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Reciprocity of poly-Dedekind-type DC sums involving poly-Euler functions

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Abstract

The classical Dedekind sums appear in the transformation behavior of the logarithm of the Dedekind eta-function under substitutions from the modular group. The Dedekind sums and their generalizations are defined in terms of Bernoulli functions and their generalizations, and are shown to satisfy some reciprocity relations. In contrast, Dedekind-type DC (Daehee and Changhee) sums and their generalizations are defined in terms of Euler functions and their generalizations. The purpose of this paper is to introduce the poly-Dedekind-type DC sums, which are obtained from the Dedekind-type DC sums by replacing the Euler function by poly-Euler functions of arbitrary indices, and to show that those sums satisfy, among other things, a reciprocity relation.

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1 Introduction

Apostol [1, 2] considered the generalized Dedekind sums given by

$$S_p(h, m) = \sum_{\mu=1}^{m-1} \frac{\mu}{m} \bar{B}_p\left(\frac{h\mu}{m}\right) \quad (1)$$

and showed that they satisfy a reciprocity relation. Here $\bar{B}_p(x) = B_p(x - [x])$ are the Bernoulli functions with Bernoulli polynomials $B_p(x)$ given by

$$\frac{t}{e^t - 1} e^{xt} = \sum_{p=0}^{\infty} B_p(x) \frac{t^p}{p!}.$$

We remark that the Dedekind sum $S(h, m) = S_1(h, m)$ appears in the transformation behavior of the logarithm of the Dedekind eta-function under substitutions from the modular group, and a reciprocity law of that was demonstrated by Dedekind in 1892.

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As an extension of the sums in (1), the poly-Dedekind sums given by

$$S_p^{(k)}(h, m) = \sum_{\mu=1}^{m-1} \frac{\mu}{m} \bar{B}_p^{(k)}\left(\frac{h\mu}{m}\right)$$

were considered, and a reciprocity law for those sums was shown in [16, 19]. Here $B_p^{(k)}(x)$ are the type 2 poly-Bernoulli polynomials of index k , $\bar{B}_p^{(k)}(x) = B_p^{(k)}(x - [x])$ (see [16]), and $B_p^{(1)}(x) = B_p(x)$.

The Dedekind-type DC sums (see (8)) were first introduced and shown to satisfy a reciprocity relation in [13]. The aim of this paper is introducing the poly-Dedekind-type DC sums (see (2)), which are obtained from the Dedekind-type DC sums by replacing the Euler function by poly-Euler functions of arbitrary indices, and showing that those sums satisfy, among other things, a reciprocity relation (see (3)). The motivation of this paper is to explore our new sums in connection with modular forms, zeta functions, and trigonometric sums, just as in the cases of Apostol–Dedekind sums, their generalizations, and some related sums. Indeed, Simsek [22] found trigonometric representations of the Dedekind-type DC sums and their relations to the Clausen functions, polylogarithm function, Hurwitz zeta function, generalized Lambert series (G-series), and Hardy–Berndt sums. In addition, Bayad and Simsek [3] studied three new shifted sums of Apostol–Dedekind–Rademacher type. These sums generalize the classical Dedekind–Rademacher sums and can be expressed in terms of Jacobi modular forms or cotangent functions or special values of the Barnes multiple zeta functions. They found reciprocity laws for these sums and demonstrated that some well-known reciprocity laws can be deduced from their results. As applications of our results, we plan to carry out this line of research in a subsequent paper.

In this paper, we consider the poly-Dedekind-type DC sums defined by

$$T_p^{(k)}(h, m) = 2 \sum_{\mu=1}^{m-1} (-1)^\mu \frac{\mu}{m} \bar{E}_p^{(k)}\left(\frac{h\mu}{m}\right), \tag{2}$$

where $h, m, p \in \mathbb{N}$, and $\bar{E}_p^{(k)}$ are the poly-Euler functions of index k given by $\bar{E}_p^{(k)}(x) = E_p^{(k)}(x - [x])$ (see (12), (17)). We show the following reciprocity relation for the poly-Dedekind-type DC sums given by (see Theorem 9)

$$m^p T_p^{(k)}(h, m) + h^p T_p^{(k)}(m, h) = 2 \sum_{\mu=0}^{m-1} \sum_{l=0}^p \sum_{v=0}^{h-1} \sum_{j=1}^{p+1-l} (-1)^{\mu+v} \frac{(mh)^{l-1} \binom{p}{l} S_1(p-l+1, j)}{(p-l+1)j^{k-1}} \times ((\mu h)m^{p-l} + (vm)h^{p-l}) \bar{E}_l\left(\frac{v}{h} + \frac{\mu}{m}\right), \tag{3}$$

where $m, h, p \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$ and $h \equiv 1 \pmod{2}$, and $k \in \mathbb{Z}$.

For $k = 1$, this reciprocity relation for the poly-Dedekind-type DC sums reduces to that for the Dedekind-type DC sums given by (see Corollary 4)

$$m^p T_p(h, m) + h^p T_p(m, h) = 2(mh)^{p-1} \sum_{\mu=0}^{m-1} \sum_{\nu=0}^{h-1} (-1)^{\mu+\nu} (\mu h + \nu m) \bar{E}_p\left(\frac{\nu}{h} + \frac{\mu}{m}\right),$$

where $m, h, p \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$ and $h \equiv 1 \pmod{2}$.

For the rest of this section, we recall some necessary facts. It is well known that Euler polynomials are defined by

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad (\text{see [1-3, 5-7, 10-17, 19, 21, 22]}). \tag{4}$$

When $x = 0$, $E_n = E_n(0)$ are called the Euler numbers.

From (4), we note that

$$E_n(x) = \sum_{l=0}^n \binom{n}{l} E_l x^{n-l}, \quad (n \geq 0), \quad (\text{see [1-3, 5-7, 10-17, 19-22]}). \tag{5}$$

The first few of Euler numbers are $E_0 = 1, E_1 = -\frac{1}{2}, E_2 = 0, E_3 = \frac{1}{4}, E_4 = 0, E_5 = -\frac{1}{2}, \dots$, and $E_{2k} = 0$ for $k = 1, 2, \dots$

From (4) we note that $E_0 = 1$ and $E_n(1) + E_n = 2\delta_{0,n}$ ($n \geq 0$), where $\delta_{n,k}$ is the Kronecker symbol. The Euler functions $\bar{E}_n(x)$ are defined by

$$\bar{E}_n(x) = E_n(x - [x]) \quad (n \geq 0) \quad (\text{see [2, 6, 13, 22]}), \tag{6}$$

where $[x]$ denotes the greatest integer not exceeding x .

From (4) we can easily derive the following identity:

$$2 \sum_{k=0}^{n-1} (-1)^k k^l = (-1)^{n-1} E_l(n) + E_l \quad (n \in \mathbb{N}). \tag{7}$$

It is known that Dedekind-type DC sums are given by

$$T_p(h, m) = 2 \sum_{\mu=0}^{m-1} (-1)^\mu \frac{\mu}{m} \bar{E}_p\left(\frac{h\mu}{m}\right) \quad (h, m \in \mathbb{N}) \quad (\text{see [13, 22]}). \tag{8}$$

Note that

$$T_1(h, m) = 2 \sum_{\mu=0}^{m-1} (-1)^\mu \left(\left(\frac{\mu}{m}\right)\right) \left(\left(\frac{h\mu}{m}\right)\right) \quad (\text{see [1, 2, 6, 12, 14, 21]}),$$

where $((x))$ is defined by

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \text{ is not an integer,} \\ 0 & \text{if } x \text{ is an integer.} \end{cases}$$

The Genocchi polynomials are defined by

$$\frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} \quad (\text{see [7, 11, 17]}). \tag{9}$$

When $x = 0$, $G_n = G_n(0)$ are called the Genocchi numbers.

Note that $G_0 = 0$, $G_1 = 1$, $G_2 = -1$, $G_3 = 0$, $G_4 = 1$, $G_5 = 0$, $G_6 = -3, \dots$, and $G_{2k+1} = 0$ for $k = 1, 2, 3, \dots$.

By (4) and (9) we get

$$\frac{G_{n+1}(x)}{n + 1} = E_n(x), \quad \frac{G_{n+1}}{n + 1} = E_n \quad (n \geq 0).$$

The degenerate Hardy polyexponential function of index k is defined by

$$\text{Ei}_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{x^n (1)_{n,\lambda}}{n^k (n-1)!} \quad (k \in \mathbb{Z}) \quad (\text{see [15]}), \tag{10}$$

where $(x)_{0,\lambda} = 1$ and $(x)_{n,\lambda} = x(x-\lambda) \cdots (x-(n-1)\lambda)$, $(n \geq 1)$.

Recently, the degenerate poly-Genocchi polynomials of index k were defined in terms of the degenerate Hardy polyexponential function of index k by

$$\frac{2\text{Ei}_{k,\lambda}(\log_{\lambda}(1+t))}{e_{\lambda}(t) + 1} e_{\lambda}^x(t) = \sum_{n=0}^{\infty} G_n^{(k)}(x) \frac{t^n}{n!} \quad (\text{see [17]}), \tag{11}$$

where

$$e_{\lambda}^x(t) = \sum_{n=0}^{\infty} \frac{(x)_{n,\lambda}}{n!} t^n, \quad e_{\lambda}(t) = e_{\lambda}^1(t), \quad \text{and} \quad \log_{\lambda}(t) = \frac{1}{\lambda}(t^{\lambda} - 1)$$

is the compositional inverse to $e_{\lambda}(t)$ satisfying $e_{\lambda}(\log_{\lambda}(t)) = \log_{\lambda}(e_{\lambda}(t)) = t$. Taking $\lambda \rightarrow 0$ in (11), we get the poly-Genocchi polynomials of index k given by

$$\frac{2\text{Ei}_k(\log(1+t))}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x) \frac{t^n}{n!} \quad (\text{see [7, 17]}), \tag{12}$$

where $G_n^{(k)}(x) = \lim_{\lambda \rightarrow 0} G_{n,\lambda}^{(k)}(x)$ ($n \geq 0$), and

$$\text{Ei}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k (n-1)!} \quad (\text{see [10, 15]}) \tag{13}$$

is the polyexponential function of index k .

When $x = 0$, $G_n^{(k)} = G_n^{(k)}(0)$, ($n \geq 0$) are called the poly-Genocchi numbers of index k . By (12) we easily get $G_0^{(k)} = 0$, $G_1^{(k)} = 1$, $G_2^{(k)} = -2 + 2^{1-k}, \dots$. Also, from (12) we note that

$$G_n^{(k)}(x) = \sum_{l=0}^n \binom{n}{l} G_l^{(k)} x^{n-l} \quad (n \geq 0). \tag{14}$$

Remark 1 The polyexponential functions were first considered by Hardy and are given by

$$e(x, a|s) = \sum_{n=0}^{\infty} \frac{x^n}{(n+a)^s n!} \quad (\operatorname{Re}(a) > 0) \text{ (see [4, 8, 9]).}$$

Komatsu [18], defined the polylogarithm factorial function $Lif_k(x)$ by $xLif_k(x) = xe(x, 1|k) = Ei_k(x)$. So the polylogarithm factorial functions are particular cases of Hardy’s polyexponential functions, but our polyexponential functions are not. In fact, a slight difference between ours and Komatsu’s functions is crucial in defining, for example, the type 2 poly-Bernoulli polynomials (see [10, 15]) and also in constructing poly-Dedekind sums associated with such polynomials (see [16, 19]). Here we recall from [10] that the type 2 poly-Bernoulli polynomials $\beta_n^{(k)}(x)$ of index k are defined by

$$\frac{Ei_k(\log(1+t))}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} \beta_n^{(k)}(x) \frac{t^n}{n!}. \tag{15}$$

We also recall that for any integer k , the poly-Bernoulli polynomials $B_n^{(k)}(x)$ of index k are defined by

$$\frac{Li_k(1 - e^{-t})}{1 - e^{-t}} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}, \tag{16}$$

where the polylogarithm functions $Li_k(x)$ are given by $Li_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}$.

The reason why $Ei_k(x)$ is needed and $Lif_k(x)$ is not in (15) is twofold. The first reason is that $Ei_k(x)$ has order 1, so that the composition $Ei_k(\log(1+t))$ still has order 1, which is definitely required, whereas $Lif_k(x)$ has order 0, so that $Lif_k(\log(1+t))$ also has order 0. The second reason is that we want $\beta_n^{(1)}(x)$ to be the ordinary Bernoulli polynomials when $k = 1$. Indeed, $Ei_1(x) = e^x - 1$, so that $\beta_n^{(1)}(x)$ are those polynomials with $\beta_1^{(1)}(x) = x - \frac{1}{2}$.

The construction of the type 2 poly-Bernoulli polynomials is in parallel with that of the poly-Bernoulli polynomials. Note that $Li_k(x)$ has order 1, so that the composition $Li_k(1 - e^{-t})$ also has order 1. In addition, $Li_1(x) = -\log(1 - x)$, so that $B_n^{(1)}(x)$ are the ordinary Bernoulli polynomials with $B_1^{(1)}(x) = x + \frac{1}{2}$ (see (16)). Thus we may say that $Ei_k(x)$ is a kind of a compositional inverse to $Li_k(x)$.

Now we define the *poly-Euler polynomials of index k* by

$$E_n^{(k)}(x) = \frac{G_{n+1}^{(k)}(x)}{n+1} \quad (n \geq 0). \tag{17}$$

When $x = 0$, $E_n^{(k)} = E_n^{(k)}(0)$ are called the poly-Euler numbers of index k . Note that $E_n^{(1)}(x) = E_n(x)$ and $G_n^{(1)}(x) = G_n(x)$.

From (14) we note that

$$\begin{aligned}
 E_n^{(k)}(x) &= \frac{1}{n+1} G_{n+1}^{(k)}(x) = \frac{1}{n+1} \sum_{l=0}^{n+1} \binom{n+1}{l} G_l^{(k)} x^{n+1-l} \\
 &= \frac{1}{n+1} \sum_{l=1}^{n+1} \binom{n+1}{l} G_l^{(k)} x^{n+1-l} = \frac{1}{n+1} \sum_{l=0}^n \binom{n+1}{l+1} G_{l+1}^{(k)} x^{n-l} \\
 &= \sum_{l=0}^n \binom{n}{l} \frac{G_{l+1}^{(k)}}{l+1} x^{n-l} = \sum_{l=0}^n \binom{n}{l} E_l^{(k)} x^{n-l} \quad (n \geq 0).
 \end{aligned}
 \tag{18}$$

Remark 2 From (8) and (2) we see that the poly-Dedekind type DC sums are obtained from the Dedekind-type DC sums by replacing the Euler functions by poly-Euler functions of arbitrary indices. Note that the key to this generalization is the construction of poly-Euler polynomials defined in (17), which is done in an elaborate manner. First, we replace t by $Ei_k(\log(1+t))$ as in (12), so that we construct the poly-Genocchi polynomials $G_n^{(k)}(x)$ of index k such that $G_n^{(1)}(x) = G_n(x)$ are the usual Genocchi polynomials. Next, defining the poly-Euler polynomials $E_n^{(k)}(x)$ as in (17), we have the desirable property $E_n^{(1)}(x) = E_n(x)$. Consequently, for $k = 1$, the poly-Dedekind-type DC sums $T_p^{(k)}(h, m)$ in (2) reduce to the Dedekind-type DC sums $T_p(h, m)$ in (8).

In Sect. 2, we derive various facts about the poly-Genocchi and poly-Euler polynomials that will be needed in the next section. In Sect. 3, we define the poly-Dedekind-type DC sums and demonstrate, among other things, a reciprocity relation for them.

2 Poly-Genocchi polynomials and poly-Euler polynomials

By (12) we have

$$\begin{aligned}
 2Ei_k(\log(1+t)) &= \frac{2Ei_k(\log(1+t))}{e^t + 1} e^t + \frac{2Ei_k(\log(1+t))}{e^t + 1} \\
 &= \sum_{n=0}^{\infty} (G_n^{(k)}(1) + G_n^{(k)}) \frac{t^n}{n!}.
 \end{aligned}
 \tag{19}$$

On the other hand, we also have

$$\begin{aligned}
 2Ei_k(\log(1+t)) &= 2 \sum_{m=1}^{\infty} \frac{1}{m^k(m-1)!} (\log(1+t))^m \\
 &= \sum_{n=1}^{\infty} \left(2 \sum_{m=1}^n \frac{1}{m^{k-1}} S_1(n, m) \right) \frac{t^n}{n!},
 \end{aligned}
 \tag{20}$$

where $S_1(n, m)$ are the Stirling numbers of the first kind.

Therefore by (19) and (20) we get the following theorem.

Theorem 1 For $n \geq 1$, we have

$$2 \sum_{m=1}^n \frac{1}{m^{k-1}} S_1(n, m) = G_n^{(k)}(1) + G_n^{(k)}.$$

Corollary 1 For $n \geq 1$, we have

$$\frac{2}{n} \sum_{m=1}^n \frac{1}{m^{k-1}} S_1(n, m) = E_{n-1}^{(k)}(1) + E_{n-1}^{(k)}.$$

From (14) and (18) we see that

$$\frac{d}{dx} G_{n+1}^{(k)}(x) = (n + 1)G_n^{(k)}(x), \quad \frac{d}{dx} E_n^{(k)}(x) = nE_{n-1}^{(k)}(x) \quad (n \geq 1).$$

Thus we note that

$$\int_0^x G_n^{(k)}(x) dx = \frac{1}{n + 1} (G_{n+1}^{(k)}(x) - G_{n+1}^{(k)}),$$

$$\int_0^x E_{n-1}^{(k)}(x) dx = \frac{1}{n} (E_n^{(k)}(x) - E_n^{(k)}) \quad (n \geq 1).$$

From (5) and (11) we have

$$\begin{aligned} \frac{2\text{Ei}_k(\log(1 + t))}{e^t + 1} e^{xt} &= \sum_{m=1}^{\infty} \frac{(\log(1 + t))^m}{m^k(m - 1)!} \sum_{l=0}^{\infty} E_l(x) \frac{t^l}{l!} \\ &= \sum_{j=1}^{\infty} \sum_{m=1}^j \frac{S_1(j, m)}{m^{k-1}} \frac{t^j}{j!} \sum_{l=0}^{\infty} E_l(x) \frac{t^l}{l!} \\ &= \sum_{n=1}^{\infty} \left(\sum_{j=1}^n \sum_{m=1}^j \binom{n}{j} \frac{S_1(j, m)}{m^{k-1}} E_{n-j}(x) \right) \frac{t^n}{n!}. \end{aligned} \tag{21}$$

On the other hand, we also have

$$\frac{2\text{Ei}_k(\log(1 + t))}{e^t + 1} e^{xt} = \sum_{n=1}^{\infty} G_n^{(k)}(x) \frac{t^n}{n!} = \sum_{n=1}^{\infty} nE_{n-1}^{(k)}(x) \frac{t^n}{n!}. \tag{22}$$

Therefore by (21) and (22) we obtain the following theorem.

Theorem 2 For $n \in \mathbb{N}$, we have

$$E_{n-1}^{(k)}(x) = \frac{1}{n} \sum_{j=1}^n \sum_{m=1}^j \binom{n}{j} \frac{S_1(j, m)}{m^{k-1}} E_{n-j}(x).$$

For $m \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$, we have

$$\begin{aligned} \frac{2}{e^t + 1} e^{xt} &= \frac{2}{1 + e^{mt}} \sum_{i=0}^{m-1} (-1)^i e^{it} e^{xt} \\ &= \sum_{i=0}^{m-1} (-1)^i \frac{2}{e^{mt} + 1} e^{(\frac{i+x}{m})mt} \\ &= \sum_{n=0}^{\infty} \left(m^n \sum_{i=0}^{m-1} (-1)^i E_n\left(\frac{x+i}{m}\right) \right) \frac{t^n}{n!}. \end{aligned} \tag{23}$$

By (4) and (23) we get the distribution relation

$$E_n(x) = m^n \sum_{i=0}^{m-1} (-1)^i E_n\left(\frac{x+i}{m}\right), \tag{24}$$

where $m \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$, and $n \geq 0$.

For $x \in \mathbb{N}$, we have

$$\begin{aligned} 2 \sum_{i=0}^{x-1} (-1)^i e^{it} \text{Ei}_k(\log(1+t)) &= 2 \sum_{i=0}^{x-1} (-1)^i \sum_{l=0}^{\infty} i^l \frac{t^l}{l!} \sum_{j=1}^{\infty} \frac{(\log(1+t))^j}{j^k(j-1)!} \\ &= \sum_{l=0}^{\infty} 2 \sum_{i=0}^{x-1} (-1)^i i^l \frac{t^l}{l!} \sum_{m=1}^{\infty} \sum_{j=1}^m \frac{S_1(m,j)}{j^{k-1}} \frac{t^m}{m!} \\ &= \sum_{n=1}^{\infty} \left(2 \sum_{m=1}^n \sum_{j=1}^m \sum_{i=0}^{x-1} (-1)^i i^{n-m} \binom{n}{m} \frac{S_1(m,j)}{j^{k-1}} \right) \frac{t^n}{n!}. \end{aligned} \tag{25}$$

On the other hand, we also have

$$\begin{aligned} 2 \sum_{i=0}^{x-1} (-1)^i e^{it} \text{Ei}_k(\log(1+t)) &= \frac{2\text{Ei}_k(\log(1+t))}{e^t + 1} ((-1)^{x-1} e^{xt} + 1) \\ &= \sum_{n=0}^{\infty} ((-1)^{x-1} G_n^{(k)}(x) + G_n^{(k)}) \frac{t^n}{n!}. \end{aligned} \tag{26}$$

Therefore by (25) and (26) we obtain the following theorem.

Theorem 3 For $x, n \in \mathbb{N}$, we have

$$(-1)^{x-1} G_n^{(k)}(x) + G_n^{(k)} = 2 \sum_{m=1}^n \sum_{j=1}^m \sum_{i=0}^{x-1} (-1)^i i^{n-m} \binom{n}{m} \frac{S_1(m,j)}{j^{k-1}}.$$

Note that, for $k = 1$, we have

$$(-1)^{x-1} G_n^{(1)}(x) + G_n^{(1)} = 2n \sum_{i=0}^{x-1} (-1)^i i^{n-1}.$$

Corollary 2 For $x, n \in \mathbb{N}$, we have

$$(-1)^{x-1} E_{n-1}^{(k)}(x) + E_{n-1}^{(k)} = \frac{2}{n} \sum_{m=1}^n \sum_{j=1}^m \sum_{i=0}^{x-1} (-1)^i i^{n-m} \binom{n}{m} \frac{S_1(m,j)}{j^{k-1}}.$$

Note that

$$(-1)^{x-1} E_{n-1}^{(1)}(x) + E_{n-1}^{(1)} = 2 \sum_{i=0}^{x-1} (-1)^i i^{n-1} \quad (n, x \in \mathbb{N}).$$

For $m \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$, we note that

$$\begin{aligned}
 & 2 \frac{\text{Ei}_k(\log(1+t))}{e^t + 1} e^{xt} \tag{27} \\
 &= \frac{1}{m} \sum_{s=0}^{m-1} (-1)^s e^{\left(\frac{s+x}{m}\right)mt} \frac{2mt}{e^{mt} + 1} \frac{1}{t} \text{Ei}_k(\log(1+t)) \\
 &= \sum_{l=0}^{\infty} m^{l-1} \sum_{s=0}^{m-1} (-1)^s G_l \left(\frac{s+x}{m} \right) \frac{t^l}{l!} \frac{1}{t} \sum_{j=1}^{\infty} \frac{(\log(1+t))^j}{j^k(j-1)!} \\
 &= \sum_{l=0}^{\infty} m^{l-1} \sum_{s=0}^{m-1} (-1)^s G_l \left(\frac{s+x}{m} \right) \frac{t^l}{l!} \frac{1}{t} \sum_{i=1}^{\infty} \sum_{j=1}^i \frac{S_1(i,j)}{j^{k-1}} \frac{t^i}{i!} \\
 &= \sum_{l=0}^{\infty} m^{l-1} \sum_{s=0}^{m-1} (-1)^s G_l \left(\frac{s+x}{m} \right) \frac{t^l}{l!} \sum_{i=0}^{\infty} \sum_{j=1}^{i+1} \frac{1}{j^{k-1}} \frac{S_1(i+1,j)}{i+1} \frac{t^i}{i!} \\
 &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \binom{n}{l} m^{l-1} \sum_{j=1}^{n-l+1} \sum_{s=0}^{m-1} (-1)^s G_l \left(\frac{s+x}{m} \right) \frac{1}{j^{k-1}} \frac{S_1(n-l+1,j)}{n-l+1} \right) \frac{t^n}{n!}.
 \end{aligned}$$

Therefore by (27) we obtain the following theorem.

Theorem 4 For $n \geq 0$ and $m \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$, we have

$$G_n^{(k)}(x) = \sum_{l=0}^n \binom{n}{l} m^{l-1} \sum_{j=1}^{n-l+1} \sum_{s=0}^{m-1} (-1)^s G_l \left(\frac{s+x}{m} \right) \frac{1}{j^{k-1}} \frac{S_1(n-l+1,j)}{n-l+1}.$$

From Theorem 4 we have

$$\begin{aligned}
 \frac{G_n^{(k)}(x)}{n} &= \frac{1}{n} \sum_{l=1}^n \binom{n}{l} m^{l-1} \sum_{j=1}^{n-l+1} \sum_{s=0}^{m-1} (-1)^s G_l \left(\frac{s+x}{m} \right) \frac{1}{j^{k-1}} \frac{S_1(n-l+1,j)}{n-l+1} \\
 &= \frac{1}{n} \sum_{l=0}^{n-1} \binom{n}{l+1} m^l \sum_{j=1}^{n-l} \sum_{s=0}^{m-1} (-1)^s G_{l+1} \left(\frac{s+x}{m} \right) \frac{1}{j^{k-1}} \frac{S_1(n-l,j)}{n-l} \\
 &= \sum_{l=0}^{n-1} \binom{n-1}{l} m^l \sum_{j=1}^{n-l} \sum_{s=0}^{m-1} (-1)^s \frac{G_{l+1}\left(\frac{s+x}{m}\right)}{l+1} \frac{1}{j^{k-1}} \frac{S_1(n-l,j)}{n-l} \\
 &= \sum_{l=0}^{n-1} \binom{n-1}{l} m^l \sum_{j=1}^{n-l} \sum_{s=0}^{m-1} (-1)^s E_l \left(\frac{s+x}{m} \right) \frac{1}{j^{k-1}} \frac{S_1(n-l,j)}{n-l},
 \end{aligned}$$

where $n, m \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$. Thus we obtain the important corollary, which will be used in deriving the reciprocity law in Theorem 9.

Corollary 3 For $n, m \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$, we have

$$E_{n-1}^{(k)}(x) = \sum_{l=0}^{n-1} \binom{n-1}{l} m^l \sum_{j=1}^{n-l} \sum_{s=0}^{m-1} (-1)^s E_l \left(\frac{s+x}{m} \right) \frac{1}{j^{k-1}} \frac{S_1(n-l,j)}{n-l}.$$

Note that

$$E_{n-1}^{(1)}(x) = m^{n-1} \sum_{s=0}^{m-1} (-1)^s E_{n-1} \left(\frac{s+x}{m} \right),$$

where $n, m \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$.

For $p, s \in \mathbb{N}$ with $s < p$, we have

$$\frac{d^s}{dx^s} (xE_p^{(k)}(x))|_{x=1} = s! \binom{p}{s} E_{p-s}^{(k)}(1) + s! \binom{p}{s-1} E_{p-s+1}^{(k)}(1). \tag{28}$$

On the other hand, by (18) we get

$$\frac{d^s}{dx^s} (xE_p^{(k)}(x))|_{x=1} = s! \sum_{v=0}^p \binom{p-v+1}{s} \binom{p}{v} E_v^{(k)}. \tag{29}$$

Therefore by (28) and (29) we obtain the following lemma.

Lemma 1 For $p, s \in \mathbb{N}$ with $s < p$, we have

$$\sum_{v=0}^p \binom{p-v+1}{s} \binom{p}{v} E_v^{(k)} = \binom{p}{s} E_{p-s}^{(k)}(1) + \binom{p}{s-1} E_{p-s+1}^{(k)}(1).$$

In particular, for $k = 1$ and $p, s \in \mathbb{N}$ with $p \equiv 1 \pmod{2}$ and $s \equiv 0 \pmod{2}$, we have

$$\sum_{v=0}^p \binom{p-v+1}{s} \binom{p}{v} E_v = \binom{p}{s} E_{p-s}(1) = -\binom{p}{s} E_{p-s}.$$

Note that

$$\begin{aligned} \int_0^1 xE_p^{(k)}(x) dx &= \frac{1}{p+1} E_{p+1}^{(k)}(1) - \frac{1}{p+1} \int_0^1 E_{p+1}^{(k)}(x) dx \\ &= \frac{1}{p+1} E_{p+1}^{(k)}(1) - \frac{1}{(p+1)(p+2)} (E_{p+2}^{(k)}(1) - E_{p+2}^{(k)}). \end{aligned} \tag{30}$$

On the other hand, from (18) we have

$$\begin{aligned} \int_0^1 xE_p^{(k)}(x) dx &= \sum_{v=0}^p \binom{p}{v} E_v^{(k)} \int_0^1 x^{p-v+1} dx \\ &= \sum_{v=0}^p \binom{p}{v} \frac{E_v^{(k)}}{p-v+2}. \end{aligned} \tag{31}$$

Therefore by (30) and (31) we obtain the following lemma.

Lemma 2 For $p \in \mathbb{N}$, we have

$$\sum_{v=0}^p \binom{p}{v} \frac{E_v^{(k)}}{p-v+2} = \frac{1}{p+1} E_{p+1}^{(k)}(1) - \frac{1}{(p+1)(p+2)} E_{p+2}^{(k)}(1) + \frac{1}{(p+1)(p+2)} E_{p+2}^{(k)}.$$

In particular, for $p \in \mathbb{N}$ with $p \equiv 1 \pmod{2}$ and $k = 1$, we get

$$\sum_{v=0}^p \binom{p}{v} \frac{E_v}{p-v+2} = \frac{2}{(p+1)(p+2)} E_{p+2}.$$

3 Poly-Dedekind type DC sums

The Dedekind type DC sums are defined by

$$T_p(h, m) = 2 \sum_{\mu=1}^{m-1} (-1)^\mu \frac{\mu}{m} \bar{E}_p\left(\frac{h\mu}{m}\right) \quad (h, m \in \mathbb{N}), \tag{32}$$

where $\bar{E}_p(x)$ is the p th Euler function (see [13, 22]).

For $p \in \mathbb{N}$ with $p \equiv 1 \pmod{2}$ and relative prime positive integers m, h with $m \equiv 1 \pmod{2}$ and $h \equiv 1 \pmod{2}$, the reciprocity law of $T_p(h, m)$ is given by

$$\begin{aligned} & m^p T_p(h, m) + h^p T_p(m, h) \\ &= 2 \sum_{\mu} \left(mh \left(E + \frac{\mu}{m} \right) + m \left(E + h - \left[\frac{h\mu}{m} \right] \right) \right)^p \\ & \quad + (hE + mE)^p + (p+2)E_p, \end{aligned}$$

where μ runs over all integers satisfying $0 \leq \mu \leq m-1$ and $\mu - \lfloor \frac{h\mu}{m} \rfloor \equiv 1 \pmod{2}$, and

$$(hE + mE)^p = \sum_{l=0}^p \binom{p}{l} h^l E_l h^{p-l} m^{p-l} E_{p-l}.$$

For the rest of our discussion, we assume that k is any integer. In light of (32), we define the poly-Dedekind-type DC sums by

$$T_p^{(k)}(h, m) = 2 \sum_{\mu=1}^{m-1} \binom{\mu}{m} (-1)^\mu \bar{E}_p^{(k)}\left(\frac{h\mu}{m}\right), \tag{33}$$

where $h, m, p \in \mathbb{N}$, and $\bar{E}_p^{(k)}$ are the poly-Euler functions

$$\bar{E}_p^{(k)}(x) = E_p^{(k)}(x - [x]).$$

By (32) and (33) we get

$$T_p^{(1)}(h, m) = 2 \sum_{\mu=1}^{m-1} (-1)^\mu \binom{\mu}{m} \bar{E}_p\left(\frac{\mu}{m}\right) = T_p(h, m). \tag{34}$$

Let us take $h = 1$. Then we have

$$T_p^{(k)}(1, m) = 2 \sum_{\mu=0}^{m-1} (-1)^\mu \binom{\mu}{m} \sum_{v=0}^p \binom{p}{v} \binom{\mu}{m}^{p-v} E_v^{(k)} \tag{35}$$

$$= \sum_{\nu=0}^p \binom{p}{\nu} E_{\nu}^{(k)} m^{-(p-\nu+1)} \left(2 \sum_{\mu=1}^{m-1} (-1)^{\mu} \mu^{p-\nu+1} \right).$$

Assume that $m \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$. Then, by (7) and (35) we get

$$\begin{aligned} T_p^{(k)}(1, m) &= \sum_{\nu=0}^p \binom{p}{\nu} E_{\nu}^{(k)} m^{-(p+1-\nu)} (E_{p+1-\nu}(m) + E_{p+1-\nu}) \tag{36} \\ &= \sum_{\nu=0}^p \binom{p}{\nu} E_{\nu}^{(k)} m^{-(p+1-\nu)} \left(\sum_{i=0}^{p+1-\nu} \binom{p+1-\nu}{i} m^{p+1-\nu-i} E_i + E_{p+1-\nu} \right) \\ &= \sum_{\nu=0}^p \binom{p}{\nu} E_{\nu}^{(k)} m^{-(p+1-\nu)} \sum_{i=0}^{p-\nu} \binom{p-\nu+1}{i} m^{p+1-\nu-i} E_i \\ &\quad + 2 \sum_{\nu=0}^p \binom{p}{\nu} E_{\nu}^{(k)} E_{p+1-\nu} m^{-(p+1-\nu)}. \end{aligned}$$

From (36) we have

$$m^p T_p^{(k)}(1, m) = \sum_{\nu=0}^p \binom{p}{\nu} E_{\nu}^{(k)} \sum_{i=0}^{p-\nu} \binom{p-\nu+1}{i} m^{p-i} E_i + 2 \sum_{\nu=0}^p \binom{p}{\nu} E_{\nu}^{(k)} E_{p+1-\nu} m^{\nu-1}. \tag{37}$$

Let us define $S_p^{(k)}(1, m)$ as

$$m^p T_p^{(k)}(1, m) - 2 \sum_{\nu=0}^p \binom{p}{\nu} E_{\nu}^{(k)} E_{p+1-\nu} m^{\nu-1} = S_p^{(k)}(1, m), \tag{38}$$

where $p, m \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$.

Therefore, by (37) and (38) we obtain the following theorem.

Theorem 5 For $m, p \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$, we have

$$S_p^{(k)}(1, m) = \sum_{\nu=0}^p \binom{p}{\nu} E_{\nu}^{(k)} \sum_{i=0}^{p-\nu} \binom{p-\nu+1}{i} E_i m^{p-i}. \tag{39}$$

Now, we assume that $p \geq 3$ is an odd integer, so that $E_{p-1} = 0$. Interchanging the order of summation in (39), we have

$$\begin{aligned} S_p^{(k)}(1, m) &= \sum_{i=0}^p \sum_{\nu=0}^{p-i} \binom{p}{\nu} \binom{p-\nu+1}{i} E_{\nu}^{(k)} E_i m^{p-i} \tag{40} \\ &= \sum_{i=1}^{p-2} \sum_{\nu=0}^{p-i} \binom{p}{\nu} \binom{p-\nu+1}{i} E_{\nu}^{(k)} E_i m^{p-i} + \binom{p+1}{p} E_p \\ &\quad + \sum_{\nu=0}^p \binom{p}{\nu} E_{\nu}^{(k)} m^p + \sum_{\nu=0}^1 \binom{p}{\nu} E_{\nu}^{(k)} \binom{p-\nu+1}{p-1} E_{p-1} m \\ &= \sum_{i=1}^{p-2} \sum_{\nu=0}^{p-i} \binom{p}{\nu} \binom{p-\nu+1}{i} E_{\nu}^{(k)} E_i m^{p-i} + (p+1) E_p + \sum_{\nu=0}^p \binom{p}{\nu} E_{\nu}^{(k)} m^p \end{aligned}$$

$$= \sum_{i=1}^{p-2} \sum_{v=0}^{p-i} \binom{p}{v} \binom{p-v+1}{i} E_v^{(k)} E_i m^{p-i} + (p+1)E_p + m^p E_p^{(k)}(1).$$

Therefore we obtain the following theorem.

Theorem 6 For $m \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$ and $p \equiv 1 \pmod{2}$ with $p > 1$, we have

$$S_p^{(k)}(1, m) = \sum_{i=1}^{p-2} \sum_{v=0}^{p-i} \binom{p}{v} \binom{p-v+1}{i} E_v^{(k)} E_i m^{p-i} + (p+1)E_p + m^p E_p^{(k)}(1).$$

In other words, we have

$$\begin{aligned} m^p T_p^{(k)}(1, m) & \tag{41} \\ &= \sum_{i=1}^{p-2} \sum_{v=0}^{p-i} \binom{p}{v} \binom{p-v+1}{i} E_v^{(k)} E_i m^{p-i} + (p+1)E_p + m^p E_p^{(k)}(1) \\ & \quad + 2 \sum_{v=0}^p \binom{p}{v} E_v^{(k)} E_{p+1-v} m^{v-1}. \end{aligned}$$

We observe that

$$\begin{aligned} \sum_{v=0}^p \binom{p-v+1}{s} \binom{p}{v} E_v^{(k)} &= \sum_{v=0}^{p-s+1} \binom{p-v+1}{s} \binom{p}{v} E_v^{(k)} \\ &= \sum_{v=0}^{p-s} \binom{p-v+1}{s} \binom{p}{v} E_v^{(k)} + \binom{p}{s-1} E_{p-s+1}^{(k)}. \end{aligned} \tag{42}$$

From (42) and Lemma 1 we have

$$\begin{aligned} \sum_{v=0}^{p-s} \binom{p-v+1}{s} \binom{p}{v} E_v^{(k)} & \tag{43} \\ &= \sum_{v=0}^p \binom{p-v+1}{s} \binom{p}{v} E_v^{(k)} - \binom{p}{s-1} E_{p-s+1}^{(k)} \\ &= \binom{p}{s} E_{p-s}^{(k)}(1) + \binom{p}{s-1} E_{p-s+1}^{(k)}(1) - \binom{p}{s-1} E_{p-s+1}^{(k)}. \end{aligned}$$

By (43) we get

$$\begin{aligned} \sum_{i=1}^{p-2} \sum_{v=0}^{p-i} \binom{p}{v} \binom{p-v+1}{i} E_v^{(k)} E_i m^{p-i} & \tag{44} \\ &= \sum_{i=1}^{p-2} \binom{p}{i} E_{p-i}^{(k)}(1) E_i m^{p-i} + \sum_{i=1}^{p-2} \binom{p}{i-1} (E_{p-i+1}^{(k)}(1) - E_{p-i+1}) E_i m^{p-i}. \end{aligned}$$

From (41) and (44) we note that

$$\begin{aligned}
 m^p T_p^{(k)}(1, m) &= \sum_{i=1}^{p-2} \binom{p}{i} E_{p-i}^{(k)}(1) E_i m^{p-i} + \sum_{i=1}^{p-2} \binom{p}{i-1} (E_{p-i+1}^{(k)}(1) - E_{p-i+1}^{(k)}) E_i m^{p-i} \\
 &\quad + (p+1)E_p + m^p E_p^{(k)}(1) + 2 \sum_{v=0}^p \binom{p}{v} E_v^{(k)} E_{p+1-v} m^{v-1}.
 \end{aligned} \tag{45}$$

It is easy to show that

$$E_1^{(k)}(1) - E_1^{(k)} = 1. \tag{46}$$

By (45) and (46) we get

$$\begin{aligned}
 m^p T_p^{(k)}(1, m) &= \sum_{i=0}^p \binom{p}{i} E_{p-i}^{(k)}(1) E_i m^{p-i} + \sum_{i=1}^p \binom{p}{i-1} (E_{p-i+1}^{(k)}(1) - E_{p-i+1}^{(k)}) m^{p-i} E_i \\
 &\quad + 2 \sum_{v=0}^p \binom{p}{v} E_v^{(k)} E_{p+1-v} m^{v-1}.
 \end{aligned} \tag{47}$$

Therefore by (47) we obtain the following theorem.

Theorem 7 For $m \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$ and $p \equiv 1 \pmod{2}$ with $p > 1$, we have

$$\begin{aligned}
 m^p T_p^{(k)}(1, m) &= \sum_{i=0}^p \binom{p}{i} E_{p-i}^{(k)}(1) E_i m^{p-i} + \sum_{i=1}^p (E_{p-i+1}^{(k)}(1) - E_{p-i+1}^{(k)}) m^{p-i} E_i \binom{p}{i-1} \\
 &\quad + 2 \sum_{v=0}^p \binom{p}{v} E_v^{(k)} E_{p+1-v} m^{v-1}.
 \end{aligned}$$

Now we employ the symbolic notations $E_n(x) = (E + x)^n$, $E_n^{(k)}(x) = (E^{(k)} + x)^n$ ($n \geq 0$).

Then we first observe that

$$\begin{aligned}
 m^p \sum_{\mu=0}^{m-1} (-1)^\mu \sum_{s=0}^p \binom{p}{s} h^s E_s^{(k)}\left(\frac{\mu}{m}\right) E_{p-s}\left(h - \left\lfloor \frac{h\mu}{m} \right\rfloor\right) &= m^p \sum_{\mu=0}^{m-1} (-1)^\mu \left(h \left(E^{(k)} + \frac{\mu}{m} \right) + \left(E + h - \left\lfloor \frac{h\mu}{m} \right\rfloor \right) \right)^p \\
 &= m^p \sum_{\mu=0}^{m-1} (-1)^\mu \left(h E^{(k)} + E + h + \frac{1}{2} - \frac{1}{2} + h\mu m^{-1} - \left\lfloor \frac{h\mu}{m} \right\rfloor \right)^p \\
 &= m^p \sum_{\mu=0}^{m-1} (-1)^\mu \left(h E^{(k)} + E + h + \frac{1}{2} + \bar{E}_1 \left(\frac{h\mu}{m} \right) \right)^p.
 \end{aligned} \tag{48}$$

Assume that h, m are relatively prime positive integers. Then, as the index μ ranges over the values $\mu = 0, 1, 2, \dots, m - 1$, the product $h\mu$ does over a complete residue system modulo m , and due to the periodicity of $\bar{E}_1(x)$, the term $\bar{E}_1(\frac{h\mu}{m})$ may be replaced by $\bar{E}_1(\frac{\mu}{m})$ without alternating the sum over μ .

For $m \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$, by (48) and (24) we get

$$\begin{aligned}
 & m^p \sum_{\mu=0}^{m-1} (-1)^\mu \sum_{s=0}^p \binom{p}{s} h^s E_s^{(k)}\left(\frac{\mu}{m}\right) E_{p-s}\left(h - \left\lfloor \frac{h\mu}{m} \right\rfloor\right) \tag{49} \\
 &= m^p \sum_{\mu=0}^{m-1} (-1)^\mu \left(hE^{(k)} + h + E + \frac{1}{2} + \bar{E}_1\left(\frac{\mu}{m}\right) \right)^p \\
 &= m^p \sum_{\mu=0}^{m-1} (-1)^\mu \left(h(E^{(k)} + 1) + E + \frac{\mu}{m} \right)^p \\
 &= m^p \sum_{\mu=0}^{m-1} (-1)^\mu \sum_{s=0}^p \binom{p}{s} \left(E + \frac{\mu}{m} \right)^s h^{p-s} (E^{(k)} + 1)^{p-s} \\
 &= m^p \sum_{\mu=0}^{m-1} (-1)^\mu \sum_{s=0}^p \binom{p}{s} E_s\left(\frac{\mu}{m}\right) h^{p-s} E_{p-s}^{(k)}(1) \\
 &= \sum_{s=0}^p \binom{p}{s} m^{p-s} \left(m^s \sum_{\mu=0}^{m-1} (-1)^\mu E_s\left(\frac{\mu}{m}\right) \right) h^{p-s} E_{p-s}^{(k)}(1) \\
 &= \sum_{s=0}^p \binom{p}{s} (mh)^{p-s} E_s E_{p-s}^{(k)}(1).
 \end{aligned}$$

Therefore by (48) and (49) we obtain the following theorem.

Theorem 8 For $h, m, p \in \mathbb{N}$ with $(h, m) = 1$ and $m \equiv 1 \pmod{2}$, we have

$$\begin{aligned}
 & m^p \sum_{\mu=0}^{m-1} (-1)^\mu \sum_{s=0}^p \binom{p}{s} h^s E_s^{(k)}\left(\frac{\mu}{m}\right) E_{p-s}\left(h - \left\lfloor \frac{h\mu}{m} \right\rfloor\right) \\
 &= \sum_{s=0}^p \binom{p}{s} (mh)^{p-s} E_s E_{p-s}^{(k)}(1).
 \end{aligned}$$

From Corollary 3 we note that

$$E_n^{(k)}(x) = \sum_{l=0}^n \sum_{j=1}^{n+1-l} \sum_{s=0}^{m-1} \binom{n}{l} m^l (-1)^s E_l\left(\frac{s+x}{m}\right) \frac{S_1(n+1-l, j)}{j^{k-1}(n+1-l)}, \tag{50}$$

where $m \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$, and $n \geq 0$.

For $m, h \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$ and $h \equiv 1 \pmod{2}$, by (50) we get

$$\begin{aligned}
 & m^p T_p^{(k)}(h, m) + h^p T_p^{(k)}(m, h) \tag{51} \\
 &= 2m^p \sum_{\mu=0}^{m-1} \frac{\mu}{m} (-1)^\mu \bar{E}_p^{(k)}\left(\frac{h\mu}{m}\right) + 2h^p \sum_{\nu=0}^{h-1} \frac{\nu}{h} (-1)^\nu \bar{E}_p^{(k)}\left(\frac{m\nu}{h}\right)
 \end{aligned}$$

$$\begin{aligned}
 &= 2m^p \sum_{\mu=0}^{m-1} (-1)^\mu \frac{\mu}{m} \sum_{l=0}^p \binom{p}{l} h^l \sum_{v=0}^{h-1} \sum_{j=1}^{p+1-l} (-1)^v \bar{E}_l \left(\frac{v}{h} + \frac{\mu}{m} \right) \frac{S_1(p+1-l, j)}{j^{k-1}(p+1-l)} \\
 &\quad + 2h^p \sum_{v=0}^{h-1} (-1)^v \frac{v}{h} \sum_{l=0}^p \binom{p}{l} m^l \sum_{\mu=0}^{m-1} \sum_{j=1}^{p+1-l} (-1)^\mu \bar{E}_l \left(\frac{\mu}{m} + \frac{v}{h} \right) \frac{S_1(p+1-l, j)}{j^{k-1}(p+1-l)} \\
 &= 2 \sum_{\mu=0}^{m-1} (-1)^\mu \frac{\mu}{m} \sum_{l=0}^p m^{p-l} (mh)^l \binom{p}{l} \sum_{v=0}^{h-1} \sum_{j=1}^{p+1-l} (-1)^v \bar{E}_l \left(\frac{\mu}{m} + \frac{v}{h} \right) \frac{S_1(p-l+1, j)}{(p-l+1)j^{k-1}} \\
 &\quad + 2 \sum_{v=0}^{h-1} (-1)^v \frac{v}{h} \sum_{l=0}^p h^{p-l} (mh)^l \binom{p}{l} \sum_{\mu=0}^{m-1} \sum_{j=1}^{p+1-l} (-1)^\mu \bar{E}_l \left(\frac{v}{h} + \frac{\mu}{m} \right) \frac{S_1(p-l+1, j)}{(p-l+1)j^{k-1}} \\
 &= 2 \sum_{\mu=0}^{m-1} \sum_{l=0}^p \sum_{v=0}^{h-1} \sum_{j=1}^{p+1-l} (-1)^{\mu+v} (\mu h) (mh)^{-1} m^{p-l} (mh)^l \binom{p}{l} \bar{E}_l \left(\frac{\mu}{m} + \frac{v}{h} \right) \frac{S_1(p-l+1, j)}{(p-l+1)j^{k-1}} \\
 &\quad + 2 \sum_{\mu=0}^{m-1} \sum_{l=0}^p \sum_{v=0}^{h-1} \sum_{j=1}^{p+1-l} (-1)^{\mu+v} (vm) (mh)^{-1} h^{p-l} (mh)^l \binom{p}{l} \bar{E}_l \left(\frac{v}{h} + \frac{\mu}{m} \right) \frac{S_1(p-l+1, j)}{(p-l+1)j^{k-1}} \\
 &= 2 \sum_{\mu=0}^{m-1} \sum_{l=0}^p \sum_{v=0}^{h-1} \sum_{j=1}^{p+1-l} (-1)^{\mu+v} \frac{(mh)^{l-1} \binom{p}{l} S_1(p-l+1, j)}{(p-l+1)j^{k-1}} \\
 &\quad \times \left((\mu h) m^{p-l} + (vm) h^{p-l} \right) \bar{E}_l \left(\frac{v}{h} + \frac{\mu}{m} \right).
 \end{aligned}$$

Therefore by (51) we obtain the following reciprocity relation.

Theorem 9 For $m, h, p \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$ and $h \equiv 1 \pmod{2}$ and $k \in \mathbb{Z}$, we have

$$\begin{aligned}
 &m^p T_p^{(k)}(h, m) + h^p T_p^{(k)}(m, h) \\
 &= 2 \sum_{\mu=0}^{m-1} \sum_{l=0}^p \sum_{v=0}^{h-1} \sum_{j=1}^{p+1-l} (-1)^{\mu+v} \frac{(mh)^{l-1} \binom{p}{l} S_1(p-l+1, j)}{(p-l+1)j^{k-1}} \\
 &\quad \times \left((\mu h) m^{p-l} + (vm) h^{p-l} \right) \bar{E}_l \left(\frac{v}{h} + \frac{\mu}{m} \right).
 \end{aligned}$$

In case of $k = 1$, we obtain the following reciprocity relation for the Dedekind type DC sums.

Corollary 4 For $m, h, p \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$ and $h \equiv 1 \pmod{2}$, we have

$$\begin{aligned}
 &m^p T_p(h, m) + h^p T_p(m, h) \\
 &= 2(mh)^{p-1} \sum_{\mu=0}^{m-1} \sum_{v=0}^{h-1} (-1)^{\mu+v} (\mu h + vm) \bar{E}_p \left(\frac{v}{h} + \frac{\mu}{m} \right).
 \end{aligned}$$

4 Conclusions

The generalized Dedekind sums considered by Apostol are given by

$$S_p(h, m) = \sum_{\mu=1}^{m-1} \frac{\mu}{m} \overline{B}_p\left(\frac{h\mu}{m}\right)$$

and satisfy a reciprocity relation (see [1, 2]), where $\overline{B}_p(x)$ is the p th Bernoulli function.

Recently, the type 2 poly-Bernoulli polynomials of index k were defined in terms of the polyexponential function of index k by

$$\frac{\text{Ei}_k(\log(1+t))}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!} \quad (k \in \mathbb{Z}).$$

As a further extension of the generalized Dedekind sums, the poly-Dedekind sums defined by

$$S_p^{(k)}(h, m) = \sum_{\mu=1}^{m-1} \frac{\mu}{m} \overline{B}_p^{(k)}\left(\frac{h\mu}{m}\right)$$

were considered and shown to satisfy a reciprocity relation in [16], where $\overline{B}_p^{(k)}(x) = B_p^{(k)}(x - [x])$ are the type 2 poly-Bernoulli functions of index k , and $S_p^{(1)}(h, m) = S_p(h, m)$.

The Dedekind-type DC sums defined by

$$T_p(h, m) = 2 \sum_{\mu=1}^{m-1} (-1)^\mu \frac{\mu}{m} \overline{E}_p\left(\frac{h\mu}{m}\right)$$

were introduced and shown to satisfy a reciprocity relation in [13], where $\overline{E}_p(x)$ is the p th Euler function. Simsek found trigonometric representations of the Dedekind-type DC sums and their relations to Clausen functions, polylogarithm function, Hurwitz zeta function, generalized Lambert series (G-series), and Hardy–Berndt sums.

In this paper, as a further generalization of the Dedekind type DC sums, we considered the poly-Dedekind-type DC sums

$$T_p^{(k)}(h, m) = 2 \sum_{\mu=1}^{m-1} (-1)^\mu \frac{\mu}{m} \overline{E}_p^{(k)}\left(\frac{h\mu}{m}\right)$$

and showed, among other things, that they satisfy a reciprocity relation in Theorem 9.

Finally, we defined the Dedekind sums and their generalizations in terms of Bernoulli functions and their generalizations and the Dedekind-type DC sums and their generalizations in terms of Euler functions and their generalizations.

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Authors' contributions

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