Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Lie symmetry reductions and conservation laws for fractional order coupled KdV system

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Abstract

Lie symmetry analysis is achieved on a new system of coupled KdV equations with fractional order, which arise in the analysis of several problems in theoretical physics and numerous scientific phenomena. We determine the reduced fractional ODE system corresponding to the governing factional PDE system.

In addition, we develop the conservation laws for the system of fractional order coupled KdV equations.

MSC: 35R11; 76M60

Keywords: Reduced fractional system; Coupled KdV system; Riemann–Liouville derivative; Lie symmetry method; Conservation laws

1 Introduction

Fractional partial differential equations (FPDEs) have a significant role to play in various fields such as chemistry, physics, fluid dynamics and biology, therefore obtaining solutions of such FPDEs is unavoidable [1, 2]. There are many numerical and theoretical methods for solving fractional order differential equations [1–4].

The Lie symmetry technique is one of the most useful techniques to conclude to solutions of nonlinear FPDEs, generally, Lie symmetries might be used to reduce the order of the original equation (system of equations) as well as the number of independent variables [5–11].

Lie symmetry analysis and conservation low have been applied to different type of fractional PDEs such as the time-fractional Caudrey–Dodd–Gibbon–Sawada–Kotera equation, time-fractional third-order evolution equation, the space-time-fractional nonlinear evolution equations, the time-fractional modified Zakharov–Kuznetsov equation, the time-fractional generalized Burgers–Huxley equation and the time-fractional dispersive long-wave equation [12–17]. In [8] the new coupled KdV system

$$\begin{cases} u_t + u_{xxx} + 3uu_x + 3ww_x = 0, \\ v_t + v_{xxx} + 3vv_x + 3ww_x = 0, \\ w_t + w_{xxx} + \frac{3}{2}(uw)_x + \frac{3}{2}(vw)_x = 0, \end{cases} \tag{1}$$

was derived and examined by the Lie symmetry method.

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Unlike the case of ordinary partial differential equations (PDEs) symmetries of FPDEs have not considered comprehensively. The study of FPDEs through symmetries is remarkable and interesting [18–22].

In this paper, we consider the new coupled KdV system (1) of fractional order given by

$$\begin{cases} D_t^\alpha u + u_{xxx} + 3uu_x + 3ww_x = 0, \\ D_t^\alpha v + v_{xxx} + 3vv_x + 3ww_x = 0, \\ D_t^\alpha w + w_{xxx} + \frac{3}{2}(uw)_x + \frac{3}{2}(vw)_x = 0, \end{cases} \tag{2}$$

where $\alpha \in (0,2)$.

The article is organized as follows. In Sect. 2, some definitions and properties of Lie group scheme to analysis of (2) are given. In Sect. 3, we find Lie point symmetries of system(2) and reduced system of this system. The conservation laws of (2) are obtained in Sect. 4. Discussion and conclusions are summarized in Sect. 5.

2 The symmetry group analysis of (2)

In this section, we briefly review some key definitions and properties of the fractional Lie group scheme to obtain infinitesimal function of the FPDE system.

Definition 1 The Riemann–Liouville fractional derivative of order α [1, 2] is defined by

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{\partial^n u(x,t)}{\partial t^n}; & \alpha = n \in \mathbb{N}, \\ \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{u(x,\tau)}{(t-\tau)^{\alpha+1-n}} \, \mathrm{d}\tau; & n-1 < \alpha < n. \end{cases}$$

For a fractional PDE system like (2) with two independent variables we have

$$\begin{cases} \frac{\partial^\alpha \mathbf{u}(x,t)}{\partial t^\alpha} = F(x,t,\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{u}_{(1)},\mathbf{v}_{(1)},\mathbf{w}_{(1)},\ldots), \\ \frac{\partial^\alpha \mathbf{v}(x,t)}{\partial t^\alpha} = G(x,t,\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{u}_{(1)},\mathbf{v}_{(1)},\mathbf{w}_{(1)},\ldots), \\ \frac{\partial^\alpha \mathbf{w}(x,t)}{\partial t^\alpha} = H(x,t,\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{u}_{(1)},\mathbf{v}_{(1)},\mathbf{w}_{(1)},\ldots), \quad 0 < \alpha < 2. \end{cases} \tag{3}$$

Throughout the article we use $(\bar{x}, \bar{\mathbf{u}})$ instead of $(x,t,\mathbf{u},\mathbf{v},\mathbf{w})$.

Assume that (3) is invariant under the one parameter Lie group of infinitesimal transformations,

$$\begin{aligned} t^\star &= t + \epsilon\tau(\bar{x},\bar{\mathbf{u}}) + O(\epsilon^2), \\ x^\star &= x + \epsilon\xi(\bar{x},\bar{\mathbf{u}}) + O(\epsilon^2), \\ \mathbf{u}^\star &= \mathbf{u} + \epsilon\eta^u(\bar{x},\bar{\mathbf{u}}) + O(\epsilon^2), \\ \mathbf{v}^\star &= \mathbf{v} + \epsilon\eta^v(\bar{x},\bar{\mathbf{u}}) + O(\epsilon^2), \\ \mathbf{w}^\star &= \mathbf{w} + \epsilon\eta^w(\bar{x},\bar{\mathbf{u}}) + O(\epsilon^2), \\ D_{t^\star}^\alpha \mathbf{u}^\star &= D_t^\alpha \mathbf{u} + \epsilon\eta^{(\alpha)\mathbf{u}}(\bar{x},\bar{\mathbf{u}}) + O(\epsilon^2), \\ D_{t^\star}^\alpha \mathbf{v}^\star &= D_t^\alpha \mathbf{v} + \epsilon\eta^{(\alpha)\mathbf{v}}(\bar{x},\bar{\mathbf{u}}) + O(\epsilon^2), \\ D_{t^\star}^\alpha \mathbf{w}^\star &= D_t^\alpha \mathbf{w} + \epsilon\eta^{(\alpha)\mathbf{w}}(\bar{x},\bar{\mathbf{u}}) + O(\epsilon^2), \end{aligned}$$

$$\begin{aligned}
&\frac{\partial^j \mathbf{u}^\star}{\partial x^{\star j}} = \frac{\partial^j \mathbf{u}}{\partial x^j} + \epsilon \eta^{(j)\mathbf{u}}(\bar{x}, \bar{\mathbf{u}}) + O(\epsilon^2), \\
&\frac{\partial^j \mathbf{v}^\star}{\partial x^{\star j}} = \frac{\partial^j \mathbf{v}}{\partial x^j} + \epsilon \eta^{(j)\mathbf{v}}(\bar{x}, \bar{\mathbf{u}}) + O(\epsilon^2), \\
&\frac{\partial^j \mathbf{w}^\star}{\partial x^{\star j}} = \frac{\partial^j \mathbf{w}}{\partial x^j} + \epsilon \eta^{(j)\mathbf{w}}(\bar{x}, \bar{\mathbf{u}}) + O(\epsilon^2), \quad j = 1, 2, \ldots,
\end{aligned} \tag{4}$$

where ξ, τ, $\eta^{\mathbf{u}}$, $\eta^{\mathbf{v}}$, $\eta^{\mathbf{w}}$ are infinitesimals and $\eta^{(\alpha)\mathbf{u}}$, $\eta^{(\alpha)\mathbf{v}}$, $\eta^{(\alpha)\mathbf{w}}$, $\eta^{(j)\mathbf{u}}$, $\eta^{(j)\mathbf{v}}$, $\eta^{(j)\mathbf{w}}$ are extended infinitesimals. ϵ is the group parameter.

According to Lie's algorithm, the infinitesimal generator of (2) is given by

$$\begin{aligned}
X = {}& \xi(\bar{x}, \bar{\mathbf{u}})\frac{\partial}{\partial x} + \tau(\bar{x}, \bar{\mathbf{u}})\frac{\partial}{\partial t} + \eta^{\mathbf{u}}(\bar{x}, \bar{\mathbf{u}})\frac{\partial}{\partial \mathbf{u}} + \eta^{\mathbf{v}}(\bar{x}, \bar{\mathbf{u}})\frac{\partial}{\partial \mathbf{v}} \\
& + \eta^{\mathbf{w}}(\bar{x}, \bar{\mathbf{u}})\frac{\partial}{\partial \mathbf{w}}.
\end{aligned} \tag{5}$$

The coupled KdV system of fractional order has at most αth-order derivatives, therefore, the α-prolongation of the generator should be considered in the form

$$\begin{aligned}
X^{(\alpha)} = {}& \xi(\bar{x}, \bar{\mathbf{u}})\frac{\partial}{\partial x} + \tau(\bar{x}, \bar{\mathbf{u}})\frac{\partial}{\partial t} + \eta^{\mathbf{u}}(\bar{x}, \bar{\mathbf{u}})\frac{\partial}{\partial \mathbf{u}} + \eta^{\mathbf{v}}(\bar{x}, \bar{\mathbf{u}})\frac{\partial}{\partial \mathbf{v}} \\
& + \eta^{\mathbf{w}}(\bar{x}, \bar{\mathbf{u}})\frac{\partial}{\partial w} + \eta_i^{(1)\mathbf{u}}(x, t, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{u}_{(i)}, \mathbf{v}_{(i)}, \mathbf{w}_{(i)})\frac{\partial}{\partial \mathbf{u}_i} \\
& + \eta_i^{(1)\mathbf{v}}(\bar{x}, \bar{\mathbf{u}}, \mathbf{u}_{(i)}, \mathbf{v}_{(i)}, \mathbf{w}_{(i)})\frac{\partial}{\partial v_i} + \eta_i^{(1)\mathbf{w}}(\bar{x}, \bar{\mathbf{u}}, \mathbf{u}_{(i)}, \mathbf{v}_{(i)}, \mathbf{w}_{(i)})\frac{\partial}{\partial \mathbf{w}_i} + \cdots \\
& + \eta_{i_1\cdots i_k}^{(k)\mathbf{u}}(\bar{x}, \bar{\mathbf{u}}, \mathbf{u}_{(1)}, \mathbf{v}_{(1)}, \mathbf{w}_{(1)}, \ldots, \mathbf{u}_{(k)}, \mathbf{v}_{(k)}, \mathbf{w}_{(k)})\frac{\partial}{\partial \mathbf{u}_{i_1,\ldots,i_k}} \\
& + \eta_{i_1\cdots i_k}^{(k)\mathbf{v}}(\bar{x}, \bar{\mathbf{u}}, \mathbf{u}_{(1)}, \mathbf{v}_{(1)}, \mathbf{w}_{(1)}, \ldots, \mathbf{u}_{(k)}, \mathbf{v}_{(k)}, \mathbf{w}_{(k)})\frac{\partial}{\partial \mathbf{v}_{i_1,\ldots,i_k}} \\
& + \eta_{i_1\cdots i_k}^{(k)\mathbf{w}}(\bar{x}, \bar{\mathbf{u}}, \mathbf{u}_{(1)}, \mathbf{v}_{(1)}, \mathbf{w}_{(1)}, \ldots, \mathbf{u}_{(k)}, \mathbf{v}_{(k)}, \mathbf{w}_{(k)})\frac{\partial}{\partial \mathbf{w}_{i_1,\ldots,i_k}} \\
& + \eta_t^{(\alpha)\mathbf{u}}(\bar{x}, \bar{\mathbf{t}}, \ldots, \mathbf{u}_{(\alpha)}, \ldots)\frac{\partial}{\partial \mathbf{u}_t^\alpha} + \eta_t^{(\alpha)\mathbf{v}}(\bar{x}, \bar{\mathbf{u}}, \ldots, \mathbf{v}_{(\alpha)}, \ldots)\frac{\partial}{\partial \mathbf{v}_t^\alpha} \\
& + \eta_t^{(\alpha)\mathbf{w}}(\bar{x}, \bar{\mathbf{u}}, \ldots, \mathbf{w}_{(\alpha)}, \ldots)\frac{\partial}{\partial \mathbf{w}_t^\alpha},
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
&\eta_t^{(\alpha)\mathbf{u}} = D_{1t}^\alpha(\eta^{\mathbf{u}}) + \xi D_{1t}^\alpha(\mathbf{u}_x) - D_{1t}^\alpha(\xi \mathbf{u}_x) + D_{1t}^\alpha(D_{1t}(\tau)u) - D_{1t}^{\alpha+1}(\tau u) + \tau D_{1t}^{\alpha+1}\mathbf{u}, \\
&\eta_t^{(\alpha)\mathbf{v}} = D_{2t}^\alpha(\eta^{\mathbf{v}}) + \xi D_{2t}^\alpha(\mathbf{v}_x) - D_{2t}^\alpha(\xi \mathbf{v}_x) + D_{2t}^\alpha(D_{2t}(\tau)v) - D_{2t}^{\alpha+1}(\tau v) + \tau D_{2t}^{\alpha+1}v, \\
&\eta_t^{(\alpha)\mathbf{w}} = D_{3t}^\alpha(\eta^{w}) + \xi D_{3t}^\alpha(\mathbf{w}_x) - D_{3t}^\alpha(\xi \mathbf{w}_x) + D_{3t}^\alpha(D_{3t}(\tau)w) - D_{3t}^{\alpha+1}(\tau w) + \tau D_{3t}^{\alpha+1}\mathbf{w}.
\end{aligned}$$

D_{1t}, D_{2t} and D_{3t} are the total derivative operators defined as

$$D_{1t} = \frac{\partial}{\partial t} + \mathbf{u}_t\frac{\partial}{\partial \mathbf{u}} + \mathbf{u}_{xt}\frac{\partial}{\partial \mathbf{u}_x} + \mathbf{u}_{tt}\frac{\partial}{\partial \mathbf{u}_t} + \mathbf{u}_{xxt}\frac{\partial}{\partial \mathbf{u}_{xx}} + \cdots,$$

$$D_{2t} = \frac{\partial}{\partial t} + \mathbf{v}_t \frac{\partial}{\partial \mathbf{v}} + \mathbf{v}_{xt} \frac{\partial}{\partial \mathbf{v}_x} + \mathbf{v}_{tt} \frac{\partial}{\partial \mathbf{v}_t} + \mathbf{v}_{xxt} \frac{\partial}{\partial \mathbf{v}_{xx}} + \cdots,$$
$$D_{3t} = \frac{\partial}{\partial t} + \mathbf{w}_t \frac{\partial}{\partial \mathbf{w}} + \mathbf{w}_{xt} \frac{\partial}{\partial \mathbf{w}_x} + \mathbf{w}_{tt} \frac{\partial}{\partial \mathbf{w}_t} + \mathbf{w}_{xxt} \frac{\partial}{\partial \mathbf{w}_{xx}} + \cdots.$$

Definition 2 A vector X given by (5) is said to be a Lie point symmetry vector field for system (2), if

$$X^{(\alpha)}\left[D_t^\alpha \mathbf{u} + \mathbf{u}_{xxx} + 3u\mathbf{u}_x + 3w\mathbf{w}_x\right] = 0,$$
$$X^{(\alpha)}\left[D_t^\alpha \mathbf{v} + \mathbf{v}_{xxx} + 3v\mathbf{v}_x + 3w\mathbf{w}_x\right] = 0,$$
$$X^{(\alpha)}\left[D_t^\alpha \mathbf{w} + \mathbf{w}_{xxx} + \frac{3}{2}(\mathbf{uw})_x + \frac{3}{2}(\mathbf{vw})_x\right] = 0.$$

3 Lie symmetries and similarity reductions for (2)

We apply the α-prolongation of $X^{(\alpha)}$ to Eq. (2). It gives the following claim.

Theorem 1 *Lie symmetry group of* (2) *is spanned by the following vector fields*:

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \alpha x \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t} - 2\alpha \mathbf{u} \frac{\partial}{\partial \mathbf{u}} - 2\alpha \mathbf{v} \frac{\partial}{\partial \mathbf{v}} - 2\alpha \mathbf{w} \frac{\partial}{\partial \mathbf{w}}. \tag{7}$$

Proof Let us consider the one parameter Lie group of infinitesimal transformation in x, t, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ given by

$$t \longrightarrow t + \epsilon \xi^t(\bar{x}, \bar{\mathbf{u}}),$$
$$x \longrightarrow x + \epsilon \xi^x(\bar{x}, \bar{\mathbf{u}}),$$
$$\mathbf{v} \longrightarrow \mathbf{v} + \epsilon \eta^{\mathbf{v}}(\bar{x}, \bar{\mathbf{u}}),$$
$$\mathbf{u} \longrightarrow \mathbf{u} + \epsilon \eta^{\mathbf{u}}(\bar{x}, \bar{\mathbf{u}}),$$
$$\mathbf{w} \longrightarrow \mathbf{w} + \epsilon \eta^{\mathbf{w}}(\bar{x}, \bar{\mathbf{u}}),$$

now we find the coefficient functions ξ, τ, $\eta^{\mathbf{u}}$, $\eta^{\mathbf{v}}$, $\eta^{\mathbf{w}}$.

By applying the $X^{(\alpha)}$ to both sides of (2),we have

$$X^{(\alpha)}\left[D_t^\alpha u + \mathbf{u}_{xxx} + 3u\mathbf{u}_x + 3w\mathbf{w}_x\right] = 0,$$
$$X^{(\alpha)}\left[D_t^\alpha v + \mathbf{v}_{xxx} + 3v\mathbf{v}_x + 3w\mathbf{w}_x\right] = 0,$$
$$X^{(\alpha)}\left[D_t^\alpha w + \mathbf{w}_{xxx} + \frac{3}{2}(uw)_x + \frac{3}{2}(vw)_x\right] = 0. \tag{8}$$

We obtain the Lie point symmetries by expanding (8), and solving the resulting system using Maple as follows:

$$\xi(\bar{x}, \bar{\mathbf{u}}) = c_1 + c_2 \alpha x, \qquad \tau(\bar{x}, \bar{\mathbf{u}}) = 3c_2 t,$$
$$\eta^{\mathbf{u}}(\bar{x}, \bar{\mathbf{u}}) = -2c_2\alpha \mathbf{u}, \qquad \eta^{\mathbf{v}}(\bar{x}, \bar{\mathbf{u}}) = -2c_2\alpha \mathbf{v}, \qquad \eta^{\mathbf{w}}(\bar{x}, \bar{\mathbf{u}}) = -2c_2\alpha \mathbf{w},$$

here c_1 and c_2 are arbitrary constants. Thus, the corresponding vector fields are

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \alpha x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2\alpha u\frac{\partial}{\partial \mathbf{u}} - 2\alpha v\frac{\partial}{\partial \mathbf{v}} - 2\alpha w\frac{\partial}{\partial \mathbf{w}}. \qquad \square$$

Here we want to obtain symmetry reductions of (2), then the system (2) transforms into a system of fractional ODE.

In order to solve the following associated Lagrange equations:

$$\frac{dx}{\xi(\bar{x},\bar{\mathbf{u}})} = \frac{dt}{\tau(\bar{x},\bar{\mathbf{u}})} = \frac{d\mathbf{u}}{\eta^{\mathbf{u}}(\bar{x},\bar{\mathbf{u}})} = \frac{d\mathbf{v}}{\eta^{\mathbf{v}}(\bar{x},\bar{\mathbf{u}})} = \frac{d\mathbf{w}}{\eta^{\mathbf{w}}(\bar{x},\bar{\mathbf{u}})}.$$

We consider the following cases.

- Case 1: $X_1 = \frac{\partial}{\partial x}$.

  In this case the symmetry X_1 gives rise to the group-invariant solution:

$$r = t, \qquad \mathbf{u} = F(r), \qquad \mathbf{v} = G(r), \qquad \mathbf{w} = H(r), \tag{9}$$

  substituting (9) into (2) results in the fact that $F(r)$, $G(r)$ and $H(r)$ fulfill the following differential equations:

$$\frac{d^\alpha F(t)}{dt^\alpha} = 0, \qquad \frac{d^\alpha G(t)}{dt^\alpha} = 0, \qquad \frac{d^\alpha H(t)}{dt^\alpha} = 0.$$

  By using a Laplace transformation we get

$$F(t) = \frac{k_1}{\Gamma(\alpha)}t^{\alpha-1}, \qquad G(t) = \frac{k_2}{\Gamma(\alpha)}t^{\alpha-1}, \qquad H(t) = \frac{k_3}{\Gamma(\alpha)}t^{\alpha-1},$$

  where k_1, k_2 and k_3 are constant; therefore

$$\mathbf{u}(x,t) = \frac{k}{\Gamma(\alpha)}t^{\alpha-1}, \qquad \mathbf{v}(x,t) = \frac{k}{\Gamma(\alpha)}t^{\alpha-1}, \qquad \mathbf{w}(x,t) = \frac{k}{\Gamma(\alpha)}t^{\alpha-1}.$$

- Case 2: $X_2 = \alpha x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2\alpha \mathbf{u}\frac{\partial}{\partial \mathbf{u}} - 2\alpha \mathbf{v}\frac{\partial}{\partial \mathbf{v}} - 2\alpha \mathbf{w}\frac{\partial}{\partial \mathbf{w}}$.

  In this case, the group-invariant solution is

$$r = tx^{\frac{-3}{\alpha}}, \qquad \mathbf{u} = F(r)x^{-2}, \qquad \mathbf{v} = G(r)x^{-2}, \qquad \mathbf{w} = H(r)x^{-2}, \tag{10}$$

  substituting (10) into (2) leads to the following fractional ODE system:

$$\begin{cases} D_r^\alpha F + k_1F(r) + k_2rF'(r) + k_3r^2F''(r) + k_4r^3F^{(3)}(r) + k_5F^2(r) + k_6rF(r)F'(r) \\ \quad + k_7H^2(r) + k_8rH(r)H'(r) = 0, \\ D_r^\alpha G + k_1'G(r) + k_2'rG'(r) + k_3'r^2G''(r) + k_4'r^3G^{(3)}(r) + k_5'G^2(r) + k_6'rG(r)G'(r) \\ \quad + k_7'H^2(r) + k_8'rH(r)H'(r) = 0, \\ D_r^\alpha H + k_1''H(r) + k_2''rH'(r) + k_3''r^2H''(r) + k_4''r^3H^{(3)}(r) + k_5''F(r)H(r) + k_6''rF'(r)H(r) \\ \quad + k_7''rF(r)H''(r) + k_8''G(r)H(r) + k_9''rG'(r)H(r) + k_{10}''rG(r)H''(r) = 0, \end{cases}$$

where $k_i = h_i(\alpha)$, $k_i' = g_i(\alpha)$, $(i = 1,2,\dots,8)$ and $k_j'' = m_j(\alpha)$, $(j = 1,2,\dots,10)$ are constants.

Note. For $\alpha = 1$, the Lie point symmetries provide similar results to those obtained by Adem and Khalique in [8].

4 Conservation laws

Now, we construct conservation laws for system (2) by using the Lie point symmetry (7).

The vectors $C_i = (C_i^t, C_i^x)$, $(i = 1, 2, 3)$ are called conserved vectors for system (2), if they satisfy the conservation equations,

$$\begin{aligned}
&D_t\big(C_1^t\big) + D_x\big(C_1^x\big)|_{D_t^\alpha \mathbf{u}+\mathbf{u}_{xxx}+3u\mathbf{u}_x+3\mathbf{w}\mathbf{w}_x=0} = 0,\\
&D_t\big(C_2^t\big) + D_x\big(C_2^x\big)|_{D_t^\alpha \mathbf{v}+\mathbf{v}_{xxx}+3v\mathbf{v}_x+3\mathbf{w}\mathbf{w}_x=0} = 0,\\
&D_t\big(C_3^t\big) + D_x\big(C_3^x\big)|_{D_t^\alpha \mathbf{w}+\mathbf{w}_{xxx}+\frac{3}{2}(\mathbf{u}\mathbf{w})_x+\frac{3}{2}(\mathbf{v}\mathbf{w})_x=0} = 0.
\end{aligned}$$

For system (2), a formal Lagrangian can be introduced as

$$\begin{aligned}
L ={}& \Lambda^1(x,t)\big[D_t^\alpha u + \mathbf{u}_{xxx} + 3u\mathbf{u}_x + 3w\mathbf{w}_x\big] + \Lambda^2(x,t)\big[D_t^\alpha v + \mathbf{v}_{xxx} + 3v\mathbf{v}_x + 3\mathbf{w}\mathbf{w}_x\big]\\
&+ \Lambda^3(x,t)\left[D_t^\alpha w + \mathbf{w}_{xxx} + \frac{3}{2}(\mathbf{u}\mathbf{w})_x + \frac{3}{2}(\mathbf{v}\mathbf{w})_x\right] = 0,
\end{aligned} \tag{11}$$

where $\Lambda^i(x,t), i = 1, 2, 3$, are new dependent variables.

The Euler–Lagrange operators are defined by

$$\begin{aligned}
\frac{\delta}{\delta u} &= \frac{\partial}{\partial \mathbf{u}} + \big(D_t^\alpha\big)^\star \frac{\partial}{\partial D_t^\alpha \mathbf{u}} - D_x\frac{\partial}{\partial \mathbf{u}_x} + D_x^2\frac{\partial}{\partial \mathbf{u}_{xx}} - D_x^3\frac{\partial}{\partial \mathbf{u}_{xxx}},\\
\frac{\delta}{\delta v} &= \frac{\partial}{\partial \mathbf{v}} + \big(D_t^\alpha\big)^\star \frac{\partial}{\partial D_t^\alpha \mathbf{v}} - D_x\frac{\partial}{\partial \mathbf{v}_x} + D_x^2\frac{\partial}{\partial \mathbf{v}_{xx}} - D_x^3\frac{\partial}{\partial \mathbf{v}_{xxx}},\\
\frac{\delta}{\delta w} &= \frac{\partial}{\partial \mathbf{w}} + \big(D_t^\alpha\big)^\star \frac{\partial}{\partial D_t^\alpha \mathbf{w}} - D_x\frac{\partial}{\partial \mathbf{w}_x} + D_x^2\frac{\partial}{\partial \mathbf{w}_{xx}} - D_x^3\frac{\partial}{\partial \mathbf{w}_{xxx}},
\end{aligned}$$

here $(D_t^\alpha)^\star$ is the adjoint operator of D_t^α.

For the RL-fractional operators

$$\big(D_t^\alpha\big)^\star = (-1)^n I_T^{n-\alpha}\big(D_t^n\big) =_t^C D_T^\alpha,$$

where

$$I_T^{n-\alpha} f(t,x) = \frac{1}{\Gamma(n-\alpha)}\int_t^\tau \frac{f(\tau,x)}{(\tau-t)^{1+\alpha-n}}\,\mathrm{d}\tau, \quad n = [\alpha] + 1.$$

The adjoint equations to the system (2) are written as

$$F_1^\star = \frac{\delta L}{\delta u} = 0, \qquad F_2^\star = \frac{\delta L}{\delta v} = 0, \qquad F_3^\star = \frac{\delta L}{\delta w} = 0. \tag{12}$$

Replacing the formal Lagrangian (11) into (12), we have

$$\begin{aligned}
F_1^\star &= \big(D_t^\alpha\big)^\star\Lambda^1 - 3u\Lambda_x^1 - \Lambda_{xxx}^1 + \frac{3}{2}\mathbf{w}_x\Lambda^3 - \frac{3}{2}w\Lambda_x^3 = 0,\\
F_2^\star &= \big(D_t^\alpha\big)^\star\Lambda^2 - 3v\Lambda_x^2 - \Lambda_{xxx}^2 + \frac{3}{2}\mathbf{w}_x\Lambda^3 - \frac{3}{2}w\Lambda_x^3 = 0,\\
F_3^\star &= \big(D_t^\alpha\big)^\star\Lambda^3 - 3w\Lambda_x^1 - 3w\Lambda_x^2 + \frac{3}{2}(\mathbf{u}_x + \mathbf{v}_x)\Lambda^3 - \frac{3}{2}(u+v)\Lambda_x^3 - \Lambda_{xxx}^3 = 0.
\end{aligned} \tag{13}$$

Since in the system(2), there are no fractional derivatives involved w.r.t. x, we have

$$X^{(\alpha)} + D_{1t}(\tau)L + D_{1x}(\xi)L = \mathbf{w}_i \frac{\partial}{\partial \mathbf{u}} + D_{1t}N_1^t + D_{1x}N_1^x,$$

$$X^{(\alpha)} + D_{2t}(\tau)L + D_{2x}(\xi)L = \mathbf{w}_i \frac{\partial}{\partial \mathbf{v}} + D_{2t}N_2^t + D_{2x}N_2^x,$$

$$X^{(\alpha)} + D_{3t}(\tau)L + D_{3x}(\xi)L = \mathbf{w}_i \frac{\partial}{\partial \mathbf{w}} + D_{3t}N_3^t + D_{3x}N_3^x,$$

where

$$W_i = (\eta^{\mathbf{u}} + \eta^{\mathbf{v}} + \eta^{\mathbf{w}}) - \xi_i(\mathbf{u}_x + \mathbf{v}_x + \mathbf{w}_x) - \tau_i(\mathbf{u}_t + \mathbf{v}_t + \mathbf{w}_t), \quad i = 1, 2,$$

are the Lie characteristic functions corresponding to the Lie symmetries X_1 and X_2.

If we have the RL-time-fractional derivative in the system (2) then the operators N^t are given by

$$N_1^t = \sum_{k=0}^{n-1} (-1)^k D_{1t}^{\alpha-1-k}(\mathbf{w}_i) D_{1t}^k \frac{\partial}{(\partial D_t^\alpha \mathbf{u})} - (-1)^n J\left(W_i, D_{1t}^n \frac{\partial}{(\partial D_t^\alpha u)}\right),$$

$$N_2^t = \sum_{k=0}^{n-1} (-1)^k D_{2t}^{\alpha-1-k}(W_i) D_{2t}^k \frac{\partial}{(\partial D_t^\alpha \mathbf{v})} - (-1)^n J\left(W_i, D_{2t}^n \frac{\partial}{(\partial D_t^\alpha v)}\right),$$

$$N_3^t = \sum_{k=0}^{n-1} (-1)^k D_{1t}^{\alpha-1-k}(W_i) D_{3t}^k \frac{\partial}{(\partial D_t^\alpha \mathbf{w})} - (-1)^n J\left(W_i, D_{3t}^n \frac{\partial}{(\partial D_t^\alpha w)}\right),$$

where J is the integral

$$J(f,g) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \int_t^\tau \frac{f(\tau,x)g(\mu,x)}{(\mu-\tau)^{\alpha+1-n}}\, \mathrm{d}\mu\, \mathrm{d}\tau;$$

the operators N^x are defined by

$$N_1^x = W_i \frac{\partial L}{\partial \mathbf{u}_x} + D_{1x}(W_i)\frac{\partial L}{\partial \mathbf{u}_{xx}} + D_{1x}^2(W_i)\frac{\partial L}{\partial \mathbf{u}_{xxx}},$$

$$N_2^x = W_i \frac{\partial L}{\partial \mathbf{v}_x} + D_{2x}(W_i)\frac{\partial L}{\partial \mathbf{v}_{xx}} + D_{2x}^2(W_i)\frac{\partial L}{\partial \mathbf{v}_{xxx}},$$

$$N_3^x = W_i \frac{\partial L}{\partial \mathbf{w}_x} + D_{3x}(W_i)\frac{\partial L}{\partial \mathbf{w}_{xx}} + D_{3x}^2(W_i)\frac{\partial L}{\partial \mathbf{w}_{xxx}}.$$

For any generator X of system (2), we have

$$\left(X^{(\alpha)}L + D_{1t}(\tau)L + D_{1x}(\xi)L\right)|_{D_t^\alpha u + \mathbf{u}_{xxx} + 3u\mathbf{u}_x + 3w\mathbf{w}_x = 0} = 0,$$

$$\left(X^{(\alpha)}L + D_{2t}(\tau)L + D_{2x}(\xi)L\right)|_{D_t^\alpha v + \mathbf{v}_{xxx} + 3v\mathbf{v}_x + 3w\mathbf{w}_x = 0} = 0,$$

$$\left(X^{(\alpha)}L + D_{3t}(\tau)L + D_{3x}(\xi)L\right)|_{D_t^\alpha w + \mathbf{w}_{xxx} + \frac{3}{2}(uw)_x + \frac{3}{2}(vw)_x = 0} = 0.$$

These equalities yield the conservation laws

$$D_{1t}(N_1^t L) + D_{1x}(N_1^x L) = 0,$$

$$D_{2t}\big(N_2^t L\big) + D_{2x}\big(N_2^x L\big) = 0,$$
$$D_{3t}\big(N_3^t L\big) + D_{3x}\big(N_3^x L\big) = 0.$$

For the case, when $\alpha \in (0,1)$, using N_i^t and N_i^x $(i = 1,2,3)$, one can get the components of the conserved vectors

$$C_1^t = (-1)^0 D_{1t}^{\alpha-1}(W_i) D_{1t}^0 \frac{\partial L}{\partial D_t^\alpha u} - (-1)^1 J\left(W_i, D_{1t}^1 \frac{\partial L}{\partial D_t^\alpha u}\right) = \Lambda^1 D_{1t}^{\alpha-1}(W_i) + J\big(W_i, \Lambda_t^1\big),$$
$$C_2^t = (-1)^0 D_{2t}^{\alpha-1}(W_i) D_{2t}^0 \frac{\partial L}{\partial D_t^\alpha v} - (-1)^1 J\left(W_i, D_{2t}^1 \frac{\partial L}{\partial D_t^\alpha v}\right) = \Lambda^2 D_{2t}^{\alpha-1}(W_i) + J\big(W_i, \Lambda_t^2\big),$$
$$C_3^t = (-1)^0 D_{3t}^{\alpha-1}(W_i) D_{3t}^0 \frac{\partial L}{\partial D_t^\alpha w} - (-1)^1 J\left(W_i, D_{3t}^1 \frac{\partial L}{\partial D_t^\alpha w}\right) = \Lambda^3 D_{1t}^{\alpha-1}(W_i) + J\big(W_i, \Lambda_t^3\big),$$

and

$$\begin{aligned} C_1^x &= W_i\left(\frac{\partial L}{\partial \mathbf{u}_x} - D_{1x}\frac{\partial L}{\partial \mathbf{u}_{xx}} + D_{1x}^2 \frac{\partial L}{\partial \mathbf{u}_{xxx}}\right) + D_{1x}(W_i)\left(\frac{\partial L}{\partial \mathbf{u}_{xx}} - D_{1x}\frac{\partial L}{\partial \mathbf{u}_{xxx}}\right) \\ &\quad + D_{1x}^2(W_i)\frac{\partial}{\partial \mathbf{u}_{xxx}} \\ &= W_i\left(3u\Lambda^1 + \frac{3}{2}w\Lambda^3 + \Lambda_{xx}^1\right) - D_{1x}(W_i)\Lambda_x^1 + D_{1x}^2(W_i)\Lambda^1, \end{aligned} \tag{14}$$
$$\begin{aligned} C_2^x &= W_i\left(\frac{\partial L}{\partial \mathbf{v}_x} - D_{2x}\frac{\partial L}{\partial \mathbf{v}_{xx}} + D_{2x}^2 \frac{\partial L}{\partial \mathbf{v}_{xxx}}\right) + D_{2x}(W_i)\left(\frac{\partial L}{\partial \mathbf{v}_{xx}} - D_{2x}\frac{\partial L}{\partial \mathbf{v}_{xxx}}\right) \\ &\quad + D_{2x}^2(W_i)\frac{\partial}{\partial \mathbf{v}_{xxx}} \\ &= \mathbf{w}_i\left(3v\Lambda^2 + \frac{3}{2}w\Lambda^3 + \Lambda_{xx}^2\right) - D_{2x}(W_i)\Lambda_x^2 + D_{2x}^2(W_i)\Lambda^2, \end{aligned} \tag{15}$$
$$\begin{aligned} C_3^x &= W_i\left(\frac{\partial L}{\partial \mathbf{w}_x} - D_{3x}\frac{\partial L}{\partial \mathbf{w}_{xx}} + D_{3x}^2 \frac{\partial L}{\partial \mathbf{w}_{xxx}}\right) + D_{3x}(W_i)\left(\frac{\partial L}{\partial \mathbf{w}_{xx}} - D_{3x}\frac{\partial L}{\partial \mathbf{w}_{xxx}}\right) \\ &\quad + D_{3x}^2(W_i)\frac{\partial}{\partial \mathbf{w}_{xxx}} \\ &= W_i\left(3w\Lambda^1 + 3w\Lambda^2 + \frac{3}{2}u\Lambda^3 + \frac{3}{2}v\Lambda^3 + \Lambda_{xx}^3\right) - D_{3x}(W_i)\Lambda_x^3 + D_{3x}^2(W_i)\Lambda^3, \end{aligned} \tag{16}$$

where $i = 1,2$ and the functions W_i are

$$\begin{aligned} W_1 &= -(\mathbf{u}_x + \mathbf{v}_x + \mathbf{w}_x), \\ W_2 &= -2\alpha u - 2\alpha v - 2\alpha w - \alpha x(\mathbf{u}_x + \mathbf{v}_x + \mathbf{w}_x) - 3t(\mathbf{u}_t + \mathbf{v}_t + \mathbf{w}_t). \end{aligned} \tag{17}$$

Also, when $\alpha \in (1,2)$, we get the components of the conserved vectors

$$C_1^t = \Lambda^1 D_{1t}^{\alpha-1}(\mathbf{w}_i) + J\big(W_i, \Lambda_t^1\big) - \Lambda_t^1 D_{1t}^{\alpha-2}(W_i) - J\big(W_i, \Lambda_{tt}^1\big),$$
$$C_2^t = \Lambda^2 D_{2t}^{\alpha-1}(W_i) + J\big(W_i, \Lambda_t^2\big) - \Lambda_t^2 D_{2t}^{\alpha-2}(W_i) - J\big(W_i, \Lambda_{tt}^2\big),$$
$$C_3^t = \Lambda^3 D_{3t}^{\alpha-1}(W_i) + J\big(W_i, \Lambda_t^3\big) - \Lambda_t^3 D_{3t}^{\alpha-2}(W_i) - J\big(W_i, \Lambda_{tt}^3\big),$$

where $i = 1, 2$ and the functions W_i in the form (17); also the conserved vectors C_1^x, C_2^x, C_3^x coincide with (14), (15) and (16).

5 Conclusions

In this paper, Lie symmetries and conservation laws have been studied for fractional order coupled KdV system (2). First, we obtained the fractional Lie point symmetries to the KdV system (2) with Riemann–Liouville derivative and we have shown that system (2) can be reduced to a nonlinear system of FDEs. Finally, conservation laws are constructed for system (2), the calculated conserved vectors, might be used for creating the particular solutions for the KdV system by the given method in [23, 24].

Acknowledgements
We would like thank the editor and reviewers for their time spent on reviewing this manuscript and their comments, helping us to improve the article.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 19 May 2020 Accepted: 29 November 2020 Published online: 11 December 2020

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