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Global existence, energy decay and blow-up of solutions for wave equations with time delay and logarithmic source

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Abstract

In this paper, we study the wave equation with frictional damping, time delay in the velocity, and logarithmic source of the form

$$u_{tt}(x,t) - \Delta u(x,t) + \alpha u_t(x,t) + \beta u_t(x,t-\tau) = u(x,t)\ln\left|u(x,t)\right|^{\gamma}.$$

There is much literature on wave equations with a polynomial nonlinear source, but not much on the equations with logarithmic source. We show the local and global existence of solutions using Faedo–Galerkin's method and the logarithmic Sobolev inequality. And then we investigate the decay rates and infinite time blow-up for the solutions through the potential well and perturbed energy methods.

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Keywords: Wave equation; Logarithmic source; Time delay; Energy decay; Blow-up

1 Introduction

We consider the following wave equation with frictional damping, time delay in the velocity, and logarithmic source:

$$u_{tt} - \Delta u + \alpha u_t(t) + \beta u_t(x,t-\tau) = u \ln|u|^{\gamma} \quad \text{for } (x,t) \in \Omega \times (0,\infty), \tag{1.1}$$

$$u(x,t) = 0 \quad \text{for } (x,t) \in \partial\Omega \times (0,\infty), \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) \quad \text{for } x \in \Omega, \tag{1.3}$$

$$u_t(x,t) = j_0(x,t) \quad \text{for } (x,t) \in \Omega \times (-\tau,0), \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded domain with smooth boundary $\partial\Omega$. $\tau > 0$ is time delay, α, β, and γ are real numbers that will be specified later. Equation (1.1) is related to a relativistic version of logarithmic quantum mechanics and many branches of physics such as nuclear physics, optics and geophysics [3, 10, 15].

One of the important theories addressing the existence and nonexistence of solutions for problems with source terms is the potential well method, which was devised by Sattinger

[29]. Based on the method, the interaction between the damping and the source terms was firstly considered by Levine [16]. Since then, the damped wave equation with polynomial nonlinear source of the form

$$u_{tt} - \Delta u + h(u_t) = |u|^{p-2}u \tag{1.5}$$

has been studied extensively on existence, nonexistence, stability, and blow-up of solutions (see [4, 12, 13, 30] and the references therein). Recently, much attention has been paid to the study of nonlinear models of hyperbolic and parabolic equations with logarithmic source nonlinearity [1, 2, 5–8, 17, 18, 22]. For the strongly damped wave equation

$$u_{tt} - \Delta u - a\Delta u_t + bu_t = u \ln |u|^{\gamma},$$

Ma and Fang [22] showed the global existence and infinite time blow-up of solutions when $\gamma = 2$, $a = 1$, and $b = 0$. They used a family of potential wells that is related to the logarithmic nonlinearity, which was introduced by Chen et al. [7]. Lian and Xu [18] proved the global existence, energy decay and infinite time blow-up of solutions when $\gamma = 1$, $a \geq 0$, and $b > -a\lambda$, where λ is the first eigenvalue of the operator $-\Delta$ under homogeneous Dirichlet boundary conditions. In [1], the authors considered the plate equation

$$u_{tt}(x,t) + \Delta^2 u(x,t) + u(x,t) + u_t(x,t) = u(x,t) \ln\left|u(x,t)\right|^{\gamma}.$$

They proved the global existence of solutions and showed that the solutions decay exponentially for a suitable initial data. Later, they extended the results to the case of nonlinear damping in the work [2]. There is not much literature for wave equations with time delay and logarithmic nonlinear source. Thus, in this paper, we intend to study such problem; see (1.1)–(1.4). When $\gamma = 0$ in (1.1), Nicaise and Pignotti [24] proved that the energy decays exponentially under the condition $0 < \beta < \alpha$, and then improved the result to the case of time varying delay in [25]. For related work on problems with time delay, we also refer to [9, 14, 27, 31, 32] and the references therein. Inspired by these results, we discuss the solutions for problem (1.1)–(1.4). To the best of our knowledge, there is little work that takes into account wave equations with time delay and logarithmic source. Thus, we prove the local existence of solutions for problem (1.1)–(1.4) via Faedo–Galerkin's method and the logarithmic Sobolev inequality, and then show the global existence and energy estimates of solutions using the perturbed energy method. Moreover, we establish an infinite time blow-up result by applying the ideas presented in [20, 23, 26] with some necessary modification.

The outline of this paper is as follows. In Sect. 2, we give some notations and material needed for our work. In Sect. 3, we prove the local existence for problem (1.1)–(1.4). In Sect. 4, we provide the global existence and energy decay rates of solutions. Finally, in Sect. 5, we show that the solution occurs with an infinite time blow-up.

2 Preliminaries

We denote the norm of X by $\|\cdot\|_X$ for a Banach space X. We denote the scalar product in $L^2(\Omega)$ by $(\cdot,\cdot)$. For brevity, we denote $\|\cdot\|_2$ by $\|\cdot\|$. Let B_1 be the optimal constant of the embedding inequality

$$\|u\|^2 \le B_1 \|\nabla u\|^2 \quad \text{for } u \in H_0^1(\Omega) \tag{2.1}$$

With regard to problem (1.1)–(1.4), we impose the following assumptions:

(H_1) The weights of dissipation and delay satisfy

$$0 < |\beta| < \alpha. \tag{2.2}$$

(H_2) The constant γ in (1.1) satisfies

$$0 < \gamma < \pi e^{\frac{2(N+1)}{N}}. \tag{2.3}$$

Let us list some lemmas for our work.

Lemma 2.1 (Logarithmic Sobolev inequality [7, 11]) *For any $u \in H_0^1(\Omega)$ and any positive real number k,*

$$\int_\Omega u^2 \ln|u|\,dx \le \frac{1}{2}\|u\|^2 \ln\|u\|^2 + \frac{k^2}{2\pi}\|\nabla u\|^2 - \frac{N}{2}(1+\ln k)\|u\|^2. \tag{2.4}$$

Remark 2.1 Even though the inequality (2.4) holds for all $k > 0$, for the computations throughout this work, we take the constant k satisfying

$$\rho := \max\left\{ e^{-\frac{N+1}{N}}, \mu^{\frac{1}{N}}\sqrt{\frac{\pi}{\gamma}} \right\} < k < \sqrt{\frac{\pi}{\gamma}}, \tag{2.5}$$

where μ is any real number with

$$0 < \mu < 1. \tag{2.6}$$

Lemma 2.2 (Logarithmic Gronwall inequality [5]) *Let $c > 0$ and $l \in L^1(0,T;\mathbb{R}^+)$. If a function $f\colon [0,T] \to [1,\infty)$ satisfies*

$$f(t) \le c\left(1 + \int_0^t l(s) f(s) \ln f(s)\,ds\right), \quad 0 \le t \le T,$$

then

$$f(t) \le c e^{c\int_0^t l(s)\,ds}, \quad 0 \le t \le T.$$

For $v \in H_0^1(\Omega)$, we define

$$J(v) = \frac{1}{2}\|\nabla v\|^2 - \frac{1}{2}\int_\Omega v^2(x) \ln\left|v(x)\right|^\gamma\,dx + \frac{\gamma}{4}\|v\|^2, \tag{2.7}$$

$$I(v) = \|\nabla v\|^2 - \int_\Omega v^2(x) \ln\left|v(x)\right|^\gamma\,dx, \tag{2.8}$$

then

$$J(v) = \frac{1}{2}I(v) + \frac{\gamma}{4}\|v\|^2. \tag{2.9}$$

Let

$$d = \inf_{v \in H_0^1(\Omega)\setminus\{0\}} \sup_{\lambda \geq 0} J(\lambda v), \tag{2.10}$$

then it satisfies, see e.g. [6, 21, 28],

$$0 < d = \inf_{v \in \mathcal{N}} J(v), \tag{2.11}$$

where $\mathcal{N}$ is the well-known Nehari manifold, given by

$$\mathcal{N} = \{v \in H_0^1(\Omega) \setminus \{0\} \mid I(v) = 0\}.$$

Lemma 2.3 *For any $v \in H_0^1(\Omega)$ with $\|v\| \neq 0$, the functions I and J satisfy*

$$I(\lambda v) = \lambda \frac{\partial J(\lambda v)}{\partial \lambda} \begin{cases} > 0, & 0 < \lambda < \lambda^*, \\ = 0, & \lambda = \lambda^*, \\ < 0, & \lambda > \lambda^*, \end{cases} \tag{2.12}$$

where

$$\lambda^* = \exp\left(\frac{\|\nabla v\|^2 - \int_\Omega v^2(x) \ln |v(x)|^\gamma \, dx}{\gamma \|v\|^2}\right).$$

Proof By direct computation, we have, for $\lambda \geq 0$,

$$\begin{aligned} \lambda \frac{\partial}{\partial \lambda} J(\lambda v) &= \lambda \left\{ \lambda \|\nabla v\|^2 - \lambda \int_\Omega v^2(x) \ln |v(x)|^\gamma \, dx + \frac{\gamma \lambda}{2} \|v\|^2 \right. \\ &\quad \left. - \lambda \int_\Omega v^2(x) \ln |\lambda|^\gamma \, dx - \frac{\gamma \lambda}{2} \int_\Omega v^2(x) \, dx \right\} \\ &= \lambda^2 \left(\|\nabla v\|^2 - \int_\Omega v^2(x) \ln |v(x)|^\gamma \, dx - \gamma \ln |\lambda| \int_\Omega v^2(x) \, dx \right) \\ &= I(\lambda v), \end{aligned}$$

and hence we get the desired result. □

Remark 2.2 For a given $v \in H_0^1(\Omega)$, $J(\lambda v)$ has the absolute maximum value at λ^*, that is,

$$\sup_{\lambda \geq 0} J(\lambda v) = J(\lambda^* v) = \exp\left(\frac{2\|\nabla v\|^2 - 2\int_\Omega v^2(x) \ln |v(x)|^\gamma \, dx}{\gamma \|v\|^2}\right) \frac{\gamma}{4} \|v\|^2. \tag{2.13}$$

Lemma 2.4 *The potential depth d in (2.10) satisfies*

$$d \geq \frac{\gamma}{4} e^N \left(\frac{\pi}{\gamma}\right)^{\frac{N}{2}} := E_1.$$

Proof From Lemma 2.1, (2.1), and (2.5), we get

$$\begin{aligned} I(v) &\geq \left(1-\frac{k^2\gamma}{2\pi}\right)\|\nabla v\|^2+\frac{N\gamma}{2}(1+\ln k)\|v\|^2-\frac{\gamma}{2}\|v\|^2\ln\|v\|^2 \\ &> \frac{N\gamma}{2}(1+\ln k)\|v\|^2-\frac{\gamma}{2}\|v\|^2\ln\|v\|^2. \end{aligned}$$

Taking the limit $k\to\sqrt{\frac{\pi}{\gamma}}^{-}$, we have

$$I(v)\geq\left\{\frac{N\gamma}{2}\left(1+\ln\sqrt{\frac{\pi}{\gamma}}\right)-\frac{\gamma}{2}\ln\|v\|^2\right\}\|v\|^2.$$

Considering this and (2.12), we have

$$0=I\bigl(\lambda^*v\bigr)\geq\left\{\frac{N\gamma}{2}\left(1+\ln\sqrt{\frac{\pi}{\gamma}}\right)-\frac{\gamma}{2}\ln\bigl\|\lambda^*v\bigr\|^2\right\}\bigl\|\lambda^*v\bigr\|^2,$$

and hence

$$\bigl\|\lambda^*v\bigr\|^2\geq e^N\left(\frac{\pi}{\gamma}\right)^{\frac{N}{2}}.$$

Thus, we obtain from (2.13) and (2.9)

$$\sup_{\lambda\geq0}J(\lambda v)=J\bigl(\lambda^*v\bigr)=\frac{1}{2}I\bigl(\lambda^*v\bigr)+\frac{\gamma}{4}\bigl\|\lambda^*v\bigr\|^2=\frac{\gamma}{4}\bigl\|\lambda^*v\bigr\|^2\geq\frac{\gamma}{4}e^N\left(\frac{\pi}{\gamma}\right)^{\frac{N}{2}}.$$

By the definition of d given in (2.10), we get the desired result. □

3 Local existence of solutions

In this section we prove the local existence of solutions by applying the ideas in [1, 24]. Using the function

$$y(x,\eta,t)=u_t(x,t-\eta\tau)\quad\text{for }(x,\eta,t)\in\Omega\times[0,1]\times(0,\infty), \tag{3.1}$$

problem (1.1)–(1.4) is rewritten as

$$\begin{aligned} &u_{tt}(x,t)-\Delta u(x,t)+\alpha u_t(x,t)+\beta y(x,1,t) \\ &\quad=u(x,t)\ln\bigl|u(x,t)\bigr|^\gamma\quad\text{for }(x,t)\in\Omega\times(0,\infty), &(3.2)\\ &\tau y_t(x,\eta,t)+y_\eta(x,\eta,t)=0\quad\text{for }(x,\eta,t)\in\Omega\times(0,1)\times(0,\infty), &(3.3)\\ &u(x,t)=0\quad\text{for }(x,t)\in\partial\Omega\times(0,\infty), &(3.4)\\ &u(x,0)=u_0(x),\qquad u_t(x,0)=u_1(x)\quad\text{for }x\in\Omega, &(3.5)\\ &y(x,\eta,0)=j_0(x,-\eta\tau):=y_0(x,\eta)\quad\text{for }(x,\eta)\in\Omega\times(0,1). &(3.6) \end{aligned}$$

Definition 3.1 Let $T > 0$. We say that (u, y) is a local solution of problem (3.2)–(3.6) if it satisfies the following:

$$
\begin{aligned}
& u \in C\big([0,T]; H_0^1(\Omega)\big) \cap C^1\big([0,T]; L^2(\Omega)\big) \cap C^2\big([0,T); H^{-1}(\Omega)\big), \\
& \big(u_{tt}(t), v\big) + \big(\nabla u(t), \nabla v\big) + \alpha\big(u_t(t), v\big) + \beta\big(y(1,t), v\big) \\
& \quad = \big(u(t)\ln\big|u(t)\big|^{\gamma}, v\big) \quad \text{for any } v \in H_0^1(\Omega), \\
& \tau \int_0^1 \big(y_t(\eta,t), \varphi(\eta)\big)\, d\eta + \int_0^1 \big(y_\eta(\eta,t), \varphi(\eta)\big)\, d\eta = 0 \quad \text{for any } \varphi \in L^2\big(\Omega \times (0,1)\big),
\end{aligned}
$$

and

$$
u(0) = u_0 \quad \text{in } H_0^1(\Omega), \qquad u_t(0) = u_1 \quad \text{in } L^2(\Omega), \qquad y(0) = y_0 \quad \text{in } L^2\big(\Omega \times (0,1)\big).
$$

Theorem 3.1 *Assume that (H_1) and (H_2) hold. Then, for the initial data $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, $y_0 \in L^2(\Omega \times (0,1))$, there exists a local solution (u, y) of problem* (3.2)–(3.6).

Proof Let $\{v_i\}_{i\in\mathbb{N}}$ be orthogonal basis of $H_0^1(\Omega)$ which is orthonormal in $L^2(\Omega)$. Defining $\varphi_i(x,0) = v_i(x)$, we can extend $\varphi_i(x,0)$ by $\varphi_i(x,\eta)$ over $L^2(\Omega \times (0,1))$. We denote $V_n = \operatorname{span}\{v_1, v_2, \dots, v_n\}$ and $W_n = \operatorname{span}\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ for $n \geq 1$. We consider the Faedo–Galerkin approximation solution $(u^n, y^n) \in V_n \times W_n$ of the form

$$
u^n(x,t) = \sum_{i=1}^{n} h_i^n(t) v_i(x) \quad \text{and} \quad y^n(x,\eta,t) = \sum_{i=1}^{n} g_i^n(t) \varphi_i(x,\eta), \quad n = 1, 2, \dots,
$$

solving the approximate system

$$
\begin{aligned}
& \big(u_{tt}^n(t), v\big) + \big(\nabla u^n(t), \nabla v\big) + \alpha\big(u_t^n(t), v\big) + \beta\big(y^n(1,t), v\big) \\
& \quad = \int_\Omega u^n(x,t)\ln\big|u^n(x,t)\big|^{\gamma} v(x)\, dx \quad \text{for } v \in V_n, \qquad (3.7) \\
& \tau \int_0^1 \big(y_t^n(\eta,t), \varphi(\eta)\big)\, d\eta + \int_0^1 \big(y_\eta^n(\eta,t), \varphi(\eta)\big)\, d\eta = 0 \quad \text{for } \varphi \in W_n, \qquad (3.8) \\
& u^n(0) = u_0^n, \qquad u_t^n(0) = u_1^n, \qquad y^n(0) = y_0^n, \qquad (3.9)
\end{aligned}
$$

where

$$
u_0^n \to u_0 \quad \text{in } H_0^1(\Omega), \qquad u_1^n \to u_1 \quad \text{in } L^2(\Omega), \qquad y_0^n \to y_0 \quad \text{in } L^2\big(\Omega \times (0,1)\big).
$$

Since problem (3.7)–(3.9) is a normal system of ordinary differential equations, there exists a solution (u^n, y^n) on the interval $[0, t_n)$, $t_n \in (0, T]$. The extension of this solution to the whole interval $[0, T)$ is a consequence of the estimate below.

Replacing v by $u_t^n(t)$ in (3.7) and using the relation

$$
\int_\Omega u^n(x,t)\ln\big|u^n(x,t)\big|^{\gamma} u_t^n(x,t)\, dx = \frac{d}{dt}\left\{\frac{1}{2}\int_\Omega \big(u^n(x,t)\big)^2 \ln\big|u^n(x,t)\big|^{\gamma}\, dx - \frac{\gamma}{4}\big\|u^n(t)\big\|^2\right\},
$$

we have

$$\frac{d}{dt}\left\{\frac{1}{2}\left\|u_t^n(t)\right\|^2+\frac{1}{2}\left\|\nabla u^n(t)\right\|^2+\frac{\gamma}{4}\left\|u^n(t)\right\|^2-\frac{1}{2}\int_\Omega\left(u^n(x,t)\right)^2\ln\left|u^n(x,t)\right|^\gamma\,dx\right\}$$
$$=-\alpha\left\|u_t^n(t)\right\|^2-\beta\left(y^n(1,t),u_t^n(t)\right). \tag{3.10}$$

Replacing φ by $\omega y^n(\eta,t)$ in (3.8), one sees

$$\frac{\omega\tau}{2}\frac{d}{dt}\int_\Omega\int_0^1\left(y^n(x,\eta,t)\right)^2\,d\eta\,dx=-\frac{\omega}{2}\left\|y^n(1,t)\right\|^2+\frac{\omega}{2}\left\|y^n(0,t)\right\|^2. \tag{3.11}$$

Collecting (3.10) and (3.11), we get

$$\frac{d}{dt}E^n(t)=-\alpha\left\|u_t^n(t)\right\|^2-\beta\left(y^n(1,t),u_t^n(t)\right)-\frac{\omega}{2}\left\|y^n(1,t)\right\|^2+\frac{\omega}{2}\left\|y^n(0,t)\right\|^2,$$

where

$$E^n(t)=\frac{1}{2}\left\|u_t^n(t)\right\|^2+\frac{1}{2}\left\|\nabla u^n(t)\right\|^2+\frac{\gamma}{4}\left\|u^n(t)\right\|^2$$
$$-\frac{1}{2}\int_\Omega\left(u^n(x,t)\right)^2\ln\left|u^n(x,t)\right|^\gamma\,dx+\frac{\omega\tau}{2}\left\|y^n(t)\right\|^2_{L^2(\Omega\times(0,1))},$$

here

$$|\beta|<\omega<2\alpha-|\beta|. \tag{3.12}$$

By Young's inequality and the fact $y^n(x,0,t)=u_t^n(x,t)$, we get

$$\frac{d}{dt}E^n(t)\le-\left(\alpha-\frac{|\beta|}{2}-\frac{\omega}{2}\right)\left\|u_t^n(t)\right\|^2-\left(\frac{\omega}{2}-\frac{|\beta|}{2}\right)\left\|y^n(1,t)\right\|^2\le 0 \tag{3.13}$$

and

$$E^n(t)+C_1\int_0^t\left\|u_t^n(s)\right\|^2\,ds+C_2\int_0^t\left\|y^n(1,s)\right\|^2\,ds\le E^n(0), \tag{3.14}$$

where

$$C_1=\alpha-\frac{|\beta|}{2}-\frac{\omega}{2}>0\quad\text{and}\quad C_2=\frac{\omega}{2}-\frac{|\beta|}{2}>0. \tag{3.15}$$

From this and Lemma 2.1, we observe

$$\left\|u_t^n(t)\right\|^2+\left(1-\frac{\gamma k^2}{2\pi}\right)\left\|\nabla u^n(t)\right\|^2+\frac{\gamma}{2}\left(1+N(1+\ln k)\right)\left\|u^n(t)\right\|^2$$
$$+2C_1\int_0^t\left\|u_t^n(s)\right\|^2\,ds+2C_2\int_0^t\left\|y^n(1,s)\right\|^2\,ds+\omega\tau\left\|y^n(t)\right\|^2_{L^2(\Omega\times(0,1))}$$
$$\le 2E^n(0)+\frac{\gamma}{2}\left\|u^n(t)\right\|^2\ln\left\|u^n(t)\right\|^2. \tag{3.16}$$

Thanks to (2.5), we have

$$1-\frac{\gamma k^2}{2\pi}>0 \quad \text{and} \quad \frac{\gamma}{2}\bigl(1+N(1+\ln k)\bigr)>0,$$

and hence

$$\begin{aligned}&\bigl\|u_t^n(t)\bigr\|^2+\bigl\|\nabla u^n(t)\bigr\|^2+\bigl\|u^n(t)\bigr\|^2+\int_0^t\bigl\|u_t^n(s)\bigr\|^2\,ds\\&\quad+\int_0^t\bigl\|y^n(1,s)\bigr\|^2\,ds+\bigl\|y^n(t)\bigr\|^2_{L^2(\Omega\times(0,1))}\\&\quad\le c_1\bigl(1+\bigl\|u^n(t)\bigr\|^2\ln\bigl\|u^n(t)\bigr\|^2\bigr),\end{aligned}\tag{3.17}$$

here and in the sequel c_j, $j=1,2,\dots$, denotes a generic positive constant. On the other hand, it is noted that

$$u^n(x,t)=u^n(x,0)+\int_0^t u_t^n(x,s)\,ds.$$

Applying Cauchy–Schwarz' inequality and (3.17), we get

$$\begin{aligned}\bigl\|u^n(t)\bigr\|^2&=2\bigl\|u^n(0)\bigr\|^2+2T\int_0^t\bigl\|u_t^n(s)\bigr\|^2\,ds\\&\le 2\bigl\|u^n(0)\bigr\|^2+2T\int_0^t c_1\bigl(1+\bigl\|u^n(s)\bigr\|^2\ln\bigl\|u^n(s)\bigr\|^2\bigr)\,ds\\&\le c_2\left(1+\int_0^t\bigl\|u^n(s)\bigr\|^2\ln\bigl\|u^n(s)\bigr\|^2\,ds\right).\end{aligned}$$

By Lemma 2.2, we find

$$\bigl\|u^n(t)\bigr\|^2\le c_3e^{c_4T}.\tag{3.18}$$

Since the function $f(s)=s\ln s$ is continuous $(0,\infty)$, $\lim_{s\to 0^+}f(s)=0$, $\lim_{s\to+\infty}f(s)=+\infty$, and f decreases on $(0,e^{-1})$ and increases on $(e^{-1},+\infty)$, we have from (3.18) and (3.17)

$$\begin{aligned}&\bigl\|u_t^n(t)\bigr\|^2+\bigl\|\nabla u^n(t)\bigr\|^2+\bigl\|u^n(t)\bigr\|^2+\int_0^t\bigl\|u_t^n(s)\bigr\|^2\,ds\\&\quad+\int_0^t\bigl\|y^n(1,s)\bigr\|^2\,ds+\bigl\|y^n(t)\bigr\|^2_{L^2(\Omega\times(0,1))}\le c_5.\end{aligned}\tag{3.19}$$

So, there exists a subsequence of $\{(u^n,y^n)\}$, which we still denote $\{(u^n,y^n)\}$, such that

$$u^n\to u \quad \text{weakly star in } L^\infty\bigl(0,T;H_0^1(\Omega)\bigr),\tag{3.20}$$

$$u_t^n\to u_t \quad \text{weakly star in } L^\infty\bigl(0,T;L^2(\Omega)\bigr),\tag{3.21}$$

$$y^n\to y \quad \text{weakly star in } L^\infty\bigl(0,T;L^2\bigl(\Omega\times(0,1)\bigr)\bigr),\tag{3.22}$$

$$y^n(1)\to y(1) \quad \text{weakly in } L^2\bigl(0,T;L^2(\Omega)\bigr).\tag{3.23}$$

By Aubin–Lions' compactness theorem, we find

$$u^n \to u \quad \text{strongly in } L^2\big(0,T;L^2(\Omega)\big)$$

and

$$u^n(x,t) \to u(x,t) \quad \text{a.e. in } \Omega \times (0,T).$$

Since the function $s \to s\ln|s|^\gamma$ is continuous on $\mathbb{R}$,

$$u^n(x,t)\ln\big|u^n(x,t)\big|^\gamma \to u(x,t)\ln\big|u(x,t)\big|^\gamma \quad \text{a.e. in } \Omega \times (0,T). \tag{3.24}$$

Now, we let

$$\Omega_1 = \big\{x \in \Omega \mid \big|u^n(x,t)\big| < 1\big\} \quad \text{and} \quad \Omega_2 = \big\{x \in \Omega \mid \big|u^n(x,t)\big| \geq 1\big\}.$$

Then we have

$$\begin{aligned} &\int_\Omega \big(u^n(x,t)\ln\big|u^n(x,t)\big|^\gamma\big)^2\,dx \\ &\quad = \gamma^2\left\{\int_{\Omega_1}\big(u^n(x,t)\ln\big|u^n(x,t)\big|\big)^2\,dx + \int_{\Omega_2}\big(u^n(x,t)\ln\big|u^n(x,t)\big|\big)^2\,dx\right\} \\ &\quad \leq \gamma^2\left\{e^{-2}|\Omega_1| + e^{-2}\left(\frac{2}{q-2}\right)^2\int_{\Omega_2}\big(u^n(x,t)\big)^q\,dx\right\} \quad \text{for any } q > 2, \end{aligned} \tag{3.25}$$

here we used the fact

$$|s\ln s| \leq \frac{1}{e} \quad \text{for } 0 < s < 1 \quad \text{and} \quad s^{-\kappa}\ln s \leq \frac{1}{e\kappa} \quad \text{for } s \geq 1 \text{ and } \kappa > 0.$$

From (3.25) and (3.17), we arrive at

$$\int_\Omega \big(u^n(x,t)\ln\big|u^n(x,t)\big|^\gamma\big)^2\,dx \leq \gamma^2\left\{e^{-2}|\Omega_1| + e^{-2}\left(\frac{2}{q-2}\right)^2 B_2^q\big\|\nabla u^n\big\|^q\right\} \leq c_6, \tag{3.26}$$

where B_2 is the best Sobolev imbedding constant of

$$H_0^1(\Omega) \subset L^q(\Omega) \quad \text{for } q > 2, \text{if } N = 1,2; 2 < q < \frac{2N}{N-2}, \text{ if } N \geq 3.$$

Thus, we have from (3.26)

$$u^n\ln\big|u^n\big|^\gamma \text{ is uniformly bounded in } L^\infty\big(0,T;L^2(\Omega)\big). \tag{3.27}$$

By the Lebesgue bounded convergence theorem, (3.24), and (3.27), we infer

$$u^n\ln\big|u^n\big|^\gamma \to u\ln|u|^\gamma \quad \text{strongly in } L^2\big(0,T;L^2(\Omega)\big).$$

Now, we are ready to pass to the limit $m \to \infty$ in (3.7) and (3.8). The proof of the remainder is standard and can be done as in [1, 19]. □

4 Global existence and energy decay estimate

In this section, we prove the global existence and energy decay rates of solutions to problem (3.2)–(3.6). For this, we define the energy of problem (3.2)–(3.6) as

$$\begin{aligned} E(t) &:= E\big(u(t)\big) \\ &= \frac{1}{2}\big\|u_t(t)\big\|^2 + \frac{1}{2}\big\|\nabla u(t)\big\|^2 + \frac{\gamma}{4}\big\|u(t)\big\|^2 \\ &\quad - \frac{1}{2}\int_\Omega u^2(x,t)\ln\big|u(x,t)\big|^\gamma\,dx + \frac{\omega\tau}{2}\big\|y(t)\big\|^2_{L^2(\Omega\times(0,1))}, \end{aligned} \tag{4.1}$$

where ω is the positive constant given in (3.12). It is noted that

$$\begin{aligned} E(t) &= \frac{1}{2}\big\|u_t(t)\big\|^2 + J\big(u(t)\big) + \frac{\omega\tau}{2}\big\|y(t)\big\|^2_{L^2(\Omega\times(0,1))} \\ &= \frac{1}{2}\big\|u_t(t)\big\|^2 + \frac{1}{2}I\big(u(t)\big) + \frac{\gamma}{4}\big\|u(t)\big\|^2 + \frac{\omega\tau}{2}\big\|y(t)\big\|^2_{L^2(\Omega\times(0,1))}. \end{aligned} \tag{4.2}$$

By the same arguments as of (3.13), we can deduce

$$\frac{d}{dt}E(t) \le -C_1\big\|u_t(t)\big\|^2 - C_2\big\|y(1,t)\big\|^2 \le 0, \tag{4.3}$$

where C_1 and C_2 are positive constants given in (3.15).

Lemma 4.1 *Assume that* (H_1) *and* (H_2) *hold. If* $E(0) < d$ *and* $I(u_0) > 0$, *then the solution* u *of problem* (1.1)–(1.4) *satisfies*

$$I\big(u(t)\big) > 0 \quad \textit{for } t \in [0,T), \tag{4.4}$$

where T *is the maximal existence time of the solutions.*

Proof Since $I(u_0) > 0$ and u is continuous on $[0,T)$, we know that

$$I\big(u(t)\big) > 0 \quad \text{for some interval } [0,t_1) \subset [0,T). \tag{4.5}$$

Let t_0 be the maximum of t_1 satisfying (4.5). Suppose $t_0 < T$, then $I(u(t_0)) = 0$, that is,

$$u(t_0) \in \mathcal{N}.$$

Thus, we have from (2.11)

$$J\big(u(t_0)\big) \ge \inf_{v\in\mathcal{N}} J(v) = d.$$

But this is contradiction to the following relation:

$$J\big(u(t_0)\big) \le E(t_0) \le E(0) < d. \qquad \square$$

It is noted that $E(t)$ is a nonincreasing positive function from (4.3) and Lemma 4.1.

Theorem 4.1 *Under the conditions of Lemma 4.1, the solution u is global.*

Proof It suffices to show that $\|u_t(t)\|^2 + \|\nabla u(t)\|^2$ is bounded independent of t. From Lemma 4.1, (4.2), and (4.3), we have

$$\|u_t(t)\|^2 \le \|u_t(t)\|^2 + I(u(t)) \le 2E(t) \le 2E(0) < 2d. \tag{4.6}$$

Similarly, we see

$$\|u(t)\|^2 < \|u(t)\|^2 + \frac{2}{\gamma}I(u(t)) = \frac{4}{\gamma}J(u(t)) \le \frac{4}{\gamma}E(t) \le \frac{4}{\gamma}E(0) < \frac{4d}{\gamma}. \tag{4.7}$$

From Lemma 2.1 and (2.8), we infer

$$\begin{aligned}\|\nabla u(t)\|^2 &= I(u(t)) + \gamma \int_\Omega u^2(x,t)\ln|u(x,t)|\,dx \\ &\le 2E(t) + \frac{\gamma}{2}\|u(t)\|^2 \ln\|u(t)\|^2 + \frac{k^2\gamma}{2\pi}\|\nabla u(t)\|^2 - \frac{N\gamma}{2}(1+\ln k)\|u(t)\|^2.\end{aligned}$$

Taking the limit $k \to \rho^+$ in this inequality and using (4.7), we get

$$\begin{aligned}\left(1 - \frac{\rho^2\gamma}{2\pi}\right)\|\nabla u(t)\|^2 &\le 2E(t) + \frac{\gamma}{2}\left(\ln\|u(t)\|^2 - N(1+\ln\rho)\right)\|u(t)\|^2 \\ &< 2d + \frac{\gamma}{2}\left(\ln\left(\frac{4d}{\gamma}\right) - N(1+\ln\rho)\right)\|u(t)\|^2 \\ &= 2d + \frac{\gamma}{2}\left\{\ln\left(\frac{4d}{\gamma}e^{-N}\rho^{-N}\right)\right\}\|u(t)\|^2.\end{aligned} \tag{4.8}$$

From Lemma 2.4 and (2.5), we get

$$\ln\left(\frac{4d}{\gamma}e^{-N}\rho^{-N}\right) \ge \ln\left(\left(\frac{\pi}{\gamma}\right)^{\frac{N}{2}}\rho^{-N}\right) = \ln\left(\left(\sqrt{\frac{\pi}{\gamma}}\rho^{-1}\right)^N\right) > \ln 1 = 0.$$

Thus, we observe from (4.8) and (4.7) that

$$\left(1 - \frac{\rho^2\gamma}{2\pi}\right)\|\nabla u(t)\|^2 < 2d + 2d\ln\left(\frac{4d}{\gamma}e^{-N}\rho^{-N}\right).$$

This gives

$$\|\nabla u(t)\|^2 < 2d\left(1 - \frac{\rho^2\gamma}{2\pi}\right)^{-1}\left(1 + \ln\left(\frac{4d}{\gamma}e^{-N}\rho^{-N}\right)\right). \tag{4.9}$$

We complete the proof from (4.6) and (4.9). □

In order to establish asymptotic behavior for the global solution, let us define the perturbed energy by

$$\mathcal{E}(t) = E(t) + \varepsilon\Phi(t) + \varepsilon\Xi(t),$$

where $\varepsilon > 0$, $\Phi(t) = (u_t(t), u(t))$, and $\Xi(t) = \int_\Omega \int_0^1 e^{-\tau\eta}y^2(x,\eta,t)\,d\eta\,dx$.

Lemma 4.2 *If the conditions of Lemma 4.1 hold, there exist positive constants C_3 and C_4 such that*

$$C_3E(t) \le \mathcal{E}(t) \le C_4E(t).$$

Proof Young's inequality and Lemma 4.1 imply

$$\begin{aligned}
\left|\Phi(t) + \Xi(t)\right| &\le \frac{1}{2}\left\|u_t(t)\right\|^2 + \frac{1}{2}\left\|u(t)\right\|^2 + \left\|y(t)\right\|^2_{L^2(\Omega\times(0,1))} \\
&\le \frac{1}{2}\left\|u_t(t)\right\|^2 + \frac{2}{\gamma}\left(\frac{\gamma}{4}\left\|u(t)\right\|^2 + \frac{1}{2}I\big(u(t)\big)\right) + \left\|y(t)\right\|^2_{L^2(\Omega\times(0,1))} \\
&= \frac{1}{2}\left\|u_t(t)\right\|^2 + \frac{2}{\gamma}J\big(u(t)\big) + \left\|y(t)\right\|^2_{L^2(\Omega\times(0,1))} \\
&\le c_7E(t).
\end{aligned}$$

Taking $\varepsilon > 0$ suitably small, we complete the proof. □

Theorem 4.2 *Let (H_1) and (H_2) hold. Assume that $E(0) < E_1$ and $I(u_0) > 0$. Then there exist positive constants C_0 and C_5 such that*

$$0 < E(t) \le C_0e^{-C_5t} \quad \textit{for } t \ge 0.$$

Proof Using (3.2) and Young's inequality, we have

$$\begin{aligned}
\Phi'(t) &= \left\|u_t(t)\right\|^2 - \left\|\nabla u(t)\right\|^2 - \alpha\big(u_t(t), u(t)\big) - \beta\big(y(1,t), u(t)\big) \\
&\quad + \int_\Omega u^2(x,t)\ln\left|u(x,t)\right|^\gamma\,dx \\
&\le \left\|u_t(t)\right\|^2 - \frac{1}{2}\left\|\nabla u(t)\right\|^2 + \alpha^2B_1\left\|u_t(t)\right\|^2 + \beta^2B_1\left\|y(1,t)\right\|^2 \\
&\quad + \int_\Omega u^2(x,t)\ln\left|u(x,t)\right|^\gamma\,dx.
\end{aligned}$$

From (3.3) and the integration by parts, we get

$$\begin{aligned}
\Xi'(t) &= -\frac{2}{\tau}\int_\Omega\int_0^1 e^{-\tau\eta}y(x,\eta,t)y_\eta(x,\eta,t)\,d\eta\,dx \\
&= -\frac{1}{\tau}\int_\Omega\int_0^1 e^{-\tau\eta}\frac{\partial}{\partial\eta}y^2(x,\eta,t)\,d\eta\,dx \\
&= -\frac{e^{-\tau}}{\tau}\left\|y(1,t)\right\|^2 + \frac{1}{\tau}\left\|y(0,t)\right\|^2 - \int_\Omega\int_0^1 e^{-\tau\eta}y^2(x,\eta,t)\,d\eta\,dx \\
&\le \frac{1}{\tau}\left\|u_t(t)\right\|^2 - e^{-\tau}\int_\Omega\int_0^1 y^2(x,\eta,t)\,d\eta\,dx.
\end{aligned}$$

Collecting these and (4.3), we have

$$\begin{aligned}\mathcal{E}'(t) &\le -\left(C_1 - \varepsilon - \varepsilon\alpha^2 B_1 - \frac{\varepsilon}{\tau}\right)\|u_t(t)\|^2 - \frac{\varepsilon}{2}\|\nabla u(t)\|^2 - \left(C_2 - \varepsilon\beta^2 B_1\right)\|y(1,t)\|^2 \\ &\quad + \varepsilon\int_\Omega u^2(x,t)\ln|u(x,t)|^\gamma\,dx - \varepsilon e^{-\tau}\|y(t)\|^2_{L^2(\Omega\times(0,1))}.\end{aligned}$$

Subtracting and adding $\xi E(t)$ with $0 < \xi < 2\varepsilon$, we have

$$\begin{aligned}\mathcal{E}'(t) &\le -\xi E(t) - \left(C_1 - \varepsilon - \varepsilon\alpha^2 B_1 - \frac{\varepsilon}{\tau} - \frac{\xi}{2}\right)\|u_t(t)\|^2 - \left(\frac{\varepsilon}{2} - \frac{\xi}{2} - \frac{\xi\gamma B_1}{4}\right)\|\nabla u(t)\|^2 \\ &\quad - \left(C_2 - \varepsilon\beta^2 B_1\right)\|y(1,t)\|^2 + \left(\varepsilon - \frac{\xi}{2}\right)\int_\Omega u^2(x,t)\ln|u(x,t)|^\gamma\,dx \\ &\quad - \left(\varepsilon e^{-\tau} - \frac{\xi\omega\tau}{2}\right)\|y(t)\|^2_{L^2(\Omega\times(0,1))}.\end{aligned}$$

From the logarithmic Sobolev inequality, we obtain

$$\begin{aligned}\mathcal{E}'(t) &\le -\xi E(t) - \left(C_1 - \varepsilon - \varepsilon\alpha^2 B_1 - \frac{\varepsilon}{\tau} - \frac{\xi}{2}\right)\|u_t(t)\|^2 \\ &\quad - \left\{\varepsilon\left(\frac{1}{2} - \frac{\gamma k^2}{2\pi}\right) - \frac{\xi}{2}\left(1 - \frac{\gamma k^2}{2\pi}\right) - \frac{\xi\gamma B_1}{4}\right\}\|\nabla u(t)\|^2 \\ &\quad + \frac{\gamma}{2}\left(\varepsilon - \frac{\xi}{2}\right)\left\{\ln\|u(t)\|^2 - N(1+\ln k)\right\}\|u(t)\|^2 \\ &\quad - \left(C_2 - \varepsilon\beta^2 B_1\right)\|y(1,t)\|^2 - \left(\varepsilon e^{-\tau} - \frac{\xi\omega\tau}{2}\right)\|y(t)\|^2_{L^2(\Omega\times(0,1))}.\end{aligned}$$

First, we choose $\varepsilon > 0$ small such that

$$C_1 - \varepsilon - \varepsilon\alpha^2 B_1 - \frac{\varepsilon}{\tau} > 0 \quad \text{and} \quad C_2 - \varepsilon\beta^2 B_1 > 0.$$

Then, taking $\xi > 0$ sufficiently small and noting that $\frac{1}{2} - \frac{\gamma k^2}{2\pi} > 0$ (see (2.5)), we arrive at

$$\mathcal{E}'(t) \le -\xi E(t) + \frac{\gamma}{2}\left(\varepsilon - \frac{\xi}{2}\right)\left\{\ln\|u(t)\|^2 - N(1+\ln k)\right\}\|u(t)\|^2. \tag{4.10}$$

Since $0 < E(0) < E_1$, there exists $0 < \mu < 1$ such that $E(0) = \mu E_1$. Thus, we have from (4.7)

$$\ln\|u(t)\|^2 < \ln\left(\frac{4}{\gamma}E(t)\right) \le \ln\left(\frac{4}{\gamma}E(0)\right) = \ln\left(\frac{4\mu E_1}{\gamma}\right) = \ln\left(\mu e^N\left(\frac{\pi}{\gamma}\right)^{\frac{N}{2}}\right).$$

Thus, we infer from (2.5) that

$$\begin{aligned}\ln\|u(t)\|^2 - N(1+\ln k) &\le \ln\left(\mu e^N\left(\frac{\pi}{\gamma}\right)^{\frac{N}{2}}\right) - N(1+\ln k) \\ &= N\ln\left(\mu^{\frac{1}{N}}\sqrt{\frac{\pi}{\gamma}}k^{-1}\right) < N\ln 1 = 0.\end{aligned}$$

Substituting this into (4.10), we conclude

$$\mathcal{E}'(t) \leq -\xi E(t).$$

Consequently, we complete the proof from Lemma 4.2. □

5 Infinite time blow-up

In this section, inspired by the ideas in [20, 23, 26], we establish a blow-up result for problem (1.1)–(1.4). For this, we first give the following lemma.

Lemma 5.1 *Assume that* (H_1) *and* (H_2) *hold. If* $E(0) < E_1$ *and* $I(u_0) < 0$, *then the solution* u *of problem* (1.1)–(1.4) *satisfies*

$$I\bigl(u(t)\bigr) < 0 \quad \textit{for } t \in [0, T) \tag{5.1}$$

and

$$\bigl\|u(t)\bigr\|^2 > \frac{4E_1}{\gamma} \quad \textit{for } t \in [0, T), \tag{5.2}$$

where T *is the maximal existence time of solutions.*

Proof Since $I(u_0) < 0$ and u is continuous on $[0, T)$, we know that

$$I\bigl(u(t)\bigr) < 0 \quad \text{for some interval } [0, t_1) \subset [0, T). \tag{5.3}$$

Let t_0 be the maximal time satisfying (5.3) and suppose $t_0 < T$, then $I(u(t_0)) = 0$, that is,

$$u(t_0) \in \mathcal{N}.$$

Thus, we have

$$d \leq J\bigl(u(t_0)\bigr) = \frac{1}{2}I\bigl(u(t_0)\bigr) + \frac{\gamma}{4}\bigl\|u(t_0)\bigr\|^2 \leq E\bigl(u(t_0)\bigr) \leq E(0) < E_1.$$

This is in contradiction to Lemma 2.4. So, (5.1) is proved. From Lemma 2.4, (2.13), and (5.1), we find

$$\begin{aligned} E_1 \leq d \leq J\bigl(\lambda^* u(t)\bigr) &= \exp\biggl(\frac{2\|\nabla u(t)\|^2 - 2\int_\Omega u^2(x,t)\ln|u(x,t)|^\gamma\,dx}{\gamma\|u(t)\|^2}\biggr)\frac{\gamma}{4}\bigl\|u(t)\bigr\|^2 \\ &< \frac{\gamma}{4}\bigl\|u(t)\bigr\|^2. \end{aligned}$$

Thus, we complete the proof. □

Theorem 5.1 *Assume that* (H_1) *and* (H_2) *hold. Assume that* $E(0) < \zeta E_1$, *where* $0 < \zeta < 1$, *and* $I(u_0) < 0$. *Then the solution of problem* (1.1)–(1.4) *blows up at infinity.*

Proof We set

$$F(t) = \zeta E_1 - E(t). \tag{5.4}$$

From (4.3), we have

$$F'(t) = -E'(t) \geq C_1 \|u_t(t)\|^2 + C_2 \|y(1,t)\|^2 \geq 0. \tag{5.5}$$

From (5.5), (4.1), and (5.2), we observe

$$\begin{aligned} 0 < F(0) &\leq F(t) \leq \zeta E_1 + \frac{1}{2}\int_\Omega u^2(x,t)\ln\bigl|u(x,t)\bigr|^\gamma\,dx \\ &< \frac{\gamma}{4}\|u(t)\|^2 + \frac{1}{2}\int_\Omega u^2(x,t)\ln\bigl|u(x,t)\bigr|^\gamma\,dx. \end{aligned} \tag{5.6}$$

Now, we define

$$G(t) = F(t) + \varepsilon\bigl(u(t), u_t(t)\bigr) + \frac{\varepsilon\alpha}{2}\|u(t)\|^2.$$

Using (3.2), (4.1), we have

$$\begin{aligned} G'(t) &= F'(t) + \varepsilon\|u_t(t)\|^2 - \varepsilon\|\nabla u(t)\|^2 - \varepsilon\beta\bigl(u(t), y(1,t)\bigr) + \varepsilon\int_\Omega u^2(x,t)\ln\bigl|u(x,t)\bigr|^\gamma\,dx \\ &= F'(t) + 2\varepsilon\|u_t(t)\|^2 - \varepsilon\beta\bigl(u(t), y(1,t)\bigr) - 2\varepsilon E(t) \\ &\quad + \frac{\varepsilon\gamma}{2}\|u(t)\|^2 + \omega\tau\|y(t)\|^2_{L^2(\Omega\times(0,1))}. \end{aligned} \tag{5.7}$$

By Young's inequality and (5.5), we get

$$\beta\bigl(u(t), y(1,t)\bigr) \leq |\beta|\left(\delta\|u(t)\|^2 + \frac{1}{4\delta}\|y(1,t)\|^2\right) \leq \delta|\beta|\|u(t)\|^2 + \frac{|\beta|}{4\delta C_2}F'(t).$$

Adapting this to (5.7) and using (5.4) and (5.2), we get

$$\begin{aligned} G'(t) &\geq \left(1 - \frac{\varepsilon|\beta|}{4\delta C_2}\right)F'(t) + 2\varepsilon\|u_t(t)\|^2 + \left(\frac{\varepsilon\gamma}{2} - \varepsilon|\beta|\delta\right)\|u(t)\|^2 \\ &\quad + 2\varepsilon F(t) - 2\varepsilon\zeta E_1 + \omega\tau\|y(t)\|^2_{L^2(\Omega\times(0,1))} \\ &\geq \left(1 - \frac{\varepsilon|\beta|}{4\delta C_2}\right)F'(t) + 2\varepsilon\|u_t(t)\|^2 + \varepsilon\left((1-\zeta)\frac{\gamma}{2} - |\beta|\delta\right)\|u(t)\|^2 \\ &\quad + 2\varepsilon F(t) + \omega\tau\|y(t)\|^2_{L^2(\Omega\times(0,1))}. \end{aligned} \tag{5.8}$$

First, we fix $\delta > 0$ such that $(1-\zeta)\frac{\gamma}{2} - |\beta|\delta > 0$, then choose $\varepsilon > 0$ sufficiently small so that $1 - \frac{\varepsilon|\beta|}{4\delta C_2} > 0$. Then we have from (5.5)

$$G'(t) \geq c_8\bigl(F(t) + \|u_t(t)\|^2 + \|u(t)\|^2\bigr) \geq 0. \tag{5.9}$$

On the other hand, we can easily see that

$$G(t) \le c_9\bigl(F(t) + \|u_t(t)\|^2 + \|u(t)\|^2\bigr). \tag{5.10}$$

Let us take $\varepsilon > 0$ sufficiently small again to get

$$G(0) = F(0) + \varepsilon(u_0, u_1) + \frac{\varepsilon\alpha}{2}\|u_0\|^2 > 0. \tag{5.11}$$

Then we obtain from (5.9) and (5.11)

$$G(t) \ge G(0) > 0.$$

From (5.9) and (5.10), we observe

$$G'(t) \ge c_{10}G(t),$$

and hence

$$G(t) \ge e^{c_{10}t}G(0) > 0.$$

Thus, $G(t)$ blows up at infinity. □

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