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Eventual periodicity of the fuzzy max-difference equation $x_n = \max\{C, \frac{x_{n-m-k}}{x_{n-m}}\}$

Caihong Han[1,3], Guangwang Su[1,3*], Lue Li[1,3], Guoen Xia[2,3] and Taixiang Sun[1,3]

*Correspondence:
s1g6w3@163.com
[1]College of Information and Statistics, Guangxi University of Finance and Economics, Nanning, 530003, China
[3]Guangxi Key Laboratory Cultivation Base of Cross-border E-commerce Intelligent Information Processing, Nanning, 530003, China
Full list of author information is available at the end of the article

Abstract

In this paper, we study the eventual periodicity of the fuzzy max-type difference equation $x_n = \max\{C, \frac{x_{n-m-k}}{x_{n-m}}\}, n \in \{0, 1, \ldots\}$, where m and k are positive integers, C and the initial values are positive fuzzy numbers. Let the support $\operatorname{supp} C = \overline{\{t : C(t) > 0\}} = [C_1, C_2]$ of C. We show that: (1) if $C_1 > 1$, then every positive solution of this equation equals C eventually; (2) there exists a positive fuzzy number C with $C_1 = 1$ such that this equation has a positive solution which is not eventually periodic; (3) if $C_2 \le 1$, then this equation has a positive solution which is not eventually periodic; (4) if $C_1 < 1 < C_2$, then every positive solution of the above equation is not eventually periodic.

Keywords: Fuzzy max-type difference equation; Positive solution; Eventual periodicity

1 Introduction

It is well known that difference equations and difference equation systems are often used in the study of linear and nonlinear physical, physiological, and economical problems (for instance, see [1, 2]). In the recent years, because the max operator has a great importance in automatic control models (see [3, 4]), max-type difference equations and systems which are a special type of difference equations and difference equation systems have attracted the attention of many scholars (for instance, see [5–15]).

In [16], Mishev et al. proved that every solution of the difference equation

$$x_{n+1} = \max\left\{A, \frac{x_n}{x_{n-1}}\right\}, \quad n \in \mathbf{N}_0 \equiv \{0, 1, \ldots\},$$

is eventually periodic, where $A \in \mathbf{R}_+ \equiv (0, +\infty)$.

In [17], Fotiades and Papaschinopoulos studied the following max-type system of difference equations:

$$\begin{cases} x_n = \max\{A, \frac{y_{n-1}}{x_{n-2}}\}, \\ y_n = \max\{B, \frac{x_{n-1}}{y_{n-2}}\}, \end{cases} \quad n \in \mathbf{N}_0,$$

with $A, B \in \mathbf{R}_+$ and showed that every positive solution of the above system is eventually periodic.

Further, Su et al. [18] studied eventual periodicity of the following max-type system of difference equations:

$$\begin{cases} x_n = \max\{A_n, \frac{y_{n-1}}{x_{n-2}}\}, \\ y_n = \max\{B_n, \frac{x_{n-1}}{y_{n-2}}\}, \end{cases} \quad n \in \mathbf{N}_0,$$

where $A_n, B_n \in \mathbf{R}_+$ are periodic sequences with period 2 and the initial values $x_{-2}, y_{-2}, x_{-1}, y_{-1} \in \mathbf{R}_+$ and showed that every solution of the above system is eventually periodic.

Recently there has been a growing interest in the study of fuzzy difference equations (for instance, see [19–31]) because many models in biology, ecology, physiology physics, engineering, economics, probability theory, genetics, psychology and resource management are represented by these equations naturally. For example, fuzzy difference equations are suitable in finance problems. Chrysafis et al. [32] studied the fuzzy difference equation of finance. Their research is in finance which is about the alternative methodology to study the time value of money. In [33], Deeba and Korvin studied the second-order linear difference equation

$$x_{n+1} = x_n - ABx_{n-1} + C, \quad n \in \mathbf{N}_0,$$

where A, B, C and the initial values x_0, x_{-1} are fuzzy numbers. This fuzzy equation is a linearized model of a nonlinear model which determines the carbon dioxide (CO_2) level in the blood.

In [34], Rahmana et al. studied the qualitative behavior of the following second-order fuzzy rational difference equation:

$$x_{n+1} = \frac{x_{n-1}}{A + Bx_{n-1}x_n}, \quad n \in \mathbf{N}_0,$$

where A, B and the initial values x_0, x_{-1} are positive fuzzy numbers.

In [35], Stefanidou and Papaschinopoulos studied the periodicity of the following fuzzy max-difference equation:

$$z_{n+1} = \max\left\{\frac{A}{z_n}, \frac{A}{z_{n-1}}, \dots, \frac{A}{z_{n-k}}\right\}, \quad n \in \mathbf{N}_0,$$

and

$$z_{n+1} = \max\left\{\frac{A}{z_n}, \frac{B}{z_{n-1}}\right\}, \quad n \in \mathbf{N}_0,$$

where $k \in \mathbf{N} \equiv \{1, 2, \dots\}$, A, B and the initial values z_i ($i \in \mathbf{Z}(-k, 0)$) are positive fuzzy numbers (where $\mathbf{Z}(a, b) \equiv \{a, \dots, b\}$ for any integers a, b with $a \leq b$).

Furthermore, Stefanidou and Papaschinopoulos [36] studied the periodicity of the following fuzzy max-difference equation:

$$z_n = \max\left\{\frac{A}{z_{n-k}}, \frac{B}{z_{n-m}}\right\}, \quad n \in \mathbf{N}_0,$$

where A, B and the initial values z_i ($i \in \mathbf{Z}(-d, 0)$) with $d = \max\{k, m\}$ are positive fuzzy numbers. In [37], the authors investigated the periodicity of the positive solutions of the fuzzy max-difference equation

$$x_n = \max\left\{\frac{1}{x_{n-m}}, \frac{\alpha_n}{x_{n-r}}\right\}, \quad n \in \mathbf{N}_0,$$

where $k, m \in \mathbf{N}$, α_n is a periodic sequence of positive fuzzy numbers and x_i ($i \in \mathbf{Z}(-d, 0)$) with $d = \max\{r, m\}$ are positive fuzzy numbers, and showed that, if $\max(\operatorname{supp} \alpha_n) < 1$, then every positive fuzzy number solution of the above equation is eventually periodic with period $2m$.

Motivated by the above-mentioned studies for ordinary difference equations and corresponding fuzzy difference equations, this paper is to study the eventual periodicity of the following fuzzy max-difference equation:

$$x_n = \max\left\{C, \frac{x_{n-m-k}}{x_{n-m}}\right\}, \quad n \in \mathbf{N}_0, \tag{1.1}$$

where $m, k \in \mathbf{N}$, C and the initial values x_i ($i \in \mathbf{Z}(-m-k, -1)$) are positive fuzzy numbers.

The rest of this paper is organized as follows. We give some definitions and notations in Sect. 2 and give the main results and their proofs of this paper in Sect. 3.

2 Preliminaries and definitions

For the convenience of the reader, we give the following definitions and notations.

(1) If A is a function from $\mathbf{R} = (-\infty, +\infty)$ into the interval $[0, 1]$, then A is called a fuzzy set.

(2) A fuzzy set A is said to be fuzzy convex if $A(\lambda t_1 + (1-\lambda)t_2) \geq \min\{A(t_1), A(t_2)\}$ for any $\lambda \in [0, 1]$ and any $t_1, t_2 \in \mathbf{R}$.

(3) A fuzzy set A is said to be normal if there exists some $t \in \mathbf{R}$ such that $A(t) = 1$.

(4) If A is a fuzzy set, then by a λ-cut of A (for any $\lambda \in [0, 1]$) we mean the set $A_\lambda = \{t \in \mathbf{R} : A(t) \geq \lambda\}$.

It is well known that the λ-cuts of A determine the fuzzy set A. For a subset set B of $\mathbf{R}$ we denote by $\overline{B}$ the closure of B.

Definition 2.1 (see [38]) We say that a fuzzy set A is a fuzzy number if it satisfies the following conditions (i)–(iv):

(i) A is normal;

(ii) A is fuzzy convex;

(iii) A is upper semicontinuous;

(iv) The support of A, $\operatorname{supp} A = \overline{\bigcup_{\lambda \in (0,1]} A_\lambda} = \overline{\{t : A(t) > 0\}}$ is compact.

It is clear that A_λ is a closed interval. A fuzzy number A is said to be positive if $\min(\operatorname{supp} A) > 0$. Denote by $\mathcal{F}^+$ the set of all positive fuzzy numbers. If $B \in \mathbf{R}$, then B is a fuzzy number with $B_\lambda = [B, B]$ for any $\lambda \in [0, 1]$, which is said to be a trivial fuzzy number. By [38] we see that, for any $\lambda \in (0, 1]$,

$$[x_n]_\lambda = \max\left\{[C]_\lambda, \frac{[x_{n-m-k}]_\lambda}{[x_{n-m}]_\lambda}\right\}. \tag{2.1}$$

Proposition 2.1 *In* (2.1), *let* $[x_i]_\lambda = [y_{i,\lambda}, z_{i,\lambda}]$ $(i \in \{n, n-m, n-m-k\})$ *and* $[C]_\lambda = [C_{l,\lambda}, C_{r,\lambda}]$ *for any* $\lambda \in (0,1]$. *Then*

$$\begin{cases} y_{n,\lambda} = \max\{C_{l,\lambda}, \frac{y_{n-m-k,\lambda}}{z_{n-m,\lambda}}\}, \\ z_{n,\lambda} = \max\{C_{r,\lambda}, \frac{z_{n-m-k,\lambda}}{y_{n-m,\lambda}}\}. \end{cases} \tag{2.2}$$

Proof It follows from (2.1) that, for any $\lambda \in (0,1]$, we have

$$[y_{n,\lambda}, z_{n,\lambda}] = \max\left\{[C_{l,\lambda}, C_{r,\lambda}], \frac{[y_{n-m-k,\lambda}, z_{n-m-k,\lambda}]}{[y_{n-m,\lambda}, z_{n-m,\lambda}]}\right\}.$$

Let $a_\lambda, a'_\lambda \in [y_{n-m-k,\lambda}, z_{n-m-k,\lambda}]$, $b_\lambda, b'_\lambda \in [y_{n-m,\lambda}, z_{n-m,\lambda}]$, $c_\lambda, c'_\lambda \in [C_{l,\lambda}, C_{r,\lambda}]$ such that

$$y_{n,\lambda} = \max\left\{c_\lambda, \frac{a_\lambda}{b_\lambda}\right\}, \qquad z_{n,\lambda} = \max\left\{c'_\lambda, \frac{a'_\lambda}{b'_\lambda}\right\}.$$

Then we obtain

$$\begin{aligned} y_{n,\lambda} &= \max\left\{c_\lambda, \frac{a_\lambda}{b_\lambda}\right\} \geq \max\left\{C_{l,\lambda}, \frac{y_{n-m-k,\lambda}}{z_{n-m,\lambda}}\right\} \geq y_{n,\lambda}, \\ z_{n,\lambda} &= \max\left\{c'_\lambda, \frac{a'_\lambda}{b'_\lambda}\right\} \leq \max\left\{C_{r,\lambda}, \frac{z_{n-m-k,\lambda}}{y_{n-m,\lambda}}\right\} \leq z_{n,\lambda}, \end{aligned}$$

from which it follows that

$$\begin{cases} y_{n,\lambda} = \max\{C_{l,\lambda}, \frac{y_{n-m-k,\lambda}}{z_{n-m,\lambda}}\}, \\ z_{n,\lambda} = \max\{C_{r,\lambda}, \frac{z_{n-m-k,\lambda}}{y_{n-m,\lambda}}\}. \end{cases}$$

Proposition 2.1 is proven. □

Definition 2.2 A sequence of positive fuzzy numbers $\{x_n\}_{n=-m-k}^{\infty}$ is said to be a positive solution of Eq. (1.1) if it satisfies (1.1). $\{x_n\}_{n=-m-k}^{\infty}$ is said to be eventually periodic with period T if there exists $M \in \mathbf{N}$ such that $x_{n+T} = x_n$ for all $n \geq M$.

Proposition 2.2 *Let* $x_i \in \mathcal{F}^+$ $(i \in \mathbf{Z}(-m-k, -1))$. *Then there exists a unique positive solution* $\{x_n\}_{n=-m-k}^{\infty}$ *of* (1.1) *with initial values* x_i $(i \in \mathbf{Z}(-m-k, -1))$.

Proof The proof is similar to that of Proposition 3.1 of [39]. For any $\lambda \in (0,1]$, write

$$C_\lambda = [C_{l,\lambda}, C_{r,\lambda}] \quad \text{and} \quad [x_i]_\lambda = [y_{i,\lambda}, z_{i,\lambda}] \quad \bigl(i \in \mathbf{Z}(-m-k, -1), \lambda \in (0,1]\bigr), \tag{2.3}$$

and $\{(y_{n,\lambda}, z_{n,\lambda})\}_{n=-m-k}^{\infty}(\lambda \in (0,1])$ is the unique positive solution of the following system of difference equations:

$$y_{n,\lambda} = \max\left\{C_{l,\lambda}, \frac{y_{n-m-k,\lambda}}{z_{n-m,\lambda}}\right\}, \qquad z_{n,\lambda} = \max\left\{C_{r,\lambda}, \frac{z_{n-m-k,\lambda}}{y_{n-m,\lambda}}\right\} \tag{2.4}$$

with initial values $(y_{i,\lambda}, z_{i,\lambda})$ $(i \in \mathbf{Z}(-m-k,-1))$. Since $C, x_i \in \mathcal{F}^+$ $(i \in \mathbf{Z}(-m-k,-1))$, there exist $0 \le P_0 \le Q_0$ such that, for any $\lambda_1, \lambda_2 \in (0,1]$ with $\lambda_1 \le \lambda_2$, we have

$$P_0 \le C_{l,\lambda_1} \le C_{l,\lambda_2} \le C_{r,\lambda_2} \le C_{r,\lambda_1} \le Q_0,$$
$$P_0 \le y_{i,\lambda_1} \le y_{i,\lambda_2} \le z_{i,\lambda_2} \le z_{i,\lambda_1} \le Q_0 \quad \bigl(i \in \mathbf{Z}(-m-k,-1)\bigr).$$

It follows from (2.4) that, for any $\lambda_1, \lambda_2 \in (0,1]$ with $\lambda_1 \le \lambda_2$, we have

$$\begin{aligned} 0 < P_1 &= \max\left\{P_0, \frac{P_0}{Q_0}\right\} \\ &\le y_{0,\lambda_1} = \max\left\{C_{l,\lambda_1}, \frac{y_{-m-k,\lambda_1}}{z_{-m,\lambda_1}}\right\} \\ &\le y_{0,\lambda_2} = \max\left\{C_{l,\lambda_2}, \frac{y_{-m-k,\lambda_2}}{{}_{n-m,\lambda_2}}\right\} \\ &\le z_{0,\lambda_2} = \max\left\{C_{r,\lambda_2}, \frac{z_{-m-k,\lambda_2}}{y_{-m,\lambda_2}}\right\} \\ &\le z_{0,\lambda_1} = \max\left\{C_{r,\lambda_1}, \frac{z_{-m-k,\lambda_1}}{y_{-m,\lambda_1}}\right\} \\ &\le \max\left\{Q_0, \frac{Q_0}{P_0}\right\} = Q_1. \end{aligned}$$

It is easy to see that $y_{0,\lambda}, z_{0,\lambda}$ are left continuous on $\lambda \in (0,1]$ (see [40]) and $\overline{\bigcup_{\lambda\in(0,1]}[y_{0,\lambda}, z_{0,\lambda}]} \subset [P_1, Q_1]$ (i.e., $\overline{\bigcup_{\lambda\in(0,1]}[y_{0,\lambda}, z_{0,\lambda}]}$ is compact). Hence $[y_{0,\lambda}, z_{0,\lambda}]$ determines a unique $x_0 \in \mathcal{F}^+$ such that $[x_0]_\lambda = [y_{0,\lambda}, z_{0,\lambda}]$ for all $\lambda \in (0,1]$ (see [40]).

Moreover, by mathematical induction on n, it is easy to show that: (1) $0 < y_{n,\lambda_1} \le y_{n,\lambda_2} \le z_{n,\lambda_2} \le z_{n,\lambda_1}$ $(n \in \mathbf{N}_0)$; (2) $y_{n,\lambda}, z_{n,\lambda}$ are left continuous for all $n \in \mathbf{N}_0$ and $\lambda \in (0,1]$; (3) For any $n \in \mathbf{N}_0$, there exist $0 < P_{n+1} \le Q_{n+1} < +\infty$ such that $\overline{\bigcup_{\lambda\in(0,1]}[y_{n,\lambda}, z_{n,\lambda}]} \subset [P_{n+1}, Q_{n+1}]$ (i.e., $\overline{\bigcup_{\lambda\in(0,1]}[y_{n,\lambda}, z_{n,\lambda}]}$ is compact). Hence by [40], Theorem 2.1, we see that $[y_{n,\lambda}, z_{n,\lambda}]$ determines a sequence $\{x_n\}_{n=-m-k}^{\infty}$ of positive fuzzy numbers such that $[x_n]_\lambda = [y_{n,\lambda}, z_{n,\lambda}]$ for every $n \in \mathbf{N}_0$ and $\lambda \in (0,1]$, and by Proposition 2.1 we see that $\{x_n\}_{n=-m-k}^{\infty}$ is the unique positive solution of (1.1) with initial values x_i $(i \in \mathbf{Z}(-m-k,-1))$. The proof is complete. □

3 Main results

In the sequel, let $\{x_n\}_{n=-m-k}^{\infty}$ be a positive solution of (1.1) with initial values $x_i \in \mathcal{F}^+$ $(i \in \mathbf{Z}(-m-k,-1))$. Let $\operatorname{supp} C = [C_1, C_2]$. For any $\lambda \in (0,1]$, write

$$C_\lambda = [C_{l,\lambda}, C_{r,\lambda}], \qquad [x_n]_\lambda = [y_{n,\lambda}, z_{n,\lambda}].$$

Then it follows from Proposition 2.2 that $\{(y_{n,\lambda}, z_{n,\lambda})\}_{n=-m-k}^{\infty}(\lambda \in (0,1])$ satisfies the following system:

$$y_{n,\lambda} = \max\left\{C_{l,\lambda}, \frac{y_{n-m-k,\lambda}}{z_{n-m,\lambda}}\right\}, \qquad z_{n,\lambda} = \max\left\{C_{r,\lambda}, \frac{z_{n-m-k,\lambda}}{y_{n-m,\lambda}}\right\}, \tag{3.1}$$

with initial values $(y_{i,\lambda}, z_{i,\lambda})$ $(i \in \mathbf{Z}(-m-k,-1))$. From (3.1) one has, for any $n \in \mathbf{N}_0$,

$$y_{n,\lambda} \ge C_{l,\lambda}, \qquad z_{n,\lambda} \ge C_{r,\lambda}. \tag{3.2}$$

Theorem 3.1 *If $C_1 > 1$, then $x_n = C$ eventually.*

Proof Write $M = \max\{\sup(\operatorname{supp} x_j) : j \in \mathbf{Z}(0, m+k-1)\}$. From (3.1), (3.2) and a simple inductive argument we obtain the result that, for any $i \in \mathbf{Z}(0, m+k-1)$ and $n \in \mathbf{N}$,

$$\begin{aligned} C_{l,\lambda} \le y_{n(m+k)+i,\lambda} &= \max\left\{C_{l,\lambda}, \frac{y_{(n-1)(m+k)+i,\lambda}}{z_{n(m+k)+i-m,\lambda}}\right\} \le \max\left\{C_{l,\lambda}, \frac{y_{(n-1)(m+k)+i,\lambda}}{C_{r,\lambda}}\right\} \\ &\le \max\left\{C_{l,\lambda}, \frac{y_{(n-1)(m+k)+i,\lambda}}{C_1}\right\} \le \cdots \le \max\left\{C_{l,\lambda}, \frac{y_{i,\lambda}}{C_1^n}\right\} \\ &\le \max\left\{C_{l,\lambda}, \frac{M}{C_1^n}\right\} \end{aligned}$$

and

$$\begin{aligned} C_{r,\lambda} \le z_{n(m+k)+i,\lambda} &= \max\left\{C_{r,\lambda}, \frac{z_{(n-1)(m+k)+i,\lambda}}{y_{n(m+k)+i-m,\lambda}}\right\} \le \max\left\{C_{r,\lambda}, \frac{z_{(n-1)(m+k)+i,\lambda}}{C_{l,\lambda}}\right\} \\ &\le \max\left\{C_{r,\lambda}, \frac{z_{(n-1)(m+k)+i,\lambda}}{C_1}\right\} \le \cdots \le \max\left\{C_{r,\lambda}, \frac{z_{i,\lambda}}{C_1^n}\right\} \\ &\le \max\left\{C_{r,\lambda}, \frac{M}{C_1^n}\right\}. \end{aligned}$$

Then there exists an $N \in \mathbf{N}$ such that $M/C_1^n < 1$ for any $n \ge N$, which implies $y_{n(m+k)+i,\lambda} = C_{l,\lambda}$ and $z_{n(m+k)+i,\lambda} = C_{r,\lambda}$ for any $n \ge N$ and $\lambda \in (0,1]$ and $i \in \mathbf{Z}(0, m+k-1)$. Then $x_n = C$ eventually. The proof is complete. □

Theorem 3.2 *There exists an $C \in \mathcal{F}^+$ with $C_1 = 1$ such that (1.1) has a positive solution which is not eventually periodic.*

Proof Define $C \in \mathcal{F}^+$ by

$$C(t) = \begin{cases} 0, & t < 1, \\ 2t-2, & 1 \le t \le \frac{3}{2}, \\ 4-2t, & \frac{3}{2} \le t \le 2, \\ 0, & t > 2. \end{cases} \tag{3.3}$$

Define $x_i \in \mathcal{F}^+$ ($i \in \mathbf{Z}(-m-k, -1)$) by

$$x_i(t) = \begin{cases} 0, & t < 1, \\ 2t-2, & 1 \le t \le \frac{3}{2}, \\ 1, & \frac{3}{2} \le t \le 2e, \\ 0, & t > 2e. \end{cases} \tag{3.4}$$

Then, for any $n \in \mathbf{N}$,

$$C_{\frac{1}{n}} = \left[1 + \frac{1}{2n}, 2 - \frac{1}{2n}\right], \qquad [x_i]_{\frac{1}{n}} = [y_{i,\frac{1}{n}}, z_{i,\frac{1}{n}}] = \left[1 + \frac{1}{2n}, 2e\right] \quad \big(i \in \mathbf{Z}(-m-k, -1)\big).$$

Write $r = s(m+k) + i, s \in \mathbf{N}_0$ ($i \in \mathbf{Z}(0, m+k-1)$). Note $z_{j(m+k)+i-m,\frac{1}{n}} \geq 1$ for any $0 \leq j \leq s$. Then from (3.1) and a simple inductive argument we have

$$\begin{cases} 1 + \frac{1}{2n} \leq y_{r,\frac{1}{n}} = \max\{1 + \frac{1}{2n}, \frac{y_{i-m-k,\frac{1}{n}}}{\prod_{j=0}^{s} z_{j(m+k)+i-m,\frac{1}{n}}}\} = 1 + \frac{1}{2n}, \\ 2 - \frac{1}{2n} \leq z_{r,\frac{1}{n}} = \max\{2 - \frac{1}{2n}, \frac{z_{i-m-k,\frac{1}{n}}}{\prod_{j=0}^{s} y_{j(m+k)+i-m,\frac{1}{n}}}\} = \max\{2 - \frac{1}{2n}, \frac{2e}{(1+\frac{1}{2n})^{s+1}}\}. \end{cases} \tag{3.5}$$

Thus $z_{n,\frac{1}{n}} = 2e/(1+\frac{1}{2n})^{s_1+1}$ since $(2-1/2n)(1+\frac{1}{2n})^{s_1+1} < (2-1/2n)(1+\frac{1}{2n})^{2n} < 2e$, where $n = s_1(m+k) + i$. On the other hand, for any $n \in \mathbf{N}$, there exists an $N_1(n) \in \mathbf{N}$ such that $z_{r,\frac{1}{n}} = 2 - \frac{1}{2n}$ for every $r \geq N_1(n)$ since $\lim_{s\longrightarrow\infty} 2e/(1+\frac{1}{2n})^s = 0$. Thus $[x_r]_{\frac{1}{n}} \neq [x_n]_{\frac{1}{n}}$ for any $r > N_1(n)$, which implies $\{x_n\}_{n=-m-k}^{\infty}$ is not eventually periodic. The proof is complete. □

Theorem 3.3 *If $C_2 \leq 1$, then there exists a positive solution $\{x_n\}_{n=-m-k}^{\infty}$ of (1.1) such that every $x_n > 1$ is a trivial fuzzy number ($n \geq -m-k$) and $\lim_{n\longrightarrow\infty} x_n = 1$.*

Proof We show that the following equation:

$$w_n = \frac{w_{n-m-k}}{w_{n-m}}, \quad n \in \mathbf{N}_0 \tag{3.6}$$

has a decreasing solution which tends to 1. Indeed, we write

$$M_1 = \{(u_1, \ldots, u_{m+k}) : u_{m+k}u_{k+1} \geq u_1 \geq \cdots \geq u_{m+k} \geq 1\}$$

and

$$M_2 = \{(u_1, \ldots, u_{m+k}) : u_{m+k}u_k \geq u_1 \geq \cdots \geq u_{m+k} \geq 1\}.$$

Then $M_1 \subset M_2$ since for any $(u_1, \ldots, u_{m+k}) \in M_1$, we have $u_{m+k}u_{k+1} \geq u_1 \geq \cdots \geq u_{m+k} \geq 1$ and $u_{m+k}u_k \geq u_{m+k}u_{k+1} \geq x_1$. Now we define $T : M_1 \to M_2$, for any $(u_1, \ldots, u_{m+k}) \in M_1$, by

$$T(u_1, \ldots, u_{m+k}) = (v_1, \ldots, v_{m+k}) \equiv \left(u_2, \ldots, u_{m+k}, \frac{u_1}{u_{k+1}}\right). \tag{3.7}$$

We show that T is well defined. Indeed, it follows from (3.7) and the definition of M_1 that

$$\begin{cases} v_i = u_{i+1}, & \text{for } i \in \mathbf{Z}\,(1, \ldots, m+k-1), \\ v_{m+k} = \frac{u_1}{u_{k+1}}, \end{cases} \tag{3.8}$$

and

$$v_{m+k}v_k = \frac{u_1}{u_{k+1}}u_{k+1} = u_1 \geq u_2 = v_1 \geq \cdots \geq v_{m+k-1} = u_{m+k} \geq \frac{u_1}{u_{k+1}} = v_{m+k} \geq 1.$$

Thus $(v_1, \ldots, v_{m+k}) \in M_2$.

Now we show that T is a bijection from M_1 to M_2. Indeed, let $u = (u_1, \ldots, u_{m+k}), v = (v_1, \ldots, v_{m+k}) \in M_1$ with $u \neq v$. Then $T(u) \neq T(v)$. On the other hand, for any $v =$

$(v_1, \dots, v_{m+k}) \in M_2$, we have

$$v_{m+k}v_k \geq v_1 \geq \cdots \geq v_{m+k} \geq 1. \tag{3.9}$$

Write

$$u = (u_1, \dots, u_{m+k}) \equiv (v_{m+k}v_k, v_1, \dots, v_{m+k-1}). \tag{3.10}$$

By (3.9) and (3.10) we have

$$u_{m+k}u_{k+1} = v_{m+k-1}v_k \geq v_{m+k}v_k = u_1 \geq v_1 = u_2 \geq \cdots \geq u_{m+k} = v_{m+k-1} \geq 1,$$

which implies $u \in M_1$ and by (3.7) we have $T(u) = v$.

Furthermore, since $T^{-1}(v_1, \dots, v_{m+k}) = (v_{m+k}v_k, v_1, \dots, v_{m+k-1})$ is continuous, T is a homeomorphism.

Noting that $M_1 \subset M_2$ and T is a homeomorphism from M_1 onto M_2, we see $T^{-1}(M_1) \subset T^{-1}(M_2) = M_1$. By induction, it follows that, for every $n \in \mathbf{N}$,

$$p = (1, 1, \dots, 1) \in T^{-n}(M_1) \subset T^{-n+1}(M_1).$$

Because M_1 is a unbounded connected closed set, we see that $T^{-n}(M_1)$ is a unbounded connected closed set for every $n \in \mathbf{N}$. Write

$$Q = \bigcap_{n=0}^{\infty} T^{-n}(M_1).$$

Then Q is also a unbounded connected set.

Let $\{w_n\}_{n=-k-m}^{\infty}$ be a solution of (3.6) with the initial values $(w_{-m-k}, \dots, w_{-1}) \in Q - \{p\}$. Then, for every $n \in \mathbf{N}$,

$$T^n(w_{-k-m}, \dots, w_{-1}) = (w_{n-k-m}, \dots, w_{n-1}) \in M_1 - \{p\},$$

which implies $w_n \geq w_{n+1} > 1$ for any $n \geq -k - m$. Let $\lim_{n\longrightarrow\infty} w_n = a$. Then by (3.6) we have $a = 1$. It is easy to show that $\{(w_n, w_n)\}_{n=-k-m}^{\infty}$ is also a solution of (3.1) which is not eventually periodic. Thus $x_n = w_n$ is a solution of (1.1) such that every $x_n > 1$ $(n \geq -m - k)$ is a trivial fuzzy number and $\lim_{n\longrightarrow\infty} x_n = 1$. The proof is complete. □

Theorem 3.4 *If $C_1 < 1 < C_2$, then every positive solution $\{x_n\}_{n=-m-k}^{\infty}$ of (1.1) is not eventually periodic.*

Proof Since $C_1 < 1 < C_2$, we see $C_{l,\lambda_1} < 1 < C_{r,\lambda_1}$ for some $\lambda_1 \in (0, 1]$. For any $\lambda \in (0, \lambda_1]$, we have

$$0 < C_{l,\lambda} \leq C_{l,\lambda_1} < 1 < C_{r,\lambda_1} \leq C_{r,\lambda}.$$

Write $M = \max\{\sup(\operatorname{supp} x_j) : j \in \mathbf{Z}(0, m+k-1)\}$. From (3.1), (3.2) and a simple inductive argument we obtain, for any $i \in \mathbf{Z}(0, m+k-1)$ and $s \in \mathbf{N}_0$ and $\lambda \in (0, \lambda_1]$,

$$\begin{aligned} C_{l,\lambda} \le y_{s(m+k)+i,\lambda} &= \max\left\{C_{l,\lambda}, \frac{y_{(s-1)(m+k)+i,\lambda}}{z_{s(m+k)+i-m,\lambda}}\right\} \le \max\left\{C_{l,\lambda}, \frac{y_{(s-1)(m+k)+i,\lambda}}{C_{r,\lambda}}\right\} \\ &\le \max\left\{C_{l,\lambda}, \frac{y_{(s-1)(m+k)+i,\lambda}}{C_{r,\lambda_1}}\right\} \le \cdots \le \max\left\{C_{l,\lambda}, \frac{y_{i,\lambda}}{C^s_{r,\lambda_1}}\right\} \\ &\le \max\left\{C_{l,\lambda}, \frac{M}{C^s_{r,\lambda_1}}\right\}. \end{aligned}$$

Thus there exists an $N \in \mathbf{N}$ such that $y_{n,\lambda} = C_{l,\lambda}$ for any $n \ge N$ and $\lambda \in (0, \lambda_1]$ since $\lim_{s\longrightarrow\infty} M/C^s_{r,\lambda_1} = 0$.

By (3.1) and (3.2) we see that, for any $n \ge m+N$ and $\lambda \in (0, \lambda_1]$,

$$z_{n,\lambda} = \max\left\{C_{r,\lambda}, \frac{z_{n-m-k,\lambda}}{C_{l,\lambda}}\right\}. \tag{3.11}$$

If $z_{n,\lambda} = C_{r,\lambda} > z_{n-m-k,\lambda}/C_{l,\lambda}$ for some $n \in \mathbf{Z}(m+N, m+N+m+k-1)$, then by (3.11) we obtain $z_{n+s(m+k),\lambda} = C_{r,\lambda}/C^s_{l,\lambda}$ for any $s \in \mathbf{N}_0$. If $z_{n,\lambda} = z_{n-m-k,\lambda}/C_{l,\lambda} \ge C_{r,\lambda}$ for some $n \in \mathbf{Z}\,(m+N, m+N+m+k-1)$, then by (3.11) we obtain $z_{n+s(m+k),\lambda} = z_{n-m-k,\lambda}/C^{s+1}_{l,\lambda}$ for any $s \in \mathbf{N}_0$. Thus $\lim_{n\longrightarrow\infty} z_{n,\lambda} = +\infty$. Furthermore, we see that $\{x_n\}_{n=-m-k}^{\infty}$ is not eventually periodic. The proof is complete. □

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Author details
[1]College of Information and Statistics, Guangxi University of Finance and Economics, Nanning, 530003, China. [2]College of Business Administration, Guangxi University of Finance and Economics, Nanning, 530003, China. [3]Guangxi Key Laboratory Cultivation Base of Cross-border E-commerce Intelligent Information Processing, Nanning, 530003, China.

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