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Monotone iterative solutions for a coupled system of p-Laplacian differential equations involving the Riemann–Liouville fractional derivative

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Abstract

Applying the monotone iterative technique and the method of upper and lower solutions, we investigate the existence of extremal solutions for a nonlinear system of p-Laplacian differential equations with nonlocal coupled integral boundary conditions. We present a numerical example to illustrate the main result.

MSC: 34B15

Keywords: Fractional differential system; Nonlocal coupled integral boundary conditions; Extremal solution; p-Laplacian operator; Monotone iterative technique

1 Introduction

Consider the following fractional differential system with the nonlocal coupled integral boundary conditions:

$$\begin{cases} -D^{\beta}(\phi_p(-D^{\alpha}x(t))) = f(t,x(t),y(t),D^{\alpha}x(t),D^{\alpha}y(t)), \quad t\in(0,1], \\ -D^{\beta}(\phi_p(-D^{\alpha}y(t))) = g(t,y(t),x(t),D^{\alpha}y(t)),D^{\alpha}x(t)), \quad t\in(0,1], \\ D^{\alpha}x(0)=0, \\ D^{\beta-1}(\phi_p(-D^{\alpha}x(1))) = I^{\sigma}h(\eta,\phi_p(-D^{\alpha}x(\eta)))+a_1 \\ \qquad = \frac{1}{\Gamma(\sigma)}\int_0^{\eta}(\eta-s)^{\sigma-1}h(s,\phi_p(-D^{\alpha}x(s)))\,ds+a_1, \\ x(0)=0, \qquad D^{\alpha-1}x(1) = I^{\omega}x(\xi)+d_1 = \frac{1}{\Gamma(\omega)}\int_0^{\xi}(\xi-s)^{\omega-1}x(s)\,ds+d_1, \\ D^{\alpha}y(0)=0, \\ D^{\beta-1}(\phi_p(-D^{\alpha}y(1))) = I^{\sigma}k(\eta,\phi_p(-D^{\alpha}y(\eta)))+a_2 \\ \qquad = \frac{1}{\Gamma(\sigma)}\int_0^{\eta}(\eta-s)^{\sigma-1}k(s,\phi_p(-D^{\alpha}y(s)))\,ds+a_2, \\ y(0)=0, \qquad D^{\alpha-1}y(1) = I^{\omega}y(\xi)+d_2 = \frac{1}{\Gamma(\omega)}\int_0^{\xi}(\xi-s)^{\omega-1}y(s)\,ds+d_2, \end{cases} \tag{1.1}$$

where D^{α} and D^{β} are the standard Riemann–Liouville fractional derivatives, I^{σ} and I^{ω} are the Riemann–Liouville fractional integrals, and $1<\alpha,\beta<2$, $\sigma,\omega>1$, $0<\eta,\xi<1$,

$a_1, a_2, d_1, d_2 \in \mathbb{R}$, $a_2 \geq a_1$, $d_2 \geq d_1$, $f, g \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $h, k \in C([0,1] \times \mathbb{R}, \mathbb{R})$. The p-Laplacian operator is defined as $\phi_p(t) = |t|^{p-2}t$, $p > 1$, and $(\phi_p)^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$.

Fractional differential equations have recently gained much attention. In particular, much effort has been made toward the study of the existence of solutions for fractional differential equations with p-Laplacian operator [1–8]. The monotone iterative technique, combined with the method of upper and lower solutions, is a powerful tool for proving the existence of solutions of nonlinear differential equations; see [9, 10] and the references therein. However, only a few papers considered the upper an lower solutions method and the monotone iteration technique for p-Laplacian boundary value problems with fractional coupled systems. The purpose of this paper is developing a monotone iterative technique to show the existence of an extremal solution for the nonlinear system (1.1) with nonlocal integral boundary conditions.

The paper is organized as follows. In Sect. 2, we give sufficient conditions guaranteeing that (1.1) has an extremal solution and discuss some comparison results, which play a key role in establishing the proposed work. In Sect. 3, we give the main result. Finally, we present an example illustrating our results.

2 Preliminaries

In this section, we introduce definitions and some useful lemmas which play an important role in obtaining the main results of this paper.

Denote

$$C_\alpha[0,1] = \{u : u \in C[0,1], D^\alpha u(t) \in C[0,1]\}.$$

It is a Banach spaces with the norm $\|u\|_\alpha = \|u\| + \|D^\alpha u\|$, where $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$ and $\|D^\alpha u\| = \max_{0 \leq t \leq 1} |D^\alpha u(t)|$.

We need the following assumptions.

(H_1) There exist $x_0, y_0 \in C_\alpha[0,1]$ satisfying $D^\beta(\phi_p(-D^\alpha x_0(t))), D^\beta(\phi_p(-D^\alpha y_0(t))) \in C[0,1]$, $x_0(t) \leq y_0(t)$, and $D^\alpha y_0(t) \leq D^\alpha x_0(t)$ such that

$$\begin{cases} -D^\beta(\phi_p(-D^\alpha x_0(t))) \leq f(t, x_0(t), y_0(t), D^\alpha x_0(t), D^\alpha y_0(t)), \quad t \in (0,1], \\ D^\alpha x_0(0) = 0, \qquad D^{\beta-1}(\phi_p(-D^\alpha x_0(1))) \leq I^\sigma h(\eta, \phi_p(-D^\alpha x_0(\eta))) + a_1, \\ x_0(0) = 0, \qquad D^{\alpha-1} x_0(1) \leq I^\omega x_0(\xi) + d_1, \\ -D^\beta(\phi_p(-D^\alpha y_0(t))) \geq g(t, y_0(t), x_0(t), D^\alpha y_0(t)), D^\alpha x_0(t)), \quad t \in (0,1], \\ D^\alpha y_0(0) = 0, \qquad D^{\beta-1}(\phi_p(-D^\alpha y_0(1))) \geq I^\sigma k(\eta, \phi_p(-D^\alpha y_0(\eta))) + a_2, \\ y_0(0) = 0, \qquad D^{\alpha-1} y_0(1) \geq I^\omega y_0(\xi) + d_2. \end{cases}$$

(H_2) There exist two constants $M, N \in \mathbb{R}$, $M \geq N$, such that

$$\begin{aligned} & f\bigl(t, \overline{x(t)}, \overline{y(t)}, D^\alpha \overline{x(t)}, D^\alpha \overline{y(t)}\bigr) - f\bigl(t, x(t), y(t), D^\alpha x(t), D^\alpha y(t)\bigr) \\ & \quad \leq M\bigl[\phi_p\bigl(-D^\alpha x(t)\bigr) - \phi_p\bigl(-D^\alpha \overline{x(t)}\bigr)\bigr] + N\bigl[\phi_p\bigl(-D^\alpha y(t)\bigr) - \phi_p\bigl(-D^\alpha \overline{y(t)}\bigr)\bigr] \\ & g\bigl(t, \overline{x(t)}, \overline{y(t)}, D^\alpha \overline{x(t)}, D^\alpha \overline{y(t)}\bigr) - g\bigl(t, x(t), y(t), D^\alpha x(t), D^\alpha y(t)\bigr) \\ & \quad \leq M\bigl[\phi_p\bigl(-D^\alpha x(t)\bigr) - \phi_p\bigl(-D^\alpha \overline{x(t)}\bigr)\bigr] + N\bigl[\phi_p\bigl(-D^\alpha y(t)\bigr) - \phi_p\bigl(-D^\alpha \overline{y(t)}\bigr)\bigr], \end{aligned}$$

where $x_0(t) \leq \overline{x(t)} \leq x(t) \leq y_0(t)$, $x_0(t) \leq y(t) \leq \overline{y(t)} \leq y_0(t)$, and

$$\begin{aligned} &f\bigl(t,x(t),y(t),D^\alpha x(t),D^\alpha y(t)\bigr) - g\bigl(t,y(t),x(t),D^\alpha y(t),D^\alpha x(t)\bigr) \\ &\quad \leq M\bigl[\phi_p\bigl(-D^\alpha y(t)\bigr) - \phi_p\bigl(-D^\alpha x(t)\bigr)\bigr] + N\bigl[\phi_p\bigl(-D^\alpha x(t)\bigr) - \phi_p\bigl(-D^\alpha y(t)\bigr)\bigr] \end{aligned}$$

for $x_0(t) \leq x(t) \leq y(t) \leq y_0(t)$.

(H_3) There exists a constant $\lambda \geq 0$ such that

$$\begin{aligned} &h\bigl(t,\phi_p\bigl(-D^\alpha y(t)\bigr)\bigr) - h\bigl(t,\phi_p\bigl(-D^\alpha x(t)\bigr)\bigr) \geq \lambda\bigl[\phi_p\bigl(-D^\alpha y(t)\bigr) - \phi_p\bigl(-D^\alpha x(t)\bigr)\bigr], \\ &k\bigl(t,\phi_p\bigl(-D^\alpha y(t)\bigr)\bigr) - k\bigl(t,\phi_p\bigl(-D^\alpha x(t)\bigr)\bigr) \geq \lambda\bigl[\phi_p\bigl(-D^\alpha y(t)\bigr) - \phi_p\bigl(-D^\alpha x(t)\bigr)\bigr], \end{aligned}$$

where $x_0(t) \leq x(t) \leq y(t) \leq y_0(t)$, $D^\alpha y_0(t) \leq D^\alpha y(t) \leq D^\alpha x(t) \leq D^\alpha x_0(t)$, $t \in [0,1]$, and

$$k\bigl(t,\phi_p\bigl(-D^\alpha y(t)\bigr)\bigr) - h\bigl(t,\phi_p\bigl(-D^\alpha x(t)\bigr)\bigr) \geq \lambda\bigl[\phi_p\bigl(-D^\alpha y(t)\bigr) - \phi_p\bigl(-D^\alpha x(t)\bigr)\bigr]$$

for $x_0(t) \leq x(t) \leq y(t) \leq y_0(t)$, $D^\alpha y_0(t) \leq D^\alpha y(t) \leq D^\alpha x(t) \leq D^\alpha x_0(t)$, $t \in [0,1]$.

(H_4) $\Gamma(\beta+\sigma) > \lambda\eta^{\beta+\sigma-1}$.

(H_5) $2\Gamma(\beta+\sigma)(M+N) < \Gamma(\beta)[\Gamma(\beta+\sigma) - \lambda\eta^{\beta+\sigma-1}]$.

(H_6) For any $t \in (0,1)$, we have

$$\Gamma(2-\beta)\lambda\eta^\sigma < \Gamma(\sigma).$$

Lemma 2.1 ([11]) *Let $h \in C[0,1]$, $b \in \mathbb{R}$, and $\Gamma(\beta+\sigma) \neq \lambda\eta^{\beta+\sigma-1}$. Then the fractional boundary value problem*

$$\begin{cases} -D^\beta w(t) = h(t), \quad t \in [0,1], \\ w(0) = 0, \\ D^{\beta-1}w(1) = \lambda I^\sigma w(\eta) + b = \frac{\lambda}{\Gamma(\sigma)}\int_0^\eta (\eta-s)^{\sigma-1}w(s)\,ds + b, \end{cases} \tag{2.1}$$

has the following integral representation of the solution:

$$w(t) = \int_0^1 G(t,s)h(s)\,ds + \frac{b\Gamma(\beta+\sigma)t^{\beta-1}}{\Gamma(\beta)[\Gamma(\beta+\sigma) - \lambda\eta^{\beta+\sigma-1}]},$$

where

$$G(t,s) = \frac{1}{\Delta}\begin{cases} [\Gamma(\beta+\sigma) - \lambda(\eta-s)^{\beta+\sigma-1}]t^{\beta-1} & \\ \quad - [\Gamma(\beta+\sigma) - \lambda\eta^{\beta+\sigma-1}](t-s)^{\beta-1}, & s \leq t, s \leq \eta, \\ \Gamma(\beta+\sigma)t^{\beta-1} - \lambda(\eta-s)^{\beta+\sigma-1}t^{\beta-1}, & t \leq s \leq \eta, \\ \Gamma(\beta+\sigma)[t^{\beta-1} - (t-s)^{\beta-1}] + \lambda\eta^{\beta+\sigma-1}(t-s)^{\beta-1}, & \eta \leq s \leq t, \\ \Gamma(\beta+\sigma)t^{\beta-1}, & s \geq t, s \geq \eta, \end{cases}$$

and $\Delta = \Gamma(\beta)[\Gamma(\beta+\sigma) - \lambda\eta^{\beta+\sigma-1}]$.

Lemma 2.2 ([11]) *Let $M, b \in \mathbb{R}$, $h(t) \in C[0,1]$, $2\Gamma(\beta+\sigma)|M| < \Gamma(\beta)[\Gamma(\beta+\sigma) - \lambda\eta^{\beta+\sigma-1}]$, and (H_4) hold. Then*

$$\begin{cases} -D^{\beta} w(t) + Mw(t) = h(t), & t \in [0,1], \\ w(0) = 0, \\ D^{\beta-1} w(1) = \lambda I^{\sigma} w(\eta) + b, \end{cases}$$

has a unique solution $w(t) \in C[0,1]$.

Lemma 2.3 ([10, Lemma 2.4]) *Let $z(t) \in C[0,1]$ and $l \in \mathbb{R}$. Then the fractional value boundary problem*

$$\begin{cases} -D^{\alpha} u(t) = z(t), & 0 < t < 1, \\ u(0) = 0, & D^{\alpha-1} u(1) = l, \end{cases} \tag{2.2}$$

is equivalent to

$$u(t) = \int_0^1 H(t,s) z(s)\, ds + \frac{l t^{\alpha-1}}{\Gamma(\alpha)},$$

where

$$H(t,s) = \begin{cases} t^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}, & 0 \le t \le s \le 1. \end{cases}$$

Lemma 2.4 *Assume that $1 < \alpha, \beta < 2$, $\sigma_1, \sigma_2 \in C[0,1]$, M, N are nonnegative constants satisfying $M \ge N$, and (H_4) and (H_5) hold. Then the fractional differential system*

$$\begin{cases} -D^{\beta}(\phi_p(-D^{\alpha} x(t))) = \sigma_1(t) - M(\phi_p(-D^{\alpha} x(t))) - N(\phi_p(-D^{\alpha} y(t))), & t \in (0,1], \\ -D^{\beta}(\phi_p(-D^{\alpha} y(t))) = \sigma_2(t) - M(\phi_p(-D^{\alpha} y(t))) - N(\phi_p(-D^{\alpha} x(t))), & t \in (0,1], \\ D^{\alpha} x(0) = 0, \quad D^{\beta-1}(\phi_p(-D^{\alpha} x(1))) = \lambda I^{\sigma} \phi_p(-D^{\alpha} x(\eta)) + b_1, \\ x(0) = 0, \quad D^{\alpha-1} x(1) = l_1, \\ D^{\alpha} y(0) = 0, \quad D^{\beta-1}(\phi_p(-D^{\alpha} y(1))) = \lambda I^{\sigma} \phi_p(-D^{\alpha} y(\eta)) + b_2, \\ y(0) = 0, \quad D^{\alpha-1} y(1) = l_2, \end{cases} \tag{2.3}$$

has a unique solution in $C_{\alpha}[0,T] \times C_{\alpha}[0,T]$.

Proof Let

$$\phi_p\bigl(-D^{\alpha} x(t)\bigr) = \frac{u(t)+v(t)}{2} \quad \text{and} \quad \phi_p\bigl(-D^{\alpha} y(t)\bigr) = \frac{u(t)-v(t)}{2}, \quad \forall t \in [0,1].$$

Using (2.3), we have that

$$\begin{cases} -D^{\beta} u(t) = \sigma_1(t) + \sigma_2(t) - (M+N) u(t), \\ u(0) = 0, \\ D^{\beta-1} u(1) = \lambda I^{\sigma} u(\eta) + b_1 + b_2, \end{cases} \tag{2.4}$$

and

$$\begin{cases} -D^{\beta}v(t) = \sigma_1(t) - \sigma_2(t) - (M-N)v(t), \\ u(0) = 0, \\ D^{\beta-1}v(1) = \lambda I^{\sigma}v(\eta) + b_1 - b_2. \end{cases} \tag{2.5}$$

Since M, N are nonnegative constants and $M \geq N$, by assumption (H_5) we see that

$$2\Gamma(\beta+\sigma)(M-N) \leq 2\Gamma(\beta+\sigma)(M+N) < \Gamma(\beta)\left[\Gamma(\beta+\sigma) - \lambda\eta^{\beta+\sigma-1}\right]. \tag{2.6}$$

By (2.6) and Lemma 2.2 we know that (2.4) and (2.5) have a unique solution. In consequence, $\phi_p(-D^{\alpha}x(t))$ and $\phi_p(-D^{\alpha}y(t)))$ are also unique, that is,

$$\phi_p\bigl(-D^{\alpha}x(t)\bigr) = \omega_1(t) \in C[0,1], \qquad \phi_p\bigl(-D^{\alpha}y(t)\bigr) = \omega_2(t) \in C[0,1].$$

Then

$$-D^{\alpha}x(t) = \phi_q\bigl(\omega_1(t)\bigr), \qquad -D^{\alpha}y(t)) = \phi_q\bigl(\omega_2(t)\bigr).$$

In view of the boundary value condition (2.3), we obtain

$$\begin{cases} -D^{\alpha}x(t) = \phi_q(\omega_1(t)), \\ -D^{\alpha}y(t)) = \phi_q(\omega_2(t)), \\ x(0) = 0, \qquad D^{\alpha-1}x(1) = l_1, \\ y(0) = 0, \qquad D^{\alpha-1}y(1) = l_2. \end{cases} \tag{2.7}$$

Let

$$x(t) = \frac{p(t)+q(t)}{2} \quad \text{and} \quad y(t) = \frac{p(t)-q(t)}{2}.$$

Using (2.7), we have

$$\begin{cases} -D^{\alpha}p(t) = \phi_q(\omega_1(t)) + \phi_q(\omega_2(t)), \\ p(0) = 0, \\ D^{\alpha-1}p(1) = l_1 + l_2, \end{cases} \tag{2.8}$$

and

$$\begin{cases} -D^{\alpha}q(t) = \phi_q(\omega_1(t)) - \phi_q(\omega_2(t)), \\ q(0) = 0, \\ D^{\alpha-1}q(1) = l_1 - l_2, \end{cases} \tag{2.9}$$

By Lemma 2.3 we know that both (2.8) and (2.9) have a unique solution. In consequence, x and y are also unique. □

Lemma 2.5 ([9, Lemma 2.6]) *Let M be nonnegative constant, and let (H_6) hold. If $w(t) \in C[0,1]$ satisfies $D^\beta w(t) \in C[0,1]$ and*

$$\begin{cases} -D^\beta w(t) \geq -Mw(t), & t \in [0,1], \\ w(0) = 0, \\ D^{\beta-1} w(1) \geq \lambda I^\sigma w(\eta), \end{cases}$$

then $w(t) \geq 0$ for all $t \in [0,1]$.

Lemma 2.6 ([10, Lemma 2.7]) *If $x(t) \in C[0,1]$ satisfies*

$$\begin{cases} -D^\alpha x(t) \geq 0, & 0 < t < 1, \\ x(0) = 0, \\ D^{\alpha-1} x(1) \geq 0, \end{cases}$$

then $x(t) \geq 0$ for all $t \in [0,1]$.

Lemma 2.7 *Let M, N be nonnegative constants and $M \geq N$. If $u, v \in C[0,1]$ satisfy $D^\beta u(t), D^\beta v(t) \in C[0,1]$, and*

$$\begin{cases} -D^\beta u(t) \geq -Mu(t) + Nv(t), & t \in [0,T), \\ -D^\beta v(t) \geq -Mv(t) + Nu(t), & t \in [0,T), \\ u(0) = 0, \quad D^{\beta-1} u(1) \geq \lambda I^\sigma u(\eta), \\ v(0) = 0, \quad D^{\beta-1} v(1) \geq \lambda I^\sigma v(\eta), \end{cases} \tag{2.10}$$

then $u(t) \geq 0$ and $v(t) \geq 0$ for all $t \in [0,1]$.

Proof Let $p(t) = u(t) + v(t)$, $t \in [0,1]$. Then by (2.10) we have

$$\begin{cases} -D^\beta p(t) \geq -(M-N)p(t), & t \in [0,1], \\ p(0) = 0, \\ D^{\beta-1} p(1) \geq \lambda I^\sigma p(\eta). \end{cases} \tag{2.11}$$

Thus by (2.11) and Lemma 2.5 we have that

$$p(t) \geq 0, \quad \forall t \in [0,1], \quad \text{i.e.,} \quad u(t) + v(t) \geq 0, \quad \forall t \in [0,1]. \tag{2.12}$$

Next, we show that $u(t) \geq 0$ and $v(t) \geq 0$ for all $t \in [0,1]$. Using (2.10) and (2.12), we find that

$$\begin{cases} -D^\beta u(t) \geq -(M+N)u(t), & t \in [0,1], \\ u(0) = 0, \\ D^{\beta-1} u(1) \geq \lambda I^\sigma u(\eta), \end{cases} \tag{2.13}$$

which, in view of (2.13) and Lemma 2.5, yield $u(t) \geq 0$ for all $t \in [0,1]$. In a similar way, we can show that $v(t) \geq 0$ for all $t \in [0,1]$. □

3 Main results

Theorem 3.1 *Suppose that conditions* (H_1)–(H_6) *hold. Then there is an extremal solution* $(x^*, y^*) \in [x_0, y_0] \times [x_0, y_0]$ *of the nonlinear problem* (1.1)*. Moreover, there exist monotone iterative sequences* $\{x_n\}, \{y_n\} \subset [x_0, y_0]$ *such that* $x_n \to x^*$, $y_n \to y^*$ $(n \to \infty)$ *uniformly for* $t \in [0,1]$ *and*

$$x_0 \le x_1 \le \cdots \le x_n \le \cdots \le x^* \le y^* \le \cdots \le y_n \le \cdots \le y_1 \le y_0.$$

Moreover, we have

$$D^\alpha y_0 \le D^\alpha y_1 \le \cdots \le D^\alpha y_n \le \cdots \le D^\alpha y^* \le D^\alpha x^* \le \cdots \le D^\alpha x_n \le \cdots \le D^\alpha x_1 \le D^\alpha x_0,$$

where

$$[x_0, y_0] = \bigl\{x \in C_\alpha[0,1] : x_0(t) \le x(t) \le y_0(t), t \in [0,1]\bigr\}.$$

Proof For any $x_{n-1}, y_{n-1} \in C_\alpha[0,1]$, $n \ge 1$, we define

$$\begin{cases} \sigma_n^1(t) = f(t, x_{n-1}(t), y_{n-1}(t), D^\alpha x_{n-1}(t), D^\alpha y_{n-1}(t)) \\ \qquad\quad + M\phi_p(-D^\alpha x_{n-1}(t)) + N\phi_p(-D^\alpha y_{n-1}(t)), \\ \sigma_n^2(t) = g(t, y_{n-1}(t), x_{n-1}(t), D^\alpha y_{n-1}(t), D^\alpha x_{n-1}(t)) \\ \qquad\quad + M\phi_p(-D^\alpha y_{n-1}(t)) + N\phi_p(-D^\alpha x_{n-1}(t)). \end{cases}$$

Consider (2.3) as follows:

$$\begin{cases} -D^\beta(\phi_p(-D^\alpha x_n(t))) = \sigma_n^1(t) - M(\phi_p(-D^\alpha x_n(t))) - N(\phi_p(-D^\alpha y_n(t))), \\ -D^\beta(\phi_p(-D^\alpha y_n(t))) = \sigma_n^2(t) - M(\phi_p(-D^\alpha y_n(t))) - N(\phi_p(-D^\alpha x_n(t))), \\ D^\alpha x_n(0) = 0, \\ D^{\beta-1}(\phi_p(-D^\alpha x_n(1))) \\ \quad = I^\sigma\{h(\eta, \phi_p(-D^\alpha x_{n-1}(\eta))) + \lambda[\phi_p(-D^\alpha x_n(\eta)) - \phi_p(-D^\alpha x_{n-1}(\eta))]\} + a_1, \\ x_n(0) = 0, \qquad D^{\alpha-1}x_n(1) = I^\omega x_{n-1}(\xi) + d_1, \\ D^\alpha y_n(0) = 0, \\ D^{\beta-1}(\phi_p(-D^\alpha y_n(1))) \\ \quad = I^\sigma\{k(\eta, \phi_p(-D^\alpha y_{n-1}(\eta))) + \lambda[\phi_p(-D^\alpha y_n(\eta)) - \phi_p(-D^\alpha y_{n-1}(\eta))]\} + a_2, \\ y_n(0) = 0, \qquad D^{\alpha-1}y_n(1) = I^\omega y_{n-1}(\xi) + d_2. \end{cases} \tag{3.1}$$

In view of Lemma 2.4, problem (3.1) has a unique solution in $C_\alpha[0,1] \times C_\alpha[0,1]$.

Now we show that $\{x_n(t)\}$ and $\{y_n(t)\}$ satisfy the relations

$$x_{n-1} \le x_n \le y_n \le y_{n-1} \quad \text{and} \quad D^\alpha y_{n-1} \le D^\alpha y_n \le D^\alpha x_n \le D^\alpha x_{n-1}, \quad n = 1, 2, \ldots. \tag{3.2}$$

Let $u(t) = \phi_p(-D^\alpha x_1(t)) - \phi_p(-D^\alpha x_0(t))$, $v(t) = \phi_p(-D^\alpha y_0(t)) - \phi_p(-D^\alpha y_1(t))$. By condition (3.1) and (H_1) we have

$$\begin{aligned} -D^\beta u(t) &= -D^\beta\big(\phi_p\big(-D^\alpha x_1(t)\big)\big) + D^\beta\big(\phi_p\big(-D^\alpha x_0(t)\big)\big) \\ &\geq f\big(t, x_0(t), y_0(t), D^\alpha x_0(t), D^\alpha y_0(t)\big) + M\phi_p\big(-D^\alpha x_0(t)\big) + N\phi_p\big(-D^\alpha y_0(t)\big) \\ &\quad - M\phi_p\big(-D^\alpha x_1(t)\big) - N\phi_p\big(D^\alpha y_1(t)\big) - f\big(t, x_0(t), y_0(t), D^\alpha x_0(t), D^\alpha y_0(t)\big) \\ &= -Mu(t) + Nv(t). \end{aligned}$$

Also, $u(0) = 0$, and

$$\begin{aligned} D^{\beta-1}u(1) &= D^{\beta-1}\big(\phi_p\big(-D^\alpha x_1(1)\big)\big) - D^{\beta-1}\big(\phi_p\big(-D^\alpha x_0(1)\big)\big) \\ &\geq I^\sigma\big\{h\big(\eta, \phi_p\big(-D^\alpha x_0(\eta)\big)\big) + \lambda\big[\phi_p\big(-D^\alpha x_1(\eta)\big) - \phi_p\big(-D^\alpha x_0(\eta)\big)\big]\big\} + a_1 \\ &\quad - I^\sigma h\big(\eta, \phi_p\big(-D^\alpha x_0(\eta)\big)\big) - a_1 \\ &= \lambda I^\sigma u(\eta). \end{aligned}$$

In a similar way, we can prove that

$$-D^\beta v(t) \geq -Mv(t) + Nu(t), \qquad v(0) = 0, \qquad D^{\beta-1}v(1) \geq \lambda I^\sigma v(\eta).$$

So, from the above inequality we have

$$\begin{cases} -D^\beta u(t) \geq -Mu(t) + Nv(t), \\ -D^\beta v(t) \geq -Mv(t) + Nu(t), \\ u(0) = 0, \qquad D^{\beta-1}u(1) \geq \lambda I^\sigma u(\eta), \\ v(0) = 0, \qquad D^{\beta-1}v(1) \geq \lambda I^\sigma v(\eta). \end{cases}$$

Thus, in view of Lemma 2.7, we get $\phi_p(-D^\alpha x_1(t)) \geq \phi_p(-D^\alpha x_0(t))$, $\phi_p(-D^\alpha y_0(t)) \geq \phi_p(-D^\alpha y_1(t))$ for all $t \in [0,1]$. Since $\Phi_p(x)$ is nondecreasing, we have $D^\alpha x_1(t) \leq D^\alpha x_0(t)$ and $D^\alpha y_0(t) \leq D^\alpha y_1(t)$ for all $t \in [0,1]$.

Let $\epsilon(t) = x_1(t) - x_0(t)$, $\theta(t) = y_0(t) - y_1(t)$. From (3.1) and (H_1) we have

$$\begin{cases} -D^\alpha \epsilon(t) \geq 0, \quad t \in (0,1], \\ \epsilon(0) = 0, \\ D^{\alpha-1}\epsilon(1) \geq I^\omega x_0(\xi) + d_1 - I^\omega x_0(\xi) - d_1 = 0, \end{cases} \tag{3.3}$$

and

$$\begin{cases} -D^\alpha \theta(t) \geq 0, \quad t \in (0,1], \\ \theta(0) = 0, \\ D^{\alpha-1}\theta(1) \geq 0. \end{cases} \tag{3.4}$$

By Lemma 2.6 we have $x_1(t) \geq x_0(t)$ and $y_0(t) \geq y_1(t)$ for all $t \in [0,1]$.

Now we put $w(t) = \phi_p(-D^\alpha y_1(t)) - \phi_p(-D^\alpha x_1(t))$. Applying (H_2), (H_3), and (3.1), we obtain

$$\begin{aligned}
-D^\beta w(t) &= g\bigl(t, y_0(t), x_0(t), D^\alpha y_0(t), D^\alpha x_0(t)\bigr) + M\phi_p\bigl(-D^\alpha y_0(t)\bigr) + N\phi_p\bigl(-D^\alpha x_0(t)\bigr) \\
&\quad - M\phi_p\bigl(-D^\alpha y_1(t)\bigr) - N\phi_p\bigl(-D^\alpha x_1(t)\bigr) - f\bigl(t, x_0(t), y_0(t), D^\alpha x_0(t), D^\alpha y_0(t)\bigr) \\
&\quad - M\phi_p\bigl(-D^\alpha x_0(t)\bigr) - N\phi_p\bigl(-D^\alpha y_0(t)\bigr) + M\phi_p\bigl(-D^\alpha x_1(t)\bigr) + N\phi_p\bigl(-D^\alpha y_1(t)\bigr) \\
&\geq -M\bigl[\phi_p\bigl(-D^\alpha y_0(t)\bigr) - \phi_p\bigl(-D^\alpha x_0(t)\bigr)\bigr] - N\bigl[\phi_p\bigl(-D^\alpha x_0(t)\bigr) - \phi_p\bigl(-D^\alpha y_0(t)\bigr)\bigr] \\
&\quad + M\phi_p\bigl(-D^\alpha y_0(t)\bigr) + N\phi_p\bigl(-D^\alpha x_0(t)\bigr) - M\phi_p\bigl(-D^\alpha y_1(t)\bigr) - N\phi_p\bigl(-D^\alpha x_1(t)\bigr) \\
&\quad - M\phi_p\bigl(-D^\alpha x_0(t)\bigr) - N\phi_p\bigl(-D^\alpha y_0(t)\bigr) + M\phi_p\bigl(-D^\alpha x_1(t)\bigr) + N\phi_p\bigl(-D^\alpha y_1(t)\bigr) \\
&= -(M-N)w(t).
\end{aligned}$$

Also, $w(0) = \phi_p(-D^\alpha y_1(0)) - \phi_p(-D^\alpha x_1(0)) = 0$, and

$$\begin{aligned}
D^{\beta-1}w(1) &= I^\sigma\bigl\{k\bigl(\eta, \phi_p\bigl(-D^\alpha y_0(\eta)\bigr)\bigr) + \lambda\bigl[\phi_p\bigl(-D^\alpha y_1(\eta)\bigr) - \phi_p\bigl(-D^\alpha y_0(\eta)\bigr)\bigr]\bigr\} + a_2 \\
&\quad - I^\sigma\bigl\{h\bigl(\eta, \phi_p\bigl(-D^\alpha x_0(\eta)\bigr)\bigr) + \lambda\bigl[\phi_p\bigl(-D^\alpha x_1(\eta)\bigr) - \phi_p\bigl(-D^\alpha x_0(\eta)\bigr)\bigr]\bigr\} - a_1 \\
&\geq I^\sigma\bigl\{\lambda\bigl[\phi_p\bigl(-D^\alpha y_0(\eta)\bigr) - \phi_p\bigl(-D^\alpha x_0(\eta)\bigr)\bigr] + \lambda\bigl[\phi_p\bigl(-D^\alpha y_1(\eta)\bigr) - \phi_p\bigl(-D^\alpha y_0(\eta)\bigr)\bigr] \\
&\quad - \lambda\bigl[\phi_p\bigl(-D^\alpha x_1(\eta)\bigr) - \phi_p\bigl(-D^\alpha x_0(\eta)\bigr)\bigr]\bigr\} + (a_2 - a_1) \\
&\geq \lambda I^\sigma w(\eta).
\end{aligned}$$

In view of Lemma 2.5, we have that $w(t) \geq 0$ for all $t \in [0,1]$. Thus we have the relation $\phi_p(-D^\alpha x_1(t)) \leq \phi_p(-D^\alpha y_1(t))$, that is, $D^\alpha x_1(t) \geq D^\alpha y_1(t)$, since $\Phi_p(x)$ is nondecreasing. Therefore $D^\alpha y_0(t) \leq D^\alpha y_1(t) \leq D^\alpha x_1(t) \leq D^\alpha x_0(t)$ for all $t \in [0,1]$.

Let $\delta(t) = y_1(t) - x_1(t)$. It follows from (3.1) that

$$\begin{cases}
-D^\alpha \delta(t) = -D^\alpha y_1(t) + D^\alpha x_1(t) \geq 0, \\
\delta(0) = 0, \\
D^{\alpha-1}\delta(1) = I^\omega y_0(\xi) + d_2 - I^\omega x_0(\xi) - d_1 \geq 0.
\end{cases}$$

By Lemma 2.6 we obtain $y_1(t) \geq x_1(t)$ for all $t \in [0,1]$. Hence we have the relation $x_0(t) \leq x_1(t) \leq y_1(t) \leq y_0(t)$.

Now we assume that

$$x_{k-1} \leq x_k \leq y_k \leq y_{k-1} \quad \text{and} \quad D^\alpha y_{k-1} \leq D^\alpha y_k \leq D^\alpha x_k \leq D^\alpha x_{k-1} \quad \text{for some } k \geq 1.$$

We will prove that (3.2) is also true for $k+1$. Let

$$\begin{aligned}
&u(t) = \phi_p\bigl(-D^\alpha x_{k+1}(t)\bigr) - \phi_p\bigl(-D^\alpha x_k(t)\bigr), \qquad v(t) = \phi_p\bigl(-D^\alpha y_k(t)\bigr) - \phi_p\bigl(-D^\alpha y_{k+1}(t)\bigr), \\
&w(t) = \phi_p\bigl(-D^\alpha y_{k+1}(t)\bigr) - \phi_p\bigl(-D^\alpha x_{k+1}(t)\bigr), \qquad \epsilon(t) = x_{k+1}(t) - x_k(t), \\
&\theta(t) = y_k(t) - y_{k+1}(t), \qquad \delta(t) = y_{k+1}(t) - x_{k+1}(t).
\end{aligned}$$

By (H_2), (H_3), and (3.1) we have that

$$\begin{cases} -D^{\beta}u(t) \geq -Mu(t) + Nv(t), \\ -D^{\beta}v(t) \geq -Mv(t) + Nu(t), \\ u(0) = 0, \qquad D^{\beta-1}u(1) \geq \lambda I^{\sigma}u(\eta), \\ v(0) = 0, \qquad D^{\beta-1}v(1) \geq \lambda I^{\sigma}v(\eta), \end{cases}$$

$$\begin{cases} -D^{\alpha}\epsilon(t) \geq 0, \\ \epsilon(0) = 0, \\ D^{\alpha-1}\epsilon(1) \geq 0, \end{cases}$$

$$\begin{cases} -D^{\alpha}\theta(t) \geq 0, \\ \theta(0) = 0, \\ D^{\alpha-1}\theta(1) \geq 0, \end{cases}$$

and

$$\begin{cases} -D^{\beta}w(t) \geq -(M-N)w(t), \\ w(0) = 0, \\ D^{\beta-1}w(1) \geq \lambda I^{\sigma}w(\eta), \end{cases}$$

$$\begin{cases} -D^{\alpha}\delta(t) \geq 0, \\ \delta(0) = 0, \\ D^{\alpha-1}\delta(1) \geq 0. \end{cases}$$

In view of Lemmas 2.5–2.7, we obtain

$$x_k \leq x_{k+1} \leq y_{k+1} \leq y_k \quad \text{and} \quad D^{\alpha}y_k \leq D^{\alpha}y_{k+1} \leq D^{\alpha}x_{k+1} \leq D^{\alpha}x_k, \quad \forall t \in [0,1].$$

From the above, by induction, it is not difficult to prove that $x_0 \leq x_1 \leq \cdots \leq x_n \leq \cdots \leq y_n \leq \cdots \leq y_1 \leq y_0$ and $D^{\alpha}y_0 \leq D^{\alpha}y_1 \leq \cdots \leq D^{\alpha}y_n \leq \cdots \leq D^{\alpha}x_n \leq \cdots \leq D^{\alpha}x_1 \leq D^{\alpha}x_0$.

Since the solution space is $C_{\alpha}[0,1]$, the sequences $\{x_n\}$ and $\{y_n\}$ are uniformly bounded and equicontinuous. The Arzelà–Ascoli theorem guarantees that they are relatively compact sets in the space $C_{\alpha}[0,1]$. Therefore $\{x_n\}$ and $\{y_n\}$ converge to $x^*(t)$ and $y^*(t)$ uniformly on $[0,1]$, respectively, that is,

$$\lim_{n\to\infty} x_n(t) = x^*(t), \qquad \lim_{n\to\infty} y_n(t) = y^*(t), \quad \forall t \in [0,1],$$

and

$$\lim_{n\to\infty} D^{\alpha}x_n(t) = D^{\alpha}x^*(t), \qquad \lim_{n\to\infty} D^{\alpha}y_n(t) = D^{\alpha}y^*(t), \quad \forall t \in [0,1],$$

uniformly in $t \in [0,1]$. Moreover, from (3.1) and (3.2) we obtain that $x^*(t)$ and $y^*(t)$ are solutions of problem (1.1).

Finally, we show that (x^*, y^*) is an extremal solution of system (1.1). Let $(x, y) \in [x_0, y_0] \times [x_0, y_0]$ be any solution of problem (1.1), that is,

$$\begin{cases} -D^\beta(\phi_p(-D^\alpha x(t))) = f(t, x(t), y(t), D^\alpha x(t), D^\alpha y(t)), \quad t \in (0,1], \\ -D^\beta(\phi_p(-D^\alpha y(t))) = g(t, y(t), x(t), D^\alpha y(t)), D^\alpha x(t)), \quad t \in (0,1], \\ D^\alpha x(0) = 0, \\ D^{\beta-1}(\phi_p(-D^\alpha x(1))) = I^\sigma h(\eta, \phi_p(-D^\alpha x(\eta))) + a_1 \\ \qquad = \frac{1}{\Gamma(\sigma)} \int_0^\eta (\eta - s)^{\sigma-1} h(s, \phi_p(-D^\alpha x(s)))\,ds + a_1, \\ x(0) = 0, \qquad D^{\alpha-1}x(1) = I^\omega x(\xi) + d_1 = \frac{1}{\Gamma(\omega)} \int_0^\xi (\xi - s)^{\omega-1} x(s)\,ds + d_1, \\ D^\alpha y(0) = 0, \\ D^{\beta-1}(\phi_p(-D^\alpha y(1))) = I^\sigma k(\eta, \phi_p(-D^\alpha y(\eta))) + a_2 \\ \qquad = \frac{1}{\Gamma(\sigma)} \int_0^\eta (\eta - s)^{\sigma-1} k(s, \phi_p(-D^\alpha y(s)))\,ds + a_2, \\ y(0) = 0, \qquad D^{\alpha-1}y(1) = I^\omega y(\xi) + d_2 = \frac{1}{\Gamma(\omega)} \int_0^\xi (\xi - s)^{\omega-1} y(s)\,ds + d_2. \end{cases} \tag{3.5}$$

Applying (3.1), (3.5), (H_2), (H_3), Lemma 2.6, and Lemma 2.7, we have

$$x_n \le x, \qquad y \le y_n, \qquad D^\alpha x \le D^\alpha x_n, \qquad D^\alpha y_n \le D^\alpha y, \quad n = 1, 2, \ldots. \tag{3.6}$$

Taking the limit as $n \to \infty$ in (3.6), we have $x^* \le x$, $y \le y^*$, that is, (x^*, y^*) is an extremal solution of system (1.1) in $[x_0, y_0] \times [x_0, y_0]$. This completes the proof. □

4 Iteration procedure and a numerical example

In this section, we introduce a numerical procedure to obtain an appropriate solution of (1.1). Define

$$E(n) = \|x_n(t) - y_n(t)\|_1 = \int_0^1 |x_n(t) - y_n(t)|\,dt.$$

For the iteration Eq. (3.1), let $\phi_p(-D^\alpha x_n(t)) = u_n$. Then $-D^\alpha x_n(t) = \phi_q(u_n)$, and with the boundary conditions $x_n(0) = 0$ and $D^{\alpha-1}x_n(1) = D^{\frac{2}{3}}x_n(1) = I^\omega x_{n-1}(\xi) + d_1 = l_1$, by Lemma 2.3 we have

$$x_n(t) = \frac{l_1}{\Gamma(\alpha)} t^{\alpha-1} + \int_0^1 H(t,s)\phi_q(u_n(s))\,ds, \tag{4.1}$$

where $l_1 = \frac{1}{\Gamma(\omega)} \int_0^\xi (\xi - s)^{\omega-1} x_{n-1}(s)\,ds + d_1$ and

$$H(t,s) = \begin{cases} t^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}, & 0 \le t \le s \le 1. \end{cases}$$

We can also put $\phi_p(-D^\alpha y_n(t)) = v_n$. Then $-D^\alpha y_n(t) = \phi_q(v_n)$. In a similar way, we can prove that

$$y_n(t) = \frac{l_2}{\Gamma(\alpha)} t^{\alpha-1} + \int_0^1 H(t,s)\phi_q(v_n(s))\,ds, \tag{4.2}$$

where $l_2 = \frac{1}{\Gamma(\omega)}\int_0^\xi (\xi - s)^{\omega-1} y_{n-1}(s)\,ds + d_2 = 1.1284\int_0^{\frac{1}{2}} (\frac{1}{2} - s)^{\frac{1}{2}} y_{n-1}(s)\,ds + 0.004$. Thus the iteration Eq. (3.1) can be rewritten as

$$\begin{cases} -D^\beta u_n = -Mu_n - Nv_n + f(t, x_{n-1}, y_{n-1}, -\phi_q(u_{n-1}), -\phi_q(v_{n-1})) + Mu_{n-1} + Nv_{n-1}, \\ -D^\beta v_n = -Mv_n - Nu_n + g(t, y_{n-1}, x_{n-1}, -\phi_q(v_{n-1}), -\phi_q(u_{n-1})) + Mv_{n-1} + Nu_{n-1}, \\ u_n(0) = 0, \qquad D^{\beta-1}u_n(1) = \lambda I^\sigma u_n(\eta) + b_1, \\ v_n(0) = 0, \qquad D^{\beta-1}v_n(1) = \lambda I^\sigma v_n(\eta) + b_2. \end{cases} \tag{4.3}$$

Applying Lemma 2.1 to (4.3), we obtain

$$\begin{cases} u_n(t) = \frac{b_1\Gamma(\beta+\sigma)}{\Gamma(\beta)[\Gamma(\beta+\sigma)-\lambda\eta^{\beta+\sigma-1}]}t^{\beta-1} \\ \qquad + \int_0^1 G(t,s)[-Mu_n(s) - Nv_n(s) + f(s, x_{n-1}(s), y_{n-1}(s), -\phi_q(u_{n-1}(s)), \\ \qquad - \phi_q(v_{n-1}(s))) + Mu_{n-1}(s) + Nv_{n-1}(s)]\,ds, \\ v_n(t) = \frac{b_2\Gamma(\beta+\sigma)}{\Gamma(\beta)[\Gamma(\beta+\sigma)-\lambda\eta^{\beta+\sigma-1}]}t^{\beta-1} \\ \qquad + \int_0^1 G(t,s)[-Mv_n(s) - Nu_n(s) + g(s, y_{n-1}(s), x_{n-1}(s), -\phi_q(v_{n-1}(s)), \\ \qquad - \phi_q(u_{n-1}(s))) + Mv_{n-1}(s) + Nu_{n-1}(s)]\,ds, \end{cases} \tag{4.4}$$

where $b_1 = I^\sigma h(\eta, u_{n-1}(\eta)) - \lambda I^\sigma u_{n-1}(\eta) + a_1$, $b_2 = I^\sigma k(\eta, v_{n-1}(\eta)) - \lambda I^\sigma v_{n-1}(\eta) + a_2$, and

$$G(t,s) = \frac{1}{\Delta}\begin{cases} [\Gamma(\beta+\sigma) - \lambda(\eta - s)^{\beta+\sigma-1}]t^{\beta-1} & \\ \quad - [\Gamma(\beta+\sigma) - \lambda\eta^{\beta+\sigma-1}](t-s)^{\beta-1}, & s \le t, s \le \eta, \\ \Gamma(\beta+\sigma)t^{\beta-1} - \lambda(\eta - s)^{\beta+\sigma-1}t^{\beta-1}, & t \le s \le \eta, \\ \Gamma(\beta+\sigma)[t^{\beta-1} - (t-s)^{\beta-1}] + \lambda\eta^{\beta+\sigma-1}(t-s)^{\beta-1}, & \eta \le s \le t, \\ \Gamma(\beta+\sigma)t^{\beta-1}, & s \ge t, s \ge \eta, \end{cases}$$

$\Delta = \Gamma(\beta)[\Gamma(\beta+\sigma) - \lambda\eta^{\beta+\sigma-1}]$.

Discretize the interval $[0,1]$ with the nodes $t_i = ih$, $h = \frac{1}{K}$, $K \in \mathbb{N}$. Let $x_n^{(i)} \approx x_n(t_i)$, $u_n^{(i)} \approx u_n(t_i)$, $H(i,j) = H(t_i, s_j)$, $G(i,j) = G(t_i, s_j)$, and

$$\begin{cases} f_{n-1}^{(j)} = f(s_j, x_{n-1}(s_j), y_{n-1}(s_j), -\phi_q(u_{n-1}(s_j)), -\phi_q(v_{n-1}(s_j))) + Mu_{n-1}(s_j) + Nv_{n-1}(s_j), \\ g_{n-1}^{(j)} = g(s_j, y_{n-1}(s_j), x_{n-1}(s_j), -\phi_q(v_{n-1}(s_j)), -\phi_q(u_{n-1}(s_j))) + Mv_{n-1}(s_j) + Nu_{n-1}(s_j). \end{cases}$$

Using the trapezoidal quadrature rule to approximate the integrals in the right-hand sides of (4.4), (4.2), and (4.1), we obtain the following linear systems of equations:

$$\begin{cases} u_n^{(i)} = \frac{b_1\Gamma(\beta+\sigma)}{\Gamma(\beta)[\Gamma(\beta+\sigma)-\lambda\eta^{\beta+\sigma-1}]}t_i^{\beta-1} - \frac{h}{2}\sum_{j=0}^K MG(i,j)d_j u_n^{(j)} \\ \qquad - \frac{h}{2}\sum_{j=0}^K NG(i,j)d_j v_n^{(j)} + \frac{h}{2}\sum_{j=0}^K G(i,j)d_j f_{n-1}^{(j)}, \\ v_n^{(i)} = \frac{b_2\Gamma(\beta+\sigma)}{\Gamma(\beta)[\Gamma(\beta+\sigma)-\lambda\eta^{\beta+\sigma-1}]}t_i^{\beta-1} - \frac{h}{2}\sum_{j=0}^K MG(i,j)d_j v_n^{(j)} \\ \qquad - \frac{h}{2}\sum_{j=0}^K NG(i,j)d_j u_n^{(j)} + \frac{h}{2}\sum_{j=0}^K G(i,j)d_j g_{n-1}^{(j)}, \end{cases} \tag{4.5}$$

and

$$\begin{cases} x_n^{(i)} = \frac{l_1}{\Gamma(\alpha)}t_i^{\alpha-1} + \frac{h}{2}\sum_{j=0}^K H(i,j)d_j\phi_q(u_n^{(j)}), \\ y_n^{(i)} = \frac{l_2}{\Gamma(\alpha)}t_i^{\alpha-1} + \frac{h}{2}\sum_{j=0}^K H(i,j)d_j\phi_q(v_n^{(j)}), \end{cases} \tag{4.6}$$

for the unknown $u_n^{(i)}$, $x_n^{(i)}$, $0 \le i \le K$, where $\{d_j\}$ are the coefficients in the rule, $d_0 = d_K = 1$, and $d_j = 2$ for $1 \le j \le K-1$.

Setting $G_{ij} = \frac{h}{2}\sum_{j=0}^{K} G(i,j)d_j$, $H_{ij} = \frac{h}{2}\sum_{j=0}^{K} H(i,j)d_j$, the matrix $\Phi = (G_{ij})$, and $B = (H_{ij})$ with the identity matrix $\mathbb{I}$. Systems (4.5) and (4.6) can be written as a system of matrix–vector equations

$$\begin{cases} (I + M\Phi)\overrightarrow{U}_n + N\Phi\overrightarrow{v}_n = \frac{b_1\Gamma(\beta+\sigma)}{\Gamma(\beta)[\Gamma(\beta+\sigma)-\lambda\eta^{\beta+\sigma-1}]}S^{\beta-1} + \overrightarrow{F}_{n-1}, \\ (I + M\Phi)\overrightarrow{V}_n + N\Phi\overrightarrow{u}_n = \frac{b_2\Gamma(\beta+\sigma)}{\Gamma(\beta)[\Gamma(\beta+\sigma)-\lambda\eta^{\beta+\sigma-1}]}S^{\beta-1} + \overrightarrow{G}_{n-1}, \\ \overrightarrow{X}_n = \frac{l_1}{\Gamma(\alpha)}S^{\alpha-1} + B\phi_q(\overrightarrow{U}_n), \\ \overrightarrow{Y}_n = \frac{l_2}{\Gamma(\alpha)}S^{\alpha-1} + B\phi_q(\overrightarrow{V}_n), \end{cases} \tag{4.7}$$

where $\overrightarrow{X}_n = [x_n^{(0)}, x_n^{(1)}, \dots, x_n^{(K)}]$, $\overrightarrow{Y}_n = [y_n^{(0)}, y_n^{(1)}, \dots, y_n^{(K)}]$, $\overrightarrow{U}_n = [u_n^{(0)}, u_n^{(1)}, \dots, u_n^{(K)}]$, $\overrightarrow{V}_n = [v_n^{(0)}, v_n^{(1)}, \dots, v_n^{(K)}]$, $S = [t_0, t_1, \dots, t_K]^T$, and $\overrightarrow{F}_{n-1}$, $\overrightarrow{G}_{n-1}$ are column vectors of their components $F_{n-1}^{(i)} = \frac{h}{2}\sum_{j=0}^{K} G(i,j)d_j f_{n-1}^{(j)}$, $G_{n-1}^{(i)} = \frac{h}{2}\sum_{j=0}^{K} G(i,j)d_j g_{n-1}^{(j)}$.

Example 4.1 Consider the following problem:

$$\begin{cases} -D^{\frac{7}{4}}(\phi_4((-D^{\frac{5}{3}}x(t))) \\ \quad = \frac{1}{6}x^{\frac{1}{3}}(t)[(-D^{\frac{5}{3}}x(t))^{\frac{1}{3}} - 18 - t^{\frac{2}{9}}] - y(t)[(-D^{\frac{5}{3}}y(t)) - 2t^{\frac{2}{3}}], \quad t \in (0,1], \\ -D^{\frac{7}{4}}(\phi_4(-D^{\frac{5}{3}}y(t))) \\ \quad = \frac{1}{6}y^{\frac{1}{3}}(t)[(-D^{\frac{5}{3}}y(t))^{\frac{1}{3}} - 18 - t^{\frac{2}{9}}] - x(t)[(-D^{\frac{5}{3}}x(t)) - 2t^{\frac{2}{3}}], \quad t \in (0,1], \\ D^{\frac{5}{3}}x(0) = 0, \\ D^{\frac{3}{4}}(\phi_4(-D^{\frac{5}{3}}x(1))) = I^{\frac{5}{4}}h(\frac{1}{4}, \phi_4(-D^{\frac{5}{3}}x(\frac{1}{4}))) + 0.1 \\ \qquad = \frac{1}{\Gamma(\frac{5}{4})}\int_0^{\frac{1}{4}}(\frac{1}{4}-s)^{\frac{1}{4}}(s+1)(\phi_4(-D^{\frac{5}{3}}x(s)))\,ds + 0.1, \\ x(0) = 0, \qquad D^{\frac{2}{3}}x(1) = \frac{1}{\Gamma(\frac{3}{2})}\int_0^{\frac{1}{2}}(\frac{1}{2}-s)^{\frac{1}{2}}x(s)\,ds + 0.3, \\ D^{\frac{5}{3}}y(0) = 0, \\ D^{\frac{3}{4}}(\phi_4(-D^{\frac{5}{3}}y(1))) = I^{\frac{5}{4}}k(\frac{1}{4}, \phi_4(-D^{\frac{5}{3}}y(\frac{1}{4}))) + 0.2 \\ \qquad = \frac{1}{\Gamma(\frac{5}{4})}\int_0^{\frac{1}{4}}(\frac{1}{4}-s)^{\frac{1}{4}}(e^s+1)(\phi_4(-D^{\frac{5}{3}}y(s)))\,ds + 0.2, \\ y(0) = 0, \qquad D^{\frac{2}{3}}y(1) = \frac{1}{\Gamma(\frac{3}{2})}\int_0^{\frac{1}{2}}(\frac{1}{2}-s)^{\frac{1}{2}}y(s)\,ds + 0.4, \end{cases} \tag{4.8}$$

where $\beta = \frac{7}{4}$, $\alpha = \frac{5}{3}$, $\sigma = \frac{5}{4}$, $\omega = \frac{3}{2}$, $\eta = \frac{1}{4}$, $\xi = \frac{1}{2}$, $a_1 = 0.1$, $a_2 = 0.2$, $d_1 = 0.003$, $d_2 = 0.004$, $p = 4$, and

$$\begin{cases} f(t, x(t), y(t), D^{\frac{5}{3}}x(t), D^{\frac{5}{3}}y(t)) \\ \quad = \frac{1}{6}x^{\frac{1}{3}}(t)[(-D^{\frac{5}{3}}x(t))^{\frac{1}{3}} - 18 - t^{\frac{2}{9}}] - y(t)[(-D^{\frac{5}{3}}y(t)) - 2t^{\frac{2}{3}}], \\ g(t, y(t), x(t), D^{\frac{5}{3}}y(t), D^{\frac{5}{3}}x(t)) \\ \quad = \frac{1}{6}y^{\frac{1}{3}}(t)[(-D^{\frac{5}{3}}y(t))^{\frac{1}{3}} - 18 - t^{\frac{2}{9}}] - x(t)[(-D^{\frac{5}{3}}x(t)) - 2t^{\frac{2}{3}}], \\ h(t, \phi_4(-D^{\frac{5}{3}}x)) = (t+1)(\phi_4(-D^{\frac{5}{3}}x)), \\ k(t, \phi_4(-D^{\frac{5}{3}}y)) = (e^t+1)(\phi_4(-D^{\frac{5}{3}}y)). \end{cases}$$

Take $x_0(t) = 0$ and $y_0(t) = 3t^{\frac{2}{3}} - \frac{9\Gamma(\frac{2}{3})}{14\Gamma(\frac{1}{3})}t^{\frac{7}{3}}$. Then $-1 \le -t^{\frac{2}{3}} = D^{\frac{5}{3}}y_0(t) \le D^{\frac{5}{3}}x_0(t) = 0$. It is not difficult to verify that (H_1) holds.

Since the function $\sqrt[3]{x} + x^3$ is increasing for $x \in R$, we obtain

$$\begin{aligned}
&f\big(t,\overline{x(t)},\overline{y(t)},D^{\frac{5}{3}}\overline{x(t)},D^{\frac{5}{3}}\overline{y(t)}\big) - f\big(t,x(t),y(t),D^{\frac{5}{3}}x(t),D^{\frac{5}{3}}y(t)\big)\\
&\quad= \frac{1}{6}\overline{x(t)}^{\frac{1}{3}}\big[\big(-D^{\frac{5}{3}}\overline{x(t)}\big)^{\frac{1}{3}} - 600t^{\frac{1}{100}} - t^{\frac{1}{9}}\big] - \overline{y(t)}\big[\big(-D^{\frac{5}{3}}\overline{y(t)}\big) - 2t^{\frac{1}{3}}\big]\\
&\qquad - \frac{1}{6}x^{\frac{1}{3}}(t)\big[\big(-D^{\frac{5}{3}}x(t)\big)^{\frac{1}{3}} - 600t^{\frac{1}{100}} - t^{\frac{1}{9}}\big] + y(t)\big[\big(-D^{\frac{5}{3}}y(t)\big) - 2t^{\frac{1}{3}}\big],\\
&\quad\le \frac{1}{6}x^{\frac{1}{3}}(t)\big[\big(-D^{\frac{5}{3}}\overline{x(t)}\big)^{\frac{1}{3}} - \big(-D^{\frac{5}{3}}x(t)\big)^{\frac{1}{3}}\big]\\
&\quad\le \frac{1}{6}\sqrt[3]{3}\big[\big(-D^{\frac{5}{3}}x(t)\big)^{3} - \big(-D^{\frac{5}{3}}\overline{x(t)}\big)^{3}\big]\\
&\quad= \frac{1}{6}\sqrt[3]{3}\big[\Phi_4\big(-D^{\frac{5}{3}}x(t)\big) - \Phi_4\big(-D^{\frac{5}{3}}\overline{x(t)}\big)\big],
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
&k\big(t,\phi_4\big(-D^{\frac{5}{3}}y\big)\big) - k\big(t,\phi_4\big(-D^{\frac{5}{3}}x\big)\big)\\
&\quad= \big(e^t + 1\big)\big(\phi_4\big(-D^{\frac{5}{3}}y\big)\big) - \big(e^t + 1\big)\big(\phi_4\big(-D^{\frac{5}{3}}x\big)\big)\\
&\quad= \big(e^t + 1\big)\big[\phi_4\big(-D^{\frac{5}{3}}y\big) - \big(\phi_4\big(-D^{\frac{5}{3}}x\big)\big)\big]\\
&\quad\ge (t + 1)\big[\phi_4\big(-D^{\frac{5}{3}}y\big) - \big(\phi_4\big(-D^{\frac{5}{3}}x\big)\big)\big]\\
&\quad\ge \phi_4\big(-D^{\frac{5}{3}}y\big) - \big(\phi_4\big(-D^{\frac{5}{3}}x\big)\big),
\end{aligned} \tag{4.10}$$

where $x_0(t) \le \overline{x(t)} \le x(t) \le y_0(t)$, $x_0(t) \le y(t) \le \overline{y(t)} \le y_0(t)$, and $x_0(t) \le x(t) \le y(t) \le y_0(t)$. Thus (H_2) and (H_3) hold. From (4.9) and (4.10) we have $M = \frac{1}{6}\sqrt[3]{3}$, $N = 0$, and $\lambda = 1$. Then

$$\begin{aligned}
&\Gamma(\beta + \sigma) = \Gamma\left(\frac{7}{4} + \frac{5}{4}\right) = \Gamma(3) = 2 > \lambda\eta^{\beta+\sigma-1} = 1 \cdot \left(\frac{1}{4}\right)^2 = 0.0625,\\
&2\Gamma(\beta + \sigma)(M + N)\\
&\quad= 2 \cdot \Gamma(3) \cdot \frac{\sqrt[3]{3}}{6} \approx 0.9614 < \Gamma(\beta)\big[\Gamma(\beta + \sigma) - \lambda\eta^{\beta+\sigma-1}\big]\\
&\quad= \Gamma\left(\frac{7}{4}\right)\left[\Gamma(3) - 1 \cdot \left(\frac{1}{4}\right)^2\right] \approx 1.7808,\\
&\Gamma(2 - \beta)\lambda\eta^{\sigma} = \Gamma\left(\frac{1}{4}\right) \cdot 1 \cdot \left(\frac{1}{4}\right)^{\frac{5}{4}} \approx 0.6410 < \Gamma(\sigma) = \Gamma\left(\frac{5}{4}\right) \approx 0.9064,
\end{aligned}$$

which show that (H_4), (H_5), and (H_6) hold. Thus all conditions of Theorem 3.1 are satisfied. In consequence, the nonlinear system (4.8) has an extremal solution $(x^*, y^*) \in [x_0(t), y_0(t)] \times [x_0(t), y_0(t)]$. Moreover, for this example, we found that for $\delta = 10^{-10}$, which took $N = 16$ iterations for $E(N) < \delta$. The graphs of x_n and y_n for some values of n are shown in Table 1 and Fig. 1.

Table 1 $E(n) = 3n + 1, n = 0, 1, 2, 3, 4, 5$

n	1	4	7	10	13	16
$E(n)$	0.6696	0.0096	2.6171e–04	2.3360e–07	1.5608e–8	1.3942e–11

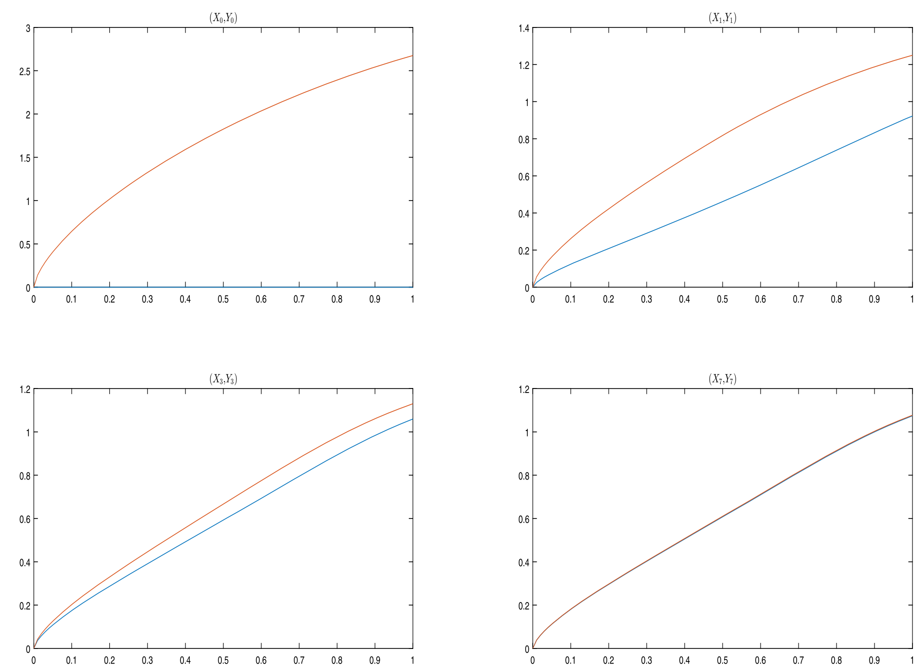

Figure 1 Graphs of X_n and Y_n

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Competing interests
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Authors' contributions
The authors have made the same contribution. Both authors read and approved the final manuscript.

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