on anti-periodic solution of QVNNs will enriches the anti-periodic solution theory of differential equations. Also, some related results complement some earlier investigations to some degree. In addition, we point out that the weighted pseudo anti-periodic solutions of neural networks [52] is a meaningful topic. However, few scholars investigate the weighted pseudo anti-periodic solutions of QVNNs. In the near future, we will focus on this aspect.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have read and approved the final manuscript.

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