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Some basic properties and fundamental relations for discrete Muckenhoupt and Gehring classes

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Abstract

In this paper, we prove some basic properties of the discrete Muckenhoupt class $\mathcal{A}^p$ and the discrete Gehring class $\mathcal{G}^q$. These properties involve the self-improving properties and the fundamental transitions and inclusions relations between the two classes.

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1 Introduction

The study of the discrete analogues in harmonic analysis became an active field of research in recent years. For example, the study of regularity and boundedness of discrete operators on l^p analogues for L^p-regularity and higher summability of sequences has been considered by some authors, see for example [3–7, 24, 25, 28] and the references cited therein. Whereas some results from Euclidean harmonic analysis admit an obvious variant in the discrete setting, others do not. The main challenge in such studies is that there are no general methods to study these questions and the discrete operators may behave differently from their continuous counterparts as is exhibited by the discrete spherical maximal operator [19]. We confine ourselves in proving some basic properties of discrete Muckenhoupt and discrete Gehring classes. For properties and the structure of classical Muckenhoupt and Gehring classes (in integral forms), the relation between them and their applications in mathematical analysis, we refer the reader to the papers [1, 2, 8–18, 20–23, 27, 29] and the references cited therein. Throughout this paper, we assume that $1 < p < \infty$ and $\mathbb{I}$ stands for a subset of $\mathbb{Z}_+$ of the form $\mathbb{I} = \{1, 2, \ldots, N\}$, where N is a fixed positive integer and n be a positive integer in $\mathbb{I}$. A discrete weight on $\mathbb{Z}_+ = \{1, 2, \ldots\}$ is a sequence $u = \{u(n)\}_{n=1}^{\infty}$ of positive real numbers. The discrete positive weight u is said to belong to the discrete Muckenhoupt class $\mathcal{A}^p(A)$ on the interval $\mathbb{I}$, for $p > 1$ and $A > 1$ (independent of p), if, for

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every $n \in \mathbb{I}$,

$$\frac{1}{n}\sum_{k=1}^{n} u(k) \le A \left(\frac{1}{n}\sum_{k=1}^{n} u^{\frac{1}{1-p}}(k) \right)^{1-p}.$$

For a given exponent $p > 1$, we define the $\mathcal{A}^p$-norm of the discrete weight u by the following quantity:

$$[u]_{\mathcal{A}^p} := \sup_{n\in\mathbb{I}} \left(\frac{1}{n}\sum_{k=1}^{n} u(k) \right) \left(\frac{1}{n}\sum_{k=1}^{n} u^{\frac{-1}{p-1}}(k) \right)^{p-1}, \tag{1}$$

where the supremum is taken over all $n \in \mathbb{I}$. The discrete weight u is said to belong to the discrete Muckenhoupt class $\mathcal{A}^1(A)$ if, for every $n \in \mathbb{I}$,

$$\frac{1}{n}\sum_{k=1}^{n} u(k) \le A \inf_{1\le k\le n} u(k),$$

for $A > 1$, holds and we define the $\mathcal{A}^1$-norm by the following quantity:

$$[u]_{\mathcal{A}^1} = \sup_{n\in\mathbb{I}} \frac{1}{n} \left(\frac{1}{\inf_{1\le k\le n} u(k)} \sum_{k=1}^{n} u(k) \right).$$

The discrete weight u is said to belong to the discrete Muckenhoupt class $\mathcal{A}^\infty(A)$ if, for every $n \in \mathbb{I}$,

$$\left(\frac{1}{n}\sum_{k=1}^{n} u(k) \right) \left(\exp \frac{1}{n}\sum_{k=1}^{n} \log \frac{1}{u(k)} \right) \le A,$$

for $A > 1$, and we define the $\mathcal{A}^\infty$-norm by the following quantity:

$$[u]_{\mathcal{A}^\infty} := \sup_{n\in\mathbb{I}} \left(\frac{1}{n}\sum_{k=1}^{n} u(k) \right) \left(\exp \frac{1}{n}\sum_{k=1}^{n} \log \frac{1}{u(k)} \right),$$

where the supremum is taken over all $n \in \mathbb{I}$. The discrete positive weight u is said to belong to the discrete Gehring class $\mathcal{G}^q(G)$ on the interval $\mathbb{I} \subset \mathbb{Z}^+$, for $q > 1$ and $G > 1$ (independent of q), if, for every $n \in \mathbb{I}$,

$$\left(\frac{1}{n}\sum_{k=1}^{n} u^q(k) \right)^{1/q} \le G\frac{1}{n}\sum_{k=1}^{n} u(k). \tag{2}$$

For a given exponent $q > 1$, we define the $\mathcal{G}^q$-norm by the following quantity:

$$[u]_{\mathcal{G}^q} := \sup_{n\in\mathbb{I}} \left[\left(\frac{1}{n}\sum_{k=1}^{n} u(k) \right)^{-1} \left(\frac{1}{n}\sum_{k=1}^{n} u^q(k) \right)^{\frac{1}{q}} \right]^{\frac{q}{q-1}},$$

where the supremum is taken over all $n \in \mathbb{I}$. The discrete weight u is said to belong to the discrete Gehring class $\mathcal{G}^{\infty}(G)$ if for every $n \in \mathbb{I}$

$$\sup_{1 \leq k \leq n} u(k) \leq G \frac{1}{n} \sum_{k=1}^{n} u(k),$$

for $G > 0$. The discrete weight is said to belong to the discrete Gehring class $\mathcal{G}^{1}(G)$ if, for every $n \in \mathbb{I}$,

$$\exp\left(\frac{1}{n} \sum_{k=1}^{n} \frac{u(k)}{\frac{1}{n}\sum_{k=1}^{n} u(k)} \log \frac{u(k)}{\frac{1}{n}\sum_{k=1}^{n} u(k)} \right) \leq G,$$

where $G > 1$ and we define the $\mathcal{G}^{1}$-norm by the following quantity:

$$[u]_{\mathcal{G}^1} := \sup_{n \in \mathbb{I}}\left[\exp\left(\frac{1}{n} \sum_{k=1}^{n} \frac{u(k)}{\frac{1}{n}\sum_{k=1}^{n} u(k)} \log \frac{u(k)}{\frac{1}{n}\sum_{k=1}^{n} u(k)} \right) \right],$$

where the supremum is taken over all $n \in \mathbb{I}$. The objective of this paper is classified as follows:

(1) Prove some basic properties of the discrete Muckenhoupt class $\mathcal{A}^p$.
(2) Prove some basic properties of the discrete Gehring class $\mathcal{G}^q$.
(3) Prove some fundamental relations between $\mathcal{A}^p$ and $\mathcal{G}^q$.
(4) Prove some fundamental relations between $\mathcal{A}^{\infty}$ and $\mathcal{G}^q$ and their norms.

The paper is organized as follows: In the next section, we state and prove some basic lemmas that are needed in the proofs of the main results. In Sect. 3, we present properties of the Muckenhoupt class which include the self-improving property. In Sect. 4, we prove the basic properties of the Gehring class which include also the self-improving property. In Sect. 5, we prove the transition and inclusion relations between the two classes which give embedding relations between $\mathcal{A}^p$ and $\mathcal{G}^q$ and also relations between $\mathcal{A}^{\infty}$ and $\mathcal{G}^q$ and their norms.

2 Basic lemmas

In what follows, all sequences in the statements of the theorems are assumed to be positive sequences defined on $\mathbb{I} \subseteq \mathbb{Z}_+$ and use the conventions $0 \cdot \infty = 0$ and $0/0 = 0$ and $\sum_{k=m}^{b} y(k) = 0$, whenever $m > b$ and

$$\Delta\left(\sum_{s=a}^{k-1} y(s) \right) = y(k), \qquad \sum_{s=a}^{k-1} \Delta y(s) = y(k) - y(a).$$

The classical Hölder inequality is given by

$$\sum_{k=1}^{n} |uv| \leq \left[\sum_{k=1}^{n} |u|^p \right]^{1/p} \left[\sum_{k=1}^{n} |v|^q \right]^{1/q}, \tag{3}$$

where $1/p + 1/q = 1$, and $p, q > 1$. This inequality is reversed for $0 < p < 1$ and if $p < 0$ or $q < 0$. For instance the inequality

$$\sum_{k=1}^{n} |uv| \geq \left[\sum_{k=1}^{n} |u|^{1/p} \right]^{p} \left[\sum_{k=1}^{n} |v|^{-1/(p-1)} \right]^{-(p-1)}, \tag{4}$$

holds if $0 < p < 1$ or $p > 1$, $q = -1/(p-1) < 0$. If $p = 1$ and $q = \infty$, the Hölder inequality is given by

$$\sum_{k=1}^{n} |uv| \leq \left(\sum_{k=1}^{n} |u| \right) \left(\sup_{n} |v| \right), \tag{5}$$

if $\sum_{k=1}^{n} |u(k)| < \infty$, and $\sup_{n} |v(n)| < \infty$. Throughout we assume that $u : \mathbb{I} \to \mathbb{R}^{+}$ is positive sequence and define

$$\mathcal{M}_{q} u(n) := \left(\frac{1}{n} \sum_{k=1}^{n} u^{q}(k) \right)^{1/q},$$

for any real number q and any $n \in \mathbb{I}$. Note that, for $q = 0$, the operator $\mathcal{M}_{q}$ takes the form

$$\mathcal{M}_{0} u = \exp \left(\frac{1}{n} \sum_{k=1}^{n} \log u(k) \right).$$

In the following lemma, we state the basic property of the operator $\mathcal{M}_{q} u$ which is proved directly by applying Jensen's inequality.

Lemma 1 *Let u be a positive weight and p and q be real numbers. If $p \leq q$, then $\mathcal{M}_{p} u \leq \mathcal{M}_{q} u$.*

We recall that the discrete positive weight u is said to belong to the discrete Gehring class $\mathcal{G}^{q}(G)$ on the interval $\mathbb{I} \subset \mathbb{Z}^{+}$, for $q > 1$ and $G > 1$, if, for every $n \in \mathbb{I}$,

$$\left(\frac{1}{n} \sum_{k=1}^{n} u^{q} \right)^{1/q} \leq G \left(\frac{1}{n} \sum_{k=1}^{n} u \right). \tag{6}$$

It is clear that the Hölder inequality (in terms of $\mathcal{M}_{q} u$) reads

$$\mathcal{M}_{q} u \geq \mathcal{M}_{1} u, \quad \text{for } q \geq 1. \tag{7}$$

The reverse of (7) is given by (6), which in terms of $\mathcal{M}$ reads

$$\mathcal{M}_{q} u \leq G \mathcal{M}_{1} u, \tag{8}$$

for some constant $G > 1$. A generalization of inequality (8) for $1 < p \leq q$, which we call the generalized reverse Hölder inequality, is given in terms of $\mathcal{M}$ by

$$\mathcal{M}_{q} u \leq G \mathcal{M}_{p} u, \tag{9}$$

for some constant $G > 1$. In [26], the authors proved the following transition properties which gives a transition relation between the class $\mathcal{A}^{p}$ and the class $\mathcal{G}^{q}$.

Theorem 2 *Let u be a positive weight and p and q be real numbers. Then $u \in \mathcal{A}^{p}$ for some p if and only if $u \in \mathcal{G}^{q}$ for some q.*

Remark 1 The equivalence in this theorem gives the transition property between the discrete Muckenhoupt and Gehring classes. The main question which is interesting is what the relation is between p and q for which the inclusions $\mathcal{A}^p \subset \mathcal{G}^q$ and $\mathcal{G}^q \subset \mathcal{A}^p$ hold, and this remains an open problem.

3 Some basic properties of Muckenhoupt weights

In this section, we prove some basic properties of Muckenhoupt weights. The first lemma proves the inclusion of the class $\mathcal{A}^p$ in the class $\mathcal{A}^\infty$ for all $p \geq 1$. That is, if $u \in \mathcal{A}^p$ for $p \geq 1$, then $u \in \mathcal{A}^\infty$.

Lemma 3 *Let u be a positive weight and p be a nonnegative real number. If $u \in \mathcal{A}^p(A)$, then the inequality*

$$\mathcal{M}_1 u \leq A \exp(\mathcal{M}_1 \log u) \tag{10}$$

holds.

Proof Since $u \in \mathcal{A}^p(A)$, then, for all $n \in \mathbb{I}$ and $A > 1$, we have

$$\frac{1}{n}\sum_{k=1}^{n} u(k) \leq A\left(\frac{1}{n}\sum_{k=1}^{n} u^{\frac{1}{1-p}}(k)\right)^{1-p}.$$

By taking the limit as p tends to ∞, then the right hand side, after using the properties of limits and L'Hôpital's rule, becomes

$$\frac{1}{n}\sum_{k=1}^{n} u \leq A \lim_{p\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n} u^{\frac{1}{1-p}}(k)\right)^{1-p} = A \exp\left(\frac{1}{n}\sum_{k=1}^{n} \log u(k)\right),$$

which is the desired inequality (10). The proof is complete. $\qquad\square$

Theorem 4 *Let u be a positive weight and p be a nonnegative real number, and $p' = p/(p-1)$ be the conjugate of p. Then $u \in \mathcal{A}^p$ if and only if $u^{1-p'} \in \mathcal{A}^{p'}$, with*

$$\left[u^{1-p'}\right]_{\mathcal{A}^{p'}} = [u]_{\mathcal{A}^p}^{p'-1}.$$

Proof From the definition of the class $\mathcal{A}^p$, and since $1 - p' = 1/(1-p) < 0$, we have for $A > 1$ and all $n \in \mathbb{I}$

$$u \in \mathcal{A}^p \quad \Leftrightarrow \quad \frac{1}{n}\sum_{k=1}^{n} u(k) \leq A\left(\frac{1}{n}\sum_{k=1}^{n} u^{\frac{1}{1-p}}(k)\right)^{1-p}$$

$$\Leftrightarrow \quad \left(\frac{1}{n}\sum_{k=1}^{n} u(k)\right)^{\frac{1}{1-p}} \geq A^{\frac{1}{1-p}}\frac{1}{n}\sum_{k=1}^{n} u^{\frac{1}{1-p}}(k)$$

$$\Leftrightarrow \quad \frac{1}{n}\sum_{k=1}^{n} u^{1-p'}(k) \leq A^{p'-1}\left(\frac{1}{n}\sum_{k=1}^{n}\left(u^{1-p'}(k)\right)^{\frac{1}{1-p'}}\right)^{1-p'}$$

$$\Leftrightarrow \quad u^{1-p'} \in \mathcal{A}^{p'}.$$

Furthermore, since $\frac{1}{1-p} = 1 - p'$ and $\frac{1}{1-p'} = 1 - p$, we have

$$
\sup_{n\in\mathbb{I}} \left(\frac{1}{n} \sum_{k=1}^{n} u^{1-p'}(k) \right) \left(\frac{1}{n} \sum_{k=1}^{n} (u^{1-p'}(k))^{\frac{1}{1-p'}} \right)^{-(1-p')}
$$

$$
= \sup_{n\in\mathbb{I}} \left(\frac{1}{n} \sum_{k=1}^{n} u^{1/(1-p)}(k) \right)^{(p'-1)(p-1)} \left(\frac{1}{n} \sum_{k=1}^{n} u(k) \right)^{-(1-p')}
$$

$$
= \sup_{n\in\mathbb{I}} \left[\left(\frac{1}{n} \sum_{k=1}^{n} u^{1/(1-p)}(k) \right)^{p-1} \left(\frac{1}{n} \sum_{k=1}^{n} u(k) \right) \right]^{p'-1}.
$$

This is equivalent to

$$
\left[u^{1-p'}\right]_{\mathcal{A}^{p'}} = [u]_{\mathcal{A}^p}^{p'-1},
$$

which is the desired result. The proof is complete.   □

In the next theorem, we prove some basic inclusion properties of Muckenhoupt classes.

Theorem 5 *Let u be a positive weight and p, q be nonnegative real numbers. Then the following inclusion relations hold*:
(1) $\mathcal{A}^1 \subset \mathcal{A}^p \subset \mathcal{A}^\infty$, *for all* $1 < p < \infty$,
(2) $\mathcal{A}^p \subset \mathcal{A}^q$ *for all* $1 < p \le q$,
(3) $\mathcal{A}^\infty = \bigcup_{1<p} \mathcal{A}^p$ *with*

$$
[u]_{\mathcal{A}^\infty} = \lim_{p\to\infty} [u]_{\mathcal{A}^p}
$$

and $\mathcal{A}^1 \subset \bigcap_{p>1} \mathcal{A}^p$.

Proof (1) Let $u \in \mathcal{A}^1$, then there exists $A > 1$ such that, for all $n \in \mathbb{I}$, we have

$$
\frac{1}{n} \sum_{k=1}^{n} u(k) \le A \inf_{1\le k\le n} u(k),
$$

or equivalently

$$
\frac{1}{n} \sum_{k=1}^{n} u(k) \le Au(k), \tag{11}
$$

for all $1 \le k \le n$. For $p > 1$, by using (11) we have, for all $n \in \mathbb{I}$,

$$
\left(\frac{1}{n} \sum_{k=1}^{n} u(k) \right) \left(\frac{1}{n} \sum_{k=1}^{n} u^{\frac{1}{1-p}}(k) \right)^{p-1} \le \left(\frac{1}{n} \sum_{k=1}^{n} u(k) \right) \left(\frac{1}{n} \sum_{k=1}^{n} \left(A^{-1} \frac{1}{n} \sum_{k=1}^{n} u(k) \right)^{\frac{1}{1-p}} \right)^{p-1}
$$

$$
= \left(\frac{1}{n} \sum_{k=1}^{n} u(k) \right) \left(A^{-1} \frac{1}{n} \sum_{k=1}^{n} u(k) \right)^{-1} = A.
$$

Hence $u \in \mathcal{A}^p$, which implies that $\mathcal{A}^1 \subset \mathcal{A}^p$. Now, assume that $u \in \mathcal{A}^p$, then Lemma 3 implies that $u \in \mathcal{A}^\infty$. That is $\mathcal{A}^p \subset \mathcal{A}^\infty$.

(2) Assume that $u \in \mathcal{A}^p$, then there exists $A > 1$ such that for all $n \in \mathbb{I}$

$$\frac{1}{n} \sum_{k=1}^{n} u(k) \left(\frac{1}{n} \sum_{k=1}^{n} u^{\frac{1}{1-p}}(k) \right)^{p-1} \leq A,$$

holds. Now, since $1 < p \leq q$, we see that $\frac{1}{p-1} \geq \frac{1}{q-1}$, and then, by using Lemma 1, we have

$$\mathcal{M}_{\frac{1}{p-1}} u^{-1} \geq \mathcal{M}_{\frac{1}{q-1}} u^{-1}.$$

Then, for all $n \in \mathbb{I}$, we obtain

$$A \geq \frac{1}{n} \sum_{k=1}^{n} u(k) \left(\frac{1}{n} \sum_{k=1}^{n} u^{\frac{1}{1-p}}(k) \right)^{p-1}$$
$$\geq \frac{1}{n} \sum_{k=1}^{n} u(k) \left(\frac{1}{n} \sum_{k=1}^{n} u^{\frac{1}{1-q}}(k) \right)^{q-1},$$

which implies that $u \in \mathcal{A}^q$.

(3) By applying Property (1), for all $1 < p < \infty$, $\mathcal{A}^p \subset \mathcal{A}^\infty$. Then,

$$\bigcup_{1 \leq p < \infty} \mathcal{A}^p \subseteq \mathcal{A}^\infty. \tag{12}$$

Conversely, we shall prove the containment by contradiction. That is, we assume that $u \in \mathcal{A}^\infty$ and assume, on the contrary, that, for all $1 \leq p < \infty$, $u \notin \mathcal{A}^p$. Then, for all $1 \leq p < \infty$, we see that

$$\sup_{n \in \mathbb{I}} \left(\frac{1}{n} \sum_{k=1}^{n} u(k) \right) \left(\frac{1}{n} \sum_{k=1}^{n} u^{\frac{1}{1-p}}(k) \right)^{p-1} = \infty,$$

which, by taking the limit as p tends to ∞ implies that

$$\sup_{n \in \mathbb{I}} \left(\frac{1}{n} \sum_{k=1}^{n} u \right) \left(\exp \frac{1}{n} \sum_{k=1}^{n} \log \frac{1}{u(k)} \right) = \infty.$$

This contradicts the assumption that $u \in \mathcal{A}^\infty$, then $u \in \mathcal{A}^\infty$ implies that, for some $1 \leq p < \infty$, $u \in \mathcal{A}^p$ and hence

$$u \in \bigcup_{1 \leq p < \infty} \mathcal{A}^p.$$

Thus

$$\mathcal{A}^\infty \subseteq \bigcup_{1 \leq p < \infty} \mathcal{A}^p. \tag{13}$$

From (12) and (13), we obtain $\mathcal{A}^\infty = \bigcup_{1 \le p < \infty} \mathcal{A}^p$. Moreover, by applying L'Hôpital's rule and some limit rules, we obtain

$$
\begin{aligned}
\lim_{p \to \infty} [u]_{\mathcal{A}^p} &= \lim_{p \to \infty} \sup_{n \in \mathbb{I}} \left(\frac{1}{n} \sum_{k=1}^n u(k) \right) \left(\frac{1}{n} \sum_{k=1}^n u^{\frac{1}{1-p}}(k) \right)^{p-1} \\
&= \sup_{n \in \mathbb{I}} \lim_{p \to \infty} \left(\frac{1}{n} \sum_{k=1}^n u(k) \right) \left(\frac{1}{n} \sum_{k=1}^n u^{\frac{1}{1-p}}(k) \right)^{p-1} \\
&= \sup_{n \in \mathbb{I}} \left(\frac{1}{n} \sum_{k=1}^n u(k) \right) \left(\exp \frac{1}{n} \sum_{k=1}^n \log \frac{1}{u(k)} \right) \\
&= [u]_{\mathcal{A}^\infty},
\end{aligned}
$$

which is the desired result. Now, assume that $u \in \mathcal{A}^1$. By Property (1), for any $p > 1$, $\mathcal{A}^1 \subset \mathcal{A}^p$, then

$$
\mathcal{A}^1 \subset \bigcap_{p>1} \mathcal{A}^p.
$$

Equality does not hold and to prove it is sufficient to provide an example of a weight u satisfies $u \in \bigcap_{p>1} \mathcal{A}^p \setminus \mathcal{A}^1$. For example: for all $p > 1$, we have $u(n) = n^\alpha \in \mathcal{A}^p$ for $\alpha > 1$ and $u(n) \notin \mathcal{A}^1$. The proof is complete. □

Remark 2 In Theorem 5, we were able to prove the containment $\mathcal{A}^1 \subset \bigcap_{p>1} \mathcal{A}^p$. In the following, we present some weights, which does not only satisfy the containment $\mathcal{A}^1 \subset \bigcap_{p>1} \mathcal{A}^p$ but also satisfy the equalities:

   (i) $u(n) = 1 \in \mathcal{A}^1$ and hence $u(n) \in \mathcal{A}^p$,
   (ii) $u(n) = n^\alpha \in \mathcal{A}^1$ for $\alpha \le 0$ and hence $u(n) \in \mathcal{A}^p$,
   (iii) $u(n) = \frac{1}{\log(n+1)} \in \mathcal{A}^1$ and hence $u(n) \in \mathcal{A}^p$.

In the following theorem, we discuss the power rule for weights in the Muckenhoupt class. That is, we discuss the cases for α which satisfies the necessity of the statement: $u \in \mathcal{A}^p$ implies that $u^\alpha \in \mathcal{A}^p$.

Theorem 6 *Let u be a positive weight, p be a nonnegative real number. Then*

   (1) *if $u \in \mathcal{A}^p$ then $u^\alpha \in \mathcal{A}^p$, for $0 \le \alpha \le 1$, with $[u^\alpha]_{\mathcal{A}^p} \le [u]_{\mathcal{A}^p}^\alpha$,*
   (2) *if $u \in \mathcal{A}^p$, then $u^\alpha \in \mathcal{A}^p$ for some $\alpha > 1$.*

Proof (1) For $0 \le \alpha \le 1$, and $u \in \mathcal{A}^p$, we have $\frac{1}{p-1} \ge \frac{\alpha}{p-1} > 0$, and hence by applying Lemma 1, we have for all $n \in \mathbb{I}$

$$
\begin{aligned}
&\left(\frac{1}{n} \sum_{k=1}^n u^\alpha(k) \right) \left(\frac{1}{n} \sum_{k=1}^n \left(u^\alpha(k) \right)^{\frac{1}{1-p}} \right)^{p-1} \\
&\quad \le \left(\frac{1}{n} \sum_{k=1}^n u(k) \right)^\alpha \left(\frac{1}{n} \sum_{k=1}^n u^{\frac{\alpha}{1-p}}(k) \right)^{p-1}
\end{aligned}
$$

$$\leq \left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)^{\alpha}\left(\frac{1}{n}\sum_{k=1}^{n}u^{\frac{1}{1-p}}(k)\right)^{\alpha(p-1)}$$

$$= \left[\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)\left(\frac{1}{n}\sum_{k=1}^{n}u^{\frac{1}{1-p}}(k)\right)^{(p-1)}\right]^{\alpha} \leq A^{\alpha},$$

that is, $u^{\alpha} \in \mathcal{A}^{p}$, with $[u^{\alpha}]_{\mathcal{A}^{p}} \leq [u]_{\mathcal{A}^{p}}^{\alpha}$, which is the desired result.

(2) Since $u \in \mathcal{A}^{p}(A)$,

$$\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right) \leq A\left(\frac{1}{n}\sum_{k=1}^{n}u^{1/(1-p)}(k)\right)^{1-p}, \quad A \geq 1. \tag{14}$$

Theorem 2 implies that $u \in \mathcal{G}^{q}(G)$. That is,

$$\left(\frac{1}{n}\sum_{k=1}^{n}u^{q}(k)\right)^{1/q} \leq G\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right). \tag{15}$$

Also, Theorem 4 implies that $u^{1/(1-p)} \in \mathcal{A}^{p'}$, again Theorem 2 implies that $u^{1/(1-p)} \in \mathcal{G}^{q_1}(G_1)$. That is,

$$\left(\frac{1}{n}\sum_{k=1}^{n}u^{q_1/(1-p)}(k)\right)^{1/q_1} \leq G_1\left(\frac{1}{n}\sum_{k=1}^{n}u^{1/(1-p)}(k)\right). \tag{16}$$

Now, by applying (14), (15), and (16), we have

$$\frac{1}{n}\sum_{k=1}^{n}u^{q}(k) \leq G^{q}\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)^{q}$$

$$\leq G^{q}\left[A\left(\frac{1}{n}\sum_{k=1}^{n}u^{1/(1-p)}(k)\right)^{1-p}\right]^{q}$$

$$= G^{q}A^{q}\left(\frac{1}{n}\sum_{k=1}^{n}u^{1/(1-p)}(k)\right)^{q(1-p)}$$

$$\leq G^{q}A^{q}\left[G_1^{-1}\left(\frac{1}{n}\sum_{k=1}^{n}u^{q_1/(1-p)}(k)\right)^{1/q_1}\right]^{q(1-p)}$$

$$= G^{q}A^{q}G_1^{q(p-1)}\left(\frac{1}{n}\sum_{k=1}^{n}u^{q_1/(1-p)}(k)\right)^{q(1-p)/q_1}.$$

If $q < q_1$, then, by applying Lemma 1, it satisfies

$$\left(\frac{1}{n}\sum_{k=1}^{n}u^{q_1/(1-p)}(k)\right)^{q(1-p)/q_1} \leq \left(\frac{1}{n}\sum_{k=1}^{n}u^{q/(1-p)}(k)\right)^{(1-p)},$$

which implies that

$$\frac{1}{n}\sum_{k=1}^{n}u^q(k) \leq G^q A^q G_1^{q(p-1)}\left(\frac{1}{n}\sum_{k=1}^{n}u^{q/(1-p)}(k)\right)^{(1-p)},$$

or equivalently, $u^q \in \mathcal{A}^p$. If $q > q_1$, then by using Hölder's inequality we see that the condition $u \in \mathcal{G}^q$ implies that

$$\left(\frac{1}{n}\sum_{k=1}^{n}u^{q_1}(k)\right)^{1/q_1} \leq G_2\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right). \tag{17}$$

By using (14), (16) and (17), we have

$$\frac{1}{n}\sum_{k=1}^{n}u^{q_1}(k) \leq G_2^{q_1}\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)^{q_1}$$

$$\leq G_2^{q_1}\left[A\left(\frac{1}{n}\sum_{k=1}^{n}u^{1/(1-p)}(k)\right)^{1-p}\right]^{q_1}$$

$$= G_2^{q_1}A^{q_1}\left(\frac{1}{n}\sum_{k=1}^{n}u^{1/(1-p)}(k)\right)^{(1-p)q_1}$$

$$\leq G_2^{q_1}A^{q_1}G_1^{p-1}\left(\frac{1}{n}\sum_{k=1}^{n}u^{q_1/(1-p)}(k)\right)^{(1-p)},$$

that is, $u^{q_1} \in \mathcal{A}^p$. This completes our proof. □

Theorem 7 *If $u_1, u_2 \in \mathcal{A}^p$, then $u_1^\alpha u_2^{1-\alpha} \in \mathcal{A}^p$, $0 \leq \alpha \leq 1$, with a constant*

$$\left[u_1^\alpha u_2^{1-\alpha}\right]_{\mathcal{A}^p} \leq [u_1]_{\mathcal{A}^p}^\alpha [u_2]_{\mathcal{A}^p}^{1-\alpha}.$$

Proof Assume that $u_1, u_2 \in \mathcal{A}^p$, then

$$\frac{1}{n}\sum_{k=1}^{n}u_1(k)\left(\frac{1}{n}\sum_{k=1}^{n}u_1^{\frac{1}{1-p}}(k)\right)^{p-1} \leq A_1 \tag{18}$$

and

$$\frac{1}{n}\sum_{k=1}^{n}u_2(k)\left(\frac{1}{n}\sum_{k=1}^{n}u_2^{\frac{1}{1-p}}(k)\right)^{p-1} \leq A_2 \tag{19}$$

hold for all $n \in \mathbb{I}$, where $A_1, A_2 > 1$. By applying Hölder's inequality (note that $0 \leq \alpha \leq 1$) with $1/\alpha$ and $1/(1-\alpha)$ and using (18) and (19), we obtain

$$\frac{1}{n}\sum_{k=1}^{n}u_1^\alpha(k)u_2^{1-\alpha}(k)$$

$$\leq \left(\frac{1}{n}\sum_{k=1}^{n}u_1(k)\right)^\alpha\left(\frac{1}{n}\sum_{k=1}^{n}u_2(k)\right)^{1-\alpha}$$

$$\leq \left(A_1 \left(\frac{1}{n} \sum_{k=1}^{n} u_1^{\frac{1}{1-p}}(k) \right)^{1-p} \right)^{\alpha} \left(A_2 \left(\frac{1}{n} \sum_{k=1}^{n} u_2^{\frac{1}{1-p}}(k) \right)^{1-p} \right)^{1-\alpha}$$

$$= A_1^{\alpha} A_2^{1-\alpha} \left[\left(\frac{1}{n} \sum_{k=1}^{n} u_1^{\frac{1}{1-p}}(k) \right)^{\alpha} \left(\frac{1}{n} \sum_{k=1}^{n} u_2^{\frac{1}{1-p}}(k) \right)^{1-\alpha} \right]^{1-p}. \tag{20}$$

Again, by applying Hölder's inequality for $1/\alpha$ and $1/(1-\alpha)$ on the term

$$\frac{1}{n} \sum_{k=1}^{n} u_1^{\frac{\alpha}{1-p}}(k) u_2^{\frac{1-\alpha}{1-p}}(k),$$

we obtain

$$\frac{1}{n} \sum_{k=1}^{n} u_1^{\frac{\alpha}{1-p}}(k) u_2^{\frac{1-\alpha}{1-p}}(k) \leq \left(\frac{1}{n} \sum_{k=1}^{n} u_1^{\frac{1}{1-p}}(k) \right)^{\alpha} \left(\frac{1}{n} \sum_{k=1}^{n} u_2^{\frac{1}{1-p}}(k) \right)^{1-\alpha}. \tag{21}$$

By substituting (21) into (20) and since $1 - p < 0$, we have

$$\frac{1}{n} \sum_{k=1}^{n} u_1^{\alpha}(k) u_2^{1-\alpha}(k) \leq A_1^{\alpha} A_2^{1-\alpha} \left[\frac{1}{n} \sum_{k=1}^{n} u_1^{\frac{\alpha}{1-p}}(k) u_2^{\frac{1-\alpha}{1-p}}(k) \right]^{1-p}$$

$$= A_1^{\alpha} A_2^{1-\alpha} \left[\frac{1}{n} \sum_{k=1}^{n} \left(u_1^{\alpha}(k) u_2^{1-\alpha}(k) \right)^{\frac{1}{1-p}} \right]^{1-p}.$$

Hence $u_1^{\alpha} u_2^{1-\alpha} \in \mathcal{A}^p$, for $0 \leq \alpha \leq 1$, with

$$\left[u_1^{\alpha} u_2^{1-\alpha} \right]_{\mathcal{A}^p} \leq [u_1]_{\mathcal{A}^p}^{\alpha} [u_2]_{\mathcal{A}^p}^{1-\alpha}.$$

The proof is complete. □

In the next theorem, we discuss the relation between an $\mathcal{A}^p$-weight and the product of two sequences in the $\mathcal{A}^1$-class of weights.

Theorem 8 *$u \in \mathcal{A}^p$ if and only if there exist $u_1, u_2 \in \mathcal{A}^1$ such that $u = u_1 u_2^{1-p}$, $1 < p < \infty$.*

Proof First, we prove that if $u \in \mathcal{A}^p$, then $u = u_1 u_2^{1-p}$ such that $u_1, u_2 \in \mathcal{A}^1$, or equivalently we prove that $\mathcal{A}^p \subset \mathcal{A}^1 (\mathcal{A}^1)^{1-p}$. Assume that $u \notin \mathcal{A}^1 (\mathcal{A}^1)^{1-p}$, then for all u_1 and u_2 satisfying $u = u_1 u_2^{1-p}$, then $u_1 \notin \mathcal{A}^1$ and $u_2 \notin \mathcal{A}^1$. Now, since $\inf_k u(k) \leq u(k)$ for all k. Then

$$\frac{1}{n} \sum_{k=1}^{n} u(k) \left(\frac{1}{n} \sum_{k=1}^{n} u^{\frac{1}{1-p}}(k) \right)^{p-1}$$

$$= \left(\frac{1}{n} \sum_{k=1}^{n} u_1(k) u_2^{1-p}(k) \right) \left(\frac{1}{n} \sum_{k=1}^{n} u_1^{\frac{1}{1-p}}(k) u_2(k) \right)^{p-1}$$

$$\leq \left(\frac{1}{n} \sum_{k=1}^{n} u_1(k) \right) \left(\inf_k u_2(k) \right)^{1-p}$$

$$\times \left(\frac{1}{n} \sum_{k=1}^{n} u_1^{\frac{1}{1-p}}(k) u_2(k) \right)^{p-1} \left(\inf_k u_1(k) \right)^{-1}$$

$$= \left(\frac{1}{\inf_k u_1(k)} \frac{1}{n} \sum_{k=1}^{n} u_1(k) \right) \left(\frac{1}{\inf_k u_2(k)} \frac{1}{n} \sum_{k=1}^{n} u_2(k) \right)^{p-1}.$$

By the assumption that $u_1 \notin \mathcal{A}^1$ and $u_2 \notin \mathcal{A}^1$, then

$$\sup_n \frac{1}{n} \sum_{k=1}^{n} u(k) \left(\frac{1}{n} \sum_{k=1}^{n} u^{\frac{1}{1-p}}(k) \right)^{p-1}$$

$$\leq \sup_n \left(\frac{1}{\inf_k u_1(k)} \frac{1}{n} \sum_{k=1}^{n} u_1(k) \right) \left(\frac{1}{\inf_k u_2(k)} \frac{1}{n} \sum_{k=1}^{n} u_2(k) \right)^{p-1}$$

$$= \infty.$$

That is $u \notin \mathcal{A}^p$. That is $\mathcal{A}^p \subset \mathcal{A}^1 (\mathcal{A}^1)^{1-p}$. Conversely, assume that $u_1, u_2 \in \mathcal{A}^1$, and $1 < p < \infty$, then for all $n \in \mathbb{I}$, we have

$$\frac{1}{n} \sum_{k=1}^{n} u_1(k) \leq A_1 u_1(k), \quad \text{and} \quad \frac{1}{n} \sum_{k=1}^{n} u_2(k) \leq A_2 u_2(k),$$

holds for all $1 \leq k \leq n$. Thus, by letting $u = u_1 u_2^{1-p}$, we have for all $n \in \mathbb{I}$ that

$$\frac{1}{n} \sum_{k=1}^{n} u(k) \left(\frac{1}{n} \sum_{k=1}^{n} u^{\frac{1}{1-p}}(k) \right)^{p-1}$$

$$= \left(\frac{1}{n} \sum_{k=1}^{n} u_1(k) u_2^{1-p}(k) \right) \left(\frac{1}{n} \sum_{k=1}^{n} u_1^{\frac{1}{1-p}}(k) u_2(k) \right)^{p-1}$$

$$\leq \left[\frac{1}{n} \sum_{k=1}^{n} u_1(k) \left(\frac{1}{A_2} \frac{1}{n} \sum_{k=1}^{n} u_2(k) \right)^{1-p} \right]$$

$$\times \left[\frac{1}{n} \sum_{k=1}^{n} \left(\frac{1}{A_1} \frac{1}{n} \sum_{k=1}^{n} u_1(k) \right)^{\frac{1}{1-p}} u_2(k) \right]^{p-1}$$

$$= A_1 A_2^{p-1} \left[\left(\frac{1}{n} \sum_{k=1}^{n} u_1(k) \right) \left(\frac{1}{n} \sum_{k=1}^{n} u_2(k) \right)^{1-p} \right]$$

$$\times \left[\left(\frac{1}{n} \sum_{k=1}^{n} u_1(k) \right)^{-1} \left(\frac{1}{n} \sum_{k=1}^{n} u_2(k) \right)^{p-1} \right]$$

$$= A_1 A_2^{p-1},$$

that is $u = u_1 u_2^{1-p} \in \mathcal{A}^p$, with $[u]_{\mathcal{A}^p} = [u_1]_{\mathcal{A}^1} [u_2]_{\mathcal{A}^1}^{p-1}$. The proof is complete. $\qquad \square$

Theorem 9 *Let u be a positive weight and p be a nonnegative real number. Then following statement holds: $u \in \mathcal{A}^p$ if and only if u and $u^{\frac{1}{1-p}}$ are in $\mathcal{A}^\infty$.*

Proof We prove this property by using by Property (3) in Theorem 5. Since

$$\mathcal{A}^{\infty} = \bigcup_{1 \le p < \infty} \mathcal{A}^p,$$

it is clear that $u \in \mathcal{A}^p$, for any $p > 1$, if and only if $u \in \mathcal{A}^{\infty}$. Now, we have by Theorem 4 that $u \in \mathcal{A}^p$ if and only if $u^{1-p'} = u^{\frac{1}{1-p}} \in \mathcal{A}^{p'}$. That is, by Property (1) in Theorem 5 ($\mathcal{A}^{p'} \subset \mathcal{A}^{\infty}$), $u \in \mathcal{A}^p$ if and only if $u^{\frac{1}{1-p}} \in \mathcal{A}^{\infty}$. The proof is complete.   □

The next theorem is a self-improving property of weights in the Muckenhoupt class.

Theorem 10 *Let u be a positive weight, p be a nonnegative number. If $u \in \mathcal{A}^p$, $p > 1$, then $u \in \mathcal{A}^{p-\epsilon}$, for some $\epsilon > 0$.*

Proof Let $u \in \mathcal{A}^p$, for $p > 1$, then, for $A > 1$ and all $n \in \mathbb{I}$, we have

$$\frac{1}{n} \sum_{k=1}^{n} u(k) \le A \left(\frac{1}{n} \sum_{k=1}^{n} u^{-p'/p}(k) \right)^{-p/p'}. \tag{22}$$

By Theorem 4, $u^{-p'/p} = u^{1-p'} \in \mathcal{A}^{p'}$. Also, by Theorem 2, $u^{-p'/p} \in \mathcal{G}^q$ for some q, or equivalently

$$\left(\frac{1}{n} \sum_{k=1}^{n} u^{-p'q/p}(k) \right)^{1/q} \le G \frac{1}{n} \sum_{k=1}^{n} u^{-p'/p}(k). \tag{23}$$

By using (22) and (23), we have

$$\left(\frac{1}{n} \sum_{k=1}^{n} u^{-p'q/p}(k) \right)^{1/q} \le G \left(A^{-1} \frac{1}{n} \sum_{k=1}^{n} u(k) \right)^{-p'/p},$$

and then

$$\frac{1}{n} \sum_{k=1}^{n} u(k) \le G^{p/p'} A \left(\frac{1}{n} \sum_{k=1}^{n} u^{-p'q/p}(k) \right)^{-p/p'q}.$$

This follows on taking $p - \epsilon - 1 = p/(p'q)$, or equivalently, taking $\epsilon = \frac{p-1}{q'}$. Then $u \in \mathcal{A}^{p-\epsilon}$ for $p > 1$ and some $\epsilon > 0$. The proof is complete.   □

4 Some basic properties of Gehring weights

In this section, we prove some basic properties of Gehring weights. In the next lemma, we present the inclusion relation of Gehring classes $\mathcal{G}^q$ in $\mathcal{G}^1$-class of weights for all $1 < q < \infty$.

Lemma 11 *Let u be a positive weight and q be a nonnegative number. If $u \in \mathcal{G}^q$ for $q > 1$, then*

$$\exp \left(\frac{1}{n} \sum_{k=1}^{n} \frac{u(k)}{\frac{1}{n} \sum_{k=1}^{n} u(k)} \log \left(\frac{u(k)}{\frac{1}{n} \sum_{k=1}^{n} u(k)} \right) \right) < \infty,$$

holds.

Proof Assume that $u \in \mathcal{G}^q$, then there exists $G > 1$ such that

$$\left[\left(\frac{1}{n} \sum_{k=1}^{n} u^q(k) \right)^{1/q} \left(\frac{1}{n} \sum_{k=1}^{n} u(k) \right)^{-1} \right]^{q/(q-1)} \leq G,$$

holds for all $n \in \mathbb{I}$, or equivalently

$$\left(\frac{1}{n} \sum_{k=1}^{n} \left(\frac{u(k)}{\frac{1}{n} \sum_{k=1}^{n} u(k)} \right)^q \right)^{1/(q-1)} \leq G,$$

holds for all $n \in \mathbb{I}$. By taking the limit as q tends to 1 and applying L'Hôpital's rule, we obtain

$$G \geq \lim_{q \to 1} \left(\frac{1}{n} \sum_{k=1}^{n} \left(\frac{u(k)}{\frac{1}{n} \sum_{k=1}^{n} u(k)} \right)^q \right)^{1/(q-1)}$$

$$= \exp \left(\frac{1}{n} \sum_{k=1}^{n} \frac{u(k)}{\frac{1}{n} \sum_{k=1}^{n} u(k)} \log \left(\frac{u(k)}{\frac{1}{n} \sum_{k=1}^{n} u(k)} \right) \right).$$

The proof is complete.   □

In the next theorem, we present some basic inclusion properties of weights in the Gehring class.

Theorem 12 *Let u be a positive weight and p and q be real nonnegative numbers such that $p, q > 1$. The following properties hold*:
  (1) $\mathcal{G}^\infty \subset \mathcal{G}^q \subset \mathcal{G}^1$ *for all* $1 < q \leq \infty$.
  (2) $\mathcal{G}^q \subset \mathcal{G}^p$ *for all* $1 < p \leq q$.
  (3) $\mathcal{G}^1 = \bigcup_{1 < q \leq \infty} \mathcal{G}^q$ *with* $[u]_{\mathcal{G}^1} = \lim_{q \to 1} [u]_{\mathcal{G}^q}$.

Proof (1) Assume that $u \in \mathcal{G}^\infty$, then, by the definition of $\mathcal{G}^\infty$, there exists $0 < C < \infty$ such that, for all $n \in \mathbb{I}$, we have

$$\sup_{1 \leq k \leq n} u(k) \leq C \left(\frac{1}{n} \sum_{k=1}^{n} u(k) \right),$$

or equivalently

$$u(k) \leq C \left(\frac{1}{n} \sum_{k=1}^{n} u(k) \right), \tag{24}$$

for all $1 \leq k \leq n$. By applying (24) for all $1 < q < \infty$, we have

$$\left(\frac{1}{n} \sum_{k=1}^{n} u^q(k) \right)^{1/q} \left(\frac{1}{n} \sum_{k=1}^{n} u(k) \right)^{-1}$$

$$\leq \left[\frac{1}{n} \sum_{k=1}^{n} \left(C \left(\frac{1}{n} \sum_{k=1}^{n} u(k) \right) \right)^q \right]^{1/q} \left(\frac{1}{n} \sum_{k=1}^{n} u(k) \right)^{-1}$$

$$= C \left(\frac{1}{n} \sum_{k=1}^{n} u(k) \right) \left(\frac{1}{n} \sum_{k=1}^{n} u(k) \right)^{-1} = C.$$

That is, $u \in \mathcal{G}^q$ and hence $\mathcal{G}^\infty \subset \mathcal{G}^q$. Now, the inclusion $\mathcal{G}^q \subset \mathcal{G}^1$ is proved in Lemma 11. This is the desired result.

(2) Assume $u \in \mathcal{G}^q$. Then there exists $G > 1$ such that, for all $n \in \mathbb{I}$,

$$\left(\frac{1}{n} \sum_{k=1}^{n} u^q(k) \right)^{1/q} \le G \frac{1}{n} \sum_{k=1}^{n} u(k). \tag{25}$$

Lemma 1 implies that $\mathcal{M}_p u \le \mathcal{M}_q u$ for all $p \le q$. Then, by substituting in (25), we have

$$\left(\frac{1}{n} \sum_{k=1}^{n} u^p(k) \right)^{1/p} \le \left(\frac{1}{n} \sum_{k=1}^{n} u^q(k) \right)^{1/q} \le G \frac{1}{n} \sum_{k=1}^{n} u(k).$$

That is, $u \in \mathcal{G}^p$, which completes the proof of the second case.

(3) Property (1) states that, for all $1 < q \le \infty$, we have $\mathcal{G}^q \subset \mathcal{G}^1$, and then

$$\bigcup_{1 < q \le \infty} \mathcal{G}^q \subseteq \mathcal{G}^1. \tag{26}$$

Conversely, assume that $u \in \mathcal{G}^1$ and assume, on the contrary, for all $1 < q \le \infty$, that $u \notin \mathcal{G}^q$. That is, for all $n \in \mathbb{I}$,

$$\sup_{n \in \mathbb{I}} \left[\left(\frac{1}{n} \sum_{k=1}^{n} u^q(k) \right)^{1/q} \left(\frac{1}{n} \sum_{k=1}^{n} u(k) \right)^{-1} \right]^{q/(q-1)} = \infty. \tag{27}$$

By taking the limit of both sides of (27) as q tends to 1, we have

$$\sup_{n \in \mathbb{I}} \left[\exp \left(\frac{1}{n} \sum_{k=1}^{n} \left(\frac{u(k)}{\frac{1}{n} \sum_{k=1}^{n} u(k)} \right) \log \left(\frac{u(k)}{\frac{1}{n} \sum_{k=1}^{n} u(k)} \right) \right) \right] = \infty.$$

This contradicts the assumption $u \in \mathcal{G}^1$, which implies that, for some $1 < q \le \infty$, $u \in \mathcal{G}^q$, and

$$\mathcal{G}^1 \subseteq \bigcup_{1 < q \le \infty} \mathcal{G}^q. \tag{28}$$

By (26) and (28), we have $\mathcal{G}^1 = \bigcup_{1 < q \le \infty} \mathcal{G}^q$. Furthermore, by applying L'Hôpital's rule and limit rules, we have

$$\lim_{q \to 1} [u]_{\mathcal{G}^q} = \lim_{q \to 1} \sup_{n \in \mathbb{I}} \left[\left(\frac{1}{n} \sum_{k=1}^{n} u^q(k) \right)^{1/q} \left(\frac{1}{n} \sum_{k=1}^{n} u(k) \right)^{-1} \right]^{q/(q-1)}$$

$$= \sup_{n \in \mathbb{I}} \lim_{q \to 1} \left[\left(\frac{1}{n} \sum_{k=1}^{n} u^q(k) \right)^{1/q} \left(\frac{1}{n} \sum_{k=1}^{n} u(k) \right)^{-1} \right]^{q/(q-1)}$$

$$= \sup_{n \in \mathbb{I}} \left[\exp \left(\frac{1}{n} \sum_{k=1}^{n} \left(\frac{u(k)}{\frac{1}{n} \sum_{k=1}^{n} u(k)} \right) \log \left(\frac{u(k)}{\frac{1}{n} \sum_{k=1}^{n} u(k)} \right) \right) \right]$$

$$= [u]_{\mathcal{G}^1},$$

which is the desired result of the case (3). The proof is complete. $\square$

Theorem 13 *Let u be a positive weight, p be a nonnegative number. If $u \in \mathcal{A}^p$, then $u^{-1} \in \mathcal{G}^{p'-1}$.*

Proof Assume that $u \in \mathcal{A}^p$, then

$$\frac{1}{n} \sum_{k=1}^{n} u(k) \leq A \left(\frac{1}{n} \sum_{k=1}^{n} u^{1/(1-p)}(k) \right)^{1-p}. \tag{29}$$

By Lemma 1, we have $\mathcal{M}_{-1} u \leq \mathcal{M}_1 u$, then (29) becomes

$$\left(\frac{1}{n} \sum_{k=1}^{n} \left(u^{-1}(k) \right)^{1/(p-1)} \right)^{p-1} \leq A \frac{1}{n} \sum_{k=1}^{n} u^{-1}(k).$$

That is, $u^{-1} \in \mathcal{G}^{p'-1}$. This completes our proof. $\square$

In the next theorem, we prove the self-improving property of Gehring classes.

Theorem 14 *Let u be a positive weight, q be a nonnegative number. If $u \in \mathcal{G}^q$ for $q > 1$, then $u \in \mathcal{G}^{q+\epsilon}$, for some $\epsilon > 0$.*

Proof Assume that $u \in \mathcal{G}^q$ for $q > 1$, then

$$\left(\frac{1}{n} \sum_{k=1}^{n} u^q(k) \right)^{1/q} \leq G \frac{1}{n} \sum_{k=1}^{n} u(k).$$

By applying Theorem 2 we get $u \in \mathcal{A}^p$ for some p, and property (2) in Theorem 6 implies that $u^\alpha \in \mathcal{A}^p$ for $\alpha > 1$ the smallest number satisfying $u \in \mathcal{G}^\alpha$ and $u^{1/(1-p)} \in \mathcal{G}^\alpha$ (clearly, $q > \alpha$). Again, by applying Theorem 2, then $u^\alpha \in \mathcal{G}^s$ for some s. Without loss of generality, we choose the largest s satisfying $u^\alpha \in \mathcal{G}^s(A)$. Then, by using the condition $u \in \mathcal{G}^\alpha(B)$, it satisfies

$$\left(\frac{1}{n} \sum_{k=1}^{n} u^{\alpha s}(k) \right)^{1/\alpha s} \leq A^{1/\alpha} \left(\frac{1}{n} \sum_{k=1}^{n} u^\alpha(k) \right)^{1/\alpha} \leq A^{1/\alpha} B \frac{1}{n} \sum_{k=1}^{n} u(k).$$

That is, $u \in \mathcal{G}^{\alpha s}$. The cases $q < s$ and ($q > s$ with $q < \alpha s$) are the only valid cases of s and q as otherwise there exists $q/\alpha > s$ satisfying the condition $u^\alpha \in \mathcal{G}^{q/\alpha}$, which contradicts the assumption. Then $u \in \mathcal{G}^{q+\epsilon}$ for some $\epsilon = \alpha s - q > 0$. The proof is complete. $\square$

The next theorem presents the relation between the two classes $\mathcal{G}^1$ and $\mathcal{A}^\infty$.

Theorem 15 *The statement*

$$\mathcal{G}^1 = \mathcal{A}^\infty = \bigcup_{1 < p \leq \infty} \mathcal{A}^p = \bigcup_{1 < q \leq \infty} \mathcal{G}^q$$

holds.

Proof Assume that $u \in \mathcal{G}^1$, then, by Property (3) in Theorem 12, $u \in \mathcal{G}^q$ for some $1 < q \leq \infty$. By Theorem 2, this is true if and only if $u \in \mathcal{A}^p$ for some p, which implies that $u \in \mathcal{A}^\infty$. Hence, it is clear that $\mathcal{A}^\infty = \mathcal{G}^1$. The proof is complete. □

5 Some fundamental relations

In this section, we prove some transition and inclusion relations involving the Muckenhoupt $\mathcal{A}^\infty$ and Gehring $\mathcal{G}^q$ classes.

Theorem 16 *Let u be a positive weight and p_0 be a real number. Then $u \in \mathcal{G}^{p_0}$ if and only if $u^{p_0} \in \mathcal{A}^\infty$.*

Proof We start by assuming that $u^{p_0} \in \mathcal{A}^\infty = \bigcup_{p < \infty} \mathcal{A}^p$ for a fixed $1 < p_0 < \infty$, then $u^{p_0} \in \mathcal{A}^p$, for some $1 < p < \infty$. That is, for all $n \in \mathbb{I}$, there exists $1 < A < \infty$, such that

$$\frac{1}{n} \sum_{k=1}^n u^{p_0} \leq A \left(\frac{1}{n} \sum_{k=1}^n u^{\frac{p_0}{1-p}} \right)^{1-p}. \tag{30}$$

Since $p_0/(1-p) < 0 < 1$, Lemma 1 implies that

$$\left(\frac{1}{n} \sum_{k=1}^n u^{\frac{p_0}{1-p}} \right)^{\frac{1-p}{p_0}} \leq \frac{1}{n} \sum_{k=1}^n u. \tag{31}$$

By substituting (31) into (30), we have

$$\frac{1}{n} \sum_{k=1}^n u^{p_0} \leq A \left(\frac{1}{n} \sum_{k=1}^n u \right)^{p_0},$$

or equivalently

$$\left(\frac{1}{n} \sum_{k=1}^n u^{p_0} \right)^{1/p_0} \leq A^{1/p_0} \left(\frac{1}{n} \sum_{k=1}^n u \right).$$

Hence, $u \in \mathcal{G}^{p_0}$. Conversely, if we assume that $u \in \mathcal{G}^{p_0}$, for a fixed $1 < p_0 < \infty$. Then there exists $1 < G < \infty$, such that for all $n \in \mathbb{I}$

$$\left(\frac{1}{n} \sum_{k=1}^n u^{p_0} \right)^{1/p_0} \leq G \left(\frac{1}{n} \sum_{k=1}^n u \right). \tag{32}$$

Theorem [14] implies that $u \in \mathcal{A}^\infty$. That is, there exists $1 < A < \infty$, such that, for all $n \in \mathbb{I}$, u satisfies

$$\left(\frac{1}{n} \sum_{k=1}^{n} u \right) \left(\exp \frac{1}{n} \sum_{k=1}^{n} \log \frac{1}{u} \right) \leq A. \tag{33}$$

Then, by using (32) and (33), since $p_0 > 1$, we have

$$\begin{aligned}
\left(\frac{1}{n} \sum_{k=1}^{n} u^{p_0} \right) & \left(\exp \frac{1}{n} \sum_{k=1}^{n} \log \frac{1}{u^{p_0}} \right) \\
&= \left(\frac{1}{n} \sum_{k=1}^{n} u^{p_0} \right) \left(\exp \frac{1}{n} \sum_{k=1}^{n} \log \frac{1}{u} \right)^{p_0} \\
&\leq G^{p_0} \left(\frac{1}{n} \sum_{k=1}^{n} u \right)^{p_0} \left(\exp \frac{1}{n} \sum_{k=1}^{n} \log \frac{1}{u} \right)^{p_0} \leq (GA)^{p_0}.
\end{aligned}$$

That is, $u^{p_0} \in \mathcal{A}^\infty$, which completes our proof. $\qquad\square$

Now, we give a quantitative version of the above theorem, which gives some relations between the norms of the classes $\mathcal{A}^\infty$ and $\mathcal{G}^p$.

Theorem 17 *Let u be a positive weight and $1 < p < \infty$. Then*

$$\frac{[u^p]_{\mathcal{A}^\infty}^{1/p}}{[u]_{\mathcal{A}^\infty}} \leq [u]_{\mathcal{G}^p}^{1/p'} \leq [u^p]_{\mathcal{A}^\infty}^{1/p}. \tag{34}$$

Proof By the definition of $[u]_{\mathcal{G}^p}$, we have for all $n \in \mathbb{I}$

$$\frac{1}{n} \sum_{k=1}^{n} u^p \leq [u]_{\mathcal{G}^p}^{p/p'} \left[\frac{1}{n} \sum_{k=1}^{n} u \right]^p.$$

By multiplying both sides by $((1/n) \sum_{k=1}^{n} u^{p(1-q')})^{q-1}$, for $q < \infty$, we obtain

$$\frac{1}{n} \sum_{k=1}^{n} u^p \left(\frac{1}{n} \sum_{k=1}^{n} u^{p(1-q')} \right)^{q-1} \leq [u]_{\mathcal{G}^p}^{p/p'} \left[\frac{1}{n} \sum_{k=1}^{n} u \left(\frac{1}{n} \sum_{k=1}^{n} u^{p(1-q')} \right)^{\frac{q-1}{p}} \right]^p. \tag{35}$$

By taking the supremum of (35) over all $n \in \mathbb{I}$, we have

$$\sup_{n \in \mathbb{I}} \frac{1}{n} \sum_{k=1}^{n} u^p \left(\frac{1}{n} \sum_{k=1}^{n} u^{p(1-q')} \right)^{q-1} \leq [u]_{\mathcal{G}^p}^{p/p'} \sup_{n \in \mathbb{I}} \left[\frac{1}{n} \sum_{k=1}^{n} u \left(\frac{1}{n} \sum_{k=1}^{n} u^{p(1-q')} \right)^{\frac{q-1}{p}} \right]^p,$$

that is,

$$[u^p]_{\mathcal{A}^q} \leq [u]_{\mathcal{G}^p}^{p/p'} \left(\mathcal{A}_{\frac{p+q-1}{p}}(u) \right)^p. \tag{36}$$

By taking the limit as q tends to ∞, we have

$$\frac{[u^p]_{\mathcal{A}^\infty}^{1/p}}{[u]_{\mathcal{A}^\infty}} \le [u]_{\mathcal{G}^p}^{1/p'}, \tag{37}$$

which is the first inequality. For the second inequality, by the definition of $[u^p]_{\mathcal{A}^q}^{1/p}$, we have for all $n \in \mathbb{I}$

$$\frac{(\frac{1}{n}\sum_{k=1}^n u^p)^{1/p}}{\frac{1}{n}\sum_{k=1}^n u} = \frac{(\frac{1}{n}\sum_{k=1}^n u^p)^{1/p}(\frac{1}{n}\sum_{k=1}^n u^{\frac{p}{1-q}})^{\frac{q-1}{p}}}{\frac{1}{n}\sum_{k=1}^n u(\frac{1}{n}\sum_{k=1}^n u^{\frac{p}{1-q}})^{\frac{q-1}{p}}}. \tag{38}$$

Now, by Jensen's inequality since $p/(1-q) < 0 < 1$, we see that

$$\frac{1}{n}\sum_{k=1}^n u^{\frac{p}{1-q}} \ge \left(\frac{1}{n}\sum_{k=1}^n u\right)^{\frac{p}{1-q}},$$

which implies that

$$\frac{1}{\frac{1}{n}\sum_{k=1}^n u(\frac{1}{n}\sum_{k=1}^n u^{\frac{p}{1-q}})^{\frac{q-1}{p}}} \le \frac{1}{\frac{1}{n}\sum_{k=1}^n u((\frac{1}{n}\sum_{k=1}^n u)^{\frac{p}{1-q}})^{\frac{q-1}{p}}} \le 1.$$

By using this in (38), we get

$$\frac{(\frac{1}{n}\sum_{k=1}^n u^p)^{1/p}}{\frac{1}{n}\sum_{k=1}^n u} \le \left(\frac{1}{n}\sum_{k=1}^n u^p \left(\frac{1}{n}\sum_{k=1}^n u^{\frac{p}{1-q}}\right)^{q-1}\right)^{1/p}.$$

Taking the supremum of (38) over all $n \in \mathbb{I}$, we have $[u]_{\mathcal{G}^p}^{1/p'} \le [u^p]_{\mathcal{A}^q}^{1/p}$. By taking the limit as q tends to ∞, we have

$$[u]_{\mathcal{G}^p}^{1/p'} \le \left[u^p\right]_{\mathcal{A}^\infty}^{1/p}.$$

The proof is complete. $\qquad\square$

Theorem 18 *Let u be a positive weight and $1 < p < \infty$. Then*

$$\max\left\{[u]_{\mathcal{A}^\infty}, \left[u^{1-p'}\right]_{\mathcal{A}^\infty}^{p-1}\right\} \le [u]_{\mathcal{A}^p} \le [u]_{\mathcal{A}^\infty}\left[u^{1-p'}\right]_{\mathcal{A}^\infty}^{p-1}. \tag{39}$$

Proof For $p \le q$, we have $[u]_{\mathcal{A}^p} \ge [u]_{\mathcal{A}^q}$, and thus

$$[u]_{\mathcal{A}^\infty} \le [u]_{\mathcal{A}^p}. \tag{40}$$

Furthermore, for $q < \infty$, we have

$$
\left[u^{1-p'}\right]_{\mathcal{A}^q}^{p-1} = \sup_{n \in \mathbb{I}} \left\{ \frac{1}{n} \sum_{k=1}^{n} u^{1-p'} \left(\frac{1}{n} \sum_{k=1}^{n} u^{(1-p')(1-q')} \right)^{q-1} \right\}^{p-1}
$$

$$
= \sup_{n \in \mathbb{I}} \left\{ \frac{\frac{1}{n} \sum_{k=1}^{n} u \left(\frac{1}{n} \sum_{k=1}^{n} u^{1-p'} \right)^{p-1}}{\frac{1}{n} \sum_{k=1}^{n} u \left(\frac{1}{n} \sum_{k=1}^{n} u^{(1-p')(1-q')} \right)^{-(1-p)(1-q)}} \right\}
$$

$$
\leq \sup_{n \in \mathbb{I}} \frac{1}{n} \sum_{k=1}^{n} u \left(\frac{1}{n} \sum_{k=1}^{n} u^{1-p'} \right)^{p-1} = [u]_{\mathcal{A}^p}. \tag{41}
$$

By taking the limit of (41) as q tends to ∞, we have

$$
\left[u^{1-p'}\right]_{\mathcal{A}^\infty}^{p-1} \leq [u]_{\mathcal{A}^p}. \tag{42}
$$

By (40) and (42), then

$$
\max \left\{ [u]_{\mathcal{A}^\infty}, \left[u^{1-p'}\right]_{\mathcal{A}^\infty}^{p-1} \right\} \leq [u]_{\mathcal{A}^p}.
$$

Now, for the second inequality, we have

$$
\frac{1}{n} \sum_{k=1}^{n} u \left(\frac{1}{n} \sum_{k=1}^{n} u^{1-p'} \right)^{p-1} = \frac{1}{n} \sum_{k=1}^{n} u \left(\frac{1}{n} \sum_{k=1}^{n} u^{1-q'} \right)^{q-1}
$$

$$
\times \left(\frac{1}{n} \sum_{k=1}^{n} u^{1-p'} \left(\frac{1}{n} \sum_{k=1}^{n} u^{1-q'} \right)^{\frac{1-q}{p-1}} \right)^{p-1}. \tag{43}
$$

Since $1 - q$ and $1 - q' < 0$, and since (see Lemma 1)

$$
\left(\frac{1}{n} \sum_{k=1}^{n} u^{-1} \right)^{-1} \leq \frac{1}{n} \sum_{k=1}^{n} u,
$$

we have

$$
\frac{1}{n} \sum_{k=1}^{n} u^{1-p'} \left(\frac{1}{n} \sum_{k=1}^{n} u^{1-q'} \right)^{\frac{1-q}{p-1}} \leq \frac{1}{n} \sum_{k=1}^{n} u^{1-p'} \left(\frac{1}{n} \sum_{k=1}^{n} u^{q'-1} \right)^{\frac{q-1}{p-1}}
$$

$$
= \frac{1}{n} \sum_{k=1}^{n} u^{1-p'} \left(\frac{1}{n} \sum_{k=1}^{n} u^{(1-p')(q'-1)/(1-p')} \right)^{\frac{q-1}{p-1}}. \tag{44}
$$

By setting $r - 1 = (q - 1)/(p - 1)$, we have

$$
r' - 1 = \frac{1}{r-1} = \frac{p-1}{q-1} = \frac{q'-1}{p'-1}.
$$

Hence, from (43) and (44), we have

$$\frac{1}{n}\sum_{k=1}^{n}u\left(\frac{1}{n}\sum_{k=1}^{n}u^{1-p'}\right)^{p-1} \leq \frac{1}{n}\sum_{k=1}^{n}u\left(\frac{1}{n}\sum_{k=1}^{n}u^{1-q'}\right)^{q-1}$$
$$\times\left[\frac{1}{n}\sum_{k=1}^{n}u^{1-p'}\left(\frac{1}{n}\sum_{k=1}^{n}u^{(1-p')(1-r')}\right)^{r-1}\right]^{p-1}.$$

Thus by taking the supremum over all $n \in \mathbb{I}$, we have

$$[u]_{\mathcal{A}^p} \leq [u]_{\mathcal{A}^q}\left[u^{1-p'}\right]_{\mathcal{A}^r}^{p-1}. \tag{45}$$

By taking the limit of both sides of (45) as q tends to ∞, we have

$$[u]_{\mathcal{A}^p} \leq [u]_{\mathcal{A}^\infty}\left[u^{1-p'}\right]_{\mathcal{A}^\infty}^{p-1}.$$

The proof is complete. □

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