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A necessary and sufficient condition for sequences to be minimal completely monotonic

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Dedicated to Professor Hari M. Srivastava on the occasion of his eightieth birthday.

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Abstract

In this article, we present a necessary and sufficient condition under which sequences are minimal completely monotonic.

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1 Introduction and the main results

We first recall some definitions and basic results on completely monotonic sequences and minimal completely monotonic sequences.

Definition 1 ([20]) A sequence $\{\mu_n\}_{n=0}^{\infty}$ is called completely monotonic if

$$(-1)^k \Delta^k \mu_n \geq 0, \quad n, k \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}, \tag{1}$$

where

$$\Delta^0 \mu_n = \mu_n \tag{2}$$

and

$$\Delta^{k+1} \mu_n = \Delta^k \mu_{n+1} - \Delta^k \mu_n. \tag{3}$$

Here in Definition 1, and throughout the paper, $\mathbb{N}$ is the set of all positive integers and $\mathbb{N}_0$ is the set of all nonnegative integers.

Widder [25] defined a sub-class of the class of completely monotonic sequences.

Definition 2 A sequence $\{\mu_n\}_{n=0}^{\infty}$ is called minimal completely monotonic if it is completely monotonic and if it will not be completely monotonic when μ_0 is replaced by a number less than μ_0.

Regarding the relationships between completely monotonic sequences and minimal completely monotonic sequences, in [6] the author proved that if the sequence $\{\mu_n\}_{n=0}^{\infty}$ is completely monotonic, then:

(1) for any $m \in \mathbb{N}$, the sequence $\{\mu_n\}_{n=m}^{\infty}$ is minimal completely monotonic, and

(2) there exists one (then only one) number μ_0^* such that the sequence

$$\{\mu_0^*, \mu_1, \mu_2, \dots\}$$

is minimal completely monotonic.

Please note that the complete monotonicity of the sequence $\{\mu_n\}_{n=1}^{\infty}$ cannot guarantee that there exists a number μ_0^* such that the sequence

$$\{\mu_0^*, \mu_1, \mu_2, \dots\} \tag{4}$$

is completely monotonic. In fact, if the sequence (4) is completely monotonic, then the sequence $\{\mu_n\}_{n=1}^{\infty}$ should be minimal completely monotonic.

In [18] the authors showed that if the sequence $\{\mu_n\}_{n=0}^{\infty}$ is completely monotonic, then, for any $m \in \mathbb{N}_0$, the series

$$\sum_{j=0}^{\infty}(-1)^j \Delta^j \mu_{m+1}$$

converges and

$$\mu_m \ge \sum_{j=0}^{\infty}(-1)^j \Delta^j \mu_{m+1}. \tag{5}$$

We also recall the following definition.

Definition 3 ([4]) A function f is said to be completely monotonic on an interval I, if $f \in C(I)$, has derivatives of all orders on I^o (the interior of I) and for all $n \in \mathbb{N}_0$

$$(-1)^n f^{(n)}(x) \ge 0, \quad x \in I^o. \tag{6}$$

Here in Definition 3 $C(I)$ is the space of all continuous functions on the interval I. The class of all completely monotonic functions on the interval I is denoted by $CM(I)$.

There is rich literature on completely monotonic functions and sequences, and their applications. For more recent works, see, for example, [1–3, 5–19, 21–24].

For sequences to be interpolated by completely monotonic functions, Widder [25] proved that there exists a function

$$f \in CM[0, \infty)$$

such that

$$f(n) = \mu_n, \quad n \in \mathbb{N}_0$$

if and only if the sequence $\{\mu_n\}_{n=0}^{\infty}$ is minimal completely monotonic. From this we see that the condition of minimal complete monotonicity is critical for a sequence $\{\mu_n\}_{n=0}^{\infty}$ to be interpolated by a completely monotonic function on the interval $[0, \infty)$.

In this article, we shall further investigate on minimal completely monotonic sequences. The main results of this article are as follows.

Theorem 4 *Suppose that the sequence $\{\mu_n\}_{n=1}^{\infty}$ is completely monotonic and that the series*

$$\sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_1 \tag{7}$$

converges. Let

$$\mu_0^* := \sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_1. \tag{8}$$

Then the sequence

$$\{\mu_0^*, \mu_1, \mu_2, \mu_3, \dots\} \tag{9}$$

is minimal completely monotonic.

Remark 5 It should be noted that the condition: "the series

$$\sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_1 \tag{10}$$

converges" in Theorem 4 cannot be dropped since the complete monotonicity of the sequence $\{\mu_n\}_{n=1}^{\infty}$ cannot guarantee the convergence of the series

$$\sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_1.$$

For example, let

$$\mu_n = \frac{1}{n}, \quad n \in \mathbb{N}.$$

We can verify that the sequence $\{\mu_n\}_{n=1}^{\infty}$ is completely monotonic and that

$$\Delta^j \mu_1 = \frac{(-1)^j}{j+1}.$$

Hence

$$\sum_{j=0}^{\infty}(-1)^j\Delta^j\mu_1=\sum_{j=0}^{\infty}\frac{1}{j+1},$$

which is divergent.

Theorem 6 *Suppose that the sequence* $\{\mu_n\}_{n=0}^{\infty}$ *is minimal completely monotonic. Then the series*

$$\sum_{j=0}^{\infty}(-1)^j\Delta^j\mu_1 \tag{11}$$

converges and

$$\mu_0=\sum_{j=0}^{\infty}(-1)^j\Delta^j\mu_1. \tag{12}$$

Theorem 7 *A necessary and sufficient condition for the sequence* $\{\mu_n\}_{n=0}^{\infty}$ *to be minimal completely monotonic is that the sequence* $\{\mu_n\}_{n=1}^{\infty}$ *is completely monotonic, the series*

$$\sum_{j=0}^{\infty}(-1)^j\Delta^j\mu_1 \tag{13}$$

converges, and

$$\mu_0=\sum_{j=0}^{\infty}(-1)^j\Delta^j\mu_1. \tag{14}$$

2 Proof of the main results

Now we are in a position to prove the main results.

Proof of Theorem 4 By Theorem 11 in [18], we see that the sequence

$$\{\mu_0^*,\mu_1,\mu_2,\mu_3,\ldots\} \tag{15}$$

is completely monotonic. By Theorem 9 in [18], if a sequence

$$\{\mu_0,\mu_1,\mu_2,\mu_3,\ldots\} \tag{16}$$

is completely monotonic, then

$$\mu_0\ge\sum_{j=0}^{\infty}(-1)^j\Delta^j\mu_1=\mu_0^*. \tag{17}$$

Hence by the definition of minimal completely monotonic sequence, we know that the sequence

$$\{\mu_0^*,\mu_1,\mu_2,\mu_3,\ldots\} \tag{18}$$

is minimal completely monotonic. The proof of Theorem 4 is completed. □

Proof of Theorem 6 Since the sequence

$$\{\mu_0, \mu_1, \mu_2, \mu_3, \ldots\} \tag{19}$$

is completely monotonic, by Theorem 9 in [18], the series

$$\sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_1 \tag{20}$$

converges and

$$\mu_0 \geq \sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_1. \tag{21}$$

By Theorem 11 in [18], we see that the sequence

$$\left\{ \sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_1, \mu_1, \mu_2, \mu_3, \ldots \right\} \tag{22}$$

is completely monotonic. Since the completely monotonic sequence

$$\{\mu_0, \mu_1, \mu_2, \mu_3, \ldots\} \tag{23}$$

is minimal, we have

$$\mu_0 \leq \sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_1. \tag{24}$$

From (21) and (24), we get our conclusion. The proof of Theorem 6 is completed. □

Proof of Theorem 7 By the definition of completely monotonic sequence, Theorem 9 in [18] and Theorem 6, we know that the condition is necessary. By Theorem 4, we see that the condition is sufficient. The proof of Theorem 7 is thus completed. □

3 Conclusion

In this paper, we investigated properties of completely monotonic sequences. We have proved a necessary condition for a sequence to be a minimal completely monotonic sequence. We also have presented a necessary and sufficient condition under which sequences are minimal completely monotonic.

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