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# On Ulam–Hyers–Rassias stability of a generalized Caputo type multi-order boundary value problem with four-point mixed integro-derivative conditions

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**Abstract**

In this research article, we turn to studying the existence and different types of stability such as generalized Ulam–Hyers stability and generalized Ulam–Hyers–Rassias stability of solutions for a new modeling of a boundary value problem equipped with the fractional differential equation which contains the multi-order generalized Caputo type derivatives furnished with four-point mixed generalized Riemann–Liouville type integro-derivative conditions. At the end of the current paper, we formulate two illustrative examples to confirm the correctness of theoretical findings from computational aspects.

**MSC:** Primary 34A08; secondary 34A12

**Keywords:** Multi-order fractional differential equation; Generalized Caputo type derivative; Ulam–Hyers stability; Ulam–Hyers–Rassias stability

## 1 Introduction

New versions of generalized fractional boundary problems have drawn much interest in recent years owing to their extensive utilization in various directions of applied sciences such as engineering, mechanics, potential theory, biology, chemistry, etc. (for example, refer to [1–15]). Many researchers play an important role in different desirable developments on the existence criteria, and some results about the uniqueness for numerous fractional differential equations have been obtained (see for instance [7, 16–24]). On the other hand, the subject of stability is a very important notion in physics since most phenomena in the real world include this concept. In fact, the stability notion of physical phenomena has an old historical context, and for the sake of such importance and applicability, one can observe a lot of work in the numerous publications not only in the last century but also before it (for example, refer to the references [25–33]). Besides, a considerable attention has been given to reviewing and investigating Hyers–Ulam stability of different functional differential and integral equations during recent decades (for example, see [34–42]).

In 2016, Niyom et al. [21] formulated the following boundary value problem supplemented with Riemann–Liouville fractional derivatives of four different orders:

$$\begin{cases} \lambda^*\mathcal{D}^{k^*}(u(t)) + (1-\lambda^*)\mathcal{D}^{\theta^*}(u(t)) = \hat{\Upsilon}(t,u(t)), \quad (t\in[0,T], k^*\in[1,2)), \\ u(0)=0, \qquad \mu_1^*\mathcal{D}^{\gamma_1^*}u(T) + (1-\mu_1^*)\mathcal{D}^{\gamma_2^*}u(T) = \delta_1^*, \end{cases} \tag{1}$$

where $\lambda^*, \mu_1^* \in (0,1)$. One year later, Ntouyas et al. [23] reviewed some existence results for the following boundary value problem furnished with multiple orders of mixed Riemann–Liouville integro-derivative operators:

$$\begin{cases} \lambda^*\mathcal{D}^{k^*}(u(t)) + (1-\lambda^*)\mathcal{D}^{\theta^*}(u(t)) = \hat{\Upsilon}(t,u(t)), \quad (t\in[0,T], k^*\in[1,2)), \\ u(0)=0, \qquad \mu_2^*\mathcal{I}^{q_1^*}u(T) + (1-\mu_2^*)\mathcal{I}^{q_2^*}u(T) = \delta_2^*, \end{cases} \tag{2}$$

where $\lambda^*, \mu_2^* \in (0,1)$.

In this position, by utilizing and mixing interesting ideas of the above-mentioned manuscripts, we intend to check some specific aims about the existence of unique solution and different types of stability for the following proposed four-point generalized Caputo type BVP including multi-order fractional integro-derivative conditions of generalized Riemann–Liouville type:

$$\begin{cases} \lambda^{*\,CC}\mathcal{D}_{t_0}^{k^*,\varrho}u(t) + {}^{CC}\mathcal{D}_{t_0}^{\theta^*,\varrho}u(t) = \hat{\Upsilon}(t,u(t)), \quad (t\in[t_0,T], k^*\in[2,3)), \\ u(t_0)=0, \\ \mu_1^{*\,CC}\mathcal{D}_{t_0}^{\gamma_1^*,\varrho}u(T) + {}^{CC}\mathcal{D}_{t_0}^{\gamma_2^*,\varrho}u(\eta) = \delta_1, \\ \mu_2^{*\,RC}\mathcal{I}_{t_0}^{q_1^*,\varrho}u(T) + {}^{RC}\mathcal{I}_{t_0}^{q_2^*,\varrho}u(\nu) = \delta_2, \end{cases} \tag{3}$$

where $\nu, \eta \in [t_0,T]$, $2<\theta^*<k^*$, $0<\lambda^*, \mu_1^*, \mu_2^* \le 1$, $0\le \gamma_1^*, \gamma_2^* < k^*-\theta^*$, $q_1^*, q_2^* \in \mathbb{R}^+$, ${}^{CC}\mathcal{D}_{t_0}^{k^*,\varrho}$ stands for the left generalized Caputo type derivative of order $\beta^* \in \{k^*, \theta^*, \gamma_1^*, \gamma_2^*\}$ with $\varrho \in (0,1]$ and $t_0 \ge 0$, ${}^{RC}\mathcal{I}_{t_0}^{q^*,\varrho}$ illustrates the left generalized Riemann–Liouville type integral of order $q* \in \{q_1^*, q_2^*\}$. Moreover, the map $\hat{\Upsilon}: [t_0,T]\times\mathbb{R}\to\mathbb{R}$ is supposed to be continuous. We draw the reader's attention to the fact that our proposed problem is unique. The novelty of this research is that we have applied fractional generalized Caputo and Riemann–Liouville type operators in such a multi-order structure for the first time, which can cover some existing works as special cases. As a special case, if we take $\varrho = 1$ and $t_0 = 0$, then our multi-order fractional operators reduce to standard fractional operators in the Caputo and Riemann–Liouville setting. The readers can find more details on the structure of new operators ${}^{CC}\mathcal{D}_{t_0}^{\cdot,\varrho}$ and ${}^{RC}\mathcal{I}_{t_0}^{\cdot,\varrho}$ in the next section. In fact, we believe that this work gives new ideas for other researchers to challenge themselves to study newer and more complex models. Compared to previous published articles in this field, the strength of this work is that the proposed construction can model different natural structures in which the boundary conditions may be designed as mixed integro-derivative conditions. It is natural that in such a case, simple modelings can satisfy as special cases in the context of our existence theorems. In this research article, we turn to study the existence and different types of stability such as generalized Ulam–Hyers stability and generalized Ulam–Hyers–Rassias stability of solutions for a new modeling of boundary value problem (3) furnished with four-point mixed generalized Riemann–Liouville type integro-derivative conditions. The contents in this research manuscript are arranged as follows. In Sect. 2, we review some fundamental and auxiliary notions and properties of fractional generalized Caputo and Riemann–Liouville type operators. In Sect. 3, the existence criteria of solutions for multi-order problem (3) are investigated with the help of some theoretical theorems based on the analytical methods. In Sect. 4, some results on different types of stability of solutions for the proposed multi-order problem (3) are reviewed. At the end of the current paper, two illustrative examples are formulated in Sect. 5 to confirm the correctness of theoretical findings from computational aspects.

## 2 Preliminaries

Now, we review some fundamental and auxiliary notions and properties of generalized Caputo and Riemann–Liouville type fractional operators. As we see in many literature works, the fractional integral operator of Riemann–Liouville type of order $k^* > 0$ for a continuous function $w : [0,+\infty) \to \mathbb{R}$ is given by ${}^{R}\mathcal{I}_0^{k^*} w(t) = \int_0^t \frac{(t-r)^{k^*-1}}{\Gamma(k^*)} w(r)\,\mathrm{d}r$ provided that the value of the integral is finite [5, 43]. Now, let us assume that $k^* \in (n-1,n)$ so that $n = [k^*]+1$. For a given function $w \in \mathcal{AC}_{\mathbb{R}}^{(n)}([0,+\infty))$, the Caputo fractional derivative operator is defined as follows:

$$ {}^{C}\mathcal{D}_0^{k^*} w(t) = \int_0^t \frac{(t-r)^{n-k^*-1}}{\Gamma(n-k^*)} w^{(n)}(r)\,\mathrm{d}r $$

so that the right-hand side integral is finite-valued [5, 43]. The left generalized derivative at the initial point $t_0$ for a function $w : [t_0,\infty) \to \mathbb{R}$ with $\varrho \in (0,1]$ is given as follows:

$$ \mathcal{D}_{t_0}^{\varrho} w(t) = \lim_{\lambda \to 0} \frac{w(t+\lambda(t-t_0)^{1-\varrho}) - w(t)}{\lambda} $$

so that the value of limit is finite [44]. Furthermore, it is evident that $\mathcal{D}_{t_0}^{\varrho} w(t) = (t-t_0)^{1-\varrho} w'(t)$ if $w$ is a differentiable function. The definition of the left generalized integral of $w$ with $\varrho \in (0,1]$ is given in the following form: $\mathcal{I}_{t_0}^{\varrho} w(t) = \int_{t_0}^t w(r) \frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}}$ whenever the right-hand side integral has finite values [44]. Next, Jarad et al. [45] extended aforementioned generalized operators to arbitrary orders in both Riemann–Liouville and Caputo settings. To see this, we assume that $k^* \in \mathbb{C}$ with $\mathrm{Re}(k^*) \geq 0$. Then the generalized Riemann–Liouville type fractional integral for a function $w$ of order $k^*$ with $\varrho \in (0,1]$ is formulated as follows:

$$ {}^{RC}\mathcal{I}_{t_0}^{k^*,\varrho} w(t) = \frac{1}{\Gamma(k^*)} \int_{t_0}^t \left( \frac{(t-t_0)^{\varrho} - (r-t_0)^{\varrho}}{\varrho} \right)^{k^*-1} w(r) \frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} $$

if the value of integral exists [45]. One can easily observe that if we take $t_0 = 0$ and $\varrho = 1$, then ${}^{RC}\mathcal{I}_{t_0}^{k^*,\varrho} w(t)$ reduces to the standard operator named the Riemann–Liouville integral ${}^{R}\mathcal{I}_0^{k^*} w(t)$. In addition, the generalized Riemann–Liouville type fractional derivative for a function $w$ of order $k^*$ with $\varrho \in (0,1]$ is illustrated as follows:

$$ \begin{aligned} {}^{RC}\mathcal{D}_{t_0}^{k^*,\varrho} w(t) &= \mathcal{D}_{t_0}^{n,\varrho} \left( {}^{RC}\mathcal{I}_{t_0}^{n-k^*,\varrho} w \right)(t) \\ &= \frac{\mathcal{D}_{t_0}^{n,\varrho}}{\Gamma(n-k^*)} \int_{t_0}^t \left( \frac{(t-t_0)^{\varrho} - (r-t_0)^{\varrho}}{\varrho} \right)^{n-k^*-1} w(r) \frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \end{aligned} $$

provided that $n = [\mathrm{Re}(k^*)] + 1$ and $\mathcal{D}_{t_0}^{n,\varrho} = \overbrace{\mathcal{D}_{t_0}^{\varrho}\mathcal{D}_{t_0}^{\varrho}\ldots\mathcal{D}_{t_0}^{\varrho}}^{n \text{ times}}$, where $\mathcal{D}_{t_0}^{\varrho}$ stands for the left generalized derivative with $\varrho \in (0,1]$ [45]. In a similar manner, it is obvious that if we take $t_0 = 0$ and $\varrho = 1$, then ${}^{RC}\mathcal{D}_{t_0}^{k^*,\varrho} w(t)$ reduces to the standard operator named Riemann–Liouville derivative ${}^{R}\mathcal{D}_{0}^{\varrho} w(t)$. In this position, to formulate a similar concept in the Caputo setting, we construct

$$\mathcal{L}_{\varrho}(t_0) := \left\{\varphi : [s_0, b] \to \mathbb{R} : \mathcal{I}_{t_0}^{\varrho}\varphi(s) \text{ exists for any } s \in [t_0, b]\right\}$$

for $\varrho \in (0,1]$ and set

$$\mathbb{I}_{\upsilon}\big([t_0, b]\big) := \left\{w : [t_0, b] \to \mathbb{R} : w(t) = \mathcal{I}_{t_0}^{\varrho}\varphi(t) + w(t_0) \text{ for some } \varphi \in \mathcal{L}_{\varrho}(t_0)\right\},$$

where $\mathcal{I}_{t_0}^{\varrho}\varphi(t) = \int_{t_0}^{t}\varphi(r)\,\mathrm{d}\upsilon(r,t_0) = \int_{t_0}^{t}\varphi(r)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}}$ is a left generalized integral of $\varphi$ [46]. For $n = 1, 2, \ldots$, we represent $\mathcal{C}_{t_0,\varrho}^{n}([t_0,b]) := \{w : [t_0,b] \to \mathbb{R} : \mathcal{D}_{t_0}^{n-1,\varrho} w \in \mathbb{I}_{\upsilon}([t_0,b])\}$. Then the generalized Caputo type fractional derivative for a function $w \in \mathcal{C}_{t_0,\upsilon}^{n}([t_0,b])$ of order $k^*$ with $\varrho \in (0,1]$ is demonstrated by

$$\begin{aligned}{}^{CC}\mathcal{D}_{t_0}^{k^*,\varrho} w(t) &= {}^{RC}\mathcal{I}_{t_0}^{n-k^*,\varrho}\big(\mathcal{D}_{t_0}^{n,\varrho} w\big)(t) \\ &= \frac{1}{\Gamma(n-k^*)}\int_{t_0}^{t}\left(\frac{(t-t_0)^{\varrho} - (r-t_0)^{\varrho}}{\varrho}\right)^{n-k^*-1}\mathcal{D}_{t_0}^{n,\varrho} w(r)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}}\end{aligned}$$

so that $n = [\mathrm{Re}(k^*)] + 1$ [45]. Evidently, ${}^{CC}\mathcal{D}_{t_0}^{k^*,\varrho} w(t) = {}^{C}\mathcal{D}_{0}^{k^*} w(t)$ if we take $t_0 = 0$ and $\varrho = 1$. In the sequel, some fundamental properties of generalized Caputo and Riemann–Liouville type fractional operators can be regarded in two next lemmas.

**Lemma 2.1** ([45]) *Suppose that* $\mathrm{Re}(k^*) > 0$, $\mathrm{Re}(\varpi) > 0$, *and* $\mathrm{Re}(\beta) > 0$. *Then, for* $\varrho \in (0,1]$ *and for any* $t > t_0$, *the following four statements are valid*:

(L1) ${}^{RC}\mathcal{I}_{t_0}^{k^*,\varrho}({}^{RC}\mathcal{I}_{t_0}^{\varpi,\varrho} w)(t) = ({}^{RC}\mathcal{I}_{t_0}^{k^*+\varpi,\varrho} w)(s)$,

(L2) ${}^{RC}\mathcal{I}_{t_0}^{k^*,\varrho}(t-t_0)^{\varrho(\beta-1)}(z) = \frac{1}{\varrho^{k^*}}\frac{\Gamma(\beta)}{\Gamma(\beta+k^*)}(z-t_0)^{\varrho(\beta+k^*-1)}$,

(L3) ${}^{RC}\mathcal{D}_{t_0}^{k^*,\varrho}(t-t_0)^{\varrho(\beta-1)}(z) = \varrho^{k^*}\frac{\Gamma(\beta)}{\Gamma(\beta-k^*)}(z-t_0)^{\varrho(\beta-k^*-1)}$,

(L4) ${}^{RC}\mathcal{D}_{t_0}^{k^*,\varrho}({}^{RC}\mathcal{I}_{t_0}^{\varpi,\varrho} w)(t) = ({}^{RC}\mathcal{I}_{t_0}^{\varpi-k^*,\varrho} w)(t)$, $(\mathrm{Re}(k^*) < \mathrm{Re}(\varpi))$.

**Lemma 2.2** ([45]) *Let* $n-1 < \mathrm{Re}(k^*) < n$ *and* $w \in \mathcal{C}_{t_0,\varrho}^{n}([t_0,b])$. *Then, for* $\varrho \in (0,1]$, *we have*

$${}^{RC}\mathcal{I}_{t_0}^{k^*,\varrho}\big({}^{CC}\mathcal{D}_{t_0}^{k^*,\varrho} w\big)(t) = w(t) - \sum_{j=0}^{n-1}\frac{\mathcal{D}_{t_0}^{j,\varrho} w(t_0)}{\upsilon^j j!}(t-t_0)^{j\varrho}.$$

In the light of the above lemma, one can verify that the general solution of the linear homogeneous equation $({}^{CC}\mathcal{D}_{t_0}^{k^*,\varrho} w)(t) = 0$ is computed by

$$w(t) = \sum_{j=0}^{n-1} b_j(t-t_0)^{j\varrho} = b_0 + b_1(t-t_0)^{\varrho} + b_2(s-s_0)^{2\varrho} + \cdots + b_{n-1}(t-t_0)^{(n-1)\varrho},$$

so that $n-1 < \mathrm{Re}(k^*) < n$ and $b_0, b_1, \ldots, b_{n-1} \in \mathbb{R}$.

Note that the following two theorems are utilized to derive the required existence criteria of solutions for the proposed four-point generalized Caputo type fractional BVP including multi-order fractional integro-derivative conditions of generalized Riemann–Liouville type (3).

**Theorem 2.3** ([47] Krasnoselskii's fixed point theorem) *Let $\mathcal{M}$ be a closed, bounded, convex, and nonempty subset of a Banach space $\mathcal{X}_*$. Moreover, the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ on $\mathcal{M}$ are supposed with the following properties*:

(a) $\mathcal{A}_1 u + \mathcal{A}_2 w \in \mathcal{M}$ *for all* $u, w \in \mathcal{M}$,

(b) $\mathcal{A}_1$ *is compact and continuous*,

(c) $\mathcal{A}_2$ *is a contraction*.

*Then there is* $z \in \mathcal{M}$ *such that* $z = \mathcal{A}_1 z + \mathcal{A}_2 z$.

**Theorem 2.4** ([48] Leray–Schauder's nonlinear alternative) *Let $\mathcal{X}_*$ be a Banach space, $\mathcal{B}^*$ be a closed and convex subset of $\mathcal{X}_*$, $\mathcal{U}$ be an open subset of $\mathcal{C}$, and $0 \in \mathcal{U}$. In addition, let $\mathcal{P} : \bar{\mathcal{U}} \to \mathcal{C}$ be a continuous and compact map. Then either*

(a) $\mathcal{P}$ *has a fixed point in* $\bar{\mathcal{U}}$, *or*

(b) *there are an element* $u \in \partial\mathcal{U}$ (*the boundary of* $\mathcal{U}$) *and a constant* $\tau^* \in (0,1)$ *such that* $u = \tau^* \mathcal{P}(u)$.

## 3 Existence criteria of solutions

In this part of the manuscript, we verify some existence results by applying some analytical techniques based on the fixed point theory. Let $0 \le t_0 < T$ and take $\tilde{J} = [t_0, T]$. Then one can easily confirm that $\mathcal{X}_* = \mathcal{C}^2(\tilde{J}, \mathbb{R})$ is a Banach space of continuous mappings furnished with the sup norm $\|u\| = \sup_{t \in \tilde{J}} |u(t)|$. First, we formulate the structure of the solution for the four-point multi-order generalized Caputo type fractional BVP as an equivalent generalized Riemann–Liouville type fractional integral equation in the following lemma.

**Lemma 3.1** *Let $\hat{\Upsilon} \in \mathcal{X}_*$. Then a map $\tilde{u}_0^*$ is a solution for the four-point multi-order linear generalized Caputo type fractional BVP*

$$\begin{cases} \lambda^* {}^{CC}\mathcal{D}_{t_0}^{k^*,\varrho} u(t) + {}^{CC}\mathcal{D}_{t_0}^{\theta^*,\varrho} u(t) = \hat{\Upsilon}(t), \quad (t \in \tilde{J}, k^* \in (2,3]), \\ u(t_0) = 0, \qquad \mu_1^* {}^{CC}\mathcal{D}_{t_0}^{\gamma_1^*,\varrho} u(T) + {}^{CC}\mathcal{D}_{t_0}^{\gamma_2^*,\varrho} u(\eta) = \delta_1, \\ \mu_2^* {}^{RC}\mathcal{I}_{t_0}^{q_1^*,\varrho} u(T) + {}^{RC}\mathcal{I}_{t_0}^{q_2^*,\varrho} u(\nu) = \delta_2, \end{cases} \tag{4}$$

*if and only if $\tilde{u}_0^*$ is a solution for the generalized Riemann–Liouville type integral equation*

$$\begin{aligned} u(t) = {} & \frac{1}{\lambda\Gamma(k^*)} \int_{t_0}^{t} \left( \frac{(t-t_0)^\varrho - (r-t_0)^\varrho}{\varrho} \right)^{k^*-1} \hat{\Upsilon}(r) \frac{\mathrm{d}r}{(r-t_0)^{1-k^*}} \\ & - \frac{1}{\lambda\Gamma(k^*-\theta^*)} \int_{t_0}^{t} \left( \frac{(t-t_0)^\varrho - (r-t_0)^\varrho}{\varrho} \right)^{k^*-\theta^*-1} u_0(r) \frac{\mathrm{d}r}{(r-t_0)^{1-k^*+\theta^*}} \\ & + \frac{(t-t_0)^\varrho}{\Theta^*} \left[ \frac{\mu_1^* \Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho} \hat{\Upsilon}(T) - \frac{\mu_1^* \Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho} u_0(T) \right. \\ & + \frac{\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho} \hat{\Upsilon}(\eta) - \frac{\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho} u_0(\eta) - \frac{\mu_2^* \Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho} \hat{\Upsilon}(T) \end{aligned}$$

$$
\begin{aligned}
&+ \frac{\mu_2^* \Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho} u_0(T) - \frac{\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho} \hat{\Upsilon}(\nu) \\
&+ \frac{\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho} u_0(\nu) - \delta_1 \Delta_4 + \Delta_2 \delta_2 \biggr] + \frac{(t-t_0)^{2\varrho}}{\Theta^*} \\
&\times \biggl[ -\frac{\mu_1^* \Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho} \hat{\Upsilon}(T) + \frac{\mu_1^* \Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho} u(T) \\
&- \frac{\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho} \hat{\Upsilon}(\eta) + \frac{\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho} u_0(\eta) + \frac{\mu_2^* \Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho} \hat{\Upsilon}(T) \\
&- \frac{\mu_2^* \Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho} u_0(T) + \frac{\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho} \hat{\Upsilon}(\nu) \\
&- \frac{\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho} u_0(\nu) + \delta_1 \Delta_3 - \delta_2 \Delta_1 \biggr]
\end{aligned}
\tag{5}
$$

*provided that*

$$
\begin{aligned}
&\Delta_1 = \mu_1^* \varrho^{\gamma_1^*} \frac{1}{\Gamma(2-\gamma_1^*)} (T-t_0)^{\varrho(1-\gamma_1^*)} + \varrho^{\gamma_2^*} \frac{1}{\Gamma(2-\gamma_2^*)} (\eta-t_0)^{\varrho(1-\gamma_2^*)}, \\
&\Delta_2 = \mu_1^* \varrho^{\gamma_1^*} \frac{2}{\Gamma(3-\gamma_1^*)} (T-t_0)^{\varrho(2-\gamma_1^*)} + \varrho^{\gamma_2^*} \frac{2}{\Gamma(3-\gamma_2^*)} (\eta-t_0)^{\varrho(2-\gamma_2^*)}, \\
&\Delta_3 = \frac{\mu_2^*}{\varrho^{q_1^*}} \frac{1}{\Gamma(2+q_1^*)} (T-t_0)^{\varrho(1+q_1^*)} + \frac{1}{\varrho^{q_2^*}} \frac{1}{\Gamma(2+q_2^*)} (\nu-t_0)^{\varrho(1+q_2^*)}, \\
&\Delta_4 = \frac{\mu_2^*}{\varrho^{q_1^*}} \frac{2}{\Gamma(3+q_1^*)} (T-t_0)^{\varrho(2+q_1^*)} + \frac{1}{\varrho^{q_2^*}} \frac{2}{\Gamma(3+q_2^*)} (\nu-t_0)^{\varrho(2+q_2^*)}, \\
&\Theta^* = \Delta_2 \Delta_3 - \Delta_1 \Delta_4.
\end{aligned}
\tag{6}
$$

*Proof* At the beginning, let $\tilde{u}_0^*$ be a solution for the four-point multi-order linear generalized Caputo type fractional BVP (4). Then, according to the properties of fractional generalized operators of both Riemann–Liouville and Caputo types, one can write

$$
\tilde{u}_0^*(t) = \frac{1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*,\varrho} \hat{\Upsilon}(t) - \frac{1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*,\varrho} \tilde{u}_0^*(t) + \tilde{c}_0^* + \tilde{c}_1^*(t-t_0)^{\varrho} + \tilde{c}_2^*(t-t_0)^{2\varrho}, \tag{7}
$$

where $\tilde{c}_0^*$, $\tilde{c}_1^*$, and $\tilde{c}_2^*$ are arbitrary constants. From the first condition, we get $\tilde{c}_0^* = 0$. By taking the generalized Caputo type derivative of order $\gamma \in \{\gamma_1^*, \gamma_2^*\}$, we obtain

$$
\begin{aligned}
\bigl({}^{CC}\mathcal{D}_{t_0}^{\gamma,\varrho} \tilde{u}_0^*\bigr)(t) = {}& \frac{1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma,\varrho} \hat{\Upsilon}(t) \\
&- \frac{1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma,\varrho} \tilde{u}_0^*(t) + \tilde{c}_1^* \varrho^{\gamma} \frac{1}{\Gamma(2-\gamma)} (t-t_0)^{\varrho(1-\gamma)} \\
&+ \tilde{c}_2^* \varrho^{\gamma} \frac{2}{\Gamma(3-\gamma)} (t-t_0)^{\varrho(2-\gamma)}.
\end{aligned}
\tag{8}
$$

Moreover, by taking the generalized Riemann–Liouville type integral of order $q \in \{q_1^*, q_2^*\}$, we obtain

$$
\begin{aligned}
\bigl({}^{RC}\mathcal{I}_{t_0}^{q,\varrho} \tilde{u}_0^*\bigr)(t) = {}& \frac{1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q+k^*,\varrho} \hat{\Upsilon}(t) \\
&- \frac{1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*+q-\theta^*,\varrho} \tilde{u}_0^*(t) + \frac{\tilde{c}_1^*}{\varrho^{q}} \frac{1}{\Gamma(2+q)} (t-t_0)^{\varrho(1+q)}
\end{aligned}
$$

$$+\frac{\tilde{c}_2^*}{\varrho^q}\frac{2}{\Gamma(3+q)}(t-t_0)^{\varrho(2+q)}. \tag{9}$$

By combining equations (8) and (9) with boundary conditions of four-point multi-order BVP (3), we get

$$\begin{aligned}
\tilde{c}_1^* &= \frac{1}{\Delta_2\Delta_3-\Delta_1\Delta_4} \\
&\quad\times\Bigg[\frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}\hat{\Upsilon}(T)-\frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho}\tilde{u}_0^*(T) \\
&\quad+\frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho}\hat{\Upsilon}(\eta) \\
&\quad-\frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho}\tilde{u}_0^*(\eta)-\frac{\mu_2^*\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho}\hat{\Upsilon}(T) \\
&\quad+\frac{\mu_2^*\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho}\tilde{u}_0^*(T) \\
&\quad-\frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho}\hat{\Upsilon}(\nu)+\frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho}\tilde{u}_0^*(\nu)-\delta_1\Delta_4+\Delta_2\delta_2\Bigg]
\end{aligned}$$

and

$$\begin{aligned}
\tilde{c}_2^* &= \frac{1}{\Delta_2\Delta_3-\Delta_1\Delta_4} \\
&\quad\times\Bigg[-\frac{\mu_1^*\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}\hat{\Upsilon}(T)+\frac{\mu_1^*\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho}\tilde{u}_0^*(T) \\
&\quad-\frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho}\hat{\Upsilon}(\eta) \\
&\quad+\frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho}\tilde{u}_0^*(\eta)+\frac{\mu_2^*\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho}\hat{\Upsilon}(T) \\
&\quad-\frac{\mu_2^*\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho}\tilde{u}_0^*(T) \\
&\quad+\frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho}\hat{\Upsilon}(\nu)-\frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho}\tilde{u}_0^*(\nu)+\delta_1\Delta_3-\delta_2\Delta_1\Bigg].
\end{aligned}$$

Finally, if we substitute constants $\tilde{c}_0^*$ and $\tilde{c}_1^*$ and $\tilde{c}_2^*$ in (7), then we reach the generalized Riemann–Liouville type integral equation (5). In the opposite direction, one can easily verify that $\tilde{u}_0^*$ is considered as a solution for the four-point multi-order linear generalized Caputo type fractional BVP (4) whenever $\tilde{u}_0^*$ satisfies the generalized Riemann–Liouville type integral equation (5). □

Based on the implemented calculations in Lemma 3.1, we define the operator $\tilde{\mathcal{F}}_*:\mathcal{X}_*\to\mathcal{X}_*$ in the following framework:

$$\begin{aligned}
\tilde{\mathcal{F}}_*u(t) &= \frac{1}{\lambda\Gamma(k^*)}\int_{t_0}^{t}\left(\frac{(t-t_0)^\varrho-(r-t_0)^\varrho}{\varrho}\right)^{k^*-1}\hat{\Upsilon}\big(r,u(r)\big)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \\
&\quad-\frac{1}{\lambda\Gamma(k^*-\theta^*)}\int_{t_0}^{t}\left(\frac{(t-t_0)^\varrho-(r-t_0)^\varrho}{\varrho}\right)^{k^*-\theta^*-1}u(r)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \\
&\quad+\frac{(t-t_0)^\varrho}{\Theta^*}\Bigg[\frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}\hat{\Upsilon}\big(T,u(T)\big)-\frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho}u(T)
\end{aligned} \tag{10}$$

$$
\begin{aligned}
&+ \frac{\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho} \hat{\Upsilon}\big(\eta, u(\eta)\big) - \frac{\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho} u(\eta) \\
&- \frac{\mu_2^*\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho} \hat{\Upsilon}\big(T, u(T)\big) \\
&+ \frac{\mu_2^*\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho} u(T) - \frac{\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho} \hat{\Upsilon}\big(\nu, u(\nu)\big) \\
&+ \frac{\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho} u(\nu) \\
&- \delta_1\Delta_4 + \Delta_2\delta_2 \bigg] + \frac{(t-t_0)^{2\varrho}}{\Theta^*} \bigg[ -\frac{\mu_1^*\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho} \hat{\Upsilon}\big(T, u(T)\big) \\
&+ \frac{\mu_1^*\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho} u(T) \\
&- \frac{\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho} \hat{\Upsilon}\big(\eta, u(\eta)\big) + \frac{\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho} u(\eta) \\
&+ \frac{\mu_2^*\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho} \hat{\Upsilon}\big(T, u(T)\big) \\
&- \frac{\mu_2^*\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho} u(T) + \frac{\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho} \hat{\Upsilon}\big(\nu, u(\nu)\big) \\
&- \frac{\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho} u(\nu) + \delta_1\Delta_3 - \delta_2\Delta_1 \bigg].
\end{aligned}
$$

It is notable that the four-point multi-order generalized Caputo type fractional BVP (3) has a solution $\tilde{u}_0^*$ if and only if $\tilde{u}_0^*$ is a fixed point for the self-map $\tilde{\mathcal{F}}_*$. For the sake of convenience in writing, we utilize the following simplified notations:

$$
\begin{aligned}
\mathcal{W}_1 = {} & \frac{1}{\lambda\Gamma(k^*-\theta^*+1)} \left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k^*-\theta^*} \\
&+ \frac{(T-t_0)^\varrho}{|\Theta^*|} \bigg[ \frac{\mu_1^*\Delta_4}{\lambda\Gamma(k^*-\theta^*-\gamma_1^*+1)} \left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k^*-\theta^*-\gamma_1^*} \\
&+ \frac{\Delta_4}{\lambda\Gamma(k^*-\theta^*-\gamma_2^*+1)} \left(\frac{(\eta-t_0)^\varrho}{\varrho}\right)^{k^*-\theta^*-\gamma_2^*} \\
&+ \frac{\mu_2^*\Delta_2}{\lambda\Gamma(q_1^*+k^*-\theta^*+1)} \left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{q_1^*+k^*-\theta^*} \\
&+ \frac{\Delta_2}{\lambda\Gamma(q_2^*+k^*-\theta^*+1)} \left(\frac{(\nu-t_0)^\varrho}{\varrho}\right)^{q_2^*+k^*-\theta^*} \bigg] + \frac{(T-t_0)^{2\varrho}}{|\Theta^*|} \\
&\times \bigg[ \frac{\mu_1^*\Delta_3}{\lambda\Gamma(k^*-\theta^*-\gamma_1^*+1)} \left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k^*-\theta^*-\gamma_1^*} \\
&+ \frac{\Delta_3}{\lambda\Gamma(k^*-\theta^*-\gamma_2^*+1)} \left(\frac{(\eta-t_0)^\varrho}{\varrho}\right)^{k^*-\theta^*-\gamma_2^*} \\
&+ \frac{\mu_2^*\Delta_1}{\lambda\Gamma(q_1^*+k^*-\theta^*+1)} \left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{q_1^*+k^*-\theta^*} \\
&+ \frac{\Delta_1}{\lambda\Gamma(q_2^*+k^*-\theta^*+1)} \left(\frac{(\nu-t_0)^\varrho}{\varrho}\right)^{q_2^*+k^*-\theta^*} \bigg]
\end{aligned}
\tag{11}
$$

and

$$\begin{aligned}
\mathcal{W}_2 &= \frac{1}{\lambda\Gamma(k^*+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k^*} \\
&\quad + \frac{(T-t_0)^\varrho}{|\Theta^*|}\left[\frac{\mu_1^*\Delta_4}{\lambda\Gamma(k^*-\gamma_1^*+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k^*-\gamma_1^*}\right. \\
&\quad + \frac{\Delta_4}{\lambda\Gamma(k^*-\gamma_2^*+1)}\left(\frac{(\eta-t_0)^\varrho}{\varrho}\right)^{k^*-\gamma_2^*} \\
&\quad + \frac{\mu_2^*\Delta_2}{\lambda\Gamma(q_1^*+k^*+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{q_1^*+k^*} \\
&\quad \left. + \frac{\Delta_2}{\lambda\Gamma(q_2^*+k^*+1)}\left(\frac{(\nu-t_0)^\varrho}{\varrho}\right)^{q_2^*+k^*}\right] \\
&\quad + \frac{(T-t_0)^{2\varrho}}{|\Theta^*|}\left[\frac{\mu_1^*\Delta_3}{\lambda\Gamma(k^*-\gamma_1^*+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k^*-\gamma_1^*}\right. \\
&\quad + \frac{\Delta_3}{\lambda\Gamma(k^*-\gamma_2^*+1)}\left(\frac{(\eta-t_0)^\varrho}{\varrho}\right)^{k^*-\gamma_2^*} \\
&\quad + \frac{\mu_2^*\Delta_1}{\lambda\Gamma(q_1^*+k^*+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{q_1^*+k^*} \\
&\quad \left. + \frac{\Delta_1}{\lambda\Gamma(q_2^*+k^*+1)}\left(\frac{(\nu-t_0)^\varrho}{\varrho}\right)^{q_2^*+k^*}\right].
\end{aligned} \tag{12}$$

**Theorem 3.2** *Let the real-valued mapping $\hat{\Upsilon}:\tilde{J}\times\mathcal{X}_*\to\mathbb{R}$ be continuous and there be a constant $\mathcal{L}_*>0$ such that $|\hat{\Upsilon}(t,u)-\hat{\Upsilon}(t,u')|\leq\mathcal{L}_*|u-u'|$ for all $t\in\tilde{J}$ and $u,u'\in\mathcal{X}_*$. If $\mathcal{L}_*\mathcal{W}_2+\mathcal{W}_1<1$, then the four-point multi-order linear generalized Caputo type fractional BVP (3) has a unique solution, where $\mathcal{W}_1$ and $\mathcal{W}_2$ are illustrated by (11) and (12).*

*Proof* Put $\sup_{t\in\tilde{J}}|\hat{\Upsilon}(t,0)|=\mathcal{N}^*<\infty$. We choose $\mathcal{R}^*>0$ such that

$$\frac{|\Theta^*|N^*\mathcal{W}_2+(T-t_0)^{2\varrho}(|\delta_1\Delta_3|+|\delta_2\Delta_1|)+(T-t_0)^\varrho(|\delta_1\Delta_4|+|\delta_2\Delta_2|)}{|\Theta^*|(1-\mathcal{L}_*\mathcal{W}_2-\mathcal{W}_1)}\leq\mathcal{R}^*,$$

where $\Delta_j (j=1,2,3,4)$ are illustrated by (6). Next, construct the set $\mathcal{B}^*_{\mathcal{R}^*}=\{u\in\mathcal{X}_*:\|u\|\leq\mathcal{R}^*\}$. In this case, we verify that $\tilde{\mathcal{F}}_*\mathcal{B}^*_{\mathcal{R}^*}\subset\mathcal{B}^*_{\mathcal{R}^*}$. To observe this, for each $u\in\mathcal{B}^*_{\mathcal{R}^*}$, we may write

$$\begin{aligned}
\left|\tilde{\mathcal{F}}_*u(t)\right| &\leq \frac{1}{\lambda\Gamma(k^*)}\int_{t_0}^t\left(\frac{(t-t_0)^\varrho-(r-t_0)^\varrho}{\varrho}\right)^{k^*-1} \\
&\qquad\times\left(\left|\hat{\Upsilon}\big(r,u(r)\big)-\hat{\Upsilon}(r,0)\right|+\left|\hat{\Upsilon}(r,0)\right|\right)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \\
&\qquad+\frac{1}{\lambda\Gamma(k^*-\theta^*)}\int_{t_0}^t\left(\frac{(t-t_0)^\varrho-(r-t_0)^\varrho}{\varrho}\right)^{k^*-\theta^*-1} \\
&\qquad\times\left|u(r)\right|\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}}+\frac{(T-t_0)^\varrho}{|\Theta^*|}
\end{aligned}$$

$$
\begin{aligned}
&\times \Bigg[ \frac{\mu_1^* \Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho} \big( \big| \hat{\Upsilon}\big(T, u(T)\big) - \hat{\Upsilon}(T,0) \big| + \big| \hat{\Upsilon}(T,0) \big| \big) \\
&+ \frac{\mu_1^* \Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho} \big| u(T) \big| \\
&+ \frac{\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho} \big( \big| \hat{\Upsilon}\big(\eta, u(\eta)\big) - \hat{\Upsilon}(\eta,0) \big| + \big| \hat{\Upsilon}(\eta,0) \big| \big) \\
&+ \frac{\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho} \big| u(\eta) \big| \\
&+ \frac{\mu_2^* \Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho} \big( \big| \hat{\Upsilon}\big(T, u(T)\big) - \hat{\Upsilon}(T,0) \big| + \big| \hat{\Upsilon}(T,0) \big| \big) \\
&+ \frac{\mu_2^* \Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho} \big| u(T) \big| \\
&+ \frac{\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho} \big( \big| \hat{\Upsilon}\big(\nu, u(\nu)\big) - \hat{\Upsilon}(\nu,0) \big| + \big| \hat{\Upsilon}(\nu,0) \big| \big) \\
&+ \frac{\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho} \big| u(\nu) \big| + |\delta_1 \Delta_4| + |\Delta_2 \delta_2| \Bigg] + \frac{(T-t_0)^{2\varrho}}{|\Theta^*|} \\
&\times \Bigg[ \frac{\mu_1^* \Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho} \big( \big| \hat{\Upsilon}\big(T, u(T)\big) - \hat{\Upsilon}(T,0) \big| + \big| \hat{\Upsilon}(T,0) \big| \big) \\
&+ \frac{\mu_1^* \Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho} \big| u(T) \big| \\
&+ \frac{\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho} \big( \big| \hat{\Upsilon}\big(\eta, u(\eta)\big) - \hat{\Upsilon}(\eta,0) \big| + \big| \hat{\Upsilon}(\eta,0) \big| \big) \\
&+ \frac{\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho} \big| u(\eta) \big| \\
&+ \frac{\mu_2^* \Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho} \big( \big| \hat{\Upsilon}\big(T, u(T)\big) - \hat{\Upsilon}(T,0) \big| + \big| \hat{\Upsilon}(T,0) \big| \big) \\
&+ \frac{\mu_2^* \Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho} \big| u(T) \big| \\
&+ \frac{\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho} \big( \big| \hat{\Upsilon}\big(\nu, u(\nu)\big) - \hat{\Upsilon}(\nu,0) \big| + \big| \hat{\Upsilon}(\nu,0) \big| \big) \\
&+ \frac{\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho} \big| u(\nu) \big| + |\delta_1 \Delta_3| + |\delta_2 \Delta_1| \Bigg] \\
\le{}& \big( \mathcal{L}_* \|u\| + \mathcal{N}^* \big) \mathcal{W}_2 + \|u\| \mathcal{W}_1 + \frac{(T-t_0)^{2\varrho}}{|\Theta^*|} \big( |\delta_1 \Delta_3| + |\delta_2 \Delta_1| \big) \\
&+ \frac{(T-t_0)^{\varrho}}{|\Theta^*|} \big( |\delta_1 \Delta_4| + |\delta_2 \Delta_2| \big) \\
\le{}& (\mathcal{L}_* \mathcal{W}_2 + \mathcal{W}_1) \mathcal{R} + \mathcal{N}^* \mathcal{W}_2 + \frac{(T-t_0)^{2\varrho}}{|\Theta^*|} \big( |\delta_1 \Delta_3| + |\delta_2 \Delta_1| \big) \\
&+ \frac{(T-t_0)^{\varrho}}{|\Theta^*|} \big( |\delta_1 \Delta_4| + |\delta_2 \Delta_2| \big) \le \mathcal{R}^*.
\end{aligned}
$$

Thus, we reach the inequality $\|\tilde{\mathcal{F}}_* u\| \le \mathcal{R}^*$, which means that $\tilde{\mathcal{F}}_* \mathcal{B}_{\mathcal{R}^*}^* \subset \mathcal{B}_{\mathcal{R}^*}^*$. In the next stage, let us assume that $u, u' \in \mathcal{X}_*$. For any $t \in \tilde{J}$, one can write

$$
\big| \tilde{\mathcal{F}}_* u(t) - \tilde{\mathcal{F}}_* u'(t) \big|
$$

$$
\begin{aligned}
&\leq \frac{1}{\lambda\Gamma(k^*)}\int_{t_0}^{t}\left(\frac{(t-t_0)^\varrho-(r-t_0)^\varrho}{\varrho}\right)^{k^*-1}\\
&\quad\times\left(\left|\hat{\Upsilon}\left(r,u(r)\right)-\hat{\Upsilon}\left(r,u'(r)\right)\right|\right)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}}\\
&\quad+\frac{1}{\lambda\Gamma(k^*-\theta^*)}\int_{t_0}^{t}\left(\frac{(t-t_0)^\varrho-(r-t_0)^\varrho}{\varrho}\right)^{k^*-\theta^*-1}\\
&\quad\times\left|u(r)-u'(r)\right|\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}}+\frac{(T-t_0)^\varrho}{|\Theta^*|}\\
&\quad\times\left[\frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}\left(\left|\hat{\Upsilon}\left(T,u(T)\right)-\hat{\Upsilon}\left(T,u'(T)\right)\right|\right)\right.\\
&\quad+\frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho}\left|u(T)-u'(T)\right|\\
&\quad+\frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho}\left(\left|\hat{\Upsilon}\left(\eta,u(\eta)\right)-\hat{\Upsilon}\left(\eta,u'(\eta)\right)\right|\right)\\
&\quad+\frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho}\left|u(\eta)-u'(\eta)\right|\\
&\quad+\frac{\mu_2^*\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho}\left(\left|\hat{\Upsilon}\left(T,u(T)\right)-\hat{\Upsilon}\left(T,u'(T)\right)\right|\right)\\
&\quad+\frac{\mu_2^*\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho}\left|u(T)-u'(T)\right|\\
&\quad\times\frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho}\left(\left|\hat{\Upsilon}\left(\nu,u(\nu)\right)-\hat{\Upsilon}\left(\nu,u'(\nu)\right)\right|\right)\\
&\quad\left.+\frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho}\left|u(\nu)-u'(\nu)\right|\right]\\
&\quad+\frac{(T-t_0)^{2\varrho}}{|\Theta^*|}\left[\frac{\mu_1^*\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}\left(\left|\hat{\Upsilon}\left(T,u(T)\right)-\hat{\Upsilon}\left(T,u'(T)\right)\right|\right)\right.\\
&\quad+\frac{\mu_1^*\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho}\left|u(T)-u'(T)\right|\\
&\quad+\frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho}\left(\left|\hat{\Upsilon}\left(\eta,u(\eta)\right)-\hat{\Upsilon}\left(\eta,u'(\eta)\right)\right|\right)\\
&\quad+\frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho}\left|u(\eta)-u'(\eta)\right|\\
&\quad+\frac{\mu_2^*\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho}\left(\left|\hat{\Upsilon}\left(T,u(T)\right)-\hat{\Upsilon}\left(T,u'(T)\right)\right|\right)\\
&\quad+\frac{\mu_2^*\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho}\left|u(T)-u'(T)\right|\\
&\quad+\frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho}\left(\left|\hat{\Upsilon}\left(\nu,u(\nu)\right)-\hat{\Upsilon}\left(\nu,u'(\nu)\right)\right|\right)\\
&\quad\left.+\frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho}\left|u(\nu)-u'(\nu)\right|\right]\\
&\leq\left(\mathcal{L}_*\left\|u-u'\right\|\right)\mathcal{W}_2+\left\|u-u'\right\|\mathcal{W}_1=(\mathcal{L}_*\mathcal{W}_2+\mathcal{W}_1)\left\|u-u'\right\|.
\end{aligned}
$$

This represents $\|\tilde{\mathcal{F}}_*u-\tilde{\mathcal{F}}_*u'\|\leq(\mathcal{L}_*\mathcal{W}_2+\mathcal{W}_1)\|u-u'\|$, which implies that $\tilde{\mathcal{F}}_*$ is a contraction since $\mathcal{L}_*\mathcal{W}_2+\mathcal{W}_1<1$. Hence, with due attention to the Banach principle, the operator

$\tilde{\mathcal{F}}_*$ has a unique fixed point, which means that the four-point multi-order nonlinear generalized Caputo type fractional BVP (3) has a unique solution. □

Here, by applying another method based on Krasnoselskii's fixed point theorem, we derive another kind of existence criterion of solutions for the proposed problem (3).

**Theorem 3.3** *Consider a continuous map* $\hat{\Upsilon}:\tilde{J}\times\mathcal{X}_*\to\mathbb{R}$. *Let there be a positive constant* $\mathcal{L}_*$ *so that an inequality* $|\hat{\Upsilon}(t,u)-\hat{\Upsilon}(t,u')|\le\mathcal{L}_*|u-u'|$ *holds for any* $t\in\tilde{J}$ *and* $u,u'\in\mathcal{X}_*$. *If there exists* $\mathcal{V}(t)\in\mathcal{C}_{\mathbb{R}^+}(\tilde{J})$ *provided that* $\hat{\Upsilon}(t,u)\le\mathcal{V}(t)$ *for all* $(t,u)\in\tilde{J}\times\mathcal{X}_*$ *and* $\mathcal{L}_*\mathcal{W}_2<1$, *then the four-point multi-order nonlinear generalized Caputo type fractional BVP* (3) *has at least one solution. Note that* $\mathcal{W}_2$ *is defined in* (12).

*Proof* By setting $\|\mathcal{V}\|=\sup_{t\in\tilde{J}}|\mathcal{V}(t)|$ and choosing an appropriate constant $r^*>0$, construct the nonempty set $\mathcal{B}^*_{r^*}=\{u\in\mathcal{X}_*:\|u\|\le r^*\}$, where

$$\frac{|\Theta^*|\|\mathcal{V}\|\mathcal{W}_2+(T-t_0)^{2\varrho}(|\delta_1\Delta_3|+|\delta_2\Delta_1|)+(T-t_0)^{\varrho}(|\delta_1\Delta_4|+|\delta_2\Delta_2|)}{|\Theta^*|(1-\mathcal{W}_1)}\le r^*$$

and $\Delta_1,\Delta_2,\Delta_3,\Delta_4,\mathcal{W}_1$ and $\mathcal{W}_2$ are given by (6), (11), and (12), respectively. For every $t\in\tilde{J}$, we consider two operators $\hat{\mathcal{F}}_1$ and $\hat{\mathcal{F}}_2$ on $\mathcal{B}^*_{r^*}$ by the following defined rules:

$$\begin{aligned}
\hat{\mathcal{F}}_1u(t)={}&\frac{-1}{\lambda\Gamma(k^*-\theta^*)}\int_{t_0}^{t}\left(\frac{(t-t_0)^{\varrho}-(r-t_0)^{\varrho}}{\varrho}\right)^{k^*-\theta^*-1}u(r)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}}\\
&+\frac{(t-t_0)^{\varrho}}{\Theta^*}\left[-\frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho}u(T)-\frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho}u(\eta)\right.\\
&\left.+\frac{\mu_2^*\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho}u(T)+\frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho}u(\nu)\right]\\
&+\frac{(t-t_0)^{2\varrho}}{\Theta^*}\left[\frac{\mu_1^*\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho}u(T)\frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho}u(\eta)\right.\\
&\left.-\frac{\mu_2^*\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho}u(T)-\frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho}u(\nu)\right]
\end{aligned}$$

and

$$\begin{aligned}
\hat{\mathcal{F}}_2u(t)={}&\frac{1}{\lambda\Gamma(k^*)}\int_{t_0}^{t}\left(\frac{(t-t_0)^{\varrho}-(r-t_0)^{\varrho}}{\varrho}\right)^{k^*-1}\hat{\Upsilon}\big(r,u(r)\big)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}}\\
&+\frac{(t-t_0)^{\varrho}}{\Theta^*}\left[\frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}\hat{\Upsilon}\big(T,u(T)\big)+\frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho}\hat{\Upsilon}\big(\eta,u(\eta)\big)\right.\\
&\left.-\frac{\mu_2^*\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho}\hat{\Upsilon}\big(T,u(T)\big)-\frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho}\hat{\Upsilon}\big(\nu,u(\nu)\big)-\delta_1\Delta_4+\Delta_2\delta_2\right]\\
&+\frac{(t-t_0)^{2\varrho}}{\Theta^*}\left[-\frac{\mu_1^*\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}\hat{\Upsilon}\big(T,u(T)\big)-\frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho}\hat{\Upsilon}\big(\eta,u(\eta)\big)\right.\\
&\left.+\frac{\mu_2^*\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho}\hat{\Upsilon}\big(T,u(T)\big)+\frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho}\hat{\Upsilon}\big(\nu,u(\nu)\big)+\delta_1\Delta_3-\delta_2\Delta_1\right].
\end{aligned}$$

In this position, we intend to prove that $\hat{\mathcal{F}}_1 u + \hat{\mathcal{F}}_2 u' \in \mathcal{B}^*_{r^*}$. Let $u, u' \in \mathcal{B}^*_{r^*}$. Then one can write

$$
\begin{aligned}
&\left|\hat{\mathcal{F}}_1 u(t) + \hat{\mathcal{F}}_2 u'(t)\right| \\
&\quad\le \|u\| \Bigg[ \frac{1}{\lambda\Gamma(k^* - \theta^* + 1)} \left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k^*-\theta^*} + \frac{(T-t_0)^\varrho}{|\Theta^*|} \\
&\qquad \times \Bigg[ \frac{\mu_1^* \Delta_4}{\lambda\Gamma(k^* - \theta^* - \gamma_1^* + 1)} \left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k^*-\theta^*-\gamma_1^*} \\
&\qquad + \frac{\Delta_4}{\lambda\Gamma(k^* - \theta^* - \gamma_2^* + 1)} \left(\frac{(\eta-t_0)^\varrho}{\varrho}\right)^{k^*-\theta^*-\gamma_2^*} \\
&\qquad + \frac{\mu_2^* \Delta_2}{\lambda\Gamma(q_1^* + k^* - \theta^* + 1)} \left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{q_1^*+k^*-\theta^*} \\
&\qquad + \frac{\Delta_2}{\lambda\Gamma(q_2^* + k^* - \theta^* + 1)} \left(\frac{(\nu-t_0)^\varrho}{\varrho}\right)^{q_2^*+k^*-\theta^*} \Bigg] \\
&\qquad + \frac{(T-t_0)^{2\varrho}}{|\Theta^*|} \Bigg[ \frac{\mu_1^* \Delta_3}{\lambda\Gamma(k^* - \theta^* - \gamma_1^* + 1)} \left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k^*-\theta^*-\gamma_1^*} \\
&\qquad + \frac{\Delta_3}{\lambda\Gamma(k^* - \theta^* - \gamma_2^* + 1)} \left(\frac{(\eta-t_0)^\varrho}{\varrho}\right)^{k^*-\theta^*-\gamma_2^*} \\
&\qquad + \frac{\mu_2^* \Delta_1}{\lambda\Gamma(q_1^* + k^* - \theta^* + 1)} \left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{q_1^*+k^*-\theta^*} \\
&\qquad + \frac{\Delta_1}{\lambda\Gamma(q_2^* + k^* - \theta^* + 1)} \left(\frac{(\nu-t_0)^\varrho}{\varrho}\right)^{q_2^*+k^*-\theta^*} \Bigg]\Bigg] \\
&\qquad + \|\mathcal{V}\| \Bigg[ \frac{1}{\lambda\Gamma(k^* + 1)} \left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k^*} \\
&\qquad + \frac{(T-t_0)^\varrho}{|\Theta^*|} \Bigg[ \frac{\mu_1^* \Delta_4}{\lambda\Gamma(k^* - \gamma_1^* + 1)} \left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k^*-\gamma_1^*} \\
&\qquad + \frac{\Delta_4}{\lambda\Gamma(k^* - \gamma_2^* + 1)} \left(\frac{(\eta-t_0)^\varrho}{\varrho}\right)^{k^*-\gamma_2^*} \\
&\qquad + \frac{\mu_2^* \Delta_2}{\lambda\Gamma(q_1^* + k^* + 1)} \left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{q_1^*+k^*} \\
&\qquad + \frac{\Delta_2}{\lambda\Gamma(q_2^* + k^* + 1)} \left(\frac{(\nu-t_0)^\varrho}{\varrho}\right)^{q_2^*+k^*} \Bigg] \\
&\qquad + \frac{(T-t_0)^{2\varrho}}{|\Theta^*|} \Bigg[ \frac{\mu_1^* \Delta_3}{\lambda\Gamma(k^* - \gamma_1^* + 1)} \left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k^*-\gamma_1^*} \\
&\qquad + \frac{\Delta_3}{\lambda\Gamma(k^* - \gamma_2^* + 1)} \left(\frac{(\eta-t_0)^\varrho}{\varrho}\right)^{k^*-\gamma_2^*} + \frac{\mu_2^* \Delta_1}{\lambda\Gamma(q_1^* + k^* + 1)} \left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{q_1^*+k^*} \\
&\qquad + \frac{\Delta_1}{\lambda\Gamma(q_2^* + k^* + 1)} \left(\frac{(\nu-t_0)^\varrho}{\varrho}\right)^{q_2^*+k^*} \Bigg]\Bigg] + \frac{(T-t_0)^{2\varrho}}{|\Theta^*|} \big(|\delta_1 \Delta_3| + |\delta_2 \Delta_1|\big) \\
&\qquad + \frac{(T-t_0)^\varrho}{|\Theta^*|} \big(|\delta_1 \Delta_4| + |\delta_2 \Delta_2|\big)
\end{aligned}
$$

$$\begin{aligned}
&\leq \bigl(r^*\mathcal{W}_1 + \|\mathcal{V}\|\mathcal{W}_2\bigr) + \frac{(T-t_0)^{2\varrho}}{|\Theta^*|}\bigl(|\delta_1\Delta_3| + |\delta_2\Delta_1|\bigr) \\
&\quad + \frac{(T-t_0)^{\varrho}}{|\Theta^*|}\bigl(|\delta_1\Delta_4| + |\delta_2\Delta_2|\bigr) \leq r^*,
\end{aligned}$$

and so it follows that $\hat{\mathcal{F}}_1 u + \hat{\mathcal{F}}_2 u' \in \mathcal{B}^*_{r^*}$. Now, we claim that $\hat{\mathcal{F}}_2$ is a contraction. To confirm this claim, for each two elements $u, u' \in \mathcal{B}^*_{r^*}$, we have

$$\begin{aligned}
&\bigl|\hat{\mathcal{F}}_2 u(t) - \hat{\mathcal{F}}_2 u'(t)\bigr| \\
&\quad = \Biggl|\frac{1}{\lambda\Gamma(k^*)}\int_{t_0}^{t}\biggl(\frac{(t-t_0)^{\varrho}-(r-t_0)^{\varrho}}{\varrho}\biggr)^{k^*-1} \\
&\qquad \times \hat{\Upsilon}\bigl(r,u(r)\bigr) - \hat{\Upsilon}\bigl(r,u'(r)\bigr)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \\
&\qquad + \frac{(t-t_0)^{\varrho}}{\Theta^*}\biggl[\frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}\bigl(\hat{\Upsilon}\bigl(T,u(T)\bigr) - \hat{\Upsilon}\bigl(T,u'(T)\bigr)\bigr) \\
&\qquad + \frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho}\bigl(\hat{\Upsilon}\bigl(\eta,u(\eta)\bigr) - \hat{\Upsilon}\bigl(\eta,u'(\eta)\bigr)\bigr) \\
&\qquad - \frac{\mu_2^*\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho}\bigl(\hat{\Upsilon}\bigl(T,u(T)\bigr) - \hat{\Upsilon}\bigl(T,u'(T)\bigr)\bigr) \\
&\qquad - \frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho}\bigl(\hat{\Upsilon}\bigl(\nu,u(\nu)\bigr) - \hat{\Upsilon}\bigl(\nu,u(\nu)\bigr)\bigr)\biggr] \\
&\qquad + \frac{(t-t_0)^{2\varrho}}{\Theta^*}\biggl[-\frac{\mu_1^*\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}\bigl(\hat{\Upsilon}\bigl(T,u(T)\bigr) - \hat{\Upsilon}\bigl(T,u'(T)\bigr)\bigr) \\
&\qquad - \frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho}\bigl(\hat{\Upsilon}\bigl(\eta,u(\eta)\bigr) - \hat{\Upsilon}\bigl(\eta,u'(\eta)\bigr)\bigr) \\
&\qquad + \frac{\mu_2^*\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho}\bigl(\hat{\Upsilon}\bigl(T,u(T)\bigr) - \hat{\Upsilon}\bigl(T,u'(T)\bigr)\bigr) \\
&\qquad + \frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho}\bigl(\hat{\Upsilon}\bigl(\nu,u(\nu)\bigr) - \hat{\Upsilon}\bigl(\nu,u'(\nu)\bigr)\bigr)\biggr]\Biggr| \\
&\quad \leq \mathcal{L}_*\mathcal{W}_2\bigl\|u-u'\bigr\|.
\end{aligned}$$

Since $\mathcal{L}_*\mathcal{W}_2 < 1$, thus $\mathcal{F}_2$ is a contraction. In the subsequent step, we check the continuity of $\hat{\mathcal{F}}_1$ on $\mathcal{B}^*_{r^*}$. To reach this aim, let $\{u_n\}$ be a sequence in $\mathcal{B}^*_{r^*}$ approaching a point $u \in \mathcal{B}^*_{r^*}$. Then, due to the continuity of the generalized Riemann–Liouville type operator, one can write

$$\begin{aligned}
\lim_{n\to+\infty}\hat{\mathcal{F}}_1 u_n(t) &= \frac{-1}{\lambda\Gamma(k^*-\theta^*)}\int_{t_0}^{t}\biggl(\frac{(t-t_0)^{\varrho}-(r-t_0)^{\varrho}}{\varrho}\biggr)^{k^*-\theta^*-1} \\
&\quad \times \lim_{n\to+\infty}u_n(r)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \\
&\quad + \frac{(t-t_0)^{\varrho}}{\Theta^*}\biggl[-\frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho}\lim_{n\to+\infty}u_n(T) \\
&\quad - \frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho}\lim_{n\to+\infty}u_n(\eta) \\
&\quad + \frac{\mu_2^*\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho}\lim_{n\to+\infty}u_n(T) + \frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho}\lim_{n\to+\infty}u_n(\nu)\biggr]
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{(t-t_0)^{2\varrho}}{\Theta^*}\left[\frac{\mu_1^*\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho}\lim_{n\to+\infty}u_n(T)\right.\\
&+ \frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho}\lim_{n\to+\infty}u_n(\eta)\\
&\left.- \frac{\mu_2^*\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho}\lim_{n\to+\infty}u_n(T) - \frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho}\lim_{n\to+\infty}u_n(\nu)\right]\\
&= \hat{\mathcal{F}}_1u(t)
\end{aligned}
$$

for any $t\in[t_0,T]$. This indicates that $\hat{\mathcal{F}}_1$ is a continuous operator on $\mathcal{B}_{r^*}^*$. Next, we are going to investigate that $\hat{\mathcal{F}}_1(\mathcal{B}_{r^*}^*)$ is uniformly bounded on $\mathcal{B}_{r^*}^*$. For any $u\in\mathcal{B}_{r^*}^*$, we have

$$
\begin{aligned}
\left|\hat{\mathcal{F}}_1u(t)\right| &\le \frac{1}{\lambda\Gamma(k^*-\theta^*)}\int_{t_0}^{t}\left(\frac{(t-t_0)^{\varrho}-(r-t_0)^{\varrho}}{\varrho}\right)^{k^*-\theta^*-1}\left|u(r)\right|\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}}\\
&+ \frac{(T-t_0)^{\varrho}}{\Theta^*}\left[\frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho}\left|u(T)\right| + \frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho}\left|u(\eta)\right|\right.\\
&\left.+ \frac{\mu_2^*\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho}\left|u(T)\right| + \frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho}\left|u(\nu)\right|\right]\\
&+ \frac{(T-t_0)^{2\varrho}}{\Theta^*}\left[\frac{\mu_1^*\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho}\left|u(T)\right| + \frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho}\left|u(\eta)\right|\right.\\
&\left.+ \frac{\mu_2^*\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho}\left|u(T)\right| + \frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho}\left|u(\nu)\right|\right]\\
&\le \mathcal{W}_1\|u\| = \mathcal{W}_1r^*.
\end{aligned}
$$

Thus $\|\hat{\mathcal{F}}_1\|\le\mathcal{W}_1r^*$ for all $u\in\mathcal{B}_{r^*}^*$ with $\mathcal{W}_1$ given in (11). The latter inequality confirms the fact that $\hat{\mathcal{F}}_1$ is uniformly bounded on $\mathcal{B}_{r^*}^*$. Eventually, we review another property of the operator $\mathcal{F}_1$, i.e., its equicontinuity. For each $t_1,t_2\in\tilde{J}$ with $t_1<t_2$ and each $u\in\mathcal{B}_{r^*}^*$, we have

$$
\begin{aligned}
&\left|\mathcal{F}_1u(t_2)-\mathcal{F}_1u(t_1)\right|\\
&\quad\le \frac{r^*(2|((t_2-t_0)^{\varrho}-(t_2-t_0)^{\varrho})^{k^*-\theta^*}|+|(t_2-t_0)^{\varrho(k^*-\theta^*)}-(t_1-t_0)^{\varrho(k^*-\theta^*)}|)}{\lambda^*\Gamma(k^*-\theta^*+1)}\\
&\quad+ r^*\left|\frac{(t_2-t_0)^{\varrho}-(t_1-t_0)^{\varrho}}{\Theta^*}\right|\left[\frac{\mu_1^*\Delta_4}{\lambda\Gamma(k^*-\theta^*-\gamma_1^*+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{k^*-\theta^*-\gamma_1^*}\right.\\
&\quad+ \frac{\Delta_4}{\lambda\Gamma(k^*-\theta^*-\gamma_2^*+1)}\left(\frac{(\eta-t_0)^{\varrho}}{\varrho}\right)^{k^*-\theta^*-\gamma_2^*}\\
&\quad+ \frac{\mu_2^*\Delta_2}{\lambda\Gamma(q_1^*+k^*-\theta^*+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{q_1^*+k^*-\theta^*}\\
&\quad\left.+ \frac{\Delta_2}{\lambda\Gamma(q_2^*+k^*-\theta^*+1)}\left(\frac{(\nu-t_0)^{\varrho}}{\varrho}\right)^{q_2^*+k^*-\theta^*}\right]\\
&\quad+ r^*\left|\frac{(t_2-t_0)^{2\varrho}-(t_1-t_0)^{2\varrho}}{\Theta^*}\right|\\
&\quad\times\left[\frac{\mu_1^*\Delta_3}{\lambda\Gamma(k^*-\theta^*-\gamma_1^*+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{k^*-\theta^*-\gamma_1^*}\right.
\end{aligned}
$$

$$+ \frac{\Delta_3}{\lambda\Gamma(k^* - \theta^* - \gamma_2^* + 1)} \left(\frac{(\eta - t_0)^\varrho}{\varrho}\right)^{k^*-\theta^*-\gamma_2^*}$$
$$+ \frac{\mu_2^*\Delta_1}{\lambda\Gamma(q_1^* + k^* - \theta^* + 1)} \left(\frac{(T - t_0)^\varrho}{\varrho}\right)^{q_1^*+k^*-\theta^*}$$
$$+ \frac{\Delta_1}{\lambda\Gamma(q_2^* + k^* - \theta^* + 1)} \left(\frac{(\nu - t_0)^\varrho}{\varrho}\right)^{q_2^*+k^*-\theta^*}\Bigg].$$

As you observe, the RHS of the latter inequality approaches zero independently of $u$ whenever $t_1 \to t_2$. Hence, the operator $\mathcal{F}_1$ is equicontinuous, and so $\mathcal{F}_1$ is relatively compact on $\mathcal{B}^*_{r^*}$. Consequently, by invoking the Arzela–Ascoli theorem, $\mathcal{F}_1$ is compact on $\mathcal{B}^*_{r^*}$. In conclusion, by taking into account Theorem 2.3, the four-point multi-order nonlinear generalized Caputo type fractional BVP (3) has at least one solution. □

Here, with due attention to the Leray–Schauder theorem, we provide another criterion for the existence of solutions for the proposed problem (3).

**Theorem 3.4** *Let $\hat{\Upsilon} : \tilde{J} \times \mathcal{X}_* \to \mathbb{R}$ be continuous and there exist a nondecreasing continuous function $\Psi : [0,\infty) \to (0,\infty)$ and $\Phi \in \mathcal{C}_{\mathbb{R}^+}(\tilde{J})$ such that $|\hat{\Upsilon}(t,u)| \leq \Phi(t)\Psi(\|u\|)$ for each $(t,u) \in \tilde{J} \times \mathcal{X}_*$. Moreover, suppose that there is a constant $\mathcal{Q}^* > 0$ such that*

$$\frac{\mathcal{Q}^*|\Theta^*|}{\mathcal{Q}^*|\Theta^*|\mathcal{W}_1 + \Psi(\mathcal{Q}^*)\|\Phi\||\Theta^*|\mathcal{W}_2 + (T-t_0)^{2\varrho}(|\delta_1\Delta_3| + |\delta_2\Delta_1|) + (T-t_0)^\varrho(|\delta_1\Delta_4| + |\delta_2\Delta_2|)} > 1, \tag{13}$$

*where $\mathcal{W}_1$ and $\mathcal{W}_2$ are represented by (11) and (12), respectively. Then the four-point multi-order nonlinear generalized Caputo type fractional BVP (3) has at least one solution.*

*Proof* Consider the operator $\tilde{\mathcal{F}}_*$ formulated by (10). We intend to verify that $\tilde{\mathcal{F}}_*$ maps bounded sets into bounded subsets of $\mathcal{X}_*$. Select an appropriate constant $\rho^* > 0$ and build a bounded ball $\mathcal{B}^*_{\rho^*} = \{u \in \mathcal{X}_* : \|u\| \leq \rho^*\}$ in $\mathcal{X}_*$. Then, for each $t \in \tilde{J}$, we have

$$\begin{aligned}
&\big|\tilde{\mathcal{F}}_* u(t)\big| \\
&\leq \sup_{t\in\tilde{J}}\Bigg|\frac{1}{\lambda\Gamma(k^*)}\int_{t_0}^t \left(\frac{(t-t_0)^\varrho - (r-t_0)^\varrho}{\varrho}\right)^{k^*-1} \hat{\Upsilon}\big(r,u(r)\big)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \\
&\quad - \frac{1}{\lambda\Gamma(k^*-\theta^*)}\int_{t_0}^t \left(\frac{(t-t_0)^\varrho - (r-t_0)^\varrho}{\varrho}\right)^{k^*-\theta^*-1} u(r)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \\
&\quad + \frac{(t-t_0)^\varrho}{\Theta^*}\Bigg[\frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}\hat{\Upsilon}\big(T,u(T)\big) - \frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho}u(T) \\
&\quad + \frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho}\hat{\Upsilon}\big(\eta,u(\eta)\big) - \frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho}u(\eta) \\
&\quad - \frac{\mu_2^*\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho}\hat{\Upsilon}\big(T,u(T)\big) \\
&\quad + \frac{\mu_2^*\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho}u(T) - \frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho}\hat{\Upsilon}\big(\nu,u(\nu)\big) \\
&\quad + \frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho}u(\nu)
\end{aligned}$$

$$
\begin{aligned}
&- \delta_1\Delta_4 + \Delta_2\delta_2\Bigg] + \frac{(t-t_0)^{2\varrho}}{\Theta^*}\Bigg[-\frac{\mu_1^*\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}\hat{\Upsilon}\big(T,u(T)\big) \\
&+ \frac{\mu_1^*\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho}u(T) \\
&- \frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho}\hat{\Upsilon}\big(\eta,u(\eta)\big) + \frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho}u(\eta) \\
&+ \frac{\mu_2^*\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho}\hat{\Upsilon}\big(T,u(T)\big) \\
&- \frac{\mu_2^*\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho}u(T) + \frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho}\hat{\Upsilon}\big(\nu,u(\nu)\big) \\
&- \frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho}u(\nu) + \delta_1\Delta_3 - \delta_2\Delta_1\Bigg]\Bigg| \\
\le{}& \|\Phi\|\Psi\big(\|u\|\big)\mathcal{W}_2 + \|u\|\mathcal{W}_1 \\
&+ \frac{(T-t_0)^{2\varrho}}{|\Theta^*|}\big(|\delta_1\Delta_3| + |\delta_2\Delta_1|\big) + \frac{(T-t_0)^{\varrho}}{|\Theta^*|}\big(|\delta_1\Delta_4| + |\delta_2\Delta_2|\big),
\end{aligned}
$$

and consequently,

$$
\begin{aligned}
\big\|\tilde{\mathcal{F}}_*(t)\big\| \le{}& \|\Phi\|\Psi\big(\|u\|\big)\mathcal{W}_2 + \|u\|\mathcal{W}_1 + \frac{(T-t_0)^{2\varrho}}{|\Theta^*|}\big(|\delta_1\Delta_3| + |\delta_2\Delta_1|\big) \\
&+ \frac{(T-t_0)^{\varrho}}{|\Theta^*|}\big(|\delta_1\Delta_4| + |\delta_2\Delta_2|\big).
\end{aligned}
$$

Now, we continue to prove that the operator $\tilde{\mathcal{F}}_*$ maps bounded sets (balls) into equicontinuous sets of $\mathcal{X}_*$. Assuming $t_1, t_2 \in \tilde{J}$ with $t_1 < t_2$ and $u \in \mathcal{B}_{\rho^*}^*$, we have

$$
\begin{aligned}
&\big|\tilde{\mathcal{F}}_*u(t_2) - \tilde{\mathcal{F}}_*u(t_1)\big| \\
&\le \frac{\Phi(t)\Psi(\|u\|)(2|((t_2-t_0)^{\varrho} - (t_2-t_0)^{\varrho})^{k^*}| + |(t_2-t_0)^{\varrho k^*} - (t_1-t_0)^{\varrho k^*}|)}{\lambda^*\Gamma(k^*+1)} \\
&+ \frac{\|u\|(2|((t_2-t_0)^{\varrho} - (t_2-t_0)^{\varrho})^{k^*-\theta}| + |(t_2-t_0)^{\varrho(k^*-\theta)} - (t_1-t_0)^{\varrho(k^*-\theta)}|)}{\lambda^*\Gamma(k^*-\theta+1)} \\
&+ \frac{|(t_2-t_0)^{\varrho} - (t_1-t_0)^{\varrho}|}{|\Theta^*|}\Bigg[\bigg|\frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}\hat{\Upsilon}\big(T,u(T)\big)\bigg| \\
&+ \bigg|\frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho}u(T) \\
&+ \bigg|\frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho}\hat{\Upsilon}\big(\eta,u(\eta)\big)\bigg| + \bigg|\frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho}u(\eta)\bigg| \\
&+ \bigg|\frac{\mu_2^*\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho}\hat{\Upsilon}\big(T,u(T)\big)\bigg| \\
&+ \bigg|\frac{\mu_2^*\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho}u(T)\bigg| + \bigg|\frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho}\hat{\Upsilon}\big(\nu,u(\nu)\big)\bigg| \\
&+ \bigg|\frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho}u(\nu)\bigg| \\
&+ |\delta_1\Delta_4| + |\Delta_2\delta_2|\Bigg] + \frac{|(t_2-t_0)^{2\varrho} - (t_1-t_0)^{2\varrho}|}{|\Theta^*|}
\end{aligned}
$$

$$\begin{aligned}
&\times \Bigg[\bigg|\frac{\mu_1^*\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}\hat{\Upsilon}\big(T,u(T)\big)\bigg| \\
&+ \bigg|\frac{\mu_1^*\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho}u(T)\bigg|\bigg|\frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho}\hat{\Upsilon}\big(\eta,u(\eta)\big)\bigg| \\
&+ \bigg|\frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho}u(\eta)\bigg| \\
&+ \bigg|\frac{\mu_2^*\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho}\hat{\Upsilon}\big(T,u(T)\big)\bigg|\bigg|\frac{\mu_2^*\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho}u(T)\bigg| \\
&+ \bigg|\frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho}\hat{\Upsilon}\big(\nu,u(\nu)\big)\bigg| \\
&+ \bigg|\frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho}u(\nu)\bigg| + |\delta_1\Delta_3| + |\delta_2\Delta_1|\Bigg].
\end{aligned}$$

If $t_1 \to t_2$, then the RHS of the above inequality approaches 0 independently of $u \in \mathcal{B}_{\rho^*}^*$. This implies the equicontinuity of $\tilde{\mathcal{F}}_*$, and so the relative compactness of $\tilde{\mathcal{F}}_*$ on $\mathcal{B}_{\rho^*}^*$. Hence from the Arzela–Ascoli theorem it follows that $\tilde{\mathcal{F}}_*$ is completely continuous, and so $\tilde{\mathcal{F}}_*$ is compact on $\mathcal{B}_{\rho^*}^*$. The desired result is completed from the Leray–Schauder theorem 2.4 once we can verify the boundedness of the set of solutions for an equation $u = \omega^*\tilde{\mathcal{F}}_*u$ for some $\omega^* \in (0,1)$. To reach this goal, let us assume that $u$ is a solution for the latter equation. For any $t \in \tilde{J}$, we obtain

$$\begin{aligned}
\big|u(t)\big| &\le \|\Phi\|\Psi\big(\|u\|\big)\mathcal{W}_2 + \|u\|\mathcal{W}_1 + \frac{(T-t_0)^{2\varrho}}{|\Theta^*|}\big(|\delta_1\Delta_3| + |\delta_2\Delta_1|\big) \\
&\quad + \frac{(T-t_0)^{\varrho}}{|\Theta^*|}\big(|\delta_1\Delta_4| + |\delta_2\Delta_2|\big),
\end{aligned}$$

and so

$$\begin{aligned}
&\frac{\|u\||\Theta^*|}{\|u\||\Theta^*|\mathcal{W}_1 + \Psi(\|u\|)\|\Phi\||\Theta^*|\mathcal{W}_2 + (T-t_0)^{2\varrho}(|\delta_1\Delta_3| + |\delta_2\Delta_1|) + (T-t_0)^{\varrho}(|\delta_1\Delta_4| + |\delta_2\Delta_2|)} \\
&\quad < 1.
\end{aligned}$$

Select the constant $\mathcal{Q}^*$ with $\|u\| \neq \mathcal{Q}^*$. Put $\mathcal{U} = \{x \in \mathcal{X}_* : \|u\| < \mathcal{Q}^*\}$. Then one can realize that the operator $\tilde{\mathcal{F}}_* : \bar{\mathcal{U}} \to \mathcal{X}_*$ is continuous and completely continuous. By considering the choice of $\mathcal{U}$, there is no $u \in \partial\mathcal{U}$ satisfying $u = \omega^*\tilde{\mathcal{F}}_*u$ for some $\omega^* \in (0,1)$. Therefore by utilizing the Leray–Schauder theorem, it is deduced that $\tilde{\mathcal{F}}_*$ is an operator having a fixed point $u \in \bar{\mathcal{U}}$ which is a solution for the four-point multi-order nonlinear generalized Caputo type fractional BVP (3). □

## 4 Stability analysis

In the current section, we are going to investigate some well-known stability results such as Ulam–Hyers, generalized Ulam–Hyers, Ulam–Hyers–Rassias, and generalized Ulam–Hyers–Rassias stability of solutions for the four-point multi-order nonlinear generalized Caputo type fractional BVP (3). First, we state the relevant concepts in this regard, which are adapted from paper [49].

**Definition 4.1** ([49]) We say that the four-point multi-order nonlinear generalized Caputo type fractional BVP (3) is Ulam–Hyers stable whenever there is a positive constant

$\kappa \in \mathbb{R}$ such that, for $\varepsilon > 0$ and for each solution function $\nu \in \mathcal{X}_*$ of the following inequality

$$\left|\lambda^{*CC}\mathcal{D}_{t_0}^{k^*,\varrho}\nu(t) + {}^{CC}\mathcal{D}_{t_0}^{\theta^*,\varrho}\nu(t) - \hat{\Upsilon}\big(t,\nu(t)\big)\right| \le \varepsilon, \quad \big(t \in [t_0,T]\big), \tag{14}$$

there is another solution function $u \in \mathcal{X}_*$ for the four-point multi-order nonlinear generalized Caputo type fractional BVP (3) with the following property:

$$\left|\nu(t) - u(t)\right| \le \kappa\varepsilon, \quad \big(t \in [t_0,T]\big).$$

**Definition 4.2** ([49]) We say that the four-point multi-order nonlinear generalized Caputo type fractional BVP (3) is generalized Ulam–Hyers stable if there exists $\psi_{\hat{\Upsilon}} \in \mathcal{C}_{\mathbb{R}^+}(\mathbb{R}^+)$ with $\psi_{\hat{\Upsilon}}(0) = 0$ provided that, for each solution function $\nu \in \mathcal{X}_*$ of inequality (14), there is another solution function $u \in \mathcal{X}_*$ for the four-point multi-order nonlinear generalized Caputo type fractional BVP (3) satisfying the following inequality:

$$\left|\nu(t) - u(t)\right| \le \psi_{\hat{\Upsilon}}(\varepsilon), \quad \big(t \in [t_0,T]\big).$$

**Definition 4.3** ([49]) The four-point multi-order nonlinear generalized Caputo type fractional BVP (3) is defined to be Ulam–Hyers–Rassias stable depending on $\varphi : [t_0,T] \to \mathbb{R}^+$ if there is a positive constant $\kappa_\varphi \in \mathbb{R}$ such that, for each $\varepsilon > 0$ and for every solution function $\nu \in \mathcal{X}_*$ of inequality

$$\left|\lambda^{*CC}\mathcal{D}_{t_0}^{k^*,\varrho}\nu(t) + {}^{CC}\mathcal{D}_{t_0}^{\theta^*,\varrho}\nu(t) - \hat{\Upsilon}\big(t,\nu(t)\big)\right| \le \varepsilon\varphi(t), \quad \big(t \in [t_0,T]\big), \tag{15}$$

there exists another solution function $u \in \mathcal{X}_*$ of the four-point multi-order nonlinear generalized Caputo type fractional BVP (3) satisfying the following inequality:

$$\left|\nu(t) - u(t)\right| \le \kappa_\varphi\varepsilon\varphi(t), \quad \big(t \in [t_0,T]\big).$$

**Definition 4.4** ([49]) The four-point multi-order nonlinear generalized Caputo type fractional BVP (3) is said to be generalized Ulam–Hyers–Rassias stable depending on the function $\varphi : [t_0,T] \to \mathbb{R}^+$ if there is a positive constant $\kappa_\varphi\mathbb{R}$ provided that, for any $\varepsilon > 0$ and for every solution function $\nu \in \mathcal{X}_*$ of the inequality

$$\left|\lambda^{*CC}\mathcal{D}_{t_0}^{k^*,\varrho}\nu(t) + {}^{CC}\mathcal{D}_{t_0}^{\theta^*,\varrho}\nu(t) - \hat{\Upsilon}\big(t,\nu(t)\big)\right| \le \varphi(t), \quad \big(t \in [t_0,T]\big), \tag{16}$$

there is another solution function $u \in \mathcal{X}_*$ for the four-point multi-order nonlinear generalized Caputo type fractional BVP (3) which satisfies the following inequality:

$$\left|\nu(t) - u(t)\right| \le \kappa_\varphi\varphi(t), \quad \big(t \in [t_0,T]\big).$$

*Remark* 4.5 Notice that the function $\nu \in \mathcal{X}_*$ is called a solution for inequality (14) if and only if there is a function $g \in \mathcal{X}_*$ depending on $\nu$ such that

(i) $|g(t)| < \varepsilon, (t \in [t_0,T])$;
(ii) ${}^{CC}\mathcal{D}_{t_0}^{k^*,\varrho}\nu(t) + {}^{CC}\mathcal{D}_{t_0}^{\theta^*,\varrho}\nu(t) = \hat{\Upsilon}(t,\nu(t)) + g(t), (t \in [t_0,T])$.

Now, in the light of Remark 4.5, the solution function of equation

$$ {}^{CC}\mathcal{D}_{t_0}^{k^*,\varrho}\nu(t) + {}^{CC}\mathcal{D}_{t_0}^{\theta^*,\varrho}\nu(t) = \hat{\Upsilon}\big(t,\nu(t)\big) + g(t), \quad \big(t\in[t_0,T]\big) \tag{17} $$

can be represented by the following generalized Riemann–Liouville type fractional integral equation:

$$ \begin{aligned} \nu(t) ={}& \frac{1}{\lambda\Gamma(k^*)}\int_{t_0}^{t}\left(\frac{(t-t_0)^\varrho-(r-t_0)^\varrho}{\varrho}\right)^{k^*-1}\hat{\Upsilon}\big(r,\nu(r)\big)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \\ &+ \frac{1}{\lambda\Gamma(k^*)}\int_{t_0}^{t}\left(\frac{(t-t_0)^\varrho-(r-t_0)^\varrho}{\varrho}\right)^{k^*-1}g(r)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \\ &- \frac{1}{\lambda\Gamma(k^*-\theta^*)}\int_{t_0}^{t}\left(\frac{(t-t_0)^\varrho-(r-t_0)^\varrho}{\varrho}\right)^{k^*-\theta^*-1}\nu(r)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \\ &+ \frac{(t-t_0)^\varrho}{\Theta^*}\Bigg[\frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}\hat{\Upsilon}\big(T,\nu(T)\big) \\ &+ \frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}g(T) - \frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho}\nu(T) \\ &+ \frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho}\hat{\Upsilon}\big(\eta,\nu(\eta)\big) + \frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho}g(\eta) \\ &- \frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho}\nu(\eta) \\ &- \frac{\mu_2^*\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho}\hat{\Upsilon}\big(T,\nu(T)\big) - \frac{\mu_2^*\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho}g(T) \\ &+ \frac{\mu_2^*\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho}\nu(T) \\ &- \frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho}\hat{\Upsilon}\big(\nu,\nu(\nu)\big) - \frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho}g(\nu) \\ &+ \frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho}\nu(\nu) - \delta_1\Delta_4 + \Delta_2\delta_2\Bigg] \\ &+ \frac{(t-t_0)^{2\varrho}}{\Theta^*}\Bigg[-\frac{\mu_1^*\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}\hat{\Upsilon}\big(T,\nu(T)\big) - \frac{\mu_1^*\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}g(T) \\ &+ \frac{\mu_1^*\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho}\nu(T) \\ &- \frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho}\hat{\Upsilon}\big(\eta,\nu(\eta)\big) - \frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho}g(\eta) + \frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho}\nu(\eta) \\ &+ \frac{\mu_2^*\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho}\hat{\Upsilon}\big(T,\nu(T)\big) + \frac{\mu_2^*\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho}g(T) \\ &- \frac{\mu_2^*\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho}\nu(T) \\ &+ \frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho}\hat{\Upsilon}\big(\nu,\nu(\nu)\big) + \frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho}g(\nu) \\ &- \frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho}\nu(\nu) + \delta_1\Delta_3 - \delta_2\Delta_1\Bigg]. \end{aligned} $$

In the sequel, by considering the above findings and by implementing some routine computations, we obtain the following estimate:

$$
\begin{aligned}
&\Bigg|\nu(t)-\Bigg\{\frac{1}{\lambda\Gamma(k^*)}\int_{t_0}^{t}\left(\frac{(t-t_0)^{\varrho}-(r-t_0)^{\varrho}}{\varrho}\right)^{k^*-1}\hat{\Upsilon}\big(r,\nu(r)\big)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}}\\
&\quad+\frac{1}{\lambda\Gamma(k^*)}\int_{t_0}^{t}\left(\frac{(t-t_0)^{\varrho}-(r-t_0)^{\varrho}}{\varrho}\right)^{k^*-1}g(r)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}}\\
&\quad-\frac{1}{\lambda\Gamma(k^*-\theta^*)}\int_{t_0}^{t}\left(\frac{(t-t_0)^{\varrho}-(r-t_0)^{\varrho}}{\varrho}\right)^{k^*-\theta^*-1}\nu(r)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}}\\
&\quad+\frac{(t-t_0)^{\varrho}}{\Theta^*}\Bigg[\frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}\hat{\Upsilon}\big(T,\nu(T)\big)+\frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}g(T)\\
&\quad-\frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho}\nu(T)\\
&\quad+\frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho}\hat{\Upsilon}\big(\eta,\nu(\eta)\big)+\frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho}g(\eta)-\frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho}\nu(\eta)\\
&\quad-\frac{\mu_2^*\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho}\hat{\Upsilon}\big(T,\nu(T)\big)-\frac{\mu_2^*\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho}g(T)\\
&\quad+\frac{\mu_2^*\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho}\nu(T)\\
&\quad-\frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho}\hat{\Upsilon}\big(\nu,\nu(\nu)\big)-\frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho}g(\nu)\\
&\quad+\frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho}\nu(\nu)-\delta_1\Delta_4+\Delta_2\delta_2\Bigg]\\
&\quad+\frac{(t-t_0)^{2\varrho}}{\Theta^*}\Bigg[-\frac{\mu_1^*\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}\hat{\Upsilon}\big(T,\nu(T)\big)-\frac{\mu_1^*\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}g(T)\\
&\quad+\frac{\mu_1^*\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho}\nu(T)\\
&\quad-\frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho}\hat{\Upsilon}\big(\eta,\nu(\eta)\big)-\frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho}g(\eta)\\
&\quad+\frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho}\nu(\eta)\\
&\quad+\frac{\mu_2^*\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho}\hat{\Upsilon}\big(T,\nu(T)\big)+\frac{\mu_2^*\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho}g(T)\\
&\quad-\frac{\mu_2^*\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho}\nu(T)\\
&\quad+\frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho}\hat{\Upsilon}\big(\nu,\nu(\nu)\big)+\frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho}g(\nu)\\
&\quad-\frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho}\nu(\nu)+\delta_1\Delta_3-\delta_2\Delta_1\Bigg]\Bigg\}\Bigg|\\
&\leq\left|\frac{1}{\lambda\Gamma(k^*)}\int_{t_0}^{t}\left(\frac{(t-t_0)^{\varrho}-(r-t_0)^{\varrho}}{\varrho}\right)^{k^*-1}g(r)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}}\right|\\
&\quad+\frac{(t-t_0)^{\varrho}}{|\Theta^*|}\Bigg[\left|\frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}g(T)\right|+\left|\frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho}g(\eta)\right|\\
&\quad+\left|\frac{\mu_2^*\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho}g(T)\right|+\left|\frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho}g(\nu)\right|\Bigg]
\end{aligned}
$$

$$+\frac{(t-t_0)^{2\varrho}}{\Theta^*}\Bigg[\left|\frac{\mu_1^*\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}g(T)\right|$$
$$+\left|\frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho}\hat{\Upsilon}g(\eta)\right|$$
$$+\left|\frac{\mu_2^*\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho}\hat{\Upsilon}g(T)\right|+\left|\frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho}g(\nu)\right|\Bigg]\le\varepsilon\mathcal{W}_2.$$

Now, we are ready to check the establishment of Ulam–Hyers stability for the proposed problem (3).

**Theorem 4.6** *Assume that the map $\hat{\Upsilon}:\tilde{J}\times\mathfrak{X}_*\to\mathbb{R}$ is continuous and there is a positive constant $\mathcal{L}_*\in\mathbb{R}$ provided that $|\hat{\Upsilon}(t,u)-\hat{\Upsilon}(t,u')|\le\mathcal{L}_*|u-u'|$ for each $t\in\tilde{J}$ and $u,u'\in\mathfrak{X}_*$ with $\mathcal{L}_*\mathcal{W}_2+\mathcal{W}_1<1$. Then the four-point multi-order nonlinear generalized Caputo type fractional BVP (3) is Ulam–Hyers stable and consequently is generalized Ulam–Hyers stable on $\tilde{J}=[t_0,T]$.*

*Proof* Assume that $\nu\in\mathcal{X}_*$ is a solution of inequality (14). Also, let $u\in\mathcal{X}_*$ be a unique solution for the four-point multi-order nonlinear generalized Caputo type fractional BVP (3). Then one can write

$$\begin{aligned}
&\bigl|\nu(t)-u(t)\bigr|\\
&\le\Bigg|\nu(t)-\Bigg\{\frac{1}{\lambda\Gamma(k^*)}\int_{t_0}^t\left(\frac{(t-t_0)^\varrho-(r-t_0)^\varrho}{\varrho}\right)^{k^*-1}\hat{\Upsilon}\bigl(r,\nu(r)\bigr)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}}\\
&\quad+\frac{1}{\lambda\Gamma(k^*)}\int_{t_0}^t\left(\frac{(t-t_0)^\varrho-(r-t_0)^\varrho}{\varrho}\right)^{k^*-1}g(r)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}}\\
&\quad-\frac{1}{\lambda\Gamma(k^*-\theta^*)}\int_{t_0}^t\left(\frac{(t-t_0)^\varrho-(r-t_0)^\varrho}{\varrho}\right)^{k^*-\theta^*-1}\nu(r)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}}\\
&\quad+\frac{(t-t_0)^\varrho}{\Theta^*}\Bigg[\frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}\hat{\Upsilon}\bigl(T,\nu(T)\bigr)+\frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}g(T)\\
&\quad-\frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho}\nu(T)\\
&\quad+\frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho}\hat{\Upsilon}\bigl(\eta,\nu(\eta)\bigr)+\frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho}g(\eta)-\frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho}\nu(\eta)\\
&\quad-\frac{\mu_2^*\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho}\hat{\Upsilon}\bigl(T,\nu(T)\bigr)-\frac{\mu_2^*\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho}g(T)\\
&\quad+\frac{\mu_2^*\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho}\nu(T)\\
&\quad-\frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho}\hat{\Upsilon}\bigl(\nu,\nu(\nu)\bigr)-\frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho}g(\nu)\\
&\quad+\frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho}\nu(\nu)-\delta_1\Delta_4+\Delta_2\delta_2\Bigg]\\
&\quad+\frac{(t-t_0)^{2\varrho}}{\Theta^*}\Bigg[-\frac{\mu_1^*\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}\hat{\Upsilon}\bigl(T,\nu(T)\bigr)\\
&\quad-\frac{\mu_1^*\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}g(T)+\frac{\mu_1^*\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho}\nu(T)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho} \hat{\Upsilon}\big(\eta, \nu(\eta)\big) - \frac{\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho} g(\eta) + \frac{\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho} \nu(\eta) \\
&+ \frac{\mu_2^*\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho} \hat{\Upsilon}\big(T, \nu(T)\big) + \frac{\mu_2^*\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho} g(T) \\
&- \frac{\mu_2^*\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho} \nu(T) \\
&+ \frac{\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho} \hat{\Upsilon}\big(\nu, \nu(\nu)\big) + \frac{\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho} g(\nu) \\
&- \frac{\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho} \nu(\nu) + \delta_1\Delta_3 - \delta_2\Delta_1 \bigg]\bigg\} \\
&+ \bigg\{ \frac{1}{\lambda\Gamma(k^*)} \int_{t_0}^{t} \bigg( \frac{(t-t_0)^\varrho - (r-t_0)^\varrho}{\varrho} \bigg)^{k^*-1} \hat{\Upsilon}\big(r, \nu(r)\big) \frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \\
&+ \frac{1}{\lambda\Gamma(k^*)} \int_{t_0}^{t} \bigg( \frac{(t-t_0)^\varrho - (r-t_0)^\varrho}{\varrho} \bigg)^{k^*-1} g(r) \frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \\
&- \frac{1}{\lambda\Gamma(k^*-\theta^*)} \int_{t_0}^{t} \bigg( \frac{(t-t_0)^\varrho - (r-t_0)^\varrho}{\varrho} \bigg)^{k^*-\theta^*-1} \nu(r) \frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \\
&+ \frac{(t-t_0)^\varrho}{\Theta^*} \bigg[ \frac{\mu_1^*\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho} \hat{\Upsilon}\big(T, \nu(T)\big) \\
&+ \frac{\mu_1^*\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho} g(T) - \frac{\mu_1^*\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho} \nu(T) \\
&+ \frac{\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho} \hat{\Upsilon}\big(\eta, \nu(\eta)\big) + \frac{\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho} g(\eta) \\
&- \frac{\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho} \nu(\eta) \\
&- \frac{\mu_2^*\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho} \hat{\Upsilon}\big(T, \nu(T)\big) \\
&- \frac{\mu_2^*\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho} g(T) + \frac{\mu_2^*\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho} \nu(T) \\
&- \frac{\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho} \hat{\Upsilon}\big(\nu, \nu(\nu)\big) - \frac{\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho} g(\nu) \\
&+ \frac{\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho} \nu(\nu) - \delta_1\Delta_4 + \Delta_2\delta_2 \bigg] \\
&+ \frac{(t-t_0)^{2\varrho}}{\Theta^*} \bigg[ -\frac{\mu_1^*\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho} \hat{\Upsilon}\big(T, \nu(T)\big) \\
&- \frac{\mu_1^*\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho} g(T) + \frac{\mu_1^*\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho} \nu(T) \\
&- \frac{\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho} \hat{\Upsilon}\big(\eta, \nu(\eta)\big) - \frac{\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho} g(\eta) + \frac{\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho} \nu(\eta) \\
&+ \frac{\mu_2^*\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho} \hat{\Upsilon}\big(T, \nu(T)\big) + \frac{\mu_2^*\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho} g(T) \\
&- \frac{\mu_2^*\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho} \nu(T) \\
&+ \frac{\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho} \hat{\Upsilon}\big(\nu, \nu(\nu)\big) + \frac{\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho} g(\nu)
\end{aligned}
$$

$$- \frac{\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho} \nu(\nu) + \delta_1 \Delta_3 - \delta_2 \Delta_1 \Big] \Big] \Big\} - u(t) \Big|$$
$$\leq \varepsilon \mathcal{W}_2 + (\mathcal{L}_* \mathcal{W}_2 + \mathcal{W}_1) \big| \nu(t) - u(t) \big|,$$

which yields that

$$\big| \nu(t) - u(t) \big| \leq \frac{\varepsilon \mathcal{W}_2}{1 - (\mathcal{L}_* \mathcal{W}_2 + \mathcal{W}_1)}. \tag{18}$$

For the sake of simplicity in writing, we take $\kappa = \frac{\mathcal{W}_2}{1-(\mathcal{L}\mathcal{W}_2+\mathcal{W}_1)}$, and so

$$(\mathcal{L}_* \mathcal{W}_2 + \mathcal{W}_1) \leq 1.$$

Then (18) becomes

$$\big| \nu(t) - u(t) \big| \leq \varepsilon \kappa \quad \big( t \in \tilde{J} = [t_0, T] \big).$$

Thus it is deduced that the four-point multi-order nonlinear generalized Caputo type fractional BVP (3) is Ulam–Hyers stable. Furthermore, assuming $\psi_{\hat{\Upsilon}}(\epsilon) = \varepsilon\kappa$, it is clear that $\psi_{\hat{\Upsilon}}(0) = 0$. Consequently, it follows that the solution function of the four-point multi-order nonlinear generalized Caputo type fractional BVP (3) is generalized Ulam–Hyers stable and the proof is completed. □

*Remark* 4.7 Notice that a function $\nu \in \mathcal{X}_*$ is a solution of inequality (4.3) if and only if there is another function $h^* \in \mathcal{X}_*$ depending on $\nu$ provided that

(i) $|h^*(t)| < \varepsilon\varphi(t), (t \in [t_0, T])$

(ii) ${}^{CC}\mathcal{D}_{t_0}^{k^*,\varrho} \nu(t) + {}^{CC}\mathcal{D}_{t_0}^{\theta^*,\varrho} \nu(t) = \hat{\Upsilon}(t, \nu(t)) + h^*(t), (t \in [t_0, T])$.

In the light of Remark (4.7), the solution function of the equation

$${}^{CC}\mathcal{D}_{t_0}^{k^*,\varrho} \nu(t) + {}^{CC}\mathcal{D}_{t_0}^{\theta^*,\varrho} \nu(t) = \hat{\Upsilon}\big(t, \nu(t)\big) + h^*(t), \quad \big(t \in [t_0, T]\big)$$

can be represented by

$$\begin{aligned}
\nu(t) = {} & \frac{1}{\lambda\Gamma(k^*)} \int_{t_0}^{t} \left( \frac{(t-t_0)^\varrho - (r-t_0)^\varrho}{\varrho} \right)^{k^*-1} \hat{\Upsilon}\big(r, \nu(r)\big) \frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \\
& + \frac{1}{\lambda\Gamma(k^*)} \int_{t_0}^{t} \left( \frac{(t-t_0)^\varrho - (r-t_0)^\varrho}{\varrho} \right)^{k^*-1} h^*(r) \frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \\
& - \frac{1}{\lambda\Gamma(k^*-\theta^*)} \int_{t_0}^{t} \left( \frac{(t-t_0)^\varrho - (r-t_0)^\varrho}{\varrho} \right)^{k^*-\theta^*-1} \nu(r) \frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \\
& + \frac{(t-t_0)^\varrho}{\Theta^*} \bigg[ \frac{\mu_1^* \Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho} \hat{\Upsilon}\big(T, \nu(T)\big) + \frac{\mu_1^* \Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho} h^*(T) \\
& - \frac{\mu_1^* \Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho} \nu(T) \\
& + \frac{\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho} \hat{\Upsilon}\big(\eta, \nu(\eta)\big) + \frac{\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho} h^*(\eta)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho} \nu(\eta) \\
&- \frac{\mu_2^*\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho} \hat{\Upsilon}\big(T, \nu(T)\big) \\
&- \frac{\mu_2^*\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho} h^*(T) + \frac{\mu_2^*\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho} \nu(T) \\
&- \frac{\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho} \hat{\Upsilon}\big(\nu, \nu(\nu)\big) - \frac{\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho} h^*(\nu) \\
&+ \frac{\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho} \nu(\nu) - \delta_1\Delta_4 + \Delta_2\delta_2 \bigg] \\
&+ \frac{(t-t_0)^{2\varrho}}{\Theta^*} \bigg[ -\frac{\mu_1^*\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho} \hat{\Upsilon}\big(T, \nu(T)\big) - \frac{\mu_1^*\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho} h^*(T) \\
&+ \frac{\mu_1^*\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho} \nu(T) \\
&- \frac{\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho} \hat{\Upsilon}\big(\eta, \nu(\eta)\big) - \frac{\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho} h^*(\eta) \\
&+ \frac{\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho} \nu(\eta) \\
&+ \frac{\mu_2^*\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho} \hat{\Upsilon}\big(T, \nu(T)\big) + \frac{\mu_2^*\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho} h^*(T) \\
&- \frac{\mu_2^*\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho} \nu(T) \\
&+ \frac{\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho} \hat{\Upsilon}\big(\nu, \nu(\nu)\big) + \frac{\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho} h^*(\nu) \\
&- \frac{\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho} \nu(\nu) + \delta_1\Delta_3 - \delta_2\Delta_1 \bigg].
\end{aligned}
$$

Then the following estimate holds:

$$
\begin{aligned}
&\bigg| \nu(t) - \bigg\{ \frac{1}{\lambda\Gamma(k^*)} \int_{t_0}^{t} \bigg( \frac{(t-t_0)^\varrho - (r-t_0)^\varrho}{\varrho} \bigg)^{k^*-1} \hat{\Upsilon}\big(r, \nu(r)\big) \frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \\
&+ \frac{1}{\lambda\Gamma(k^*)} \int_{t_0}^{t} \bigg( \frac{(t-t_0)^\varrho - (r-t_0)^\varrho}{\varrho} \bigg)^{k^*-1} h^*(r) \frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \\
&- \frac{1}{\lambda\Gamma(k^*-\theta^*)} \int_{t_0}^{t} \bigg( \frac{(t-t_0)^\varrho - (r-t_0)^\varrho}{\varrho} \bigg)^{k^*-\theta^*-1} \nu(r) \frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \\
&+ \frac{(t-t_0)^\varrho}{\Theta^*} \bigg[ \frac{\mu_1^*\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho} \hat{\Upsilon}\big(T, \nu(T)\big) \\
&+ \frac{\mu_1^*\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho} h^*(T) - \frac{\mu_1^*\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho} \nu(T) \\
&+ \frac{\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho} \hat{\Upsilon}\big(\eta, \nu(\eta)\big) + \frac{\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho} h^*(\eta) \\
&- \frac{\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho} \nu(\eta) \\
&- \frac{\mu_2^*\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho} \hat{\Upsilon}\big(T, \nu(T)\big) - \frac{\mu_2^*\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho} h^*(T) \\
&+ \frac{\mu_2^*\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho} \nu(T)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho} \hat{\Upsilon}\big(\nu, v(\nu)\big) - \frac{\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho} h^*(\nu) \\
&+ \frac{\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho} v(\nu) - \delta_1\Delta_4 + \Delta_2\delta_2 \bigg] \\
&+ \frac{(t-t_0)^{2\varrho}}{\Theta^*} \bigg[ - \frac{\mu_1^*\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho} \hat{\Upsilon}\big(T, v(T)\big) - \frac{\mu_1^*\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho} h^*(T) \\
&+ \frac{\mu_1^*\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho} v(T) \\
&- \frac{\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho} \hat{\Upsilon}\big(\eta, v(\eta)\big) - \frac{\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho} h^*(\eta) \\
&+ \frac{\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho} v(\eta) \\
&+ \frac{\mu_2^*\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho} \hat{\Upsilon}\big(T, v(T)\big) + \frac{\mu_2^*\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho} h^*(T) \\
&- \frac{\mu_2^*\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho} v(T) \\
&+ \frac{\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho} \hat{\Upsilon}\big(\nu, v(\nu)\big) + \frac{\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho} h^*(\nu) \\
&- \frac{\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho} v(\nu) + \delta_1\Delta_3 - \delta_2\Delta_1 \bigg]\bigg\}\bigg| \\
\leq{}& \left| \frac{1}{\lambda\Gamma(k^*)} \int_{t_0}^{t} \left( \frac{(t-t_0)^\varrho - (r-t_0)^\varrho}{\varrho} \right)^{k^*-1} h^*(r) \frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \right| \\
&+ \left| \frac{(t-t_0)^\varrho}{\Theta^*} \right| \bigg[ \left| \frac{\mu_1^*\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho} h^*(T) \right| + \left| \frac{\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho} h^*(\eta) \right| \\
&+ \left| \frac{\mu_2^*\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho} h^*(T) \right| + \left| \frac{\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho} h^*(\nu) \right| \bigg] \\
&+ \frac{(t-t_0)^{2\varrho}}{\Theta^*} \bigg[ \left| \frac{\mu_1^*\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho} h^*(T) \right| + \left| \frac{\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho} h^*(\eta) \right| \\
&+ \left| \frac{\mu_2^*\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho} h^*(T) \right| + \left| \frac{\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho} h^*(\nu) \right| \bigg] \\
\leq{}& \varepsilon\varphi(t)\mathcal{W}_2.
\end{aligned}
$$

Now, we are ready to check the establishment of Ulam–Hyers–Rassias stability for the proposed problem (3).

**Theorem 4.8** *Let* $\hat{\Upsilon} : [t_0, T] \times \mathfrak{X}_* \to \mathbb{R}$ *be a continuous function and there exist a nondecreasing continuous function* $\Psi : [0,\infty) \to (0,\infty)$ *and* $\Phi \in \mathcal{C}_{\mathbb{R}^+}(\tilde{J})$ *provided that* $|\hat{\Upsilon}(t,u)| \leq \Phi(t)\Psi(\|u\|)$ *for each* $(t,u) \in \tilde{J} \times \mathfrak{X}_*$. *If condition* (13) *is valid and there is a function* $h^*$ *which satisfies Remark* 4.7 *with* $2\mathcal{Q}^* \leq h^*(t)$ *for any* $t \in \tilde{J} = [t_0, T]$, *then the four-point multi-order nonlinear generalized Caputo type fractional BVP* (3) *is Ulam–Hyers–Rassias stable, and so it is generalized Ulam–Hyers–Rassias stable.*

*Proof* Suppose that $v \in \mathcal{X}_*$ is a solution of inequality (4.3), and also let $u \in \mathcal{X}_*$ be a solution for the four-point multi-order nonlinear generalized Caputo type fractional BVP (3). Then we get

$$
\begin{aligned}
&\left|\nu(t)-u(t)\right| \\
&\leq \Bigg|\nu(t)-\Bigg\{\frac{1}{\lambda\Gamma(k^*)}\int_{t_0}^{t}\left(\frac{(t-t_0)^{\varrho}-(r-t_0)^{\varrho}}{\varrho}\right)^{k^*-1}\hat{\Upsilon}\big(r,\nu(r)\big)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \\
&-\frac{1}{\lambda\Gamma(k^*-\theta^*)}\int_{t_0}^{t}\left(\frac{(t-t_0)^{\varrho}-(r-t_0)^{\varrho}}{\varrho}\right)^{k^*-\theta^*-1}\nu(r)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \\
&+\frac{(t-t_0)^{\varrho}}{\Theta^*}\Bigg[\frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}\hat{\Upsilon}\big(T,\nu(T)\big)-\frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho}\nu(T) \\
&+\frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho}\hat{\Upsilon}\big(\eta,\nu(\eta)\big)-\frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho}\nu(\eta) \\
&-\frac{\mu_2^*\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho}\hat{\Upsilon}\big(T,\nu(T)\big)+\frac{\mu_2^*\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho}\nu(T) \\
&-\frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho}\hat{\Upsilon}\big(\nu,\nu(\nu)\big)+\frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho}\nu(\nu)-\delta_1\Delta_4+\Delta_2\delta_2\Bigg] \\
&+\frac{(t-t_0)^{2\varrho}}{\Theta^*}\Bigg[-\frac{\mu_1^*\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}\hat{\Upsilon}\big(T,\nu(T)\big)+\frac{\mu_1^*\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho}\nu(T) \\
&-\frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho}\hat{\Upsilon}\big(\eta,\nu(\eta)\big)+\frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho}\nu(\eta) \\
&+\frac{\mu_2^*\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho}\hat{\Upsilon}\big(T,\nu(T)\big)-\frac{\mu_2^*\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho}\nu(T) \\
&+\frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho}\hat{\Upsilon}\big(\nu,\nu(\nu)\big)-\frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho}\nu(\nu)+\delta_1\Delta_3-\delta_2\Delta_1\Bigg]\Bigg\}\Bigg| \\
&+\Bigg|\Bigg\{\frac{1}{\lambda\Gamma(k^*)}\int_{t_0}^{t}\left(\frac{(t-t_0)^{\varrho}-(r-t_0)^{\varrho}}{\varrho}\right)^{k^*-1}\hat{\Upsilon}\big(r,\nu(r)\big)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \\
&-\frac{1}{\lambda\Gamma(k^*-\theta^*)}\int_{t_0}^{t}\left(\frac{(t-t_0)^{\varrho}-(r-t_0)^{\varrho}}{\varrho}\right)^{k^*-\theta^*-1}\nu(r)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \\
&+\frac{(t-t_0)^{\varrho}}{\Theta^*}\Bigg[\frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}\hat{\Upsilon}\big(T,\nu(T)\big)-\frac{\mu_1^*\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho}\nu(T) \\
&+\frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho}\hat{\Upsilon}\big(\eta,\nu(\eta)\big)-\frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho}\nu(\eta) \\
&-\frac{\mu_2^*\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho}\hat{\Upsilon}\big(T,\nu(T)\big)+\frac{\mu_2^*\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho}\nu(T) \\
&-\frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho}\hat{\Upsilon}\big(\nu,\nu(\nu)\big)+\frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho}\nu(\nu)-\delta_1\Delta_4+\Delta_2\delta_2\Bigg] \\
&+\frac{(t-t_0)^{2\varrho}}{\Theta^*}\Bigg[-\frac{\mu_1^*\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho}\hat{\Upsilon}\big(T,\nu(T)\big)+\frac{\mu_1^*\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_1^*,\varrho}\nu(T) \\
&-\frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_2^*,\varrho}\hat{\Upsilon}\big(\eta,\nu(\eta)\big)+\frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*-\theta^*-\gamma_2^*,\varrho}\nu(\eta) \\
&+\frac{\mu_2^*\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho}\hat{\Upsilon}\big(T,\nu(T)\big)-\frac{\mu_2^*\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_1^*-\theta^*,\varrho}\nu(T) \\
&+\frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q_2^*+k^*,\varrho}\hat{\Upsilon}\big(\nu,\nu(\nu)\big)-\frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho}\nu(\nu)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \delta_1\Delta_3 - \delta_2\Delta_1\Big]\Big]\Big\}\Big| + \big|u(t)\big| \\
&\le \varepsilon\varphi(t)\mathcal{W}_2 + \|\Phi\|\Psi\big(\|v\|\big)\mathcal{W}_2 + \|v\|\mathcal{W}_1 + +\|\Phi\|\Psi\big(\|u\|\big)\mathcal{W}_2 + \|u\|\mathcal{W}_1 \\
&\quad + \frac{2(T-t_0)^{2\varrho}}{|\Theta^*|}\big(|\delta_1\Delta_3| + |\delta_2\Delta_1|\big) + \frac{2(T-t_0)^{\varrho}}{|\Theta^*|}\big(|\delta_1\Delta_4| + |\delta_2\Delta_2|\big) \\
&\le \varepsilon\varphi(t)\mathcal{W}_2 + 2\mathcal{Q}^* \le \varepsilon\varphi(t)\mathcal{W}_2 + \varepsilon\varphi(t),
\end{aligned}
$$

which yields that

$$\big|v(t) - u(t)\big| \le \varepsilon(\mathcal{W}_2 + 1)\varphi(t). \tag{19}$$

For the sake of simplicity in writing, we take $\kappa_\varphi = \mathcal{W}_2 + 1$. Then (19) becomes

$$\big|v(t) - u(t)\big| \le \kappa_\varphi\varepsilon\varphi(t).$$

This means that the four-point multi-order nonlinear generalized Caputo type fractional BVP (3) is Ulam–Hyers–Rassias stable. Moreover, in the same manner, one can show that the mentioned problem (3) is generalized Ulam–Hyers–Rassias stable. □

## 5 Examples

In this part of the current paper, we formulate two illustrative examples to confirm the correctness of theoretical findings from the computational aspects. Indeed, in the following examples, we consider two cases with different functions in the proposed BVPs.

*Example* 5.1 With due attention to the proposed problem (3), we design the following four-point multi-order nonlinear generalized Caputo type fractional BVP:

$$
\begin{cases}
\frac{47}{48}\,{}^{CC}\mathcal{D}_{\frac{1}{10}}^{\frac{57}{20},0.9}u(t) + {}^{CC}\mathcal{D}_{\frac{1}{10}}^{\frac{33}{16},0.9}u(t) = \hat{\Upsilon}(t,u(t)), \quad (t \in [\frac{1}{10},\frac{1}{5}]), \\
u(\frac{1}{10}) = 0, \\
\frac{1}{33}\,{}^{CC}\mathcal{D}_{\frac{1}{10}}^{\frac{7}{15},0.9}u(\frac{1}{5}) + {}^{CC}\mathcal{D}_{\frac{1}{10}}^{\frac{77}{99},0.9}u(\frac{1}{7}) = \frac{1}{100}, \\
\frac{3}{44}\,{}^{RC}\mathcal{I}_{\frac{1}{10}}^{\frac{4}{3},0.9}u(\frac{1}{5}) + {}^{RC}\mathcal{I}_{\frac{1}{10}}^{\frac{5}{3},0.9}u(\frac{1}{6}) = \frac{1}{50},
\end{cases}
\tag{20}
$$

where $\lambda^* = 47/54$, $\varrho = 0.9$, $k^* = 57/20$, $\theta^* = 33/16$, $\gamma_1^* = 7/15$, $\gamma_2^* = 77/99$, $q_1^* = 4/3$, $q_2^* = 5/3$, $\delta_1 = 1/100$, $\delta_2 = 1/50$, $t_0 = 1/10$, $\nu = 1/7$, $\eta = 1/6$, $\mu_1^* = 1/33$, $\mu_2^* = 3/44$, and $T = 1/5$. Moreover, notice that $0 < \gamma_1^*, \gamma_2^* < 0.7875 = k^* - \theta^*$.

If we define a continuous function $\hat{\Upsilon} : [\frac{1}{10},\frac{1}{5}] \times \mathbb{R} \to \mathbb{R}$ as follows:

$$\hat{\Upsilon}\big(t,u(t)\big) = t^2\left(\frac{|u(t)|}{1+|u(t)|}\right)\sin\big(u(t)\big),$$

then we get $|\hat{\Upsilon}(t,u(t)) - \Upsilon(t,u'(t))| \le \frac{1}{25}|u(t) - u'(t)|$ with $\mathcal{L}_* = 1/25$. In addition, we have $|\hat{\Upsilon}(t,u(t))| \le t^2 = \mathcal{V}(t)$. Besides, we obtain the following values:

$$\Delta_1 \approx 0.5485, \qquad \Delta_2 \approx 0.8939, \qquad \Delta_3 \approx 6.7130 \times 10^{-4}, \qquad \Delta_4 \approx 3.7840 \times 10^{-4},$$

$$\Theta^* \approx 3.9255 \times 10^{-4}, \qquad \mathcal{W}_1 \approx 0.8593, \qquad \mathcal{W}_2 \approx 0.0014.$$

Hence, it is clear that $\mathcal{L}_* \mathcal{W}_2 + \mathcal{W}_1 \approx 0.8594 < 1$. Therefore, by considering the assumptions of Theorem 3.2, the four-point multi-order nonlinear generalized Caputo type fractional BVP (20) has a unique solution. Furthermore, by some simple computations, we find that $\kappa = \frac{\mathcal{W}_2}{1-(\mathcal{L}_* \mathcal{W}_2 + \mathcal{W}_1)} = 0.0096 > 0$. Hence, the conditions of Theorem 4.6 imply that the aforementioned problem (20) is Ulam–Hyers stable and it is also generalized Ulam–Hyers stable.

*Example* 5.2 We take the same values and parameters mentioned in the above example, and we just change the function $\hat{\Upsilon}$ given by the following form:

$$\hat{\Upsilon}\big(t, u(t)\big) = \frac{1}{t^2+5}\left(\frac{u^2(t)}{|u(t)|+1} + 4\right). \tag{21}$$

Then we have

$$\big|\hat{\Upsilon}\big(t, u(t)\big)\big| = \left|\frac{1}{t^2+5}\left(\frac{u^2(t)}{|u(t)|+1} + 4\right)\right| \le \frac{1}{t^2+5}\big(|x| + 4\big).$$

Put $\Phi(t) = \frac{1}{t^2+5}$ and $\Psi(|u|) = |x| + 4$, and select $\mathcal{Q}^* > 44.0032$ so that

$$\frac{\mathcal{Q}^*|\Theta^*|}{\mathcal{Q}^*|\Theta^*|\mathcal{W}_1 + \Psi(\mathcal{Q}^*)\|\Phi\||\Theta^*|\mathcal{W}_2 + (T-t_0)^{2\varrho}(|\delta_1\Delta_3| + |\delta_2\Delta_1|) + (T-t_0)^{\varrho}(|\delta_1\Delta_4| + |\delta_2\Delta_2|)} > 1.$$

Now, in view of the assumptions of Theorem 3.4, we deduce that the four-point multi-order nonlinear generalized Caputo type fractional BVP (20) with $\hat{\Upsilon}$ defined as in (21) has at least one solution.

Moreover, by defining $h^*(t) = 2\exp(t+2)^2$ and $\mathcal{Q}^* = 45$, we reach the inequality $2\mathcal{Q}^* \le h^*(t)$ for any $t \in [\frac{1}{10}, \frac{1}{5}]$. Now, we set $\varphi = \exp(t+2)^2$, and we obtain $\kappa_\varphi = \mathcal{W}_2 + 1 = 1.0014 > 0$. Hence, Theorem 4.8 indicates that the mentioned problem (20) with $\hat{\Upsilon}$ illustrated by (21) is Ulam–Hyers–Rassias stable and also it is generalized Ulam–Hyers–Rassias stable on $[\frac{1}{10}, \frac{1}{5}]$.

## 6 Conclusion

New versions of generalized fractional boundary problems have drawn much interest in recent years owing to their extensive utilization in various directions of applied sciences such as engineering, mechanics, potential theory, biology, chemistry, etc. The subject of stability is a very important notion in physics since most phenomena in the real world include this concept. In fact, the stability notion of physical phenomena has an old historical context, and for the sake of such importance and applicability, one can observe a lot of works in the numerous publications not only in the last century but also before it. In this research article, we turn to study the existence and different types of stability such as generalized Ulam–Hyers stability and generalized Ulam–Hyers–Rassias stability of solutions for a new modeling of a boundary value problem equipped with the fractional differential equation which contains multi-order generalized Caputo type derivatives furnished with four-point mixed generalized Riemann–Liouville type integro-derivative conditions. At the end of the current paper, we formulate two illustrative examples to confirm the correctness of theoretical findings from computational aspects.

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